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Ex-Center-Tetrahedra

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Abstract. The tetrahedron having ex-centers of a tetrahedron T as vertices is said to be the ex-center-tetrahedron of T, and let us denote it by $T^\star$.

Theorem 1 shows that the ex-center-tetrahedron of a tetrahedron T and the tetrahedron that tangles T with respect to the in-center of T are the same. So $T^\star$ is also used to denote the tetrahedron that tangles T with respect to the in-center of T.

We define that a tetrahedron is weakly reversible if the sum of a pair of two face areas is equal to the sum of the other pair of two face areas. Theorem 2 shows that T is weakly reversible if and only if T and $T^\star$ have the same volume.

While $T^\star$ may not be weakly reversible when T is weakly reversible, Theorem 3 shows that $T^\star$ is reversible when T is reversible.

Key Words: tangled tetrahedron, in-sphere, in-center, in-radius, ex-sphere, ex-center, ex-radius, ex-center-tetrahedron, deep interior of a tetrahedron, isosceles

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1 Introduction

The external bisectors of any two angles of a triangle are concurrent with the internal bisector of the third angle, and this concurrent point is an ex-center of the triangle. We can find an ex-center of a tetrahedron in a similar way. However, using the in-center of a tetrahedron, our main Theorem 1 shows an alternate way to find ex-centers of a tetrahedron from its in-center without externally bisecting dihedral angles at edges (see Remark 2 for details). Let us begin with notations and definitions.

Let A, B, C, D be distinct points in $\mathbb{R}^3$. We denote the line segment with the end points A and B by $[AB]$, its length by $|AB|$, and the line AB by $\overline{AB}$. A triangle $\triangle ABC$ is formed by three non-collinear points A, B, C. A tetrahedron $\nabla ABCD$ is a solid bounded by four triangular faces $\triangle ABC$, $\triangle ABD$, $\triangle ACD$ and $\triangle BCD$, where the points A, B, C, and D are non-coplanar. The plane containing non-collinear points U, V, W is denoted by Ω_{UVW}. For simplicity for the planes defined by the faces of $\nabla ABCD$, we write $\Omega_{ABC} = \Omega_D$, $\Omega_{ABD} = \Omega_C$, $\Omega_{ACD} = \Omega_B$, and $\Omega_{BCD} = \Omega_A$.

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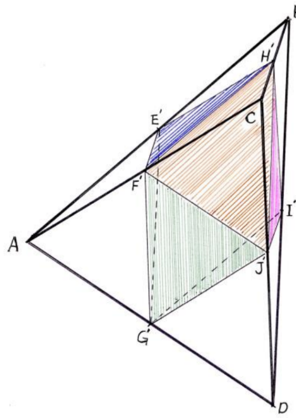

Figure 1: The deep interior of the tetrahedron $\nabla ABCD$ is the interior of the octahedron $E'F'G'H'I'J'$.

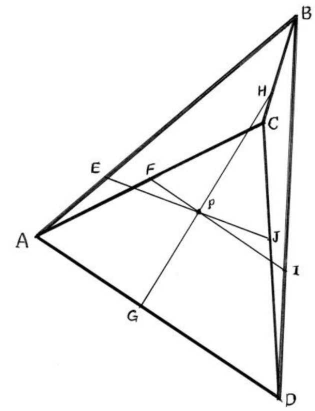

Figure 2: The in-center P and the corresponding E, F, G, H, I, J are indicated on the edges of the tetrahedron $\nabla ABCD$.

Definition 1. If E', F', G', H', I', J' are the midpoints of the edges $[AB]$, $[AC]$, $[AD]$, $[BC]$, $[BD]$, and $[CD]$, respectively, of a tetrahedron $\nabla ABCD$, then the interior of the octahedron $E'F'G'H'I'J'$ is said to be the *deep interior* of $\nabla ABCD$. See Figure 1.

Definition 2. A tetrahedron $\nabla A^*B^*C^*D^*$ is said to *tangle with* the tetrahedron $\nabla ABCD$ if $[AB] \cap [C^*D^*] = \{E\}$, $[AC] \cap [B^*D^*] = \{F\}$, $[AD] \cap [B^*C^*] = \{G\}$, $[BC] \cap [A^*D^*] = \{H\}$, $[BD] \cap [A^*C^*] = \{I\}$, and $[CD] \cap [A^*B^*] = \{J\}$ for some points E, F, G, H, I, J. In addition, if $[EJ] \cap [FI] \cap [GH] = \{P\}$ for some point P, then the tetrahedron $\nabla A^*B^*C^*D^*$ is said to *tangle with* with tetrahedron $\nabla ABCD$ *with respect to* P. See Figures 2 and 4.

Let P be a deep interior point of $\nabla ABCD$. Then it is known to have unique points E, F, G, H, I, J on the edges AB, AC, AD, BC, CD, respectively, such that $[EJ] \cap [FI] \cap [GH] = \{P\}$ (see Lemma 1(1) below and Figure 2). For simplicity, let $\Gamma_A = \Omega_{EFG}$, $\Gamma_B = \Omega_{EHI}$, $\Gamma_C = \Omega_{FHJ}$, and $\Gamma_D = \Omega_{GIJ}$. The planes Γ_B, Γ_C and Γ_D intersect (see Lemma 1(4) below and Figure 3), say at A^*. Then the three points A, P, and A^* are known to be collinear, and A and A^* are on the opposite sides of the plane Ω_A (Lemma 1, (4) and (5) below).

We have $\{A^*\} = \Gamma_B \cap \Gamma_C \cap \Gamma_D$ (see Figure 3). Similarly, let $\{B^*\} = \Gamma_A \cap \Gamma_C \cap \Gamma_D$, $\{C^*\} = \Gamma_A \cap \Gamma_B \cap \Gamma_D$, and $\{D^*\} = \Gamma_A \cap \Gamma_B \cap \Gamma_C$. The points ($A^*$ and A), (B^* and B), (C^* and C), (D^* and D) are opposite with respect to the planes Ω_A, Ω_B, Ω_C, Ω_D, respectively. It is known that the tetrahedron $\nabla A^*B^*C^*D^*$ tangles with the tetrahedron $\nabla ABCD$ with respect to P (Lemma 1(7) and see Figure 4 below). For more information on tangled tetrahedron, please see [4].

