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A Property of the Fermat-Torricelli Point for Tetrahedra and a new Characterization for Isosceles Tetrahedra

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Abstract. If the Fermat-Torricelli point A_0 is strictly inside a tetrahedron $A_1A_2A_3A_4$, we prove that the angle bisectors of $\angle A_iA_0A_j$, for $i \neq j$, $i, j = 1, 2, 3, 4$ form three bisecting lines that meet perpendicular at A_0. From this, we derive a new characterization of isosceles tetrahedra.

Key Words: Fermat-Torricelli point, angle bisector, isosceles tetrahedra

MSC 2020: 51M14 (primary), 51M20, 51M16

1 Introduction

Let $A_1A_2A_3A_4$ be a tetrahedron and let $A_i = (x_i, y_i, z_i)$, $i = 1, 2, 3, 4$. Then the Fermat's Problem states as follow:

Problem 1. Find (x, y, z) in $\mathbb{R}^3$, that minimizes:

$$f(x, y, z) = \sum_{i=1}^{4} \sqrt{(x - x_i)^2 + (y - y_i)^2 + (z - z_i)^2}. \tag{1}$$

The unsolvability of the Fermat-Torricelli problem by Euclidean constructions in $\mathbb{R}^3$ has been proved by Bajaj, Mehlhos, Melzak and Cockayne in [4, 6, 12], by applying Galois theory in some specific examples. Therefore, there is no Euclidean construction to locate the minimizing point.

Definition 1. The solution to Problem 1 is called the Fermat-Torricelli point of a tetrahedron $A_1A_2A_3A_4$ and it is denoted by A_0.

It is well known that the existence and uniqueness of the Fermat-Torricelli point A_0 in $\mathbb{R}^3$ is derived by the convexity of the Euclidean norm (distance) and compactness arguments [5, Theorem 18.3 (I), p. 237, Condition (P3), p. 238].

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Sturm and Lindelof gave a complete characterization of the solutions of the Fermat-Torricelli problem for m given points in $\mathbb{R}^n$ [11, 14]. Kupitz and Martini gave an alternative proof by using multivariate calculus [9, 10]. Eriksson and Noda, Sakai, Morimoto discovered some new characterizations for the Fermat-Torricelli point for tetrahedra in $\mathbb{R}^3$ [7, 13]. Characterizations for isosceles tetrahedra with respect to the corresponding Fermat-Torricelli point, barycenter, solid angles, dihedral angles, planar angles are given by Kupitz and Martini in [8, 10] and a complete list of references is given in Arnold's problems [3, pp. 188–191].

We shall focus on the three-dimensional case [5, Theorem 18.3, pp. 237].

We denote by $\vec{u}(j,i) \equiv \frac{\overrightarrow{A_jA_i}}{\|A_jA_i\|}$ the unit vector from A_j to A_i for $i,j = 0,1,2,3,4$.

Two cases may occur:

(I) If $\|\sum_{j=1,j\neq i}^{4} \vec{u}(j,i)\| > 1$, for each $i = 1,2,3,4$, then
  (a) A_0 is strictly inside of the tetrahedron $A_1A_2A_3A_4$,
  (b) $\sum_{i=1}^{4} \vec{u}(0,i) = \vec{0}$ (Fermat-Torricelli solution).

(II) If $\|\sum_{j=1,j\neq i}^{4} \vec{u}(j,i)\| \leq 1$ for some $i = 1,2,3,4$, then $A_0 = A_i$. (Fermat-Cavallieri solution).

Hence, there are two types for the Fermat-Torricelli point; one case is that A_0 is strictly inside of the tetrahedron, and the other case is that A_0 coincides with one of the vertices.

Abu-Abas, Abu-Saymeh and Hajja proved the non-isogonal property of the Fermat-Torricelli point for a tetrahedron [1, 2].

