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More Variations on Fermat Analogue of the Steiner-Lehmus Theorem

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Abstract. The celebrated Steiner-Lehmus theorem states that if the internal bisectors of two angles of a triangle are equal, then the corresponding sides have equal lengths. In this paper, we consider the triangle ABC whose all angles are less than 120°, F is its Fermat point, and per(ABC), $[ABC]$ stand for its perimeter and area, respectively. In Theorem 1, we prove the Fermat analogue of Steiner-Lehmus Theorem that states that if the cevians from B and C through the Fermat point F meet AC and AB at B' and C' respectively, then $BB' = CC'$ is equivalent to $AB = AC$. More stronger forms are also proved such as $AB > AC$ is equivalent to each of $BB' > CC'$ and per$(C'BC >$ per$(B'CB)$. More variations on Fermat analogue of Steiner-Lehmus Theorem are proved in Theorems 3 and 4. In Theorem 3, the cevians through F from B and C meet the external angle bisectors of C and B at D and E respectively, and it is proved that, for example, $AB = AC$ is equivalent to each of $CE = BD$, per$(EC'B) =$ per$(DB'C)$, and $[EC'B] = [DB'C]$ and more stronger forms are also proved such as $AB > AC$ is equivalent to each of $CE > BD$, per$(EC'B) >$ per$(DB'C)$, and $[EC'B] > [DB'C]$. In Theorem 4, we prove that if the angle A of the triangle ABC is not equal to 60° and the circumcevians BK and CL of the Fermat point F, that meet the circumcircle of $\triangle ABC$ at K and L, are equal, then the triangle $\triangle ABC$ is isosceles with $AB = AC$.

Key Words: Steiner-Lehmus Theorem, Fermat point, cevian, circumcevian

MSC 2020: 51M04

1 Introduction

The celebrated Steiner-Lehmus theorem states that if the internal bisectors of two angles of a triangle are equal, then the corresponding sides have equal lengths. That is to say if P is the incenter of $\triangle ABC$ and if the ray $\overrightarrow{BP}$ and the ray $\overrightarrow{CP}$ meet the sides AC and AB at B' and C' respectively, then

$$BB' = CC' \implies AB = AC.$$

An elegant proof of this theorem appeared in [4]. In [1] we considered the line AJ through the incenter P of $\triangle ABC$ that meets BC at J and proved that there is a segment XY on AJ inside of which there exists a point Q with BB' and CC' through Q that meet AC and AB or their extensions at B' and C' , respectively, and such that $BB' = CC'$ and $AB \neq AC$ and outside of which there are no such points other than the point J.

Several variations of the Steiner-Lehmus theorem have been considered in the literature. For example, in [2], we replaced the incenter by the Nagel and the Gergonne centers and proved in both cases that if the cevians from B and C of $\triangle ABC$ through the Nagel or Gergonne center meet AC and AB at B' and C' and meet the external angle bisectors of C and B at D and E, respectively, then

$$AB = AC \text{ is equivalent to each of } BB' = CC' \text{ and } BD = CE.$$

In what follows, let all angles of $\triangle ABC$ be less than 120° and F be its Fermat point(or duly Fermat-Torricelli point) defined as the unique interior point of $\triangle ABC$ whose distances from the three vertices have the minimum sum (see e.g. [3, p. 22]) and constructed by erecting externally two equilateral triangles ABQ and ACP on AB and AC, respectively, and F be the intersection of BP and CQ, as seen in Figures 1 and Figure 2. Note that if any angle of $\triangle ABC$ is greater or equal to 120°, then the vertex of this angle is the Fermat point of this triangle.

In this paper we consider in Theorem 1, the cevians from B and C through the Fermat point of $\triangle ABC$ that meet AC and AB at B' and C', respectively, and prove that if $BB' = CC'$, then $AB = AC$ and other stronger forms are proved. In Theorem 3, we consider the cevians from B and C, through the Fermat point of $\triangle ABC$, that meet the external angle bisectors of C and B at D and E, respectively. Then we prove that $CE = BD$ if and only if $AB = AC$ and also other stronger forms are proved. In Theorem 4, let the angle A of the triangle ABC be not equal to 60° and the circumcevians BK and CL of the Fermat point F be equal. Then the triangle ABC is isosceles with $AB = AC$.

Notations: Let, in all figures, F be the Fermat point of a triangle ABC and

$$\angle FBC = \phi, \quad \angle FCB = \theta, \quad \angle ABF = \delta, \quad \angle ACF = \mu, \quad \angle FAC = \xi, \quad \angle FAB = \omega$$
$$BC = a, \quad AC = b, \quad AB = c.$$

Also, for convenience, we let $[ABC]$, per(ABC) stand for the area and perimeter of $\triangle ABC$.