Definition 3. The *in-sphere* of a tetrahedron $\nabla ABCD$, denoted by S, is the sphere inside of $\nabla ABCD$, tangent to the four triangular faces $\triangle ABC$, $\triangle ABD$, $\triangle ACD$ and $\triangle BCD$. The center and the radius of S are called the *in-center* and the *in-radius,* respectively. The *ex-sphere* of the tetrahedron $\nabla ABCD$ opposite of the vertex A with respect to Ω_A is the sphere with center A' outside of $\nabla ABCD$ tangent to the planes Ω_D, Ω_B, Ω_C, and the face $\triangle BCD$, and it is denoted by $S_{A'}$. Hence, $\nabla ABCD$ has four ex-spheres $S_{A'}$, $S_{B'}$, $S_{C'}$, and $S_{D'}$. The centers A', B', C', D' are called *ex-centers* of a tetrahedron $\nabla ABCD$.

The tetrahedron $\nabla A'B'C'D'$ is said to be the *ex-center-tetrahedron* of $\nabla ABCD$. The radius of $S_{A'}$ is called *ex-radius* and denoted by $r_{A'}$.

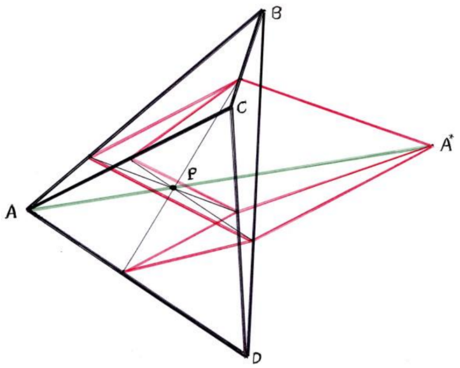

Figure 3: The construction of the point A^* from Figure 2 is indicated.

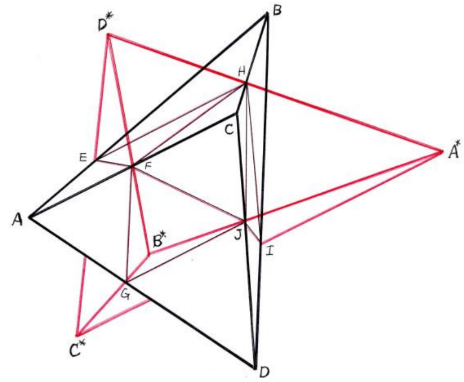

Figure 4: Tetrahedra $\nabla ABCD$ and $\nabla A^*B^*C^*D^*$ are shown.

Lemma 3 will prove that the in-center is a deep interior point of $\nabla ABCD$. Then Theorem 1 will show that a tetrahedron that tangles $\nabla ABCD$ with respect to its in-center is the ex-center-tetrahedron of $\nabla ABCD$. Therefore, the notations A', B', C', D' in Definition 3 will be replaced by A^*, B^*, C^*, D^* as in Definition 2. As shown in Remark 2, Theorem 1 gives us an alternate way to find ex-centers of a tetrahedron from its in-center without externally bisecting dihedral angles at edges.

Definition 4. Let $\nabla ABCD$ be a tetrahedron. Let the area of the faces $\triangle ABC$, $\triangle ABD$, $\triangle ACD$, $\triangle BCD$ be denoted by $\mathcal{A}_D$, $\mathcal{A}_C$, $\mathcal{A}_B$, and $\mathcal{A}_A$, respectively. Let $\mathcal{A} = \mathcal{A}_A + \mathcal{A}_B + \mathcal{A}_C + \mathcal{A}_D$, the total surface area of $\nabla ABCD$.

A tetrahedron $\nabla ABCD$ is *reversible* if ($|AB| = |CD|$ and $|AD| = |BC|$), or ($|AB| = |CD|$ and $|AC| = |BD|$), or ($|AD| = |BC|$ and $|AC| = |BD|$) holds. Klain recently proved that a tetrahedron $\nabla ABCD$ is reversible if and only if ($\mathcal{A}_A = \mathcal{A}_B$ and $\mathcal{A}_C = \mathcal{A}_D$), or ($\mathcal{A}_A = \mathcal{A}_C$ and $\mathcal{A}_B = \mathcal{A}_D$), or ($\mathcal{A}_A = \mathcal{A}_D$ and $\mathcal{A}_B = \mathcal{A}_C$) holds. (See [6, Theorem 1].)

A tetrahedron $\nabla ABCD$ is *isosceles* if $|AB| = |CD|$, $|AC| = |BD|$, and $|AD| = |BC|$. It is known that a tetrahedron $\nabla ABCD$ is isosceles if and only if $\mathcal{A}_A = \mathcal{A}_B = \mathcal{A}_C = \mathcal{A}_D$. (See [6] for more information.) An isosceles tetrahedron is reversible.

Motivated by these, we define that a tetrahedron $\nabla ABCD$ is *weakly reversible* if ($\mathcal{A}_A + \mathcal{A}_B = \mathcal{A}_C + \mathcal{A}_D$), or ($\mathcal{A}_A + \mathcal{A}_C = \mathcal{A}_B + \mathcal{A}_D$), or ($\mathcal{A}_A + \mathcal{A}_D = \mathcal{A}_B + \mathcal{A}_C$) holds.

Example 1. If a tetrahedron is reversible, then it must be weakly reversible. But a weakly reversible tetrahedron may not be reversible. In order to see this, let $A = (1, 0, 0)$, $B = (-1, 0, 0)$, $C = (0, 2, 0)$, and $D = (0, 0, 1)$ in $\mathbb{R}^3$ with the Cartesian coordinates. Then $\nabla ABCD$ is a tetrahedron such that $|AB| = 2$, $|AC| = |BC| = |DC| = \sqrt{5}$, and $|DB| = |DA| = \sqrt{2}$. Hence, $\nabla ABCD$ is not reversible. On the other hand, we can check to see that $\mathcal{A}_D = 2$ and $\mathcal{A}_C = 1$, and $\overrightarrow{AC} \times \overrightarrow{AD} = \langle -1, 2, 0\rangle \times \langle -1, 0, 1\rangle = \langle 2, 1, 2\rangle$ so that $\mathcal{A}_A = \mathcal{A}_B = \frac{1}{2}\sqrt{4+1+4} = \frac{3}{2}$. This shows that $\mathcal{A}_D + \mathcal{A}_C = 3 = \mathcal{A}_A + \mathcal{A}_B$. Thus, $\nabla ABCD$ is weakly reversible.

Let $\nabla A^*B^*C^*D^*$ be the ex-center-tetrahedron of $\nabla ABCD$. Theorem 1 will be used to prove that $\nabla ABCD$ is weakly reversible if and only if $\nabla A^*B^*C^*D^*$ and $\nabla ABCD$ have the same volume in Theorem 2. This led us to the following question: If a tetrahedron is weakly reversible, then is its ex-center-tetrahedron also weakly reversible? The answer to this is NO, as we will see in Example 2. However, Theorem 3 will show that if a tetrahedron is reversible, then its ex-center-tetrahedron must also be reversible.

