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Bending of the Torses by Changing the Regularity of the Reverse Edge Angle of Ascent

Serhiy Pylypaka[1], Tetiana Volina[1], Iryna Hryshenko[1], Oleksandra Trokhaniak[1], Iryna Taras[2]

[1] *University of Life and Environmental Sciences of Ukraine, Kyiv, Ukraine*
ps55@ukr.net, t.n.zaharova@ukr.net, hryshchenko@nubip.edu.ua, klendii_o@ukr.net

[2] *Ivano-Frankivsk National Technical University of Oil and Gas, Ivano-Frankivsk, Ukraine*
iryna.taras@nung.edu.ua

Abstract. A spatial curve can be described by two natural equations: the curvature and the torsion dependencies of its arc length. If such a curve is taken to be a torse reverse edge, its bending can be controlled by changing the curve's torsion, since the curvature does not change. However, in practice, such bending is difficult to perform, since there is no simple transition from the natural equations of the spatial curve to the parametric ones. This transition requires numerical methods for solving a system of differential equations. Another way to solve this question is to replace the dependency of the torsion of the arc length of the curve with the dependency of the angle of ascent and also of the arc length of the curve. In this case, the formulas for the transition from natural to parametric equations become much simpler and, in some cases, do not require numerical integration. This approach is used in the article for the construction of torses. The parametric equations of the torse in the general form are presented, for which the reverse edge is a spatial curve defined by the dependencies of the curvature and the angle of ascent of the length of its arc. It is shown that by changing the regularity of the angle of ascent, the process of the torse bending can be controlled. Examples are given, and the results are visualized.

Key Words: spatial curve, natural and parametric equations, torse, reverse edge, bending

1 Introduction

A spatial curve is considered to be defined if two natural equations are known: the dependency of curvature k and torsion σ of the arc length s: $k = k(s)$ and $\sigma = \sigma(s)$. To plot this curve

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in the Cartesian coordinate system, it is necessary to transform to parametric equations. For this, the kinematics of the accompanying Frenet trihedron can be used. During its movement along the curve, for the current value of the arc length s, there is a vector ω of the instantaneous axis of rotation of the trihedron. Its projections onto the trihedron's orthogonal axes can be determined by means of Euler's kinematic equations [1]:

$$\begin{aligned} \omega_\tau &= \psi' \sin\theta \sin\varphi + \theta' \cos\varphi; \\ \omega_n &= \psi' \sin\theta \cos\varphi - \theta' \sin\varphi; \\ \omega_b &= \psi' \cos\theta + \varphi', \end{aligned} \tag{1}$$

where φ, ψ, θ are Euler's angles. It is known from differential geometry [7] that $\omega_\tau = \sigma$; $\omega_n = 0$, and $\omega_b = k$. After substituting these values into (1) and solving regarding φ', ψ', and θ', it is obtained:

$$\frac{\mathrm{d}\psi}{\mathrm{d}s} = \sigma \frac{\sin\varphi}{\sin\theta}; \qquad \frac{\mathrm{d}\theta}{\mathrm{d}s} = \sigma \cos\varphi; \qquad \frac{\mathrm{d}\varphi}{\mathrm{d}s} = k - \sigma \frac{\sin\varphi}{\operatorname{tg}\theta}. \tag{2}$$

Even for the simplest case, when k and σ are constant (i.e., a given curve is a helical line), the differential equations (2) are impossible to integrate. Therefore, the dependencies $\varphi = \varphi(s)$, $\psi = \psi(s)$, and $\theta = \theta(s)$ must be found by numerical integration. The guide curve itself can be found by further numerical integration of known expressions [3]:

$$\begin{aligned} x &= \int (\cos\psi \cos\varphi - \sin\psi \sin\varphi \cos\theta)\, \mathrm{d}s; \\ y &= \int (\sin\psi \cos\varphi + \cos\psi \sin\varphi \cos\theta)\, \mathrm{d}s; \\ z &= \int \sin\varphi \sin\theta \, \mathrm{d}s. \end{aligned} \tag{3}$$

There are other approaches to the construction of curves in space [2, 6, 8] and on the surface [4]. The paper [5] proposes a transition from the equations $k = k(s)$ and $\beta = \beta(s)$, where β is the angle of ascent of the curve, to parametric equations:

$$\begin{aligned} x &= \int \cos\Big(\int \frac{\sqrt{k^2 - \beta'^2}}{\cos\beta}\, \mathrm{d}s\Big) \cos\beta \, \mathrm{d}s; \\ y &= \int \sin\Big(\int \frac{\sqrt{k^2 - \beta'^2}}{\cos\beta}\, \mathrm{d}s\Big) \cos\beta \, \mathrm{d}s; \\ z &= \int \sin\beta \, \mathrm{d}s. \end{aligned} \tag{4}$$

For some dependencies $k = k(s)$ and $\beta = \beta(s)$, the parametric equations (4) can be integrated, i.e., obtained in their final form.