2 Fermat Analogue of Steiner-Lehmus Theorem

Theorem 1. *Let all angles of $\triangle ABC$ be less than 120° and F be its Fermat point. Let $\triangle ABQ$ and $\triangle ACP$ be the equilateral triangles erected externally on AB and AC respectively, and BP and CQ intersect at F as seen in Figures 1 and 2. Let AF produced meet BC at A', BP intersect AC at B', and CQ intersect AB at C'. Then we have the following:*

(a) The statement $AB = AC$ is equivalent to each of the statements

$$\text{(i) } BB' = CC', \text{ (ii) } FB = FC, \text{ (iii) } FC' = FB', \text{ (iv) } BC' = CB', \\ \text{(v) } [C'BC] = [B'CB], \text{ (vi) } \operatorname{per}(C'BC) = \operatorname{per}(B'CB). \tag{1}$$

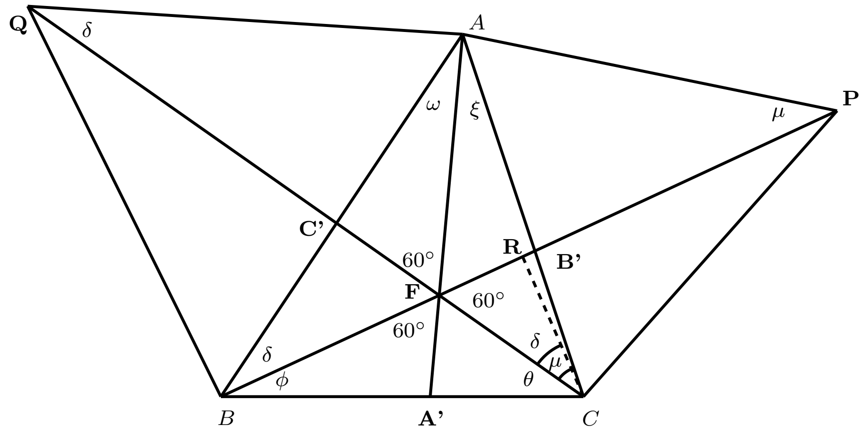


Figure 1: Illustrating the proof of Theorem 1

(b) The statement $AB > AC$ is equivalent to each of the statements

$$\begin{gathered}\textit{(i)}\ BB' > CC',\ \textit{(ii)}\ FB > FC,\ \textit{(iii)}\ FC' > FB',\ \textit{(iv)}\ BC' > CB',\\ \textit{(v)}\ [C'BC] > [B'CB],\ \textit{(vi)}\ \operatorname{per}(C'BC) > \operatorname{per}(B'CB).\end{gathered} \tag{2}$$

Proof. As shown in Figures 1 and 2, it is clear that triangles APB and ACQ are congruent by the SAS rule. Therefore

$$\begin{gathered}PB = QC, \quad \angle APB = \angle ACQ = \mu \text{ and } \angle ABP = \angle AQC = \delta,\\ \text{and hence the quadrilaterals } AFCP \text{ and } AFBQ \text{ are cyclic. Thus}\\ \angle AFC = \angle AFB = \angle BFC = 120^\circ, \quad \mu + \xi = \delta + \omega = \theta + \phi = 60^\circ,\\ PB \text{ bisects } \angle AFC,\ QC \text{ bisects } \angle AFB, \text{ and } AA' \text{ bisects } \angle BFC.\end{gathered} \tag{3}$$

So if $AB = AC$, then $\triangle PAB$ is an isosceles triangle and hence $\mu = \delta$ and by (3) we get $\xi = \omega$. Therefore AA' is the perpendicular bisector of BC. Thus we conclude that all statements in (1) hold by symmetry.

So, it is enough to show that $AB > AC$ implies that all inequalities in (2) hold and the rest of *(a)* and *(b)* follow by contradiction. To see this, notice that an implication such as

$$AB > AC \implies BB' > CC' \tag{4}$$

does indeed yield the converse implication

$$BB' > CC' \implies AB > AC. \tag{5}$$

For if $BB' > CC'$, then AB can neither be equal to AC (because this would imply that $BB' = CC'$ by symmetry), nor less than AC (because this would imply that $BB' < CC'$ by (4)). Thus (4) yields (5). Similarly (4) implies that if $BB' = CC'$, then $AB = AC$.