2 Preliminaries

We will use barycentric coordinates used in [3] to prove Theorems 1 and 2.

Definition 5. Let $\nabla ABCD$ be a tetrahedron, and its volume is denoted by $\mathcal{V} = \mathcal{V}_{ABCD}$. Let P be a point in $\mathbb{R}^3$. Then $\mathcal{V}_{PABC}$ is defined to be the volume of the tetrahedron $\nabla PABC$ if P is on the *same* side of D with respect to the plane Ω_D; and $\mathcal{V}_{PABC}$ is defined to be the *negative* of the volume of the tetrahedron $\nabla PABC$ if P is on the *opposite* side of D with respect to the plane Ω_D. Hence, for example, $\mathcal{V}_{PBCD} > 0$ if P and A are on the same side of the plane Ω_A; and $\mathcal{V}_{PBCD} < 0$ if P and A are on opposite sides of the plane Ω_A. Let $a' = \frac{\mathcal{V}_{PBCD}}{\mathcal{V}}$, $b' = \frac{\mathcal{V}_{PACD}}{\mathcal{V}}$, $c' = \frac{\mathcal{V}_{PABD}}{\mathcal{V}}$, $d' = \frac{\mathcal{V}_{PABC}}{\mathcal{V}}$. (Here, $\mathcal{V}_{PBCD}, \mathcal{V}_{PACD}$, $\mathcal{V}_{PABD}$, $\mathcal{V}_{PABC}$ are *signed* volumes.) Then the *barycentric coordinates* of P are given and denoted by $[a', b', c', d']$. Every point in $\mathbb{R}^3$ has unique barycentric coordinates. Since $\mathcal{V}_{PBCD} + \mathcal{V}_{PABC} + \mathcal{V}_{PABD} + \mathcal{V}_{PACD} = \mathcal{V}$, we have $a' + b' + c' + d = 1$.

The next Lemma 1 is a collection of results from [3].

Lemma 1. *Let P be a point* inside *of a tetrahedron $\nabla ABCD$. Then the following hold:*

(1) There are unique points E, F, G, H, I, J on the edges AB, AC, AD, BC, BD, and CD, respectively, such that $[EJ] \cap [FI] \cap [GH] = \{P\}$. (See Figure 2.) Recall that $\Omega_{EFG} = \Gamma_A$, $\Omega_{EHI} = \Gamma_B$, $\Omega_{FHJ} = \Gamma_C$, and $\Omega_{GJI} = \Gamma_D$.

(2) Let $[a', b', c', d']$ be the barycentric coordinates of P. Then $0 < a', b', c', d' < 1$. And the planes Γ_B, Γ_C, and Γ_D intersect at a point if and only if $a' \neq \frac{1}{2}$.

(3) Suppose $a' \neq \frac{1}{2}$. Let A^* be the intersection of the planes Γ_B, Γ_C and Γ_D. Then the barycentric coordinate of A^* is given by $\left[\frac{-a'}{1-2a'}, \frac{b'}{1-2a'}, \frac{c'}{1-2a'}, \frac{d'}{1-2a'}\right]$.

(4) If $a' < \frac{1}{2}$, then A^* and A are on opposite sides of the plane Ω_A. If $a' > \frac{1}{2}$, then A^* and A are on the same side of the plane Ω_A.

(5) Supposed $a' \neq \frac{1}{2}$. Then the three points A, P, A^* are collinear. (See Figure 3 when $a' < \frac{1}{2}$.)

(6) If E', F', G' are the midpoints of $[AB]$, $[AC]$, $[AD]$, respectively, then $a' = \frac{1}{2}$ if and only if P is a point on the triangle $\triangle E'F'G'$. Also, the point $P = [a', b', c', d']$ is a deep interior point if and only if $0 < a', b', c', d' < \frac{1}{2}$.

(7) Suppose P is a deep *interior point of a tetrahedron $\nabla ABCD$. Then $\Gamma_B \cap \Gamma_C \cap \Gamma_D$, $\Gamma_A \cap \Gamma_C \cap \Gamma_D$, $\Gamma_A \cap \Gamma_B \cap \Gamma_D$, and $\Gamma_A \cap \Gamma_B \cap \Gamma_C$ are all sets with one element.*

(8) Let $\Gamma_B \cap \Gamma_C \cap \Gamma_D = \{A^*\}$, $\Gamma_A \cap \Gamma_C \cap \Gamma_D = \{B^*\}$, $\Gamma_A \cap \Gamma_B \cap \Gamma_D = \{C^*\}$, and $\Gamma_A \cap \Gamma_B \cap \Gamma_C = \{D^*\}$. Then $\nabla A^*B^*C^*D^*$ is the unique tangled tetrahedron of $\nabla ABCD$ with respect to the point P.

Proof. Proof of (1): This is [3, Lemma 2]. An alternate proof is given in [4, Lemma 1].

Proof of (2) and (3): See [3, Corollary 1].

Proof of (4): By (3), the first barycentric coordinate of A^* is $\frac{-a'}{1-2a'}$. Since $\frac{-a'}{1-2a'} < 0$, the points A^* and A are on opposite sides of Ω_A. Hence, if $a' < \frac{1}{2}$, then A^* and A are on opposite sides of Ω_A. If $a' > \frac{1}{2}$, then A^* and A are on the same side of Ω_A. ([3, Remark 1] is a mistake, stated in reverse.)

Proof of (5): See [3, Theorem 1].

Proof of (6): Let E', F', G', H', I', J' be the midpoints of the edges $[AB]$, $[AC]$, $[AD]$, $[BC]$, $[BD]$, and $[CD]$, respectively. Suppose $P = [a', b', c', d']$ are the barycentric coordinates. $a' = \frac{\mathcal{V}_{PBCD}}{\mathcal{V}} = \frac{1}{2}$ if and only if the height of the tetrahedron $\nabla PBCD$ from P to Ω_A is less than half of the height of $\nabla ABCD$ from A to Ω_A. Thus, $a' = \frac{1}{2}$ if and only if P is on the triangle $\triangle E'F'G'$. Suppose P is a deep interior point. Since P is a point inside of the octahedron $E'F'G'H'I'J'$, P has to be between the planes Ω_A and $\Omega_{E'F'G'}$. Hence, the height of the tetrahedron $\nabla PBCD$ from P to Ω_A is less than the half of the height of $\nabla ABCD$ from A to Ω_A so that $\mathcal{V}_{PBCD} < \frac{1}{2}\mathcal{V}$, i.e., $a' = \frac{\mathcal{V}_{PBCD}}{\mathcal{V}} < \frac{1}{2}$. Similarly, we have b', c', $d' < \frac{1}{2}$. Conversely, if $0 < a', b', c', d' < \frac{1}{2}$, we can reverse this argument to show that $P = [a', b', c', d']$ is a deep interior point.

