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Inequalities in Triangle Geometry: From Spherical to Hyperbolic

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Abstract. This paper establishes hyperbolic analogues of ten inequalities for triangles in spherical geometry, originally presented in Mitrinović, Pečarić, and Volenec (1989). For comparison, the corresponding Euclidean versions are also included. Unified formulations of these inequalities across spherical, Euclidean, and hyperbolic geometries are provided. This work contributes to the broader effort of translating and unifying geometric results across the three classical geometries.

Key Words: Inequality, Triangle, Spherical geometry, Hyperbolic geometry

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1 Introduction

In [6], the authors established that any polynomial relation involving the side lengths, diagonal lengths, and circumradius of a Euclidean cyclic polygon also holds for hyperbolic and spherical cyclic polygons. While [6] focuses on equations, the present paper turns to inequalities.

We consider ten inequalities that hold for triangles in spherical geometry, as presented in [9], and establish their analogues in hyperbolic geometry. For comparison, corresponding forms in Euclidean geometry are also provided for some of these inequalities. These Euclidean versions are, in fact, well known.

The search for hyperbolic analogues of results originally formulated in spherical geometry is well established. For example, [10] shows how certain classical theorems in spherical geometry can be translated into theorems in hyperbolic geometry.

This paper is organized as follows. Section 2 presents ten inequalities in spherical geometry. Section 3 introduces their unified formulations, applicable to spherical, Euclidean, and hyperbolic geometries. Section 4 discusses the corresponding Euclidean analogues. Section 5 formulates the analogues in hyperbolic geometry. Section 6 provides proofs for the hyperbolic versions.

2 Inequalities for Spherical Triangles

In this section, we present ten inequalities for spherical triangles, all of which are included in [9]. The labels used for these inequalities follow those in [9], and the original references of these results are also cited there.

Using the notation from [9], let $\triangle ABC$ be a spherical triangle with the side lengths $a = BC$, $b = CA$, $c = AB$, and the angles α, β, γ at the vertices A, B, C, respectively. Let $2s = a + b + c$ and $\varepsilon = \alpha + \beta + \gamma - \pi$. Let R and r be the circumradius and inradius of $\triangle ABC$, respectively.

The sign $\{P\}$ means the equality of an inequality holds if and only if the condition P is satisfied. The sign $\mathcal{T}_R$ means $a = b = c$.

The following inequities hold [9, p. 637–640].

$$\sin\frac{a}{2}\sin\frac{b}{2}\sin\frac{c}{2} \geq \sin(s-a)\sin(s-b)\sin(s-c) \qquad \{\mathcal{T}_R\}. \tag{6.6}$$

$$\tan R \geq 2\tan r \qquad \{\mathcal{T}_R\}. \tag{6.7}$$

$$\tan R \tan r \leq \frac{2}{\sin s}\sin^3\frac{s}{3} \qquad \{\mathcal{T}_R\}. \tag{6.8}$$

$$\tan^2 r \leq \frac{1}{\sin s}\sin^3\frac{s}{3} \qquad \{\mathcal{T}_R\}. \tag{6.9}$$

$$\tan\frac{s-a}{2}\tan\frac{s-b}{2}\tan\frac{s-c}{2} \leq \tan^3\frac{s}{6} \qquad \{\mathcal{T}_R\}. \tag{6.10}$$

$$\tan^2\frac{\varepsilon}{4} \leq \tan\frac{s}{2}\tan^3\frac{s}{6} \qquad \{\mathcal{T}_R\}. \tag{6.11}$$

$$\sin^2\frac{\varepsilon}{4} \leq \frac{\sin\frac{s}{2}\sin^3\frac{s}{6}}{\cos^3\frac{s}{6}} \qquad \{\mathcal{T}_R\}. \tag{6.12}$$

$$\cos^2\frac{\varepsilon}{4} \geq \frac{\cos\frac{s}{2}\cos^3\frac{s}{6}}{\cos^3\frac{s}{3}} \qquad \{\mathcal{T}_R\}. \tag{6.13}$$

$$\frac{\sin^3\frac{s}{6}}{\cos^3\frac{s}{3}} \geq \frac{\sin\frac{s-a}{2}\sin\frac{s-b}{2}\sin\frac{s-c}{2}}{\cos\frac{a}{2}\cos\frac{b}{2}\cos\frac{c}{2}} \qquad \{\mathcal{T}_R\}. \tag{6.14}$$

$$\frac{\cos^3\frac{s}{6}}{\cos^3\frac{s}{3}} \leq \frac{\cos\frac{s-a}{2}\cos\frac{s-b}{2}\cos\frac{s-c}{2}}{\cos\frac{a}{2}\cos\frac{b}{2}\cos\frac{c}{2}} \qquad \{\mathcal{T}_R\}. \tag{6.15}$$