First, we assume that $AB > AC$ and prove that $\mu > \delta$, $\omega > \xi$, $\theta > \phi$. To achieve this we apply sine law to triangles AFB and AFC. Thus we get

$$\frac{AF}{\sin\delta} = \frac{AB}{\sin 60^\circ} = \frac{FB}{\sin\omega}, \frac{AF}{\sin\mu} = \frac{AC}{\sin 60^\circ} = \frac{FC}{\sin\xi}$$
$$\text{and hence} \quad \frac{\sin(\mu)}{\sin(\delta)} = \frac{AB}{AC} > 1 \quad \text{and} \quad \frac{FB}{FC} = \frac{AB\sin\omega}{AC\sin\xi}.$$

But $\mu+\delta<180^\circ$ and $\mu+\xi=\delta+\omega=60^\circ$ by (3). Thus

$$\mu>\delta, \quad \omega>\xi, \quad FB>FC \text{ and hence } \theta>\phi \text{ as desired.} \tag{6}$$

Note that $\mu>\delta$ also follows from the fact that $AB>AC=AP$ in $\triangle ABP$.

Next, we prove that all in *(b)* hold.

(b-i). Since $\mu>\delta$ by (6), there is a point R on FB' such that $\angle FCR=\delta$, and hence the quadrilateral $C'BCR$ is cyclic by the converse of *Euclid's proposition III-21.* Also, $\angle RCB=\theta+\delta$, $\angle C'BC=\phi+\delta<90^\circ$, and $\theta>\phi$ by (6). Therefore $\angle C'BC<90^\circ$, $\angle RCB+\angle C'BC<180^\circ$, $\angle RCB>\angle C'BC$, but the angles RCB and $C'BC$ are subtended by the chords BR and CC'. Thus $BR>CC'$. Since $BB'>BR$, it follows that $BB'>CC'$, as desired.

Second proof. Applying the sine law to triangles $C'AC, B'AB$, we get

$$\frac{BB'}{\sin(A)}=\frac{AB}{\sin(\angle AB'B)}=\frac{AB}{\sin(\mu+60^\circ)} \quad\text{and}\quad \frac{CC'}{\sin(A)}=\frac{AC}{\sin(\angle AC'C)}=\frac{AC}{\sin(\delta+60^\circ)}.$$

But by the sine law applied to $\triangle PAB$ we have $\frac{AB}{AC}=\frac{\sin\mu}{\sin\delta}$. Therefore

$$\frac{BB'}{CC'}=\frac{AB\sin(\delta+60^\circ)}{AC\sin(\mu+60^\circ)}=\frac{\sin(\mu)\sin(\delta+60^\circ)}{\sin(\delta)\sin(\mu+60^\circ)}=\frac{\sin(\mu)\sin(\delta)+\sqrt{3}\sin(\mu)\cos(\delta)}{\sin(\mu)\sin(\delta)+\sqrt{3}\sin(\delta)\cos(\mu)}.$$

But angles μ, δ are acute and $\mu>\delta$ by (6). So, $\sin\mu>\sin\delta$, $\cos\delta>\cos\mu$ and hence $BB'>CC'$ as required.

(b-ii) $FB>FC$ follows from (6).

(b-iii) Since $2[AFB]=(FA)(FB)\sin 60^\circ=FC'(FB+FA)\sin 60^\circ$ and $2[AFC]=(FA)(FC)\sin 60^\circ=FB'(FC+FA)\sin 60^\circ$, we get,

$$\frac{FC'}{FB'}=\frac{FB(FC+FA)}{FC(FB+FA)}=\frac{(FB)(FC)+(FB)(FA)}{(FC)(FB)+(FC)(FA)}.$$

But $FB>FC$ by (b-ii). Thus $FC'>FB'$ as required.

(b-iv) By applying the sine law to triangles $C'BF$ and $B'CF$, we get

$$\frac{BC'}{\sin 60^\circ}=\frac{FC'}{\sin\delta} \quad\text{and}\quad \frac{CB'}{\sin 60^\circ}=\frac{FB'}{\sin\mu}.$$

Thus we have

$$\frac{BC'}{CB'}=\frac{FC'\sin\mu}{FB'\sin\delta}.$$

But $FC'>FB'$ by (b-iii) and $\mu>\delta$ by (6). Therefore $BC'>CB'$ as desired.

It is worth mentioning here that a more lengthy proofs for *(b-i)–(b-iv)* are given in [5, Theorems 2, 4, 6, 9].

(b-v) Since

$$[C'BC]=[C'FB]+[FBC], \quad 2[C'FB]=(FB)(FC')\sin(60^\circ);$$
$$[B'CB]=[B'FC]+[FBC], \quad 2[B'FC]=(FC)(FB')\sin(60^\circ), \quad\text{and}$$
$$FB>FC \quad \text{by (b-ii) and} \quad FC'>FB' \quad \text{by (b-iii)},$$

we have $[C'FB]>[B'FC]$ and hence $[C'BC]>[B'CB]$ as desired.