Proof of (7): Since P is a deep interior point of $\nabla ABCD$, we have $0 < a', b', c', d' < \frac{1}{2}$ by (6). Hence, by part (2), $\Gamma_B \cap \Gamma_C \cap \Gamma_D$, $\Gamma_A \cap \Gamma_C \cap \Gamma_D$, $\Gamma_A \cap \Gamma_B \cap \Gamma_D$, and $\Gamma_A \cap \Gamma_B \cap \Gamma_C$ are all sets with one element.

Proof of (8): By part (4), we have that (A^* and A), (B^* and B), and (C^* and C), (D^* and D) are on the opposite side of the planes Ω_A, Ω_B, Ω_C, and Ω_D, respectively. Since $\overline{C^*D^*} = \Gamma_A \cap \Gamma_B = \Omega_{EFG} \cap \Omega_{EHI}$ so that $E \in [C^*D^*]$, we have $[AB] \cap [C^*D^*] = \{E\}$. Similarly, $[AC] \cap [B^*D^*] = \{F\}$, $[AD] \cap [B^*C^*] = \{G\}$, $[BC] \cap [A^*D^*] = \{H\}$, $[BD] \cap [A^*C^*] = \{I\}$, and $[CD] \cap [A^*B^*] = \{J\}$. Therefore, $\nabla A^*B^*C^*D^*$ is the tangled tetrahedron $\nabla ABCD$ with respect to P. The uniqueness of $\nabla A^*B^*C^*D^*$ is proved in [4, Theorem 2(3)]. □

Lemma 2. *Let $a = \frac{\mathcal{A}_A}{\mathcal{A}}$, $b = \frac{\mathcal{A}_B}{\mathcal{A}}$, $c = \frac{\mathcal{A}_C}{\mathcal{A}}$, $d = \frac{\mathcal{A}_D}{\mathcal{A}}$. Then the barycentric coordinates of the in-center of the tetrahedron $\nabla ABCD$ is given by $[a, b, c, d]$.*

Proof. Let r and P be the in-radius and in-center of $\nabla ABCD$, respectively. Recall that $\mathcal{A} = \mathcal{A}_D + \mathcal{A}_C + \mathcal{A}_B + \mathcal{A}_A$, the surface area of $\nabla ABCD$. Then $\mathcal{V}_{PBCD} = \frac{1}{3}r \cdot \mathcal{A}_A$, and $\mathcal{V} = \frac{1}{3}r \cdot \mathcal{A}$. Hence, $a = \frac{\mathcal{V}_{PBCD}}{\mathcal{V}} = \frac{\mathcal{A}_A}{\mathcal{A}}$. Similarly, we have $b = \frac{\mathcal{A}_B}{\mathcal{A}}$, $c = \frac{\mathcal{A}_C}{\mathcal{A}}$, $d = \frac{\mathcal{A}_D}{\mathcal{A}}$ so that the barycentric coordinate of P is given by $[a, b, c, d]$. □

Lemma 3. *The in-center of a tetrahedron is a deep interior point of the tetrahedron.*

Proof. Let $\nabla ABCD$ be a tetrahedron, and let S be its in-sphere. Let P and r be the center and radius of S, respectively. Let $P = [a, b, c, d]$ be the barycentric coordinates.

Let h be the height of the tetrahedron $\nabla ABCD$ from the vertex A. Let Σ be the plane through the center of S parallel to the base triangle $\triangle BCD$. Let E, F, G be the intersection with the plane Σ and the edges $[AB]$, $[AC]$, $[AD]$, respectively. Then the tetrahedron $\nabla AEFG$ has the height $h - r$ from the vertex A to the base $\triangle EFG$, and $\nabla AEFG$ also contains a hemisphere of S. This shows that $h - r > r$ or $r < \frac{1}{2}h$. Since $\mathcal{V}_{PBCD} = \frac{1}{3}r\mathcal{A}_A$ and $\mathcal{V}_{ABCD} = \frac{1}{3}h\mathcal{A}_A$, we have $a = \frac{\mathcal{V}_{PBCD}}{\mathcal{V}} = \frac{r}{h} < \frac{1}{2}$.

Alternately, since $\mathcal{A}_D + \mathcal{A}_C + \mathcal{A}_B > \mathcal{A}_A$, we have $a = \frac{\mathcal{A}_A}{\mathcal{A}} = \frac{\mathcal{A}_A}{\mathcal{A}_D + \mathcal{A}_C + \mathcal{A}_B + \mathcal{A}_A} < \frac{1}{2}$.

Similarly, we can show that $0 < b, c, d < \frac{1}{2}$. Therefore, this proves that the in-center of a tetrahedron is a deep interior point of the tetrahedron by Lemma 1(6). □

3 Ex-centers of a Tetrahedron

We need the following two lemmas to prove Theorem 1.

Lemma 4. *The barycentric coordinates of A, B, C, D are given by $A = [1, 0, 0, 0]$, $B = [0, 1, 0, 0]$, $C = [0, 0, 1, 0]$, $D = [0, 0, 0, 1]$.*

Let $P = [a, b, c, d]$ be the barycentric coordinates of the in-center of $\nabla ABCD$. Let $\nabla A^*B^*C^*D^*$ be the tangled tetrahedron of $\nabla ABCD$ with respect to P. Then

$$A^* = \left[\frac{-a}{1-2a}, \frac{b}{1-2a}, \frac{c}{1-2a}, \frac{d}{1-2a}\right], \quad B^* = \left[\frac{a}{1-2b}, \frac{-b}{1-2b}, \frac{c}{1-2b}, \frac{d}{1-2b}\right],$$
$$C^* = \left[\frac{a}{1-2c}, \frac{b}{1-2c}, \frac{-c}{1-2c}, \frac{d}{1-2c}\right], \quad D^* = \left[\frac{a}{1-2d}, \frac{b}{1-2d}, \frac{c}{1-2d}, \frac{-d}{1-2d}\right].$$