3 Unified Inequalities

Certain results in spherical, Euclidean and hyperbolic geometries can be unified using the function $\rho(x)$, along with its derivative $\rho'(x)$ and the ratio $\frac{\rho(x)}{\rho'(x)}$. In particular,

$$\rho(x) = \frac{1}{\sqrt{K}}\sin(\sqrt{K}x),$$

Table 1: Explicit expressions for the functions ρ, ρ', and $\frac{\rho}{\rho'}$ corresponding to the curvature values $K = 1,\ 0,\ -1$.

K	1	0	-1
$\rho(x)$	$\sin x$	x	$\sinh x$
$\rho'(x)$	$\cos x$	1	$\cosh x$
$\frac{\rho}{\rho'}(x)$	$\tan x$	x	$\tanh x$

where $K \in \{1, 0, -1\}$ denotes the constant curvature of the unit sphere, the Euclidean plane, or the hyperbolic plane, respectively (see, e.g., [5]). Table 1 lists the explicit expressions of functions ρ, ρ', and $\frac{\rho}{\rho'}$ for each value of K.

The inequalities in spherical geometry, from (6.6) to (6.15), suggest the following unified forms of inequalities, from (6.6^u) to (6.15^u), applicable to spherical, Euclidean, and hyperbolic geometries.

Let $\triangle ABC$ be a triangle in the unit sphere, the Euclidean plane, or the hyperbolic plane, with the side lengths $a = BC$, $b = CA$, $c = AB$, and the angles α, β, γ at the vertices A, B, C, respectively. Let $2s = a + b + c$ and $\varepsilon = \alpha + \beta + \gamma - \pi$.

For inequality (6.7^u) and (6.8^u), in the case of hyperbolic geometry, we additionally assume that the triangle $\triangle ABC$ has a circumcircle. Indeed, not every hyperbolic triangle admits a circumcircle.

The condition for the existence of a circumcircle is given by the following theorem:

Theorem 1 (Fenchel [3, p. 118]). *A hyperbolic triangle with sides of lengths a, b, and c, where $a \le b \le c$, has a circumscribed circle if and only if*

$$\sinh \frac{c}{2} < \sinh \frac{a}{2} + \sinh \frac{b}{2}.$$

An application of this theorem appears in [4].

A more general result is the following:

Theorem 2 (Beardon [2], Janson [7, (10.22)]). *Let T be a hyperbolic triangle with sides of lengths a, b, and c, where $a \le b \le c$. Then the vertices of T lie on a circle, a horocycle, or a hypercycle, respectively, according as*

$$\sinh \frac{c}{2} \begin{cases} < \sinh \frac{a}{2} + \sinh \frac{b}{2}; \\ = \sinh \frac{a}{2} + \sinh \frac{b}{2}; \\ > \sinh \frac{a}{2} + \sinh \frac{b}{2}. \end{cases}$$

Assume now $R < \infty$ and r denote the circumradius and inradius of $\triangle ABC$, respectively.

$$\rho\left(\frac{a}{2}\right)\rho\left(\frac{b}{2}\right)\rho\left(\frac{c}{2}\right) \ge \rho(s-a)\rho(s-b)\rho(s-c) \qquad \{\mathcal{T}_R\}. \tag{6.6^u}$$

$$\frac{\rho}{\rho'}(R) \ge 2\frac{\rho}{\rho'}(r) \qquad \{\mathcal{T}_R\}. \tag{6.7^u}$$

$$\frac{\rho}{\rho'}(R)\frac{\rho}{\rho'}(r) \leq \frac{2}{\rho(s)}\rho^3\left(\frac{s}{3}\right) \qquad \{\mathcal{T}_R\}. \tag{6.8^u}$$

$$\left(\frac{\rho}{\rho'}(r)\right)^2 \leq \frac{1}{\rho(s)}\rho^3\left(\frac{s}{3}\right) \qquad \{\mathcal{T}_R\}. \tag{6.9^u}$$