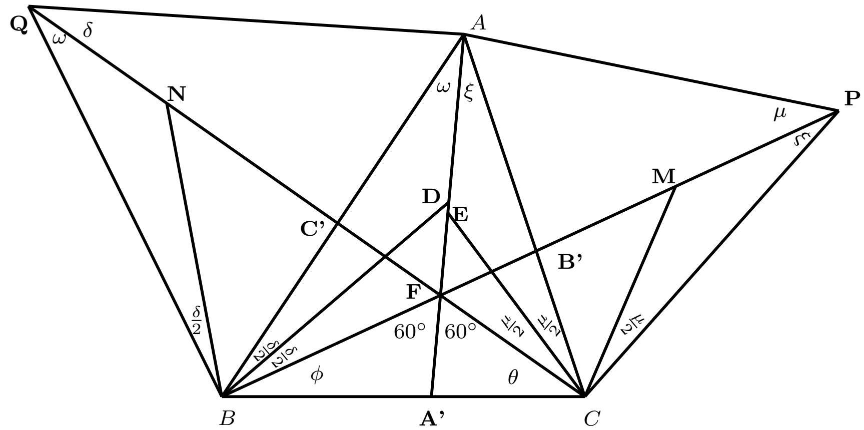


Figure 2: Illustrating the proof of Theorem 1*(b-vi)*

(b-vi) Next, to prove that $\text{per}(C'BC) > \text{per}(B'CB)$, we refer to Figure 2 where F is the Fermat point of $\triangle ABC$ and PAC, QAB are equilateral triangles. It is clear that $PB' > B'C$ and $QC' > C'B$ (because $\xi < 60°$ and $\omega < 60°$ by (3)). So, let M be a point of PB' such that $B'M = B'C$ and N a point of QC' such that $C'N = C'B$. Then $\text{per}(C'BC) = BC + CN$ and $\text{per}(B'CB) = CB + BM$. Thus

$$\begin{aligned} &\text{per}(C'BC) > \text{per}(B'CB) \iff CN = QC - QN > BM = PB - PM. \text{ But} \\ &QC = PB \text{ by (3). Therefore } \text{per}(C'BC) > \text{per}(B'CB) \iff QN < PM. \end{aligned} \tag{7}$$

Since $\angle MB'C = \angle B'FC + \angle FCB' = 60° + \mu$, $\angle NC'B = \angle C'FB + \angle FBC' = 60° + \delta$. But $MB'C$, $NC'B$ are isosceles triangles and ACP, ABQ are equilateral triangles. Therefore

$$\angle MCB' = 60° - \frac{\mu}{2}, \quad \angle MCP = \frac{\mu}{2} \quad \text{and} \quad \angle NBC' = 60° - \frac{\delta}{2}, \quad \angle NBQ = \frac{\delta}{2}. \tag{8}$$

So, to complete the proof of (b-vi) we draw the angle bisectors CE of the $\angle ACF$ and BD of the $\angle ABF$ that meet AF at E and D respectively, as seen in Figure 2. But $\angle ACF = \mu$ and $\angle ABF = \delta$. Thus $\angle ACE = \frac{\mu}{2}$ and $\angle ABD = \frac{\delta}{2}$ and hence by ASA rule for congruence of triangles, we have

$$\triangle PCM \cong \triangle ACE \quad \text{and} \quad \triangle QBN \cong \triangle ABD. \quad \text{Therefore} \quad PM = AE \quad \text{and} \quad QN = AD. \tag{9}$$

So, we deduce from (7) and (9) that

$$\text{per}(C'BC) > \text{per}(B'CB) \iff QN < PM \iff AD < AE \tag{10}$$

and to prove that $AD < AE$, we apply the angle bisector theorem and the sine law to triangles $\triangle ABF$, $\triangle ACF$, and get

$$\frac{FD}{AD} = \frac{FB}{AB} = \frac{\sin\omega}{\sin 60°}, \quad \frac{FE}{AE} = \frac{FC}{AC} = \frac{\sin\xi}{\sin 60°}.$$

But $\omega > \xi$ by (6). Thus

$$\frac{FD}{AD} > \frac{FE}{AE}.$$

So, we have

$$\frac{FD + AD}{AD} > \frac{FE + AE}{AE}.$$

Therefore

$$\frac{AF}{AD} > \frac{AF}{AE}.$$

Hence $AD < AE$ and we deduce from (10) that $\operatorname{per}(C'BC) > \operatorname{per}(B'CB)$ as required. □

Before Theorem 2 and referring to Figure 1 we investigate the relation between the remaining segments AC' and AB' of $\triangle ABC$ when $AB > AC$ and prove the following:

$$\begin{aligned} &\text{(i) } AC' = AB' \iff \angle A = 60^\circ, \quad \text{(ii) } AC' > AB' \iff \angle A > 60^\circ, \\ &\text{(iii) } AC' < AB' \iff \angle A < 60^\circ. \end{aligned} \tag{11}$$

