Journal for Geometry and Graphics
Volume 29 (2025), No. 1, 65–77

Invariant Lines and a Projective Geometric Generalisation of a Geometry Problem

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Abstract. Generalisations of mathematical problems often require innovative approaches and open up promising avenues for further research. In this article, we propose a generalisation of a geometry problem originally featured in the 1995 International Mathematical Olympiad contest. We employ projective geometric techniques to rigorously prove our generalisation, demonstrating the value of applying advanced mathematical tools to extend the boundaries of traditional problems. Even though this is a geometry problem originally intended for students, it opens up many interesting ideas for generalisation and the inclusion of more advanced tools to prove these generalisations. We define a special transformation with respect to two conic sections and a line intersecting the conics, and we prove several properties of the transformation that provide a solution to our generalised problem. Our main aim is to determine the invariant lines with respect to the transformation as a generalisation of the original IMO problem.

Key Words: Projective geometry tools, projectivity, perspectivity, invariant lines, generalisation

MSC 2020: 51M15 (primary), 51A05, 51N15

1 Introduction and organization of the paper

1.1 Introductory remarks

One interesting geometry problem appeared in the 1995 edition of the International Mathematical Olympiad (IMO) (see [3, Problem 1 on p. 275]), with two solutions presented in [3, Solution to Problem 7, p. 595-596] using the power of a point, cyclic quadrilaterals, and the

ISSN 1433-8157/

similarity of triangles. Additional solutions that employ the radical axis and the radical centre can be found in [1], while another approach employing analytic geometry tools is presented in [6].

Original IMO Geometry Problem. Let A, B, C, D be four distinct points on a line, in that order. The circles with diameters AC and BD intersect at X and Y. The line XY meets BC at Z. Let P be a point on the line XY other than Z. The line CP intersects the circle with diameter AC at C and M, and the line BP intersects the circle with diameter BD at B and N. Prove that the lines AM, DN, XY are concurrent (Figure 1).

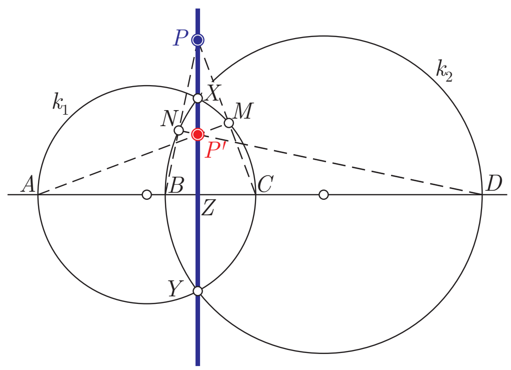

Figure 1: 36th IMO geometry problem

One possible approach to solving this problem is to prove that if P' is the intersection point of the lines AM and DN, then P' lies on the line XZ. More generally, the line XY is invariant under the given drawing process.

The line XY in the original IMO geometry problem is the radical axis of the given intersecting circles. This raised the question: what happens when, instead of intersecting circles, we have two arbitrary circles, and for the line XY, we take their radical axis? More generally, what if, instead of radical axis, we consider any line perpendicular to one connecting the centres of these circles?

This led to the following generalisation, which Kamber Hamzić, Németh, and Šabanac [5, p. 2] proved analytically.

First Generalisation of the IMO Geometry Problem. Let k_1 and k_2 be two circles, and let ℓ be the line that contains their centres. The line ℓ intersects the circle k_1 at points A_1 and A_2 and the circle k_2 at points B_1 and B_2. We assume that these points appear in either the order A_1, B_1, A_2, B_2 or A_1, A_2, B_1, B_2 along ℓ. Let p be a line perpendicular to ℓ, and let P be any point on p. The line PA_2 intersects k_1 at $P_{\mathcal{A}2}$ (in addition to A_2), and the line PB_1 intersects k_2 at $P_{\mathcal{B}1}$ (in addition to B_1). The lines $A_1P_{\mathcal{A}2}$ and $B_2P_{\mathcal{B}1}$ intersect at P'_{21}. As the position P varies, the position of P'_{21} also changes. However, P'_{21} always belongs to a line p'_{21} parallel to p (Figure 2). If p is the radical axis of k_1 and k_2, then $p'_{21} = p$.

1.2 Structure of the Paper

In this paper, we present intriguing results that emerge when, instead of considering the line ℓ passing through the centres of the given circles, we examine any line intersecting the circles at four points. Furthermore, rather than focusing solely on the line p perpendicular to ℓ, we

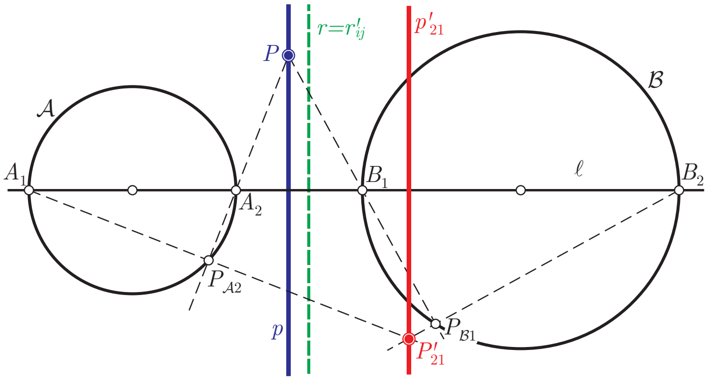

Figure 2: First generalisation of 36th IMO geometry problem

analyse any line $p \neq \ell$. Moreover, instead of limiting our study to two circles, we extend our observations to the case of any two conics.