$$\frac{\rho}{\rho'}\left(\frac{s-a}{2}\right)\frac{\rho}{\rho'}\left(\frac{s-b}{2}\right)\frac{\rho}{\rho'}\left(\frac{s-c}{2}\right) \leq \left(\frac{\rho}{\rho'}\left(\frac{s}{6}\right)\right)^3 \qquad \{\mathcal{T}_R\}. \tag{6.10^u}$$

$$\tan^2\frac{\varepsilon}{4} \leq \frac{\rho}{\rho'}\left(\frac{s}{2}\right)\left(\frac{\rho}{\rho'}\left(\frac{s}{6}\right)\right)^3 \qquad \{\mathcal{T}_R\}. \tag{6.11^u}$$

$$\sin^2\frac{\varepsilon}{4} \leq \frac{\rho\left(\frac{s}{2}\right)\rho^3\left(\frac{s}{6}\right)}{\rho'^3\left(\frac{s}{6}\right)} \qquad \{\mathcal{T}_R\}. \tag{6.12^u}$$

$$\cos^2\frac{\varepsilon}{4} \geq \frac{\rho'\left(\frac{s}{2}\right)\rho'^3\left(\frac{s}{6}\right)}{\rho'^3\left(\frac{s}{3}\right)} \qquad \{\mathcal{T}_R\}. \tag{6.13^u}$$

$$\frac{\rho^3\left(\frac{s}{6}\right)}{\rho'^3\left(\frac{s}{3}\right)} \geq \frac{\rho\left(\frac{s-a}{2}\right)\rho\left(\frac{s-b}{2}\right)\rho\left(\frac{s-c}{2}\right)}{\rho'\left(\frac{a}{2}\right)\rho'\left(\frac{b}{2}\right)\rho'\left(\frac{c}{2}\right)} \qquad \{\mathcal{T}_R\}. \tag{6.14^u}$$

$$\frac{\rho'^3\left(\frac{s}{6}\right)}{\rho'^3\left(\frac{s}{3}\right)} \leq \frac{\rho'\left(\frac{s-a}{2}\right)\rho'\left(\frac{s-b}{2}\right)\rho'\left(\frac{s-c}{2}\right)}{\rho'\left(\frac{a}{2}\right)\rho'\left(\frac{b}{2}\right)\rho'\left(\frac{c}{2}\right)} \qquad \{\mathcal{T}_R\}. \tag{6.15^u}$$

4 Inequalities for Euclidean Triangles

In Euclidean geometry, the general inequalities from (6.6^u) to (6.15^u) are specified as follows.

$$\frac{abc}{8} \geq (s-a)(s-b)(s-c) \qquad \{\mathcal{T}_R\}. \tag{6.6^e}$$

$$R \geq 2r \qquad \{\mathcal{T}_R\}. \tag{6.7^e}$$

$$Rr \leq \frac{2}{27}s^2 \qquad \{\mathcal{T}_R\}. \tag{6.8^e}$$

$$r \leq \frac{1}{3\sqrt{3}}s \qquad \{\mathcal{T}_R\}. \tag{6.9^e}$$

$$(s-a)(s-b)(s-c) \leq \frac{s^3}{27} \qquad \{\mathcal{T}_R\}. \tag{6.10^e}$$

(6.6^e) is equivalent to inequality (4) in p. 30 of [9].
(6.7^e) is Euler inequality.
(6.8^e) is equivalent to $abc \leq (\frac{a+b+c}{3})^3$.
(6.9^e) is inequality (7) in p. 47 of [9].
(6.10^e) is equivalent to inequality (2) in p. 30 of [9].
The analogues of (6.11), (6.12), (6.13) are trivial since $\varepsilon = 0$ for Euclidean triangles.
The analogue of (6.14) is also (6.10^e).
The analogue of (6.15) is trivial.

5 Inequalities for Hyperbolic Triangles

In hyperbolic geometry, the general inequalities from (6.6^u) to (6.15^u) are specified as follows.