In this paper, we will prove the following using vectors:

- The six angle bisectors of the angles $\angle A_iA_0A_j$ for $i,j = 1,2,3,4$ and $i \neq j$ form three bisecting lines, which meet perpendicularly at A_0 (Section 2, Theorem 1).
- The segments formed by the midpoints of the opposite edges of a tetrahedron $A_1A_2A_3A_4$ intersect at the right angles at their midpoints, if and only if the tetrahedron $A_1A_2A_3A_4$ is isosceles (Section 2, Theorem 2).

2 A property of the Fermat-Torricelli point for four points forming a tetrahedron in $\mathbb{R}^3$

We are interested in the case when A_0 is strictly inside a tetrahedron $A_1A_2A_3A_4$.

We denote by $\angle(ij) \equiv \angle A_iA_0A_j$.

We denote by P_{jk}^{i} the orthogonal projection of A_i to the plane defined by $\triangle A_jA_0A_k$.

We consider proper tetrahedra (four non-coplanar points). Then P_{jk}^{i} is different from A_0 (c.f. Proposition 1 below).

We denote by $a_i = \|A_0A_i\|$, for $i = 1,2,3,4$.

We need the following well known lemma [12, 15], in order to prove Proposition 1 and Theorem 1:

Lemma 1 (see in [12, Property 3, p. 154] or [15, Formulas (6,2), (6.3), p. 8])**.** *The following relations hold:*

$$\cos\angle(12) = \cos\angle(34), \tag{2}$$

$$\cos\angle(23) = \cos\angle(14), \tag{3}$$

$$\cos\angle(13) = \cos\angle(24) \tag{4}$$

and

$$1 + \cos\angle(12) + \cos\angle(13) + \cos\angle(14) = 0. \tag{5}$$

Proposition 1. *If there is some i, such that $P^i_{jk} = A_0$, then $A_1A_2A_3A_4$ degenerates to a quadrilateral whose diagonals intersect at a right angle at A_0.*

Proof. Case (I): Without loss of generality, we assume that $P^4_{12} = A_0$ and $P^3_{12} \neq A_0$. Taking into account Lemma 1 and by substituting $\angle(14) = \angle(24) = \frac{\pi}{2}$ in (2), (3), (4), (5), we obtain $\angle(13) = \frac{\pi}{2}$, $\angle(12) = \angle(34) = \pi$. Therefore, we derive that $A_1A_3A_2A_4$ is a quadrilateral whose diagonals intersect at a right angle at A_0.

Case (II): Without loss of generality, we assume that $P^3_{12} = P^4_{12} = A_0$. Taking into account Lemma 1 and by substituting $\angle(14) = \angle(24) = \angle(13) = \angle(23) = \frac{\pi}{2}$ in (2), (3), (4), (5), we obtain $\angle(12) = \angle(34) = \pi$. Therefore, we derive that $A_1A_3A_2A_4$ is a quadrilateral whose diagonals intersect at a right angle at A_0 □

We denote by $\angle(i, jk) \equiv \angle A_iA_0P^i_{jk}$, for $i, j, k = 1, 2, 3, 4$.

We denote by $\omega^i_{jk} = \angle P^i_{jk}A_0A_1$.

Taking into account Proposition 1, $\angle(i, jk)$, $\omega_{i,jk}$ are well defined.

Theorem 1. *The six angle bisectors of $\angle(ij)$ for $i, j = 1, 2, 3, 4$ and $i \neq j$ form three bisecting lines, which meet perpendicularly at A_0.*

Proof. Without loss of generality, we express the unit vectors $\vec{u}(0, i)$ for $i = 1, 2, 3, 4$ in the following form:

$$\vec{u}(0,1) = (1, 0, 0), \tag{6}$$

$$\vec{u}(0,2) = (\cos\angle(12), \sin\angle(12), 0), \tag{7}$$

$$\vec{u}(0,3) = (\cos\angle(3,12)\cos\omega^3_{12}, \cos\angle(3,12)\sin\omega^3_{12}, \sin\angle(3,12)), \tag{8}$$

$$\vec{u}(0,4) = (\cos\angle(4,12)\cos\omega^4_{12}, \cos\angle(4,12)\sin\omega^4_{12}, \sin\angle(4,12)). \tag{9}$$