In Section 1.2, we introduce the necessary terminology and present our results, which generalise the IMO geometry problem. In Section 2, we provide rigorous proofs of these generalisations. Finally, in Section 3, we summarise our findings on invariant lines and propose further exploration of the transformation IMO_{ij} (see Definition 1.1 below for details), particularly in relation to conic sections, raising questions about the existence of invariant conics and conditions under which their images remain conics.

sectionFurther Generalisations of the IMO Geometry Problem

Before generalising the IMO geometry problem further, we begin this section by defining the drawing process we will be using throughout the paper.

Definition 1.1 (IMO drawing)**.** Let $\mathcal{A}$ and $\mathcal{B}$ be two distinct circles (or, more generally, conics), and let ℓ be a line that intersects $\mathcal{A}$ and $\mathcal{B}$ at points A_1, A_2, and B_1, B_2, respectively, where all intersection points exist and are distinct (we suppose that ℓ always intersects each conic at exactly two points). The orientations of A_1, A_2 and B_1, B_2 are assumed to be the same (e.g., without restrictions, from left to right). Let P be a given point. Define $P_{\mathcal{A}i}$ and $P_{\mathcal{B}j}$ as the second intersection points of the lines PA_i and PB_j with $\mathcal{A}$ and $\mathcal{B}$, respectively, where $i, j \in \{1, 2\}$ (see Figure 3 for the case $i = 1$ and $j = 2$). Let P'_{ij} be the intersection point of the lines $A_{i^\star}P_{\mathcal{A}i}$ and $B_{j^\star}P_{\mathcal{B}j}$, where $i + i^\star = j + j^\star = 3$. The process of obtaining P'_{ij} (if it exists) from P is called the IMO_{ij} *drawing*. Then, P'_{ij} is the image of the point P with respect to the IMO_{ij} drawing.

Let t_{A_i} and t_{B_i} be the tangent lines at A_i and B_i, respectively, for $i \in \{1, 2\}$, to the corresponding circles (or conics) $\mathcal{A}$ and $\mathcal{B}$. Let T_{ij} denote the intersection points of t_{A_i} and t_{B_j}.

It is easy to see that no point on ℓ has an image. Moreover, the image of all points on the lines t_{A_i} and t_{B_j}, except for their intersection point T_{ij}, is $B_{j^\star}$ and $A_{i^\star}$, respectively. For example, in Figure 3, if X is on t_{A_1}, then its image with respect to IMO_{12} is $X'_{12} = B_1$, as $X_{\mathcal{A}i} = A_1$.

Definition 1.2. Define p'_{ij} to be the image of a given line p, where $p \neq \ell$. That is, p'_{ij} is the locus of the images of the points of p under IMO_{ij}.

Now, it is straightforward to verify the following proposition.

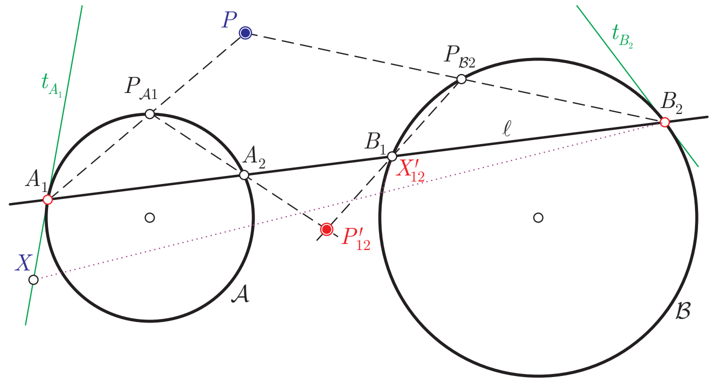

Figure 3: Image of P with respect to IMO_{12} drawing

Proposition 1.3. *The image of the point P'_{ij} with respect to $\mathrm{IMO}_{i^\star j^\star}$ is the point P. Moreover, the image of p'_{ij} with respect to $\mathrm{IMO}_{i^\star j^\star}$ is the line p.*

We extend the Euclidean plane by adding points at infinity so that T_{ij} can also be at infinity. In Figure 2, for all i, j, the points T_{ij} are at infinity and coincide. Moreover, let W_∞ denote the point at infinity on the line p. We denote the intersection point of lines p and ℓ by S. (S can be at infinity.)

Using Definition 1.1, we now can state the First Generalisation of the IMO Geometry Problem in the following equivalent form.

Equivalent form of the First Generalisation of the IMO Geometry Problem. Let $\mathcal{A}$ and $\mathcal{B}$ be two circles that do not necessarily intersect, and let ℓ be the line containing their centres. Let p be a line perpendicular to ℓ. Then, for all $i, j \in \{1, 2\}$, the image of the line p with respect to the IMO_{ij} drawing is a line parallel to p. If p is the radical axis of the circles, then p is invariant, so $p'_{ij} = p$.

In Figure 2, the line r is the radical axis of $\mathcal{A}$ and $\mathcal{B}$, and the figure illustrates the IMO_{21} drawing.

In this section, we further generalise the IMO geometry problem in two ways. First, we consider ℓ as a general line (not necessarily passing through the centres of the circles). Second, we extend our analysis to conics instead of circles. Our main objective is to determine invariant lines with respect to an IMO_{ij} drawing.

There are four possible drawing processes. In this article, for the case of circles, we assume — without restrictions — that the centres of $\mathcal{A}$ and $\mathcal{B}$, as well as the points A_1, A_2, and B_1, B_2, are arranged from left to right. In our theorems and figures, we primarily focus on the case $i = 2$, $j = 1$, which we refer to as the IMO drawing. However, similar statements hold for the other cases, which we occasionally present as well.

Moreover, we omit the indices i and j from $P_{\mathcal{A}i}$, $P_{\mathcal{B}j}$, P'_{ij}, and p'_{ij} when this does not cause confusion. We simply write them as $P_{\mathcal{A}}$, $P_{\mathcal{B}}$, P', and p'.