Proof. See [3, Example 1] for the barycentric coordinates of A, B, C, D. Since the in-center P is a deep interior point of the tetrahedron by Lemma 3, the barycentric coordinates of A^* are given by Lemma 1(3). The barycentric coordinates of B^*, C^*, D^* are applications of Lemma 1(3). □

Lemma 5. *Let $S = [s_1, s_2, s_3, s_4]$, $T = [t_1, t_2, t_3, t_4]$, $U = [u_1, u_2, u_3, u_4]$, $V = [v_1, v_2, v_3, v_4]$ be points in $\mathbb{R}^3$ given in the barycentric coordinates with respect to a tetrahedron $\nabla ABCD$. Let $\mathcal{V}$ be the volume of the tetrahedron $\nabla ABCD$, and let*

$$\delta = \begin{vmatrix} s_1, & s_2, & s_3, & s_4 \\ t_1, & t_2, & t_3, & t_4 \\ u_1, & u_2, & u_3, & u_4 \\ v_1, & v_2, & v_3, & v_4 \end{vmatrix},$$

the determinant of the matrix. Then the volume $\mathcal{V}'$ of the tetrahedron $\nabla STUV$ is given by $|\delta|\mathcal{V}$, i.e., $\mathcal{V}' = |\delta|\mathcal{V}$. (Here, we mean the usual volume, and it is not the signed volume of $\nabla STUV$ used in Definition 5 of barycentric coordinates.)

Proof. This is [3, Lemma 3]. □

We are ready to prove our first main theorem.

Theorem 1. *Let P be in-center of a tetrahedron $\nabla ABCD$. Then the tetrahedron tangles the tetrahedron $\nabla ABCD$ with respect to P and the ex-center-tetrahedron of $\nabla ABCD$ are identical.*

Proof. Suppose a tetrahedron $\nabla A^*B^*C^*D^*$ tangles $\nabla ABCD$ with respect to P. Since P is in the deep interior of $\nabla ABCD$ by Lemma 3, the existence of $\nabla A^*B^*C^*D^*$ makes sense by Lemma 1(8). Let $[a, b, c, d]$ be the barycentric coordinates of the in-center P. We will prove that A^* is the ex-center of $\nabla ABCD$ by showing $d(A^*, \Omega_D) = d(A^*, \Omega_C) = d(A^*, \Omega_B) = d(A^*, \Omega_A)$. Here, $d(A^*, \Omega_D)$ is the distance between the point A^* and the plane Ω_D.

We have the following determinants.

$$\begin{vmatrix} 0, & 1, & 0, & 0 \\ 0, & 0, & 1, & 0 \\ 0, & 0, & 0, & 1 \\ -a, & b, & c, & d \end{vmatrix} = a, \quad \begin{vmatrix} 1, & 0, & 0, & 0 \\ 0, & 0, & 1, & 0 \\ 0, & 0, & 0, & 1 \\ -a, & b, & c, & d \end{vmatrix} = b, \quad \begin{vmatrix} 1, & 0, & 0, & 0 \\ 0, & 1, & 0, & 0 \\ 0, & 0, & 0, & 1 \\ -a, & b, & c, & d \end{vmatrix} = c, \quad \begin{vmatrix} 1, & 0, & 0, & 0 \\ 0, & 1, & 0, & 0 \\ 0, & 0, & 1, & 0 \\ -a, & b, & c, & d \end{vmatrix} = d.$$

By Lemmas 4 and 5, since A and A^* are on opposite sides of Ω_A, and since $\mathcal{V}_{A^*BCD}$ is a signed volume, we have $\mathcal{V}_{A^*BCD} = -\frac{a}{1-2a}\mathcal{V}$. Since $a = \frac{\mathcal{A}_A}{\mathcal{A}}$, we have $\mathcal{V}_{A^*BCD} = -\frac{a}{1-2a}\mathcal{V} = -\frac{\mathcal{A}_A}{\mathcal{A}\cdot(1-2a)}\mathcal{V}$.

On the other hand, we also have $\mathcal{V}_{A^*BCD} = -\frac{1}{3}\mathcal{A}_A \cdot d(A^*, \Omega_A)$ since $\mathcal{V}_{A^*BCD}$ is a signed volume. Hence, $-\frac{\mathcal{A}_A}{\mathcal{A}\cdot(1-2a)}\mathcal{V} = -\frac{1}{3}\mathcal{A}_A \cdot d(A^*, \Omega_A)$.

Since $\mathcal{A}_A \neq 0$, we have $d(A^*, \Omega_A) = \frac{3\mathcal{V}}{\mathcal{A}\cdot(1-2a)}$.

Similarly, we have

$$\mathcal{V}_{A^*ACD} = -\frac{b}{1-2a}\mathcal{V} = -\frac{\mathcal{A}_B}{\mathcal{A}\cdot(1-2a)}\mathcal{V} \quad \text{and} \quad \mathcal{V}_{A^*ACD} = -\frac{1}{3}\mathcal{A}_B \cdot d(A^*, \Omega_B)$$

so that $d(A^*, \Omega_B) = \frac{3\mathcal{V}}{\mathcal{A}\cdot(1-2a)}$,

$$\mathcal{V}_{A^*ABD} = -\frac{c}{1-2a}\mathcal{V} = -\frac{\mathcal{A}_C}{\mathcal{A}\cdot(1-2a)}\mathcal{V} \quad \text{and} \quad \mathcal{V}_{A^*ABD} = -\frac{1}{3}\mathcal{A}_C \cdot d(A^*, \Omega_C)$$

so that $d(A^*, \Omega_C) = \frac{3\mathcal{V}}{\mathcal{A}\cdot(1-2a)}$, and

$$\mathcal{V}_{A^*ABC} = -\tfrac{d}{1-2a}\mathcal{V} = -\tfrac{\mathcal{A}_D}{\mathcal{A}\cdot(1-2a)}\mathcal{V} \quad \text{and} \quad \mathcal{V}_{A^*ABC} = -\tfrac{1}{3}\mathcal{A}_D \cdot d(A^*, \Omega_D)$$

so that $d(A^*, \Omega_D) = \frac{3\mathcal{V}}{\mathcal{A}\cdot(1-2a)}$.

Therefore, $d(A^*, \Omega_D) = d(A^*, \Omega_C) = d(A^*, \Omega_B) = d(A^*, \Omega_A)$. This proves that A^* is the center of the ex-sphere S_{A^*} (and $r_A = \frac{3\mathcal{V}}{\mathcal{A}\cdot(1-2a)}$ is the radius of ex-sphere S_{A^*}) of $\nabla ABCD$. Similarly, we can show that B^*, C^*, D^* are the ex-centers of $\nabla ABCD$. Therefore, this shows that the ex-center-tetrahedron of $\nabla ABCD$ is the tetrahedron that tangles $\nabla ABCD$ with respect to P.