We note that $P^4_{12}, P^3_{12} \neq A_0$ taking into account Proposition 1. The vector $\vec{\delta}_{ij}$ of the angle bisector that connects A_0 with the midpoint of the segment A_iA_j is given by:

$$\vec{\delta}_{ij} = \vec{u}(0,i) + \vec{u}(0,j) \tag{10}$$

for $i, j = 1, 2, 3, 4, i \neq j$. By replacing (6), (7), (8), (9) in (10), we get:

$$\vec{\delta}_{12} = (1 + \cos\angle(12), \sin\angle(12), 0), \tag{11}$$

$$\vec{\delta}_{13} = (1 + \cos\angle(3,12)\cos\omega^3_{12}, \cos\angle(3,12)\sin\omega^3_{12}, \sin\angle(3,12)), \tag{12}$$

$$\vec{\delta}_{14} = (1 + \cos\angle(4,12)\cos\omega^4_{12}, \cos\angle(4,12)\sin\omega^4_{12}, \sin\angle(4,12)), \tag{13}$$

$$\vec{\delta}_{23} = (\cos\angle(12) + \cos\angle(3,12)\cos\omega^3_{12}, \sin\angle(12) + \cos\angle(3,12)\sin\omega^3_{12}, \sin\angle(3,12)), \tag{14}$$

$$\vec{\delta}_{24} = (\cos\angle(12) + \cos\angle(4,12)\cos\omega^4_{12}, \sin\angle(12) + \cos\angle(4,12)\sin\omega^4_{12}, \sin\angle(4,12)), \tag{15}$$

$$\begin{aligned}\vec{\delta}_{34} = (&\cos\angle(3,12)\cos\omega^3_{12} + \cos\angle(4,12)\cos\omega^4_{12}, \cos\angle(3,12)\sin\omega^3_{12} \\ &+ \cos\angle(4,12)\sin\omega^4_{12}, \sin\angle(3,12) + \sin\angle(4,12)).\end{aligned} \tag{16}$$

The inner products $\vec{u}(0,1) \cdot \vec{u}(0,3)$, $\vec{u}(0,2) \cdot \vec{u}(0,3)$ yield, respectively:

$$\cos\angle(3,12)\cos\omega^3_{12} = \cos\angle(13), \tag{17}$$

$$\cos\angle(12)\cos\angle(3,12)\cos\omega^3_{12} + \sin\angle(12)\cos\angle(3,12)\sin\omega^3_{12} = \cos\angle(23). \tag{18}$$

By considering (11), (12), (14), we calculate the inner products $\vec{\delta}_{12}\cdot\vec{\delta}_{23}$ and $\vec{\delta}_{12}\cdot\vec{\delta}_{13}$.

$$\begin{aligned}\vec{\delta}_{12}\cdot\vec{\delta}_{23} &= (1+\cos\angle(12))(\cos\angle(12)+\cos\angle(3,12)\cos\omega_{12}^{3})\\ &\quad+\sin\angle(12)(\sin\angle(12)+\cos\angle(3,12)\sin\omega_{12}^{3})\\ &= 1+\cos\angle(12)+\cos\angle(12)\cos\angle(3,12)\cos\omega_{12}^{3}\\ &\quad+\sin\angle(12)\cos\angle(3,12)\sin\omega_{12}^{3}+\cos\angle(3,12)\cos\omega_{12}^{3}.\end{aligned}$$

Taking into account (17), (18), we obtain that:

$$\vec{\delta}_{12}\cdot\vec{\delta}_{23} = 1+\cos\angle(12)+\cos\angle(13)+\cos\angle(23). \tag{19}$$

$$\begin{aligned}\vec{\delta}_{12}\cdot\vec{\delta}_{13} &= (1+\cos\angle(12))(1+\cos\angle(3,12)\cos\omega_{12}^{3})\\ &\quad+\sin\angle(12)\cos\angle(3,12)\sin\omega_{12}^{3}\\ &= 1+\cos\angle(12)+\cos\angle(12)\cos\angle(3,12)\cos\omega_{12}^{3}\\ &\quad+\sin\angle(12)\cos\angle(3,12)\sin\omega_{12}^{3}++\cos\angle(3,12)\cos\omega_{12}^{3}.\end{aligned}$$