Inequality (6.7^h) was previously established in [11]; for convenience and completeness, we include a proof in the next section.

$$\sinh\frac{a}{2}\sinh\frac{b}{2}\sinh\frac{c}{2} \geq \sinh(s-a)\sinh(s-b)\sinh(s-c) \qquad \{\mathcal{T}_R\}. \tag{6.6^h}$$

$$\tanh R \geq 2\tanh r \qquad \{\mathcal{T}_R\}. \tag{6.7^h}$$

$$\tanh R\tanh r \leq \frac{2}{\sinh s}\sinh^3\frac{s}{3} \qquad \{\mathcal{T}_R\}. \tag{6.8^h}$$

$$\tanh^2 r \leq \frac{1}{\sinh s}\sinh^3\frac{s}{3} \qquad \{\mathcal{T}_R\}. \tag{6.9^h}$$

$$\tanh\frac{s-a}{2}\tanh\frac{s-b}{2}\tanh\frac{s-c}{2} \leq \tanh^3\frac{s}{6} \qquad \{\mathcal{T}_R\}. \tag{6.10^h}$$

$$\tan^2\frac{\varepsilon}{4} \leq \tanh\frac{s}{2}\tanh^3\frac{s}{6} \qquad \{\mathcal{T}_R\}. \tag{6.11^h}$$

$$\sin^2\frac{\varepsilon}{4} \leq \frac{\sinh\frac{s}{2}\sinh^3\frac{s}{6}}{\cosh^3\frac{s}{6}} \qquad \{\mathcal{T}_R\}. \tag{6.12^h}$$

$$\cos^2\frac{\varepsilon}{4} \geq \frac{\cosh\frac{s}{2}\cosh^3\frac{s}{6}}{\cosh^3\frac{s}{3}} \qquad \{\mathcal{T}_R\}. \tag{6.13^h}$$

$$\frac{\sinh^3\frac{s}{6}}{\cosh^3\frac{s}{3}} \geq \frac{\sinh\frac{s-a}{2}\sinh\frac{s-b}{2}\sinh\frac{s-c}{2}}{\cosh\frac{a}{2}\cosh\frac{b}{2}\cosh\frac{c}{2}} \qquad \{\mathcal{T}_R\}. \tag{6.14^h}$$

$$\frac{\cosh^3\frac{s}{6}}{\cosh^3\frac{s}{3}} \leq \frac{\cosh\frac{s-a}{2}\cosh\frac{s-b}{2}\cosh\frac{s-c}{2}}{\cosh\frac{a}{2}\cosh\frac{b}{2}\cosh\frac{c}{2}} \qquad \{\mathcal{T}_R\}. \tag{6.15^h}$$

6 Proofs

Proof of (6.6^h). The inequality

$$\sinh x\sinh y \leq \sinh^2\frac{x+y}{2} \qquad \{x=y\}$$

holds for any $x, y \geq 0$.

One approach to verify it is to observe that the function $f(x) = \ln\sinh x$ satisfies $f''(x) < 0$ on $x \geq 0$. By Jensen's inequality, we have

$$\frac{f(x)+f(y)}{2} \leq f(\frac{x+y}{2}) \qquad \{x=y\}.$$

An alternative approach is as follows.

$$\begin{aligned}\sinh^2\frac{x+y}{2} &= \frac{1}{2}(\cosh(x+y)-1)\\ &\geq \frac{1}{2}(\cosh(x+y)-\cosh(x-y)) \qquad \{x=y\}\\ &= \sinh x\sinh y.\end{aligned}$$

With $x=s-a, y=s-b, z=s-c$, multiplying

$$\sinh x\sinh y \leq \sinh^2\frac{x+y}{2}, \sinh y\sinh z \leq \sinh^2\frac{y+z}{2}, \sinh z\sinh x \leq \sinh^2\frac{z+x}{2}$$

yields (6.6^h). □

Proof of (6.7^h). Since

$$\tanh R = \frac{2\sinh\frac{a}{2}\sinh\frac{b}{2}\sinh\frac{c}{2}}{\sqrt{\sinh s\sinh(s-a)\sinh(s-b)\sinh(s-c)}},$$

$$\tanh r = \sqrt{\frac{\sinh(s-a)\sinh(s-b)\sinh(s-c)}{\sinh s}},$$

we have

$$\frac{\tanh R}{\tanh r} = \frac{2\sinh\frac{a}{2}\sinh\frac{b}{2}\sinh\frac{c}{2}}{\sinh(s-a)\sinh(s-b)\sinh(s-c)} \geq 2$$

by (6.6^h). □

Proof of (6.8^h).

$$\tanh R\tanh r = \frac{2}{\sinh s}\sinh\frac{a}{2}\sinh\frac{b}{2}\sinh\frac{c}{2}.$$