By applying the sine law to $\triangle AFC'$ and $\triangle AFB'$, we get

$$\frac{AC'}{\sin(60^\circ)} = \frac{AF}{\sin(C')} = \frac{AF}{\sin(60^\circ + \delta)}, \frac{AB'}{\sin(60^\circ)} = \frac{AF}{\sin(B')} = \frac{AF}{\sin(60^\circ + \mu)}.$$

Therefore

$$\frac{AC'}{AB'} = \frac{\sin(60^\circ + \mu)}{\sin(60^\circ + \delta)}.$$

Notice that by (6) we have $\mu > \delta$ and hence $60^\circ + \mu > 60^\circ + \delta$. Therefore $\sin(60^\circ + \mu) = \sin(60^\circ + \delta) \iff \mu + \delta = 60^\circ$, $\sin(60^\circ + \mu) > \sin(60^\circ + \delta) \iff \mu + \delta < 60^\circ$, and $\sin(60^\circ + \mu) < \sin(60^\circ + \delta) \iff \mu + \delta > 60^\circ$. Since $\theta + \phi = 60^\circ$, we get $\angle B + \angle C = 60^\circ + \mu + \delta$.

Thus we deduce from the previous note that

$$\begin{aligned} &\text{(i) } AC' = AB' \iff \mu + \delta = 60^\circ \iff \angle B + \angle C = 120^\circ \iff \angle A = 60^\circ, \\ &\text{(ii) } AC' > AB' \iff \mu + \delta < 60^\circ \iff \angle B + \angle C < 120^\circ \iff \angle A > 60^\circ, \\ &\text{(iii) } AC' < AB' \iff \mu + \delta > 60^\circ \iff \angle B + \angle C > 120^\circ \iff \angle A < 60^\circ, \end{aligned}$$

as required.

3 More Variations

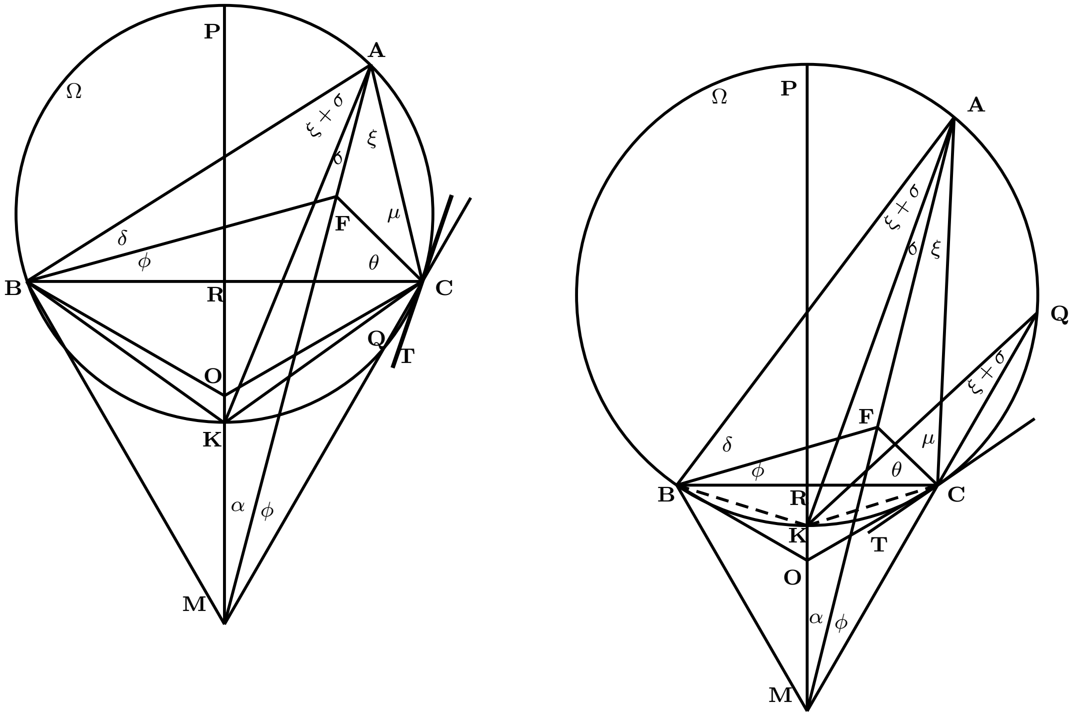
In this section we consider the two variations of the Fermat analogue of Steiner-Lehmus theorem when the cevians through F from B and C meet the external angle bisectors of C and B and when they meet the circumcircle of the triangle ABC. First, we prove in the next Theorem 2 the additional relation $\theta - \phi > \mu - \delta$ to (6) for any triangle ABC with $AB > AC$ that is of interest by itself and is needed to prove Theorem 3.