2 Construction of the Spatial Curve According to the Given Dependencies *k=k(s)* and β=β*(s)*

At $\beta =$ const., the parametric equations (4) describe a slope curve with a constant lifting angle. A slope curve with constant curvature $k =$ const. is a helical line with a constant pitch. Let's consider the construction of a curve with a constant curvature and a linear

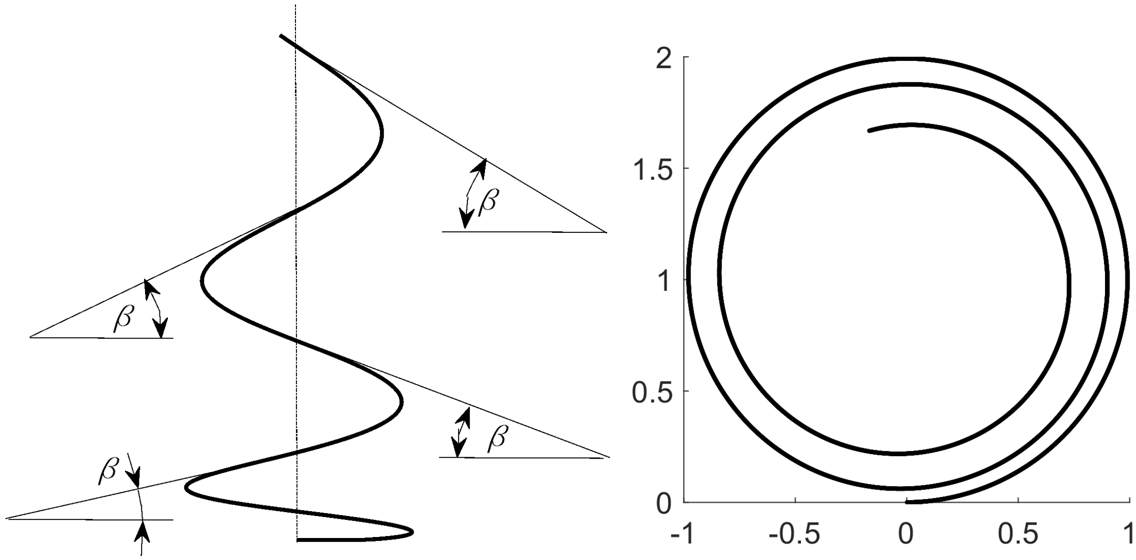

Figure 1: Projections of a constant curvature curve with a linear law of increasing the angle of ascent: a) frontal projection; b) horizontal projection.

dependence of the angle of ascent $\beta = a \cdot s$, where a is a constant value. After substituting these dependencies into (4) and partial integration, the following is obtained:

$$\begin{aligned} x &= \int \cos as \cos\left(\frac{\sqrt{k^2 - a^2}}{a} \ln\left[\mathrm{tg}\left(\tfrac{\pi}{4} + \tfrac{a}{2}s\right)\right]\right) \mathrm{d}s; \\ y &= \int \cos as \sin\left(\frac{\sqrt{k^2 - a^2}}{a} \ln\left[\mathrm{tg}\left(\tfrac{\pi}{4} + \tfrac{a}{2}s\right)\right]\right) \mathrm{d}s; \\ z &= -\frac{\cos as}{a}. \end{aligned} \tag{5}$$

As a result of the numerical integration of Equations (5), a curve was constructed for $k = 1$ and $a = 0.04$ when the arc s varies within the interval $[0, 15]$ (Fig. 1).

Fig. 1a shows that the angle β of ascent of the curve increases as its height increases.