1.3 Generalisation Involving Circles

Let $\mathcal{A}$ and $\mathcal{B}$ be two circles, and let ℓ be a general line that intersects them as described in Definition 1.1.

Theorem 1.4 (Second Generalisation of the IMO Geometry Problem)**.** *Let $p \neq \ell$ be a line. Then, the image of p with respect to the* IMO_{ij} *drawing is a conic p'_{ij} passing through the points $A_{i^\star}$, $B_{j^\star}$, $T_{i^\star j^\star}$, and W'_{ij} (the image of W_∞), where $i, j \in 1, 2$ and $i + i^\star = j + j^\star = 3$ (see Figures 4 and 5). Moreover, if p contains any of the points A_i, B_j, or T_{ij}, then p'_{ij} is a line, meaning the conic p'_{ij} is degenerate (see Figure 6, where p passes through $T_{ij} = T_{21}$).*

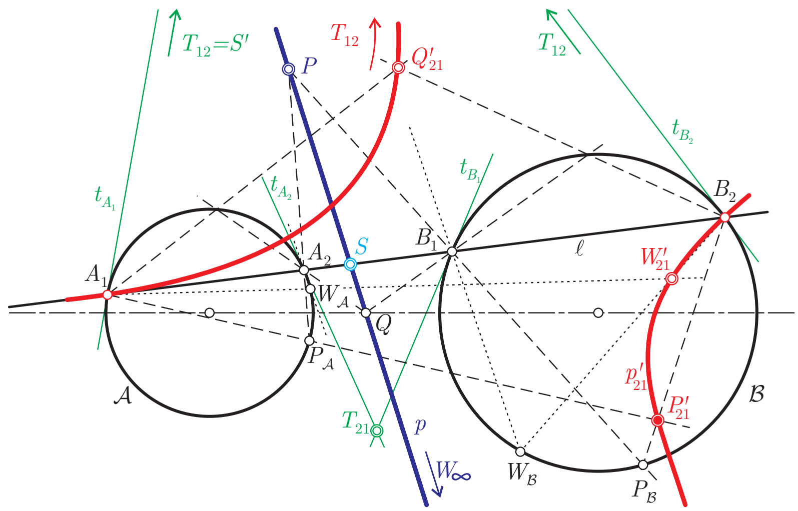

Figure 4: Line and its image with respect to IMO_{21} drawing

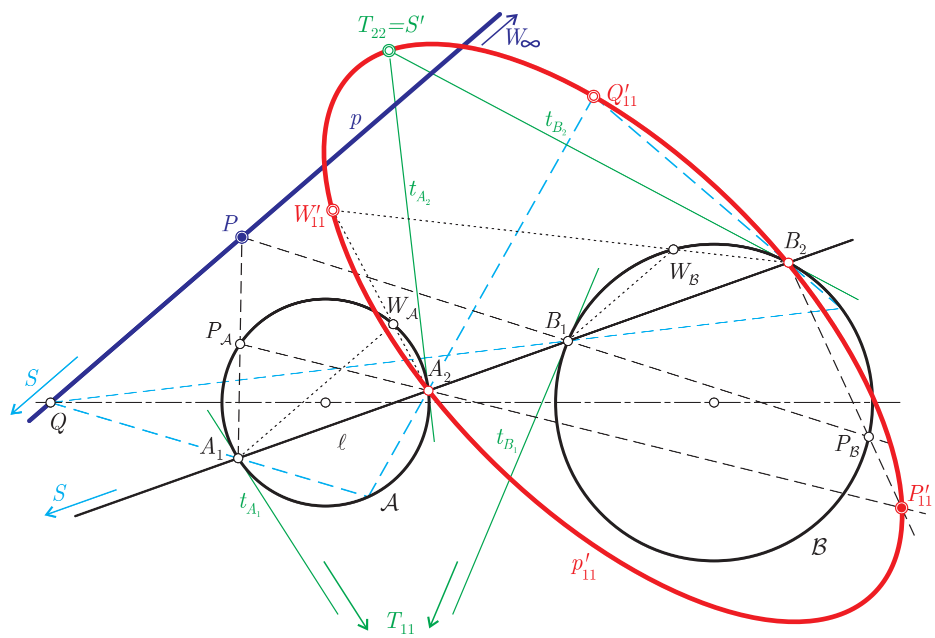

Figure 5: Line and its image with respect to IMO_{11} drawing

We note that the points $A_{i^\star}$, $B_{j^\star}$, and $T_{i^\star j^\star}$ are determined by the given circles and the line ℓ, while W'_{ij} is determined by the direction of p. We observe that the image of $S = \ell \cap p$ with respect to the IMO_{ij} drawing is $T_{i^\star j^\star}$.

Let K and $\overline{K}$ be the external and internal homothety centres, respectively, of the circles $\mathcal{A}$ and $\mathcal{B}$ in the Euclidean plane. We note that K and $\overline{K}$ do not necessarily exist, for example, in the cases of intersecting or concentric circles.