Since $f(x)=\ln\sinh x$ satisfies $f''(x)<0$ on $x\geq 0$, by Jensen's inequality, we have

$$\frac{f(\frac{a}{2})+f(\frac{b}{2})+f(\frac{c}{2})}{3} \leq f(\frac{\frac{a}{2}+\frac{b}{2}+\frac{c}{2}}{3}) \qquad \{a=b=c\},$$

or

$$\sinh\frac{a}{2}\sinh\frac{b}{2}\sinh\frac{c}{2} \leq \sinh^3\frac{s}{3} \qquad \{a=b=c\}.$$ □

Proof of (6.9^h).

$$\begin{aligned}\tanh^2 r &\leq \frac{1}{2}\tanh R\tanh r \quad \text{by } (6.7^h)\\ &\leq \frac{1}{\sin s}\sin^3\frac{s}{3} \quad \text{by } (6.8^h).\end{aligned}$$ □

Proof of (6.10^h). The inequality

$$\tanh x\tanh y \leq \tanh^2\frac{x+y}{2} \qquad \{x=y\}$$

holds for any $x,\, y\geq 0$.

One approach to verify it is to observe that the function $f(x) = \ln \tanh x$ satisfies $f''(x) < 0$ on $x \geq 0$. By Jensen's inequality, we have

$$\frac{f(x)+f(y)}{2} \leq f(\frac{x+y}{2}) \qquad \{x=y\}.$$

An alternative approach is as follows. It is equivalent to

$$\frac{\sinh x \sinh y}{\cosh x \cosh y} \leq \frac{\cosh(x+y)-1}{\cosh(x+y)+1} \qquad \{x=y\}$$

or

$$0 \leq \cosh(x+y)(\cosh(x-y)-1) \qquad \{x=y\}$$

which is true.

Applying this inequality, we have

$$\begin{aligned}
&\tanh\frac{s-a}{2}\tanh\frac{s-b}{2}\tanh\frac{s-c}{2}\tanh\frac{s}{6}\\
&\leq \tanh^2\frac{1}{2}(\frac{s-a}{2}+\frac{s-b}{2})\tanh^2\frac{1}{2}(\frac{s-c}{2}+\frac{s}{6})\\
&= \tanh^2\frac{c}{4}\tanh^2\frac{2a+2b-c}{12}\\
&\leq \tanh^4\frac{1}{2}(\frac{c}{4}+\frac{2a+2b-c}{12})\\
&= \tanh^4\frac{s}{6}.
\end{aligned}$$

Equality holds if and only if

$$s-a=s-b, \qquad s-c=\frac{s}{3}, \qquad c=\frac{2a+2b-c}{3},$$

which are equivalent to

$$a=b=c. \qquad \square$$

Proof of (6.11^h). The following equation holds [1, p. 66]

$$\tan^2\frac{\varepsilon}{4} = \tanh\frac{s}{2}\tanh\frac{s-a}{2}\tanh\frac{s-b}{2}\tanh\frac{s-c}{2}.$$

By (6.10^h),

$$\tan^2\frac{\varepsilon}{4} \leq \tanh\frac{s}{2}\tanh^3\frac{s}{6}. \qquad \square$$

Proof of (6.12^h).

$$\begin{aligned}
\sin^2\frac{\varepsilon}{4} &= \frac{1}{1+\tan^{-2}\frac{\varepsilon}{4}}\\
&\leq \frac{1}{1+\tanh^{-1}\frac{s}{2}\tanh^{-3}\frac{s}{6}} \qquad \text{by } (6.11^h)\\
&= \frac{\sinh\frac{s}{2}\sinh^3\frac{s}{6}}{\sinh\frac{s}{2}\sinh^3\frac{s}{6}+\cosh\frac{s}{2}\cosh^3\frac{s}{6}}\\
&= \frac{\sinh\frac{s}{2}\sinh^3\frac{s}{6}}{\cosh^3\frac{s}{3}}.
\end{aligned}$$

The last equation holds since

$$\sinh 3x \sinh^3 x + \cosh 3x \cosh^3 x = \cosh^3 2x$$

which can be verified as follows.

$$\begin{aligned}
&\sinh 3x \sinh^3 x + \cosh 3x \cosh^3 x \\
&= (\sinh x \cosh 2x + \sinh 2x \cosh x) \sinh^3 x + (\sinh 2x \sinh x + \cosh 2x \cosh x) \cosh^3 x \\
&= \sinh^4 x \cosh 2x + 2\cosh^2 x \sinh^4 x + 2\sinh^2 x \cosh^4 x + \cosh 2x \cosh^4 x \\
&= (\sinh^4 + \cosh^4 x) \cosh 2x + 2\cosh^2 x \sinh^2 x(\sinh^2 x + \cosh^2 x) \\
&= (\sinh^4 + \cosh^4 x + 2\cosh^2 x \sinh^2 x) \cosh 2x \\
&= \cosh^3 2x.
\end{aligned}$$