3 Constructing Torses and Their Bending by Changing the Regularity of the Angle of Ascent of the Reverse Edge

Let's take the spatial curve (4) as a reverse edge and compose the parametric equations of the torse in a general form. The directional cosines of the rectilinear generatrix are found by differentiating equations (4). The vector obtained in this way is a unit one since the curve is described as a function of the arc length s. The parametric equations of the torse are:

$$\begin{aligned} X &= \int \cos\left(\int \frac{\sqrt{k^2 - \beta'^2}}{\cos\beta}\,\mathrm{d}s\right) \cos\beta\,\mathrm{d}s + u\cos\beta\cos\left(\int \frac{\sqrt{k^2 - \beta'^2}}{\cos\beta}\,\mathrm{d}s\right); \\ Y &= \int \sin\left(\int \frac{\sqrt{k^2 - \beta'^2}}{\cos\beta}\,\mathrm{d}s\right) \cos\beta\,\mathrm{d}s + u\cos\beta\sin\left(\int \frac{\sqrt{k^2 - \beta'^2}}{\cos\beta}\,\mathrm{d}s\right); \\ Z &= \int \sin\beta\,\mathrm{d}s + u\sin\beta, \end{aligned} \tag{6}$$

where u is the second surface variable – the length of the rectilinear generatrix, which starts from a point on the reverse edge.

Let's find the first fundamental form of the surface. The partial derivatives of the surface (6) are:

$$\begin{aligned}
\tfrac{\partial X}{\partial s} &= (\cos\beta - u\beta'\sin\beta)\cos\Big(\int \frac{\sqrt{k^2-\beta'^2}}{\cos\beta}\,\mathrm{d}s\Big) - u\sqrt{k^2-\beta'^2}\sin\Big(\int \frac{\sqrt{k^2-\beta'^2}}{\cos\beta}\,\mathrm{d}s\Big);\\
\tfrac{\partial Y}{\partial s} &= \Big(\cos\beta - u\beta'\sin\beta\Big)\sin\Big(\int \frac{\sqrt{k^2-\beta'^2}}{\cos\beta}\,\mathrm{d}s\Big) + u\sqrt{k^2-\beta'^2}\cos\Big(\int \frac{\sqrt{k^2-\beta'^2}}{\cos\beta}\,\mathrm{d}s\Big);\\
\tfrac{\partial Z}{\partial s} &= \sin\beta + u\beta'\cos\beta;
\end{aligned} \tag{7}$$

$$\begin{aligned}
\tfrac{\partial X}{\partial u} &= \cos\beta\cos\Big(\int \frac{\sqrt{k^2-\beta'^2}}{\cos\beta}\,\mathrm{d}s\Big);\\
\tfrac{\partial Y}{\partial u} &= \cos\beta\sin\Big(\int \frac{\sqrt{k^2-\beta'^2}}{\cos\beta}\,\mathrm{d}s\Big);\\
\tfrac{\partial Z}{\partial u} &= \sin\beta.
\end{aligned} \tag{8}$$

Let's find the coefficients of the first fundamental form:

$$\begin{aligned}
E &= \Big(\tfrac{\partial X}{\partial u}\Big)^2 + \Big(\tfrac{\partial Y}{\partial u}\Big)^2 + \Big(\tfrac{\partial Z}{\partial u}\Big)^2 = 1;\\
F &= \tfrac{\partial X}{\partial u}\cdot\tfrac{\partial X}{\partial s} + \tfrac{\partial Y}{\partial u}\cdot\tfrac{\partial Y}{\partial s} + \tfrac{\partial Z}{\partial u}\cdot\tfrac{\partial Z}{\partial s} = 1;\\
G &= \Big(\tfrac{\partial X}{\partial s}\Big)^2 + \Big(\tfrac{\partial Y}{\partial s}\Big)^2 + \Big(\tfrac{\partial Z}{\partial s}\Big)^2 = 1 + u^2k^2.
\end{aligned} \tag{9}$$

The first fundamental form will be written:

$$\mathrm{d}S^2 = E\,\mathrm{d}u^2 + 2F\,\mathrm{d}u\,\mathrm{d}s + G\,\mathrm{d}s^2 = \mathrm{d}u^2 + 2\,\mathrm{d}u\,\mathrm{d}s + (1+u^2k^2)\,\mathrm{d}s^2. \tag{10}$$

The quadratic form (10) does not include the dependence $\beta = \beta(s)$. It means that at any dependence $\beta = \beta(s)$, Equations (6) describe the bending of the same torse. At the same time, the dependence $k = k(s)$ remains unchanged, i.e., the curvature of the reverse edge remains unchanged depending on the arc length.

4 Examples

The parametric equations (6) make it possible to construct torses – bending of the same surface with different dependencies $\beta = \beta(s)$ and the constant dependence $k = k(s)$.