Theorem 2. *Let F be the Fermat point of $\triangle ABC$ whose angles are less than 120°, $AB > AC$ and let $\angle FBC = \phi$, $\angle FCB = \theta$, $\angle ABF = \delta$, and $\angle ACF = \mu$. Then $\theta - \phi > \mu - \delta$.*

Proof. Let Ω be the circumcircle of $\triangle ABC$ and let the equilateral $\triangle MBC$ be erected externally on BC with orthocenter O and PK be the diameter of Ω bisecting BC at R and produced to M, as seen in Figures 3(a) and 3(b). Then it is easy to see that

$$\begin{aligned} &\angle BOC = 120^\circ, \angle A + \angle BKC = 180^\circ \text{ and hence } K = O \iff \angle A = 60^\circ, \\ &K \text{ is between } O \text{ and } M \iff 60^\circ < \angle A < 120^\circ \text{ and } 120^\circ > \angle BKC > 60^\circ, \text{ and} \\ &K \text{ is between } R \text{ and } O \iff \angle A < 60^\circ \text{ and } \angle BKC > 120^\circ. \end{aligned} \tag{12}$$

So, we distinguish two cases:



(a) (b)

Figure 3: Illustrating the proof of Theorem 2

Case (a): $60^\circ \leq \angle A < 120^\circ$, $120^\circ \geq \angle BKC > 60^\circ$ and we refer to Figure 3(a).

Case (b): $\angle A < 60^\circ$ and we refer to Figure 3(b).

Let TC be the tangent to Ω at the point C. Then $\angle TCB = \angle A$ (in either case) by alternate segment theorem, $\angle TCB \geq 60^\circ$ in Case (a), and $\angle TCB < 60^\circ$ in Case (b). So, the ray $\overrightarrow{MC}$ intersects Ω in Case (a) at a point Q of the minor arc KC between K and C when $\angle A > 60^\circ$, $Q = C$ when $\angle A = 60^\circ$, and at a point Q of the minor arc CA between C and A in Case (b) because $\angle ACB < 120^\circ$, as seen in Figures 3(a) and 3(b).

To complete the proof put (in either case)

$$\angle AMK = \alpha, \quad \angle MAK = \sigma, \quad \angle BAM = \omega, \quad \angle CAM = \xi.$$

Since (in either case) $KB = KC$ and the quadrilateral $FBMC$ is cyclic, it follows that $\angle CFM = \angle CBM = 60^\circ$. But $\angle AFC = 120^\circ$. So, AM passes through F, AK bisects $\angle A$, and

$$\mu + \xi = 60^\circ = \delta + \omega = \phi + \theta, \quad \omega = \xi + 2\sigma, \quad \angle FMC = \angle FBC = \phi, \quad \phi + \alpha = 30^\circ,$$

Thus

$$\theta = 30^\circ + \alpha, \quad \phi = 30^\circ - \alpha, \quad \theta - \phi = 2\alpha, \quad \mu - \delta = \omega - \xi = 2\sigma.$$

Therefore (in either case)

$$\theta - \phi > \mu - \delta \iff \alpha > \sigma \iff KA > KM. \tag{13}$$

To prove that $KA > KM$ we need the Euclid's proposition III-7 that states if PK is a diameter of a circle Ω with center J and V is a point of the radius JK which is not the center J, VA and VD both fall upon the semicircle $KDAP$ of Ω from V and $\angle PVA < \angle PVD$, then $VA > VD$. This proposition follows from the open mouth theorem applied to triangles AJV and DJV where $AJ = DJ$, $JV = JV$ and $\angle AJV > \angle DJV$.

First, if $\angle A = 60°$, then $K = O$ by (12), $PO = PK$ is a diameter of Ω and $KC = KM$. Also, $\triangle PBC$ is equilateral and its orthocenter is the center J of Ω and it follows from Euclid's proposition III-7 that $KA > KC = KM$.

Next, we prove (in either case) that if $\angle A \neq 60°$), then $KA > KM$.

Since (in Case (a)) KP is a diameter of Ω, $K \neq O$, $KA > KC$ by Euclid's proposition III-7, and $KC > OC = OM > KM$. because $\angle COK = 120°$, it follows that $KA > KM$.

Also, in Case (b), we have $\angle A < 60°$ and KP is a diameter of Ω. Thus we get by Euclid's proposition III-7 that $KA > KQ$. Since $\angle KQM = \angle KQC = \angle KAC = \frac{1}{2}\angle A < 30°$ and $\angle KMQ = 30°$, we deduce that $KQ > KM$ in $\triangle KQM$. So, we conclude from $KA > KQ$ and $KQ > KM$ that in Case (b) we have also that $KA > KM$.