Taking into account (17), (18), we obtain that:

$$\vec{\delta}_{12}\cdot\vec{\delta}_{13} = 1+\cos\angle(12)+\cos\angle(13)+\cos\angle(23). \tag{20}$$

By applying Lemma 1 in (19), (20), we derive that:

$$\vec{\delta}_{12}\cdot\vec{\delta}_{23} = \vec{\delta}_{12}\cdot\vec{\delta}_{13} = 0,$$

which yields that $\vec{\delta}_{12}\perp\vec{\delta}_{23}\perp\vec{\delta}_{13}$. Therefore, $\vec{\delta}_{12}$, $\vec{\delta}_{23}$, $\vec{\delta}_{13}$ is an orthonormal system of unit vectors.

We need to prove that the angle bisectors of the angles $\angle(12)$ and $\angle(34)$ belong to the same line.

The inner products $\vec{u}(0,1)\cdot\vec{u}(0,4)$, $\vec{u}(0,2)\cdot\vec{u}(0,4)$ $\vec{u}(0,3)\cdot\vec{u}(0,4)$ yield, respectively:

$$\cos\angle(4,12)\cos\omega_{12}^{4} = \cos\angle(14), \tag{21}$$

$$\cos\angle(12)\cos\angle(4,12)\cos\omega_{12}^{4}+\sin\angle(12)\cos\angle(4,12)\sin\omega_{12}^{4} = \cos\angle(24), \tag{22}$$

$$\begin{aligned}&\cos\angle(3,12)\cos\omega_{12}^{3}\cos\angle(4,12)\cos\omega_{12}^{4}+\\ &\cos\angle(3,12)\sin\omega_{12}^{3}\cos\angle(4,12)\sin\omega_{12}^{4}+\sin\angle(3,12)\sin\angle(4,12) = \cos\angle(34).\end{aligned} \tag{23}$$

By considering (11), (16), we calculate the inner product $\frac{\vec{\delta}_{12}}{|\vec{\delta}_{12}|}\cdot\frac{\vec{\delta}_{34}}{|\vec{\delta}_{34}|}$:

Taking into account (11), we get:

$$|\vec{\delta}_{12}| = \sqrt{2(1+\cos\angle(12))} \tag{24}$$

Taking into account (16) and (23), we get:

$$|\vec{\delta}_{34}| = \sqrt{2(1+\cos\angle(34))}. \tag{25}$$

Moreover, we obtain that:

$$\begin{aligned}\vec{\delta}_{12}\cdot\vec{\delta}_{34} &= (1+\cos\angle(12))(\cos\angle(3,12)\cos\omega_{12}^{3}+\cos\angle(4,12)\cos\omega_{12}^{4})+\\ &\quad+\sin\angle(12)(\cos\angle(3,12)\sin\omega_{12}^{3}+\cos\angle(4,12)\sin\omega_{12}^{4}).\end{aligned}$$

Taking into account (18), (22), (17), (21), we get:

$$\vec{\delta}_{12} \cdot \vec{\delta}_{34} = \cos\angle(23) + \cos\angle(24) + \cos\angle(13) + \cos\angle(14) \tag{26}$$

By taking into account (24) and (25) and by substituting (2), (3), (4), (5) in (26), we derive:

$$\frac{\vec{\delta}_{12}}{|\vec{\delta}_{12}|} \cdot \frac{\vec{\delta}_{34}}{|\vec{\delta}_{34}|} = \frac{1}{\sqrt{2(1+\cos\angle(12))}} \frac{1}{\sqrt{2(1+\cos\angle(34))}} (-2(1+\cos\angle(12))). \tag{27}$$

By replacing $\angle(34) = \angle(12)$ (Lemma 1) in (27), we derive that:

$$\frac{\vec{\delta}_{12}}{|\vec{\delta}_{12}|} \cdot \frac{\vec{\delta}_{34}}{|\vec{\delta}_{34}|} = -1.$$

Hence, the angle bisectors of the angles $\angle(12)$ and $\angle(34)$ belong to the same line.