Example 1

First, let us consider the simplest case – $k =$ const. and $\beta =$ const.. According to Equations (6), after integration, parametric equations of the torse-helicoid can be obtained:

$$\begin{aligned}
X &= \frac{\cos\beta}{k}\sin\frac{ks}{\cos\beta} + u\cos\frac{ks}{\cos\beta};\\
Y &= -\frac{\cos\beta}{k}\cos\frac{ks}{\cos\beta} + u\sin\frac{ks}{\cos\beta};\\
Z &= s\sin\beta + u\sin\beta.
\end{aligned} \tag{11}$$

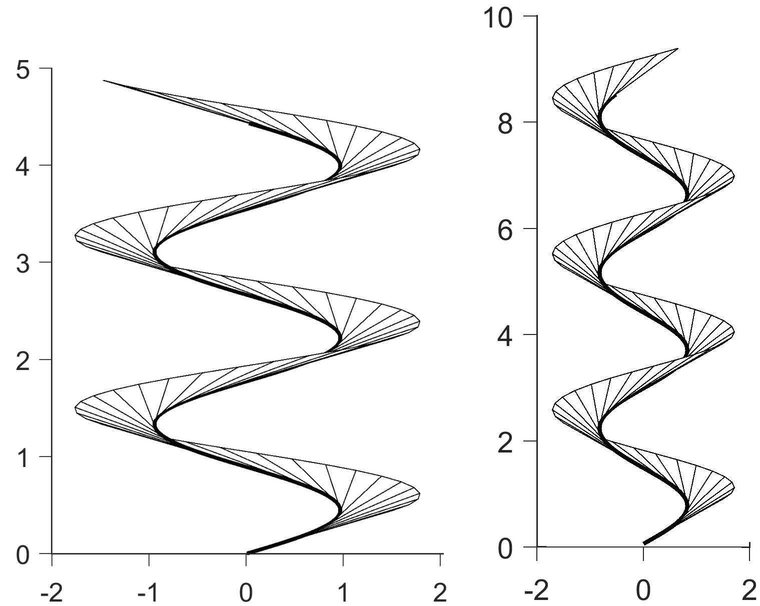

Figure 2: Frontal projections of the torses of helicoids bending on each other by changing the angle of ascent of the reverse edge: a) angle of ascent $\beta = 17°$; b) angle of ascent $\beta = 34°$.

Since the curvature of the helical line is constant, as well as of the curve in Fig. 1, these curves can be the reverse edges of two torses that bend on each other, provided that the curvatures are equal. In Fig. 2 the torses of the helicoid are plotted according to Equations (11) for different angles of the back edge when its length is 15 linear units. At $\beta = 0$, Equations (11) describe the sweep.

Example 2

Let's construct a torse for which the reverse edge is a curve with constant curvature and a linear law of increasing the angle of ascent (Fig. 1). To construct it, it is necessary to use numerical methods, by means of which the reverse edge can be constructed according to Equations (5). The intensity of the change in the angle $\beta = a \cdot s$ depends on the value of the constant a. Fig. 3 shows the torses for different values of the constant a.

With a reverse edge length of 15 linear units, the angle of ascent β at the top point is 0.3 rad (Fig. 3a) and 0.6 rad (Fig. 3b), i.e. 17° and 34°. All the shown torses can be bent on each other since the curvatures of their reverse edges are constant and equal. The process of bending in the first case (Fig. 2) and in the second (Fig. 3) has a different nature. In the first case, the torse bends simultaneously over the entire surface with the same intensity, while in the second case, it bends gradually. The torse shown in Fig. 2a, can be a continuation of the torse shown in Fig. 3a. The angle of ascent, which increases from zero to 17° (Fig. 3a), will remain constant thereafter (Fig. 3a). The same applies to the torses in Fig. 2b and 3b.

Example 3

Let us consider an example where the curvature and the angle of ascent of the reverse edge are variables. They are chosen so that Equation (6) can be fully integrated. The dependencies