Therefore (in either case) $KA > KM$ and from (13) we deduce that $\theta - \phi > \mu - \delta$ as wanted. □

Theorem 3. *Let all angles of $\triangle ABC$ be less than $120°$, the cevians from B and C through the Fermat point F meet the external angle bisectors of B and C at D and E and meet AC and AB at B' and C' respectively, as shown in Figure 4 below.*

Then we have the following:

(a) The statement $AB = AC$ is equivalent to each of the statements

$$\begin{gathered}\textit{(i) } CE = BD, \quad \textit{(ii) } BE = CD, \quad \textit{(iii) } [EC'B] = [DB'C], \\ \textit{(iv) } \operatorname{per}(EC'B) = \operatorname{per}(DB'C).\end{gathered} \tag{14}$$

(b) The statement $AB > AC$ is equivalent to each of the statements

$$\begin{gathered}\textit{(i) } CE > BD, \quad \textit{(ii) } BE > CD, \quad \textit{(iii) } [EC'B] > [DB'C], \\ \textit{(iv) } \operatorname{per}(EC'B) > \operatorname{per}(DB'C).\end{gathered} \tag{15}$$

Proof. Let $\angle B = 2\beta$, $180° - \angle B = 2\beta'$ and $\angle C = 2\gamma$, $180° - \angle C = 2\gamma'$ as shown in Figure 4. Since $60° = \mu+\xi = \omega+\delta = \phi+\theta > \theta > \phi$ by (3) and (6) and $2\beta' = \angle A+\angle C = \omega+\theta+60°$, it follows that $\beta' > \theta$. Therefore the cevian from C through F meets the external angle bisector of B at E. Similarly we have $2\gamma' = \angle A + \angle B = \xi + \phi + 60°$ and $60° > \phi$. Thus $\gamma' > \phi$ and the cevian from B through F meets the external angle bisector of C at D.

Note that if $AB = AC$, then by (1) we have $\theta = \phi$, $\mu = \delta$, $FB = FC$ and it is clear that $\gamma' = \beta'$ and hence $\triangle DBC \cong \triangle ECB$. Thus we conclude that $CE = BD$, $BE = CD$, and all equalities in (14) hold. So, in view of this note and as shown in Theorem 1, it is enough to prove that if $AB > AC$, then all the inequalities in (15) hold and the rest of (a) and (b) will follow by contradiction. Note also that $\gamma > \beta$ and $\beta' > \gamma'$.

(b-i) Applying the sine law to triangles ECB and DBC we get

$$\frac{CE}{\sin(\beta')} = \frac{BC}{\sin(\beta' - \theta)} = \frac{BE}{\sin(\theta)}, \tag{16}$$

$$\frac{BD}{\sin(\gamma')} = \frac{BC}{\sin(\gamma' - \phi)} = \frac{CD}{\sin(\phi)}. \tag{17}$$

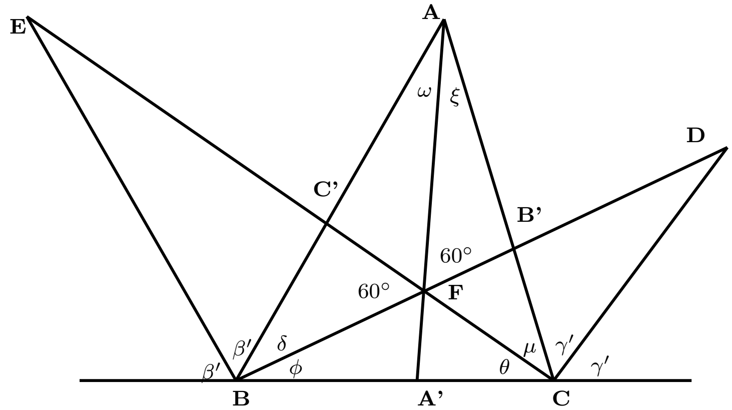


Figure 4: Illustrating the proof of Theorem 3

Since we have

$$\begin{gathered}\beta' = \frac{A+\mu+\theta}{2} > \gamma' = \frac{A+\delta+\phi}{2}, \quad \beta'+\gamma' = \frac{A}{2}+90^\circ < 180^\circ, \\ 0 < \gamma'-\phi = \frac{A+\delta-\phi}{2} < 90^\circ, \quad 0 < \beta'-\theta = \frac{A+\mu-\theta}{2} < 90^\circ \quad \text{and,} \\ \theta-\phi > \mu-\delta \text{ by Theorem 2, it follows that } \delta-\phi > \mu-\theta \text{ and hence} \\ 90^\circ > \gamma'-\phi = \frac{A+\delta-\phi}{2} > \frac{A+\mu-\theta}{2} = \beta'-\theta > 0.\end{gathered} \tag{18}$$

Thus by (16), (17), and (18) we have

$$\frac{CE}{BD} = \frac{\sin(\beta')\sin(\gamma'-\phi)}{\sin(\gamma')\sin(\beta'-\theta)} > 1.$$

Therefore $CE > BD$ as required in *(b-i)*.