By following the same process and by applying Lemma 1, we get:

$$\frac{\vec{\delta}_{23}}{|\vec{\delta}_{23}|} \cdot \frac{\vec{\delta}_{14}}{|\vec{\delta}_{14}|} = -1$$

and

$$\frac{\vec{\delta}_{13}}{|\vec{\delta}_{13}|} \cdot \frac{\vec{\delta}_{24}}{|\vec{\delta}_{24}|} = -1,$$

which yields that the angle bisectors of the angles $\angle(23)$ and $\angle(14)$ belong to the same line and the angle bisectors of the angles $\angle(13)$ and $\angle(24)$ belong to the same line, respectively. □

The following theorem was conjectured by the reviewer of this paper:

Theorem 2. *Let $A_1A_2A_3A_4$ be a tetrahedron. Let A_{ij} be the midpoint of the edge A_iA_j, for $i, j = 1, 2, 3, 4$. Then the segments $A_{12}A_{34}$, $A_{13}A_{24}$, $A_{14}A_{23}$, intersect each other at the right angle at their midpoints if, and only if, the tetrahedron $A_1A_2A_3A_4$ is isosceles.*

Proof. (I) We shall prove that if the segments $A_{12}A_{34}$, $A_{13}A_{24}$, $A_{14}A_{23}$, intersect each other at the right angle at their midpoints, then the tetrahedron $A_1A_2A_3A_4$ is isosceles.

Let A_0 be the intersection point of the segments $A_{12}A_{34}$, $A_{13}A_{24}$, $A_{14}A_{23}$, which intersect each other at a right angle. The point A_0 is inside $A_1A_2A_3A_4$. We will prove that A_0 is the Fermat-Torricelli point of $A_1A_2A_3A_4$.

We consider the vectors $\vec{a_i} \equiv \overrightarrow{A_0A_i}$, such that $\|A_0A_i\| = a_i$, for $i = 1, 2, 3, 4$:

$$\vec{a_1} = a_1(1, 0, 0), \tag{28}$$

$$\vec{a_2} = a_2(\cos\angle(12), \sin\angle(12), 0), \tag{29}$$

$$\vec{a_3} = a_3(\cos\angle(3,12)\cos\omega_{12}^3, \cos\angle(3,12)\sin\omega_{12}^3, \sin\angle(3,12)), \tag{30}$$

$$\vec{a_4} = a_4(\cos\angle(4,12)\cos\omega_{12}^4, \cos\angle(4,12)\sin\omega_{12}^4, \sin\angle(4,12)). \tag{31}$$

The vector $\vec{\delta}_{ij}$ that corresponds to each bisector that connects A_0 with the midpoint of the segment A_iA_j is given by:

$$\vec{\delta}_{ij} = \tfrac{1}{2}(a_i \vec{u}(0,i) + a_j \vec{u}(0,j)) \tag{32}$$

for $i, j = 1, 2, 3, 4$; $i \neq j$.

Taking into account that $\vec{\delta}_{12} \cdot \vec{\delta}_{23} = 0$, $\vec{\delta}_{12} \cdot \vec{\delta}_{13} = 0$, we get:

$$a_1 a_2 \cos\angle(12) + a_1 a_3 \cos\angle(13) + a_2 a_3 \cos\angle(23) = -a_1^2. \quad (33)$$

and

$$a_1 a_2 \cos\angle(12) + a_1 a_3 \cos\angle(13) + a_2 a_3 \cos\angle(23) = -a_2^2. \quad (34)$$

By subtracting (34) from (33), we derive that $a_1 = a_2$.