Conversely, suppose $\nabla A'B'C'D'$ is the ex-center-tetrahedron of $\nabla ABCD$ as in Definition 3. Let P be the in-center of $\nabla ABCD$. If $\nabla A^*B^*C^*D^*$ is the tetrahedron that tangles $\nabla ABCD$ with respect to P, then we must have $A' = A^*$, $B' = B^*$, $C' = C^*$, and $D' = D^*$ by the first part of this proof and by the uniqueness of the tangled tetrahedron with respect to P by Lemma 1(8). Therefore, the ex-center-tetrahedron $\nabla A^*B^*C^*D^*$ of $\nabla ABCD$ must be the tangled tetrahedron of $\nabla ABCD$ with respect to P. □

Remark 1. For the rest of this section and Section 4, we use P for the in-center of a tetrahedron $\nabla ABCD$ with $P = [a, b, c, d]$ being the barycentric coordinates of P. Because of Theorem 1, we can replace A', B', C', D' in Definition 3 by A^*, B^*, C^*, D^* as in Definition 2. And we will use $\nabla A^*B^*C^*D^*$ to denote the *ex-center-tetrahedron* of $\nabla ABCD$ as well as the *tetrahedron that tangles* $\nabla ABCD$ *with respect to* the in-center P. Let $\mathcal{V}$, $\mathcal{V}^*$ be the volumes of $\nabla ABCD$, and $\nabla A^*B^*C^*D^*$, respectively.

Remark 2. The interior angle between the faces $\triangle ABC$ and $\triangle ABD$ of the tetrahedron $\nabla ABCD$ is called the *dihedral angle* of $\nabla ABCD$ at the edge AB, and let us denote it by $\sphericalangle AB$. Let Δ_{AB} be the plane containing the line $\overline{AB}$ that bisect the dihedral angle $\sphericalangle AB$. Since any point on Δ_{AB} is equidistant from the planes Ω_D and Ω_C, the in-center P of $\nabla ABCD$ must be on Δ_{AB}. Therefore, $\{P\} = \Delta_{AB} \cap \Delta_{AC} \cap \Delta_{AD} \cap \Delta_{BC} \cap \Delta_{BD} \cap \Delta_{CD}$. This is a way to find P.

We can continue using exterior dihedral angle bisecting planes of $\nabla ABCD$ to find ex-centers. However, Theorem 1 gives us an alternate method of *finding ex-centers* as follows:

Let $\{E\} = [AB] \cap \Delta_{CD}$, $\{F\} = [AC] \cap \Delta_{CB}$, $\{G\} = [AD] \cap \Delta_{BC}$, $\{H\} = [BC] \cap \Delta_{AD}$, $\{I\} = [BD] \cap \Delta_{AC}$, and $\{J\} = [CD] \cap \Delta_{AB}$. Then $[EJ] \cap [FI] \cap [GH] = \{P\}$. Let $\Omega_{EFG} = \Gamma_A$, $\Omega_{EHI} = \Gamma_B$, $\Omega_{FHJ} = \Gamma_C$, and $\Omega_{GJI} = \Gamma_D$, and let $\{A^*\} = \Gamma_B \cap \Gamma_C \cap \Gamma_D$, $\{B^*\} = \Gamma_A \cap \Gamma_C \cap \Gamma_D$, $\{C^*\} = \Gamma_A \cap \Gamma_B \cap \Gamma_D$, and $\{D^*\} = \Gamma_A \cap \Gamma_B \cap \Gamma_C$. Since P is a deep interior point of $\nabla ABCD$ by Lemma 3, the points A^*, B^*, C^*, D^* exists by Lemma 1(4). By Theorem 1, A^*, B^*, C^*, D^* are the ex-centers of the tetrahedron $\nabla ABCD$.

Corollary 1. *(1) The radii of the ex-spheres S_{A^*}, S_{B^*}, S_{C^*}, S_{D^*} of $\nabla ABCD$ are given by*

$$r_{A^*} = \frac{3\mathcal{V}}{\mathcal{A}_D + \mathcal{A}_C + \mathcal{A}_B - \mathcal{A}_A}, \quad r_{B^*} = \frac{3\mathcal{V}}{\mathcal{A}_D + \mathcal{A}_C - \mathcal{A}_B + \mathcal{A}_A},$$
$$r_{C^*} = \frac{3\mathcal{V}}{\mathcal{A}_D - \mathcal{A}_C + \mathcal{A}_B + \mathcal{A}_A}, \quad r_{D^*} = \frac{3\mathcal{V}}{-\mathcal{A}_D + \mathcal{A}_C + \mathcal{A}_B + \mathcal{A}_A},$$

respectively. Also, the in-radius r of $\nabla ABCD$ is given by $r = \frac{3\mathcal{V}}{\mathcal{A}_A + \mathcal{A}_B + \mathcal{A}_C + \mathcal{A}_D}$.

(2) $\frac{1}{r_{A^*}} + \frac{1}{r_{B^*}} + \frac{1}{r_{C^*}} + \frac{1}{r_{D^*}} = \frac{2}{r}$.

Proof. Proof of (1): Since $a+b+c+d=1$, we have $\mathcal{A}\cdot(1-2a)=\mathcal{A}\cdot(d+c+b-a)=\mathcal{A}_{\mathcal{D}}+\mathcal{A}_{\mathcal{C}}+\mathcal{A}_{\mathcal{B}}-\mathcal{A}_{\mathcal{A}}$. Hence, from the above proof, the radius r_{A^*} of S_{A^*} is given by $r_{A^*}=\frac{3}{\mathcal{A}\cdot(1-2a)}\mathcal{V}=\frac{3\mathcal{V}}{\mathcal{A}_D+\mathcal{A}_C+\mathcal{A}_B-\mathcal{A}_A}$. Similar for r_{B^*}, r_{C^*}, and r_{D^*}.

Since $\mathcal{V}=\frac{1}{3}r\mathcal{A}$, we have $r=\frac{3\mathcal{V}}{\mathcal{A}}=\frac{3\mathcal{V}}{\mathcal{A}_A+\mathcal{A}_B+\mathcal{A}_C+\mathcal{A}_D}$.