□

Proof of (6.13^h).

$$\begin{aligned}
\cos^2 \frac{\varepsilon}{4} &= \frac{1}{1 + \tan^2 \frac{\varepsilon}{4}} \\
&\geq \frac{1}{1 + \tanh \frac{s}{2} \tanh^3 \frac{s}{6}} \qquad \text{by } (6.11^h) \\
&= \frac{\cosh \frac{s}{2} \cosh^3 \frac{s}{6}}{\cosh \frac{s}{2} \cosh^3 \frac{s}{6} + \sinh \frac{s}{2} \sinh^3 \frac{s}{6}} \\
&= \frac{\cosh \frac{s}{2} \cosh^3 \frac{s}{6}}{\cosh^3 \frac{s}{3}}.
\end{aligned}$$

□

Proof of (6.14^h). The following equation holds [8, Theorem 1.1 (iii)]

$$\sin^2 \frac{\varepsilon}{4} = \frac{\sinh \frac{s}{2} \sinh \frac{s-a}{2} \sinh \frac{s-b}{2} \sinh \frac{s-c}{2}}{\cosh \frac{a}{2} \cosh \frac{b}{2} \cosh \frac{c}{2}}.$$

Substituting it into (6.12^h) yields (6.14^h). □

Proof of (6.15^h). Combining the formulas of $\tan^2 \frac{\varepsilon}{4}$ and $\sin^2 \frac{\varepsilon}{4}$ gives

$$\cos^2 \frac{\varepsilon}{4} = \frac{\cosh \frac{s}{2} \cosh \frac{s-a}{2} \cosh \frac{s-b}{2} \cosh \frac{s-c}{2}}{\cosh \frac{a}{2} \cosh \frac{b}{2} \cosh \frac{c}{2}}.$$

Substituting it into (6.13^h) yields (6.15^h). □

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References

[1] D. V. Alekseevskij, E. B. Vinberg, and A. S. Solodovnikov: *Geometry of Spaces of Constant Curvature*, 1–138. Springer Berlin Heidelberg, 1993. ISBN 9783662029015. doi: 10.1007/978-3-662-02901-5_1.

[2] A. F. Beardon: *Circumscribed Hyperbolic Triangles.* Archiv der Mathematik **119**(4), 439–448, 2022. doi: 10.1007/s00013-022-01761-x.

[3] W. Fenchel: *Elementary Geometry in Hyperbolic Space*, vol. 11 of *De Gruyter Studies in Mathematics.* Walter de Gruyter & Co., Berlin, 1989. ISBN 9783110849455. doi: 10.1515/9783110849455. With an editorial by Heinz Bauer.

[4] X. Gu, R. Guo, F. Luo, J. Sun, and T. Wu: *A Discrete Uniformization Theorem for Polyhedral Surfaces II.* Journal of Differential Geometry **109**(3), 2018. doi: 10.4310/jdg/1531188190.

[5] R. Guo and F. Luo: *Rigidity of Polyhedral Surfaces, II.* Geometry & Topology **13**(3), 1265–1312, 2009. doi: 10.2140/gt.2009.13.1265.

[6] R. Guo and N. Sönmez: *Cyclic Polygons in Classical Geometry, II.* C. R. Acad. Bulg. Sci. **78**(2), 2025. doi: 10.7546/crabs.2025.02.01.

[7] S. Janson: *Euclidean, spherical and hyperbolic trigonometry*, 2015. Notes.

[8] A. D. Mednykh: *Brahmagupta formula for cyclic quadrilaterals in the hyperbolic plane.* Sib. Èlektron. Mat. Izv. **9**, 247–255, 2012.

[9] D. S. Mitrinović, J. E. Pečarić, and V. Volenec: *Recent advances in geometric inequalities.* Kluwer Academic Publishers, Dordrecht, 1989.

[10] A. Papadopoulos and W. Su: *On hyperbolic analogues of some classical theorems in spherical geometry.* Adv. Stud. Pure Math. **73**, 225–253, 2017. doi: 10.2969/aspm/07310225.

[11] D. Svrtan and D. Veljan: *Non-Euclidean versions of some classical triangle inequalities.* Forum Geom. **12**, 197–209, 2012.

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