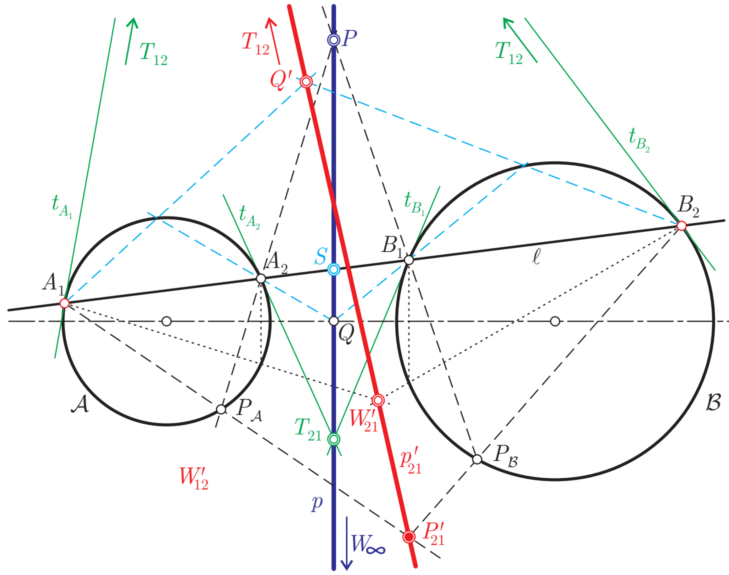

Figure 6: Line with the point T_{21} with respect to IMO_{21} drawing

Recall, according to Definition 1.1 the line ℓ always intersects $\mathcal{A}$ and $\mathcal{B}$ and the order of their indices are the same. Now we give a statement for the case when ℓ contains at least one of homothety center.

Theorem 1.5. *If the external (internal) homothety centre K ($\overline{K}$) of the circles $\mathcal{A}$ and $\mathcal{B}$ exists and ℓ passes through it, then their radical axis is an invariant line with respect to the* IMO_{12} *and* IMO_{21} *(*IMO_{11} *and* IMO_{22}*) drawings.*

We will show that Theorems 1.4 and 1.5 are corollaries of more general theorems, for which we will use projective geometry tools.

1.4 Projective Generalisation

We now consider the case of conics instead of circles, which is more general setting than the previous subsection. Let $\mathcal{A}$ and $\mathcal{B}$ be two distinct non-degenerate conics. We intersect them with a line ℓ according to Definition 1.1, as shown in Figure 7.

Theorem 1.6 (Third Generalisation of the IMO Geometry Problem)**.** *If a general line p contains none of the points A_i, B_j, or T_{ij}, then the image of p under the* IMO_{ij} *drawing is a conic p'_{ij} passing through the points $A_{i^\star}$, $B_{j^\star}$, and $T_{i^\star j^\star}$.*

Theorem 1.7. *If the line p contains at least one of the points A_i, B_j, or T_{ij}, then the image of p under the* IMO_{ij} *drawing is a line p'_{ij} (or, equivalently, p'_{ij} is a degenerate conic).*

Corollary 1.8. *If $p'_{ij} = p$, then p passes through the points T_{ij} and $T_{i^\star j^\star}$.*

Corollary 1.9. *If p passes through the points T_{ij} and $T_{i^\star j^\star}$, and $T_{i^\star j^\star}$ is a point at infinity, then p and p'_{ij} are either parallel or coincident lines.*

Let K be the intersection point of two common tangent lines of the conics $\mathcal{A}$ and $\mathcal{B}$. Then the following theorem gives the condition for the line p to be invariant with respect to the IMO_{ij} drawing.

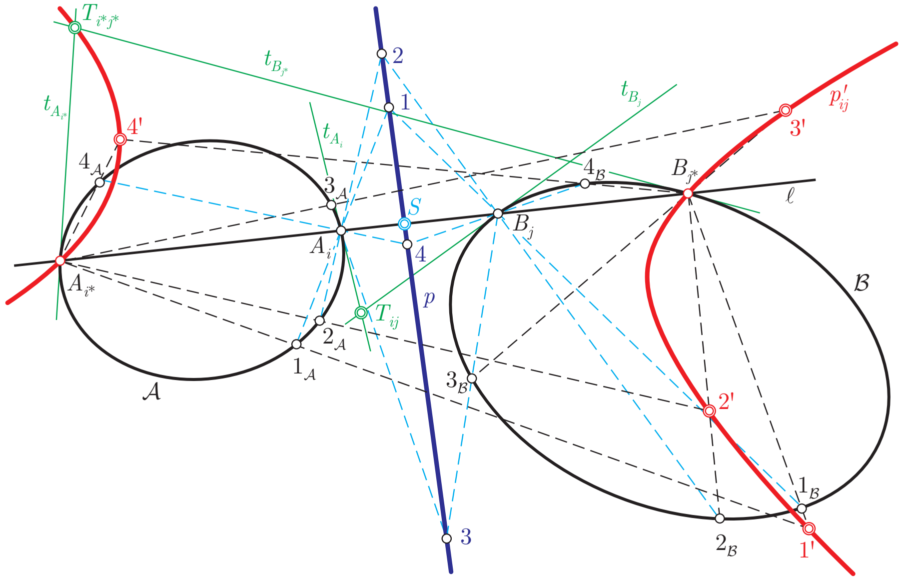

Figure 7: Generalised IMO geometry problem with conics

Theorem 1.10. *Let π be the perspective collineation defined by two non-degenerate conics $\mathcal{A}$ and $\mathcal{B}$ with center K, where A_i and B_j are corresponding points, and suppose $\mathcal{A}$ and $\mathcal{B}$ are images of each other under π. If the line ℓ passes through the center K, then the axis p of π is invariant with respect to the* IMO_{ij} *drawing.*

Moreover, the pencil of lines through K corresponds under π to the pencil of lines through $\overline{K}$ (the second center of collineation determined by $\mathcal{A}$ and $\mathcal{B}$). Consequently, there exist exactly two fixed lines in the generic case—one through K and one through $\overline{K}$—or, in exceptional degenerate cases, infinitely many fixed lines.

2 Proofs of Generalisations

In this section, we prove the aforementioned generalisations. First, we establish the generalisations for the case of conics and demonstrate that the case of circles follows as a corollary of the conic case.

Before presenting the proofs, we introduce some preliminaries on projective tools that will be used throughout.