Proof of (2):

$$\frac{1}{r_{A^*}}+\frac{1}{r_{B^*}}+\frac{1}{r_{C^*}}+\frac{1}{r_{D^*}}=\frac{1}{3\mathcal{V}}\{(\mathcal{A}_D+\mathcal{A}_C+\mathcal{A}_B-\mathcal{A}_A)+(\mathcal{A}_D+\mathcal{A}_C-\mathcal{A}_B+\mathcal{A}_A)+ (\mathcal{A}_D-\mathcal{A}_C+\mathcal{A}_B+\mathcal{A}_A)+(-\mathcal{A}_D+\mathcal{A}_C+\mathcal{A}_B+\mathcal{A}_A)\}=2\cdot\frac{(\mathcal{A}_D+\mathcal{A}_C+\mathcal{A}_B+\mathcal{A}_A)}{3\mathcal{V}}=\frac{2}{r}$$

by part (1). □

Remark 3. Corollary 1(1) can be proved by the method similar to Exercise 5 on page 13 and its hint on page 156, [2]. Corollary 1(2) is known. See Equation (5) on page 83 of [1]. If r_a, r_b, r_c are the ex-radii of a triangle $\triangle ABC$, and if r is the inradius of $\triangle ABC$, then it is known that $\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}=\frac{1}{r}$ (see Exercise 6 on page 13, [2]).

4 Weakly Reversible Tetrahedron

We will characterize a weakly reversible tetrahedron in Theorem 2 whose proof heavily depends on Theorem 1 because of its use in Lemma 6. Recall $P=[a,b,c,d]$ is the incenter of the tetrahedron $\nabla ABCD$.

Lemma 6. *Recall that $\mathcal{V}$ and $\mathcal{V}^*$ be the volumes of $\nabla ABCD$ and its ex-center-tetrahedron $\nabla A^*B^*C^*D^*$, respectively. then $\mathcal{V}^*=\frac{16abcd}{(1-2a)(1-2b)(1-2c)(1-2d)}\mathcal{V}$.*

Proof. Since $\nabla A^*B^*C^*D^*$ tangles with $\nabla ABCD$ with respect to its in-center, we can apply [3, Theorem 3(2)] by Theorem 1. □

Theorem 2. *A tetrahedron $\nabla ABCD$ is weakly reversible if and only if $\nabla ABCD$ and its ex-center-tetrahedron $\nabla A^*B^*C^*D^*$ have the same volume.*

Proof. Suppose the tetrahedron $\nabla ABCD$ is weakly reversible, say $\mathcal{A}_A+\mathcal{A}_B=\mathcal{A}_C+\mathcal{A}_D$. Then $a=\frac{\mathcal{A}_A}{\mathcal{A}}=\frac{\mathcal{A}_A}{2(\mathcal{A}_A+\mathcal{A}_B)}$. Hence, $1-2a=1-2\cdot\frac{\mathcal{A}_A}{2(\mathcal{A}_A+\mathcal{A}_B)}=\frac{(\mathcal{A}_A+\mathcal{A}_B)-\mathcal{A}_A}{\mathcal{A}_A+\mathcal{A}_B}=\frac{\mathcal{A}_B}{\mathcal{A}_A+\mathcal{A}_B}$ so that $\frac{a}{1-2a}=\frac{\mathcal{A}_A}{2(\mathcal{A}_A+\mathcal{A}_B)}\cdot\frac{(\mathcal{A}_A+\mathcal{A}_B)}{\mathcal{A}_B}=\frac{\mathcal{A}_A}{2\mathcal{A}_B}$.

Similarly, we have $\frac{b}{1-2b}=\frac{\mathcal{A}_B}{2\mathcal{A}_A}$, $\frac{c}{1-2c}=\frac{\mathcal{A}_D}{2\mathcal{A}_C}$, and $\frac{d}{1-2d}=\frac{\mathcal{A}_C}{2\mathcal{A}_D}$.

By substituting these into the formula given in Lemma 6, we have

$$\mathcal{V}^*=16\cdot\frac{a}{1-2a}\cdot\frac{b}{1-2b}\cdot\frac{c}{1-2c}\cdot\frac{d}{1-2d}\cdot\mathcal{V}=16\cdot\frac{\mathcal{A}_A}{2\mathcal{A}_B}\cdot\frac{\mathcal{A}_B}{2\mathcal{A}_A}\cdot\frac{\mathcal{A}_D}{2\mathcal{A}_C}\cdot\frac{\mathcal{A}_C}{2\mathcal{A}_D}\cdot\mathcal{V}=\mathcal{V}.$$

Hence, $\nabla ABCD$ and $\nabla A^*B^*C^*D^*$ have the same volume.

Conversely, suppose $\nabla ABCD$ and $\nabla A^*B^*C^*D^*$ have the same volume, i.e., $\mathcal{V}^*=\mathcal{V}$. By Lemma 6, we have

$$\mathcal{V}=\mathcal{V}^*=\frac{16abcd}{(1-2a)(1-2b)(1-2c)(1-2d)}\mathcal{V}.$$

Since $\mathcal{V}\neq 0$, this implies that

$$16abcd=(1-2a)(1-2b)(1-2c)(1-2d)=(-a+b+c+d)(a-b+c+d)(a+b-c+d)(a+b+c-d).$$

By multiplying and simplifying, we have

$$a^4+b^4+c^4+d^4-2(a^2b^2+a^2c^2+a^2d^2+b^2c^2+b^2d^2+c^2d^2)+8abcd=0.$$

This can be factored as

$$[a+b+c+d][(a+b)-(c+d)][(a+c)-(b+d)][(b+c)-(a+d)]=0.$$

Hence, $a+b=c+d$, $a+c=b+d$, or $a+d=b+c$. These are equivalent to $(\mathcal{A}_A+\mathcal{A}_B=\mathcal{A}_C+\mathcal{A}_D)$, $(\mathcal{A}_A+\mathcal{A}_C=\mathcal{A}_B+\mathcal{A}_D)$, or $(\mathcal{A}_A+\mathcal{A}_D=\mathcal{A}_B+\mathcal{A}_C)$. Therefore, the tetrahedron $\nabla ABCD$ is weakly reversible. □

Remark 4. As a related result, [5, Theorem 1] states that a tetrahedron T is isosceles if and only if its twin tetrahedron is the ex-center-tetrahedron of T. (Please see [5, Definition 3] or [1] for the definition of "twin".)