Similarly, by taking into account that $\vec{\delta}_{42} \cdot \vec{\delta}_{23} = 0$, $\vec{\delta}_{42} \cdot \vec{\delta}_{43} = 0$, and $\vec{\delta}_{43} \cdot \vec{\delta}_{13} = 0$, $\vec{\delta}_{41} \cdot \vec{\delta}_{13} = 0$, we derive that $a_1 = a_2 = a_3 = a_4$.

Taking into account that $\frac{\vec{\delta}_{12}}{|\vec{\delta}_{12}|} \cdot \frac{\vec{\delta}_{34}}{|\vec{\delta}_{34}|} = -1$

$$\frac{1}{\sqrt{2(1+\cos\angle(12))}} \frac{1}{\sqrt{2(1+\cos\angle(34))}} (-2(1+\cos\angle(12))) = -1,$$

which gives

$$\angle(12) = \angle(34).$$

Similarly, taking into account that $\frac{\vec{\delta}_{23}}{|\vec{\delta}_{23}|} \cdot \frac{\vec{\delta}_{14}}{|\vec{\delta}_{14}|} = -1$ we get

$$\angle(14) = \angle(23).$$

Hence, we obtain that $\sum_{i=1}^{4} \vec{u}(0,i) = 0$, which yields that the intersection point A_0 of the three bisecting lines is the Fermat-Torricelli point of $A_1A_2A_3A_4$.

We need the following auxiliary result ([8, Theorem (4.9),p. 157]), in order to prove that the tetrahedron $A_1A_2A_3A_4$ is isosceles:

> If the Fermat-Torricelli point $A_1A_2A_3A_4$ has equal distance to each vertex of $A_1A_2A_3A_4$, then $A_1A_2A_3A_4$ is an isosceles tetrahedron.

Hence, by applying this result for the Fermat-Torricelli point A_0 , we derive that $A_1A_2A_3A_4$ is an isosceles tetrahedron.

(II) We shall prove the converse of Theorem 2: If $A_1A_2A_3A_4$ is an isosceles tetrahedron, then the segments $A_{12}A_{34}$, $A_{13}A_{24}$, $A_{14}A_{23}$, intersect each other at the right angle at their midpoints.

Let $A_1A_2A_3A_4$ be a tetrahedron and A_0 is the Fermat-Torricelli point inside of $A_1A_2A_3A_4$. We will obtain a theoretical construction of an isosceles tetrahedron.

Let $B_{\mathbf{i}}$ be a point on the ray A_0A_i, such that $|A_0B_{\mathbf{i}}| = 1$, for each $\mathbf{i} = 1,2,3,4$. The balancing condition of unit vectors remain the same for $A_1A_2A_3A_4$ and $B_1B_2B_3B_4$, respectively:

$$\sum_{i=1}^{4} \vec{u}(0,i) = \sum_{i=1}^{4} \vec{u}(0,\mathbf{i}) = 0,$$

where $\vec{u}(0,\mathbf{i}) \equiv \overrightarrow{A_0B_{\mathbf{i}}}$ is the unit vector from A_0 to $B_{\mathbf{i}}$ for $\mathbf{i} = 1,2,3,4$. Thus, by taking into account Lemma 1, we get $\angle(12) = \angle(34)$, $\angle(13) = \angle(24)$, $\angle(23) = \angle(14)$ and by applying the cosine law in $\triangle B_iA_0B_j$, for $i,j = 1,2,3,4$ we derive that $B_1B_2 = B_3B_4$, $B_1B_3 = B_2B_4$, $B_2B_3 = B_1B_4$. Therefore, $B_1B_2B_3B_4$ is an isosceles tetrahedron. By applying Theorem 1 the angle bisectors $B_{12}B_{34}$, $B_{13}B_{24}$, $B_{14}B_{23}$, intersect each other at A_0 at the right angle at their midpoints and the center of the circumscribed sphere with unit radius defined by $B_1B_2B_3B_4$ coincides with the Fermat-Torricelli point A_0 of both $B_1B_2B_3B_4$ and $A_1A_2A_3A_4$. $\square$

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