2.1 Projective Tools

In this subsection, we provide a brief summary of the key terms we use. For a more detailed introduction, see, for example, the books by Coxeter [2] and Glaeser, Stachel, and Odehnal [4, Chapter 5 and 6].

Projective Plane. We extend each line of the Euclidean plane by adding an extra point, known as the point at infinity, so that parallel lines share the same infinity point. All infinity points together form a line, known as the line at infinity. With this extension, we obtain the (real) projective plane.

Range of Points. The set of all (finite and infinite) points on a (finite or infinite) line in the projective plane is called a range of points.

Pencil of Lines. The set of all lines in the projective plane passing through a common (finite or infinite) point is called a pencil of lines. The common point of a pencil of lines is called its vertex.

Perspectivity and Projectivity. Two ranges of points are called perspective if their corresponding points lie on a common line in a pencil of lines. Two pencils of lines are called perspective if the intersection points of their corresponding lines lie in a common range of lines. A range of points and a pencil of lines are called perspective if each point in the range lies on its corresponding line in the pencil [4, p. 188].

The product of arbitrarily many perspectivities is a projectivity. Projectivities preserve cross-ratios.

Steiner theorem. Two pencils of lines at different points are projective (not merely perspective) if and only if the intersection points of corresponding lines lie on a non-degenerate projective conic (see, e.g., [2, Theorem 8.51 on p. 80]).

2.2 Proof of Theorem 1.6

We consider the line p as a range of points and denote it by $[p]$. We take the pencil of lines through the points A_i, denoted by $[A_i]$, such that $[p]$ and $[A_i]$ are perspective, written as $[p] \,\overline{\overline{\wedge}}\, [A_i]$. For example, the point 1 in $[p]$ corresponds to the line A_i1 in $[A_i]$, and let $1_{\mathcal{A}}$ be its other intersection point with $\mathcal{A}$. The central projection with respect to A_i establishes a one-to-one correspondence between the points on the conic $\mathcal{A}$ and those on $[p]$, including their point at infinity (see Figure 7).

According to Steiner's theorem for projective pencils of lines, the pencil $[A_{i^\star}]$ is projective to $[A_i]$ if and only if the intersection points of their corresponding lines lie on $\mathcal{A}$. For example, the line $A_{i^\star}1_{\mathcal{A}}$ corresponds to the line $A_i1_{\mathcal{A}}$. Therefore, $[A_{i^\star}]$ and $[A_i]$ are projective, written as $[A_{i^\star}] \barwedge [A_i]$.

Similarly, we consider the pencils of lines $[B_j]$ and $[B_{j^\star}]$, so that $[p] \,\overline{\overline{\wedge}}\, [B_j]$ and $[B_j] \barwedge [B_{j^\star}]$.

Since perspectivity implies projectivity and projectivity is transitive, the pencils $[A_{i^\star}]$ and $[B_{j^\star}]$ are projective, denoted as $[A_{i^\star}] \barwedge [B_{j^\star}]$. Therefore, the intersection points of their corresponding lines lie on a conic p'_{ij}, which contains the points $A_{i^\star}$, $B_{j^\star}$, $T_{i^\star j^\star}$.

Note that Figure 7 shows the corresponding points on p, $\mathcal{A}$, $\mathcal{B}$, and p'_{ij}.

2.3 Proof of Theorem 1.7

Building on the proof of Theorem 1.6, the pencils of lines $[A_{i^\star}]$ and $[B_{j^\star}]$ are perspective, written as $[A_{i^\star}] \,\overline{\overline{\wedge}}\, [B_{j^\star}]$, since the common line $A_{i^\star}B_{j^\star}$ corresponds to itself. See the example in Figure 8, where p passes through the point T_{ij}.

As the point P on p approaches T_{ij}, both lines $A_{i^\star}P_{\mathcal{A}}$ and $B_{j^\star}P_{\mathcal{B}}$ tend towards the line $A_{i^\star}B_{j^\star} = \ell$. By Steiner's theorem for perspective pencils of lines, p'_{ij} must be a line.

The other two cases, where p passes through the points A_i or B_j, can be proved similarly.

2.4 Proof of Theorem 1.10

First, we determine how to choose the line ℓ such that p can be an invariant line with respect to the IMO$_{ij}$ drawing. We define a perspective collineation (or central collineation) between the non-degenerate conics $\mathcal{A}$ and $\mathcal{B}$, where the axis of this collineation remains invariant. We assume that $\mathcal{A}$ and $\mathcal{B}$ share common tangent lines e and f, such that their intersection point

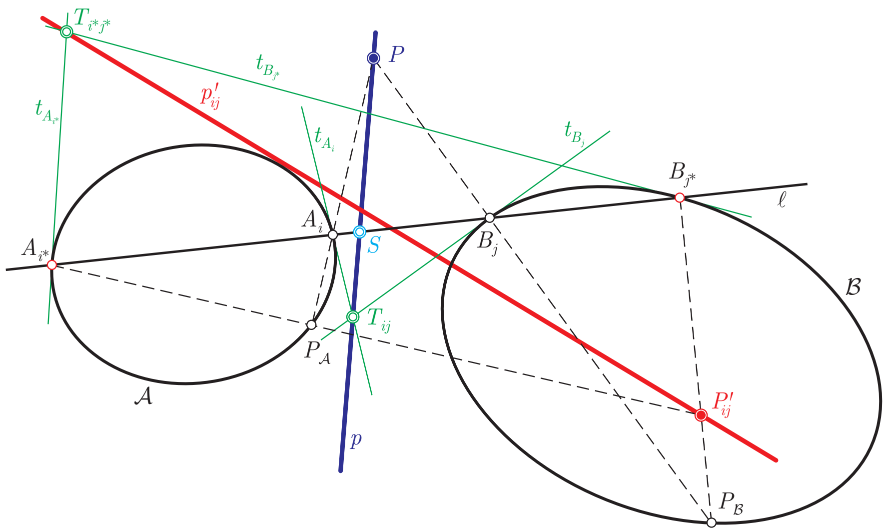

Figure 8: Line p with the point T_{ij}

K serves as the centre of the collineation. Consequently, every line passing through K and intersecting one of the conics must also intersect the other conic (see Figure 9 and [4] for further details). In the general case, there are four common tangent lines to the conics, and among their intersection points, only two can serve as centres of collineation: K and $\overline{K}$.