(b-ii) From (16), (17), (18), and $\theta > \phi$ by (6), we get

$$\frac{BE}{CD} = \frac{\sin(\theta)\sin(\gamma'-\phi)}{\sin(\phi)\sin(\beta'-\theta)} > 1.$$

Thus $BE > CD$ as required in *(b-ii)*.

(b-iii) Since $[EC'B] = \frac{1}{2}(BE)(BC')\sin(\beta')$, $[DB'C] = \frac{1}{2}(CD)(CB')\sin(\gamma')$, $BE > CD$ by *(b-ii)*, $BC' > CB'$ by (2), $\beta' > \gamma'$ and $\beta'+\gamma' < 180^\circ$ by (18), it follows that $\sin(\beta') > \sin(\gamma')$ and hence $[EC'B] > [DB'C]$.

(b-iv) Since we have by *(b-ii)* that $CE = CC' + C'E > BD = BB' + B'D$ and by (2) that $BB' > CC'$, it follows that $C'E > B'D$, we have $BE > CD$ by *(b-ii)*, and $BC' > CB'$ by (2). Thus $\text{per}(EC'B) = C'E + BE + BC' > B'D + CD + CB' = \text{per}(DB'C)$ and the proof is complete. □

In the next theorem we consider the case where the circumcevians of the Fermat point F from B and C meet the circumcircle of triangle ABC at K and L respectively.

Theorem 4. *Let Ω be the circumcircle of $\triangle ABC$ with all angles less than 120° and the $\angle A \neq 60^\circ$ and let the circumcevians BK and CL of the Fermat point F be equal, as shown in Figure 5. Then the triangle ABC is isosceles with $AB = AC$.*

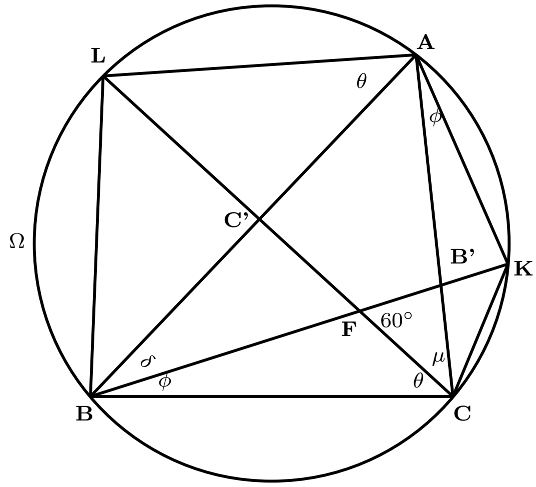


Figure 5: Illustrating the proof of Theorem 4

Proof. Since, in the circle Ω, the angles CAK and CBK subtend the chord CK and $\angle CBK = \phi$, it follows that $\angle CAK = \angle CBK = \phi$ and similarly the angles BCL and BAL subtend the chord LB and $\angle BCL = \theta$, and hence $\angle BAL = \angle BCL = \theta$.

So to complete the proof of the theorem we apply the sine law to $\triangle ABK$ and $\triangle ACL$ and get

$$\frac{BK}{\sin(A+\phi)} = \frac{AB}{\sin(K)} = \frac{AB}{\sin(C)} \text{ and } \frac{CL}{\sin(A+\theta)} = \frac{AC}{\sin(L)} = \frac{AC}{\sin(B)}.$$

Therefore

$$\frac{BK}{CL} = \frac{AB\sin(B)\sin(A+\phi)}{AC\sin(C)\sin(A+\theta)}.$$

But by the sine law applied to $\triangle ABC$ we get $AB\sin(B) = AC\sin(C)$. Thus

$$\frac{BK}{CL} = \frac{\sin(A+\phi)}{\sin(A+\theta)}.$$

But $BK = CL$. Therefore $\sin(A+\phi) = \sin(A+\theta)$. Since $\sin(A+\phi) = \sin(A+\theta)$ only if $A+\phi = A+\theta$ or $2A+\phi+\theta = 180°$ and $\phi+\theta = 60°$ by (3), it follows that $\sin(A+\phi) = \sin(A+\theta)$ only if $\phi = \theta$ or $\angle A = 60°$. But by assumption $\angle A \neq 60°$. Thus $\phi = \theta$ and hence $FB = FC$ and by (1) the triangle ABC is isosceles with $AB = AC$ as required

Note that if $AB > AC$, then $BC' > CB'$ by (2) and if also $\angle A = 60°$ then we have $AC' = AB'$ by (11) and it is clear that $\angle BKC = \angle BLC = \angle A = 60°$ for they are subtended by the chord BC and hence the triangles BLF and CKF are equilateral triangles. Therefore $BK = BF + FK = LF + FC = CL$. □

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