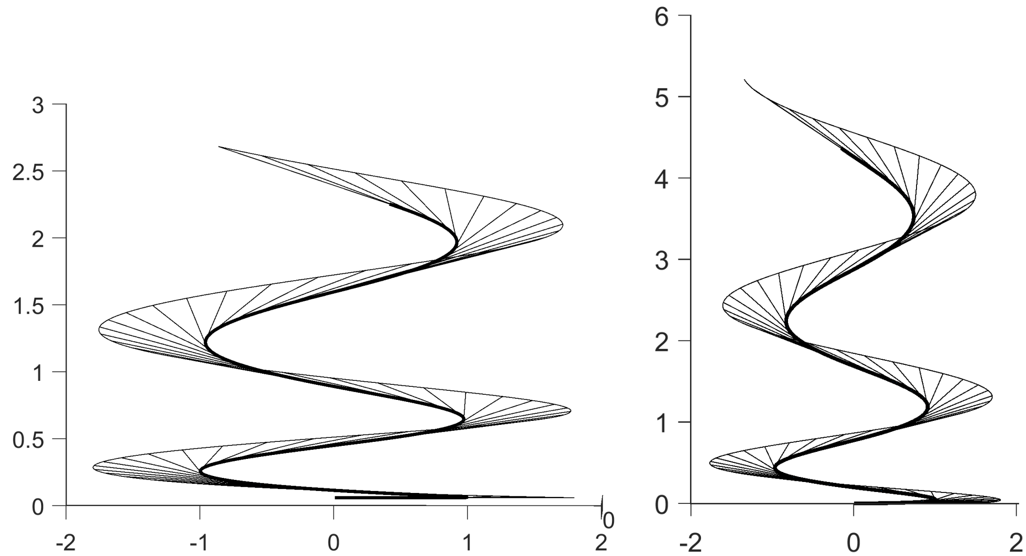

Figure 3: Frontal projections of torses bending on each other with a linear increase in the angle of ascent of the reverse edge: a) $\beta = 0.02s$; b) $\beta = 0.04s$

$k = k(s)$ and $\beta = \beta(s)$ are the following:

$$k = \frac{\sqrt{1+a^2}}{1+a^2s^2}; \qquad \beta = \operatorname{arctg}(as). \tag{12}$$

The curve given by the natural Equations (12) is located on a cylinder of the unit radius. Substitution of these equations, keeping in mind the derivative of the angle β expression in Equation (6), leads to the following result:

In Fig. 4, according to Equations (13), the torses at different values of the constant a are plotted.

$$\begin{aligned} X &= \sin\left(\frac{\operatorname{arcsinh} as}{a}\right) + \frac{u}{\sqrt{1+a^2s^2}}\cos\left(\frac{\operatorname{arcsinh} as}{a}\right); \\ Y &= \cos\left(\frac{\operatorname{arcsinh} as}{a}\right) - \frac{u}{\sqrt{1+a^2s^2}}\sin\left(\frac{\operatorname{arcsinh} as}{a}\right); \\ Z &= \frac{\sqrt{1+a^2s^2}}{a} + \frac{uas}{\sqrt{1+a^2s^2}}. \end{aligned} \tag{13}$$

During the bending of the torse, the reverse edge of which is given by the natural Equations (12), it deforms while staying on the surface of the cylinder. If a different pattern of change in the angle of ascent is specified, it will deform differently.

5 Conclusion

The bending of a torse can be controlled by changing the torsion of its reverse edge, while the curvature pattern as a function of arc length remains unchanged. However, the transition from the natural equations of curvature and torsion of the reverse edge to the parametric equations is associated with integration difficulties and cannot be practically carried out to the final form. Replacing the natural equation of the dependence of the torsion of the arc

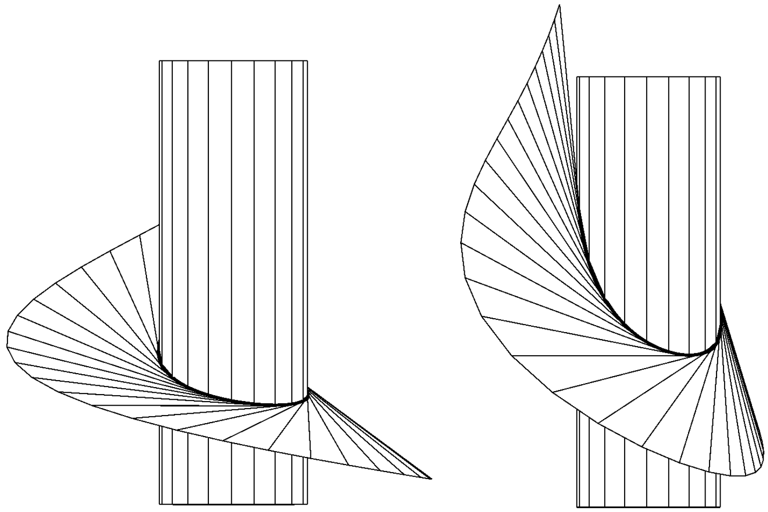

Figure 4: Frontal projections of the torses with a reverse edge on a cylinder, constructed according to parametric Equations (13): a) $a = 0.3$; b) $a = 0.6$; c) $a = 0.9$

length with the dependence of the angle of ascent on the same parameter greatly facilitates this transition. In some cases, the parametric equations of the torses can be obtained in the final form. This makes it possible to construct its intermediate positions during bending without the use of numerical methods.

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