Let L be the intersection point of the polars of K with respect to $\mathcal{A}$ and $\mathcal{B}$ (where the polar lines are the lines of tangent points with common tangent lines, respectively). Let $E_{\mathcal{A}}$, $E_{\mathcal{B}}$, $F_{\mathcal{A}}$, and $F_{\mathcal{B}}$ denote the points of tangency on the tangent lines e and f, respectively, as shown in Figure 9.

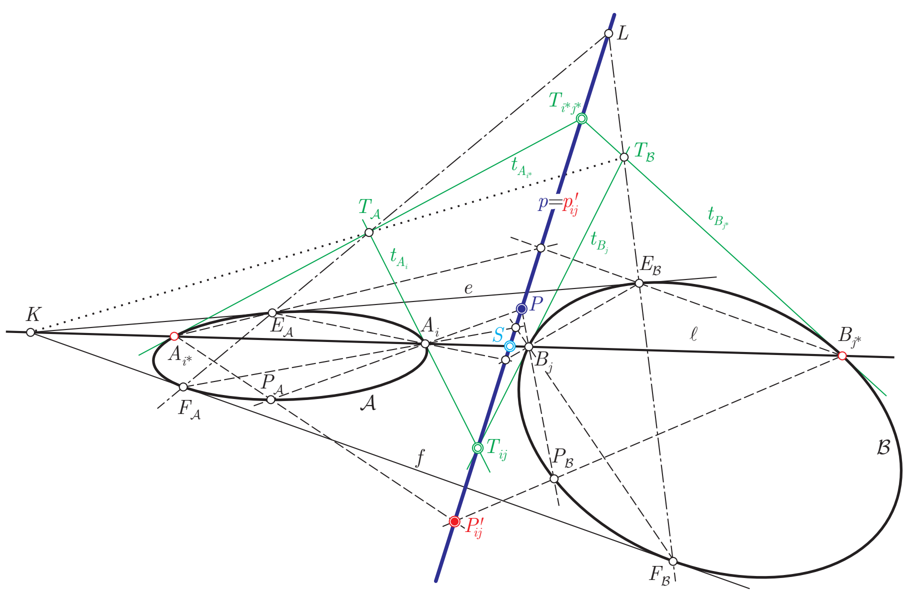

Figure 9: Perspective collineation (or central collineation) between $\mathcal{A}$ and $\mathcal{B}$

We need the following lemma.

Lemma 2.1. *Let ℓ be a line passing through the center K of a perspective collineation π between two non-degenerate conics $\mathcal{A}$ and $\mathcal{B}$. Assume that ℓ intersects $\mathcal{A}$ at points A_i and*

$A_{i^\star}$, and $\mathcal{B}$ at the corresponding points B_j and $B_{j^\star}$, in such a way that the ordering of points on ℓ is consistent with the correspondence defined by π. Then the points T_{ij}, $T_{i^\star j^\star}$, and L are collinear, and the line p containing them is the axis of π.

Proof. A perspective collineation is defined by its centre and three pairs of corresponding points. Let the centre be K, and let the corresponding point pairs be $E_\mathcal{A} - E_\mathcal{B}$ and $F_\mathcal{A} - F_\mathcal{B}$. Since the lines $E_\mathcal{A}F_\mathcal{A}$ and $E_\mathcal{B}F_\mathcal{B}$ correspond to each other under the perspective collineation, their intersection point L is a fixed point and thus lies on the axis. (If $E_\mathcal{A} = E_\mathcal{B}$ and $F_\mathcal{B} = F_\mathcal{B}$, then they automatically lie on the axis.) The perspective collineation is therefore defined. For additional corresponding points, we consider ℓ and its intersecting points with conics (as previously), which also form corresponding point pairs: $A_i - B_j$ and $A_{i^\star} - B_{j^\star}$.

The identification of these corresponding pairs depends on the order of the intersection points along ℓ relative to K and $\overline{K}$. We assume that the labels are chosen so that the ordering on ℓ is consistent with the correspondence induced by π. If the order is interleaved differently, the labels of $\{B_j, B_{j^\star}\}$ must be swapped so that (A_i, B_j) and $(A_{i^\star}, B_{j^\star})$ remain true corresponding pairs. Without this alignment, the subsequent statement about the fixed points T_{ij} and $T_{i^\star j^\star}$ may not hold.

Furthermore, their tangent lines also correspond to each other, so $t_{A_i} - t_{B_j}$ and $t_{A_{i^\star}} - t_{B_{j^\star}}$. Therefore, their intersecting points T_{ij} and $T_{i^\star j^\star}$ are fixed and lie on the axis of the perspective collineation. If ℓ contains both K and $\overline{K}$, then T_{ij}, $T_{i^\star j^\star}$, and L coincide. In this case, we must choose another pair of points and their tangents on the conics to determine the line p. □

Remark 1. Since the pencils of (possibly parallel) lines through K and through $\overline{K}$ correspond projectively, the projective correspondence of the two pencils has exactly two fixed lines (or infinitely many in the degenerate case when the correspondence is the identity). In our setting, one fixed line is the axis p through K, as constructed above, and the other passes through $\overline{K}$. This is consistent with the general theory of projective correspondences of line pencils and is also directly visible from the above construction.

Using Lemma 2.1, we now establish the proof of Theorem 1.10. Since the lines $PA_iP_\mathcal{A}$ and $PB_jP_\mathcal{B}$ are the corresponding lines with respect to π, the points $P_\mathcal{A}$ and $P_\mathcal{B}$ form a pair of corresponding points. The intersecting point P'_{ij} of the corresponding lines $A_{i^\star}P_\mathcal{A}$ and $B_{j^\star}P_\mathcal{B}$ is fixed and must be on the axis p.

2.5 Proof of Theorems 1.4 and 1.5

Theorem 1.4 follows directly from Theorem 1.6 and its related corollaries.

In Figure 4, the conic p'_{21} is determined by the points A_1, B_2, T_{12} (the image of S), Q'_{21}, and W'_{21} (the image of the point at infinity on p).

We now turn our attention to Theorem 1.5. Applying Theorem 1.10 to circles, Figure 9 transforms into Figure 10.

We need to prove that the line p is not only the axis of the central collineation but also the radical axis of the circles.

Consider the circle $\mathcal{C}$, which is tangent to $\mathcal{A}$ at A_2 and to $\mathcal{B}$ at B_1. The existence of such a circle $\mathcal{C}$ is guaranteed (see [7, Theorem 61.e on p. 41]). Since t_{A_2} is the radical axis of $\mathcal{A}$ and $\mathcal{C}$, and t_{B_1} is the radical axis of $\mathcal{C}$ and $\mathcal{B}$, their intersection point T_{21} is the radical centre of the three circles. Thus, T_{21} lies on the radical axis of $\mathcal{A}$ and $\mathcal{B}$, which is perpendicular to the line connecting their centres.

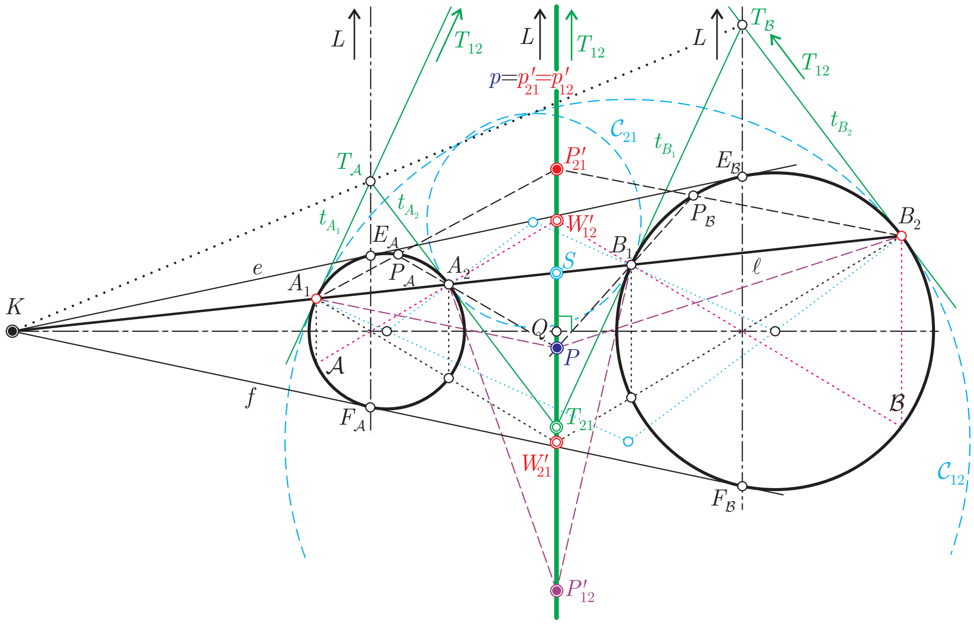

Figure 10: Invariant radical axis with respect to IMO_{21} and IMO_{12} drawings

For IMO_{21}, similarly, p is an invariant line. However, in the cases of IMO_{11} and IMO_{22}, the perspective collineation simplifies to a central homothety (symmetry), and the points T_{11}, T_{22}, and L either disappear or move to infinity. As a result, there is no invariant line (or the line at infinity becomes the invariant line).

If ℓ contains the internal homothety centre of the circles, then the situations is reversed (see Figure 11).

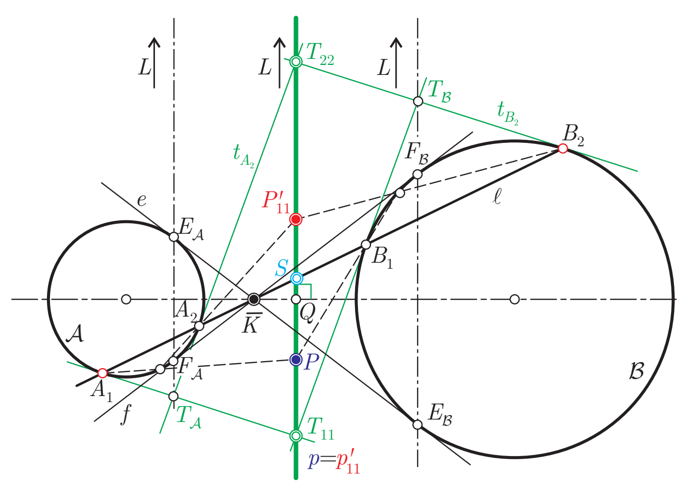

Figure 11: Invariant radical axis with internal homothety centre and with respect to IMO_{11} drawing

Moreover, if both homothety centres lie on ℓ, then in all IMO_{ij} cases, the points T_{ij} are at infinity, leading to the first generalisation of the problem [5], as shown in Figure 2.

Finally, in Figure 12, we present a case where p passes through the point B_1, with the drawing process corresponding to IMO_{21}.

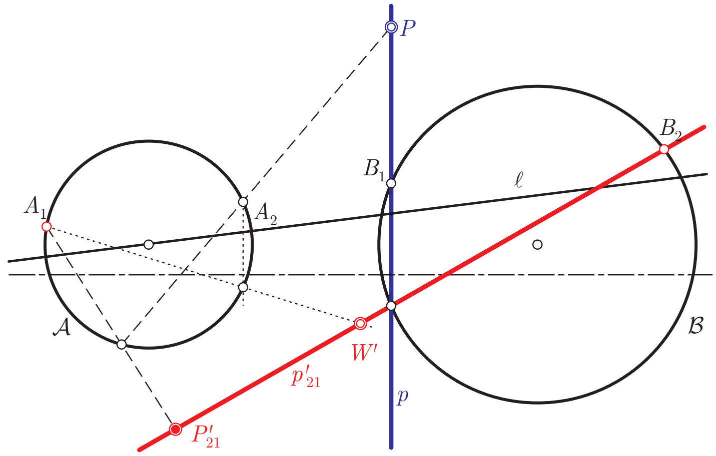

Figure 12: Line with the point B_1 and IMO_{21} drawing

3 Concluding Remarks

In this article, we identified invariant lines as a generalisation of the original IMO problem. We have achieved our main goal, yet the transformation IMO_{ij} we have introduced still holds great potential. It would be worthwhile to consider the images of conics in a similarly general way as we have done for straight lines. This raises further intriguing questions. Are there invariant conic sections under the IMO_{ij} transformation? Under what conditions does the image of a conic remain a conic? It seems that the maps IMO_{ij} are *quadratic birational maps (quadratic Cremona transformations)* which are well studied in classical geometry and also play a role in modern triangle geometry (see [4, Ch. 7.5]), The image of a conic is, in general, a quartic curve but the degree may drop if the original conic contains base points of the transformation, as observed in Figure 13, where we present the conic $\mathcal{C}$ passes through the points A_2 and B_1. In this case, its image under the IMO_{21} drawing is also a conic, containing the points A_1 and A_2.

Acknowledgment

The authors are grateful the referee for the useful comments and suggestions.

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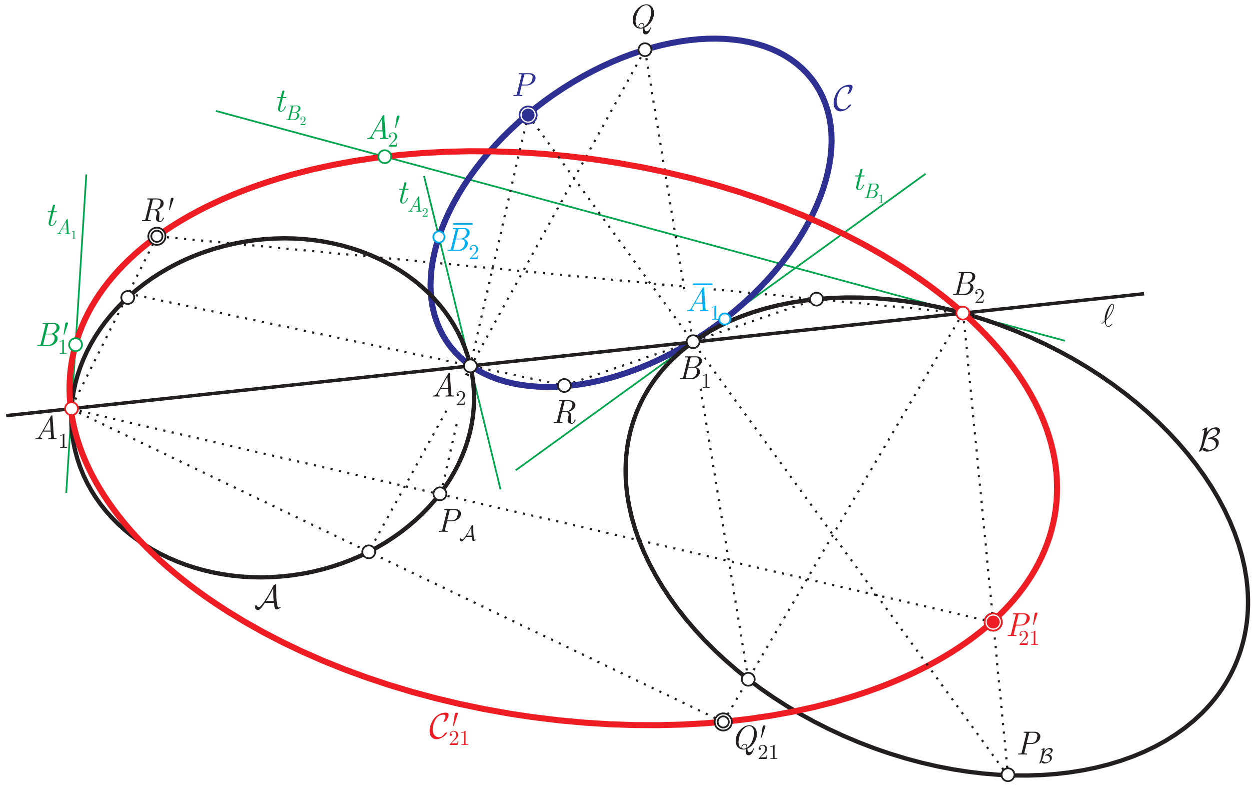

Figure 13: A conic with points A_2, B_1 and its image under the IMO_{21} drawing

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Received July 22, 2025; final form August 14, 2025.