5 Reversible Tetrahedra

Example 2. Let $A=(1,0,0)$, $B=(-1,0,0)$, $C=(0,2,0)$, and $D=(0,0,1)$ in $\mathbb{R}^3$, then the tetrahedron $\nabla ABCD$ is a weakly reversible tetrahedron (see Example 1). The normal vectors to the planes Ω_A, Ω_B, Ω_C, and Ω_D are $\vec{n}_A=\langle -2,1,2\rangle$, $\vec{n}_B=\langle 2,1,2\rangle$, $\vec{n}_C=\langle 0,1,0\rangle$, and $\vec{n}_D=\langle 0,0,1\rangle$, respectively. From these, we can check that the in-center is $P=\left(0,\frac{1}{3},\frac{1}{3}\right)$. For example,

$$d(P,\Omega_A)=\frac{|\overrightarrow{BP}\cdot\vec{n}_A|}{|\vec{n}_A|}=\frac{1}{3}=\frac{|\overrightarrow{BP}\cdot\vec{n}_C|}{|\vec{n}_C|}=d(P,\Omega_C).$$

Here, $|\overrightarrow{BP}\cdot\vec{n}_A|$ is the absolute value of the dot product, and $|\vec{n}_A|$ is the norm of the vector. Similarly, we can verify that the vertices of its ex-center-tetrahedron $\nabla A^*B^*C^*D^*$ are given by

$$A^*=\left(-1,\tfrac{2}{3},\tfrac{2}{3}\right),\quad B^*=\left(1,\tfrac{2}{3},\tfrac{2}{3}\right),\quad C^*=\left(0,-\tfrac{1}{2},\tfrac{1}{2}\right),\quad D^*=(0,1,-1).$$

Now, we can calculate the area $\mathcal{A}_{A^*B^*C^*}$ of the triangle $\triangle A^*B^*C^*$ and others as follows;

$$\mathcal{A}_{A^*B^*C^*}=\tfrac{|\overrightarrow{A^*B^*}\times\overrightarrow{A^*C^*}|}{2}=\tfrac{\sqrt{50}}{6},\quad \mathcal{A}_{A^*B^*D^*}=\tfrac{\sqrt{26}}{3},\quad \mathcal{A}_{A^*C^*D^*}=\tfrac{\sqrt{34}}{4},\quad\text{and}\quad \mathcal{A}_{B^*C^*D^*}=\tfrac{\sqrt{34}}{4}.$$

These show that no two sums of $\mathcal{A}_{A^*B^*C^*}$, $\mathcal{A}_{A^*B^*D^*}$, $\mathcal{A}_{A^*C^*D^*}$, $\mathcal{A}_{B^*C^*D^*}$ is equal to the remaining two sums. Thus, $\nabla A^*B^*C^*D^*$ is not weakly reversible. Therefore, the ex-center-tetrahedron of a weakly reversible tetrahedron may not be weakly reversible.

On the other hand, we will prove that the ex-center-tetrahedron of a reversible tetrahedron is reversible in Theorem 3 below. The proof is computational and we need the following lemma to prove it. Please note that the letters a, b, c used in this section are not related to the barycentric coordinates of the in-center P.

Lemma 7. *(1) A reversible tetrahedron $\nabla ABCD$ can be embedded in $\mathbb{R}^3$ so that $A=(-a,-b,c)$, $B=(a,b,c)$, $C=(-ka,kb,0)$, $D=(ka,-kb,0)$ for some $a,b,c>0$ and $k\geq 1$, given in Cartesian coordinates.*

(2) Let $L=\sqrt{(ac)^2+(bc)^2+4(kab)^2}$ and $M=\sqrt{(ac)^2+(bc)^2+4(ab)^2}$. And let $P=\left(0,0,\frac{kcM}{L+kM}\right)$. Then P is the in-center of $\nabla ABCD$.

(3) The vertices of its ex-center-tetrahedron $\nabla A^*B^*C^*D^*$ *are given by*

$$A^* = \left(\tfrac{kaM}{L}, \tfrac{kbM}{L}, 0\right), \quad B^* = \left(\tfrac{-kaM}{L}, \tfrac{-kbM}{L}, 0\right), \quad C^* = \left(\tfrac{aL}{M}, \tfrac{-bL}{M}, c\right), \quad D^* = \left(\tfrac{-aL}{M}, \tfrac{bL}{M}, c\right).$$

Proof. The statement (1) is [4, Lemma 11]. Normal vectors to the planes Ω_A, Ω_B, Ω_C, and Ω_D are $\vec{n}_A = \langle bc, ac, -2ab\rangle$, $\vec{n}_B = \langle -bc, -ac, -2ab\rangle$, $\vec{n}_C = \langle bc, -ac, 2kab\rangle$, and $\vec{n}_D = \langle -bc, ac, 2kab\rangle$, respectively. From these, we can check that P is the in-center of $\nabla ABCD$, and that A^*, B^*, C^*, D^* are ex-centers of the tetrahedron $\nabla ABCD$. These computational checking are similar to the above Example 2, and we leave these checking to the readers. □

Theorem 3. *If* $\nabla ABCD$ *is reversible, then so is its ex-center-tetrahedron* $A^*B^*C^*D^*$.

Proof. Using the notations and conclusions of the above lemma, we have

$$|A^*C^*|^2 = \left(\tfrac{aL}{M} - \tfrac{kaM}{L}\right)^2 + \left(\tfrac{bL}{M} + \tfrac{kbM}{L}\right)^2 + c^2 = |B^*D^*|^2$$

so that $|A^*C^*| = |B^*D^*|$. Similarly,

$$|A^*D^*|^2 = \left(\tfrac{aL}{M} + \tfrac{kaM}{L}\right)^2 + \left(\tfrac{bL}{M} - \tfrac{kbM}{L}\right)^2 + c^2 = |B^*D^*|^2$$

so that $|A^*D^*| = |B^*C^*|$. Therefore, $\nabla A^*B^*C^*D^*$ is a reversible tetrahedron. □

Remark 5. From the proof of Lemma 7, we can check that $P = \left(0, 0, \frac{kcM}{L+kM}\right)$ is the in-center of $\nabla ABCD$. Normal vectors to the planes $\Omega_{A^*B^*C^*}$ and $\Omega_{A^*B^*D^*}$ are

$$\overrightarrow{n_{A^*B^*C^*}} = \langle -bcM, acM, 2aL\rangle \quad \text{and} \quad \overrightarrow{n_{A^*B^*D^*}} = \langle bcL, acL, 2abkM\rangle.$$

Hence, we have

$$d(P, \Omega_{A^*B^*C^*}) = \frac{|\overrightarrow{A^*P} \cdot \overrightarrow{n_{A^*B^*C^*}}|}{|\overrightarrow{n_{A^*B^*C^*}}|} = \frac{2abckML}{(L+kM)\sqrt{(bcM)^2 + (acM)^2 + 4(abL)^2}},$$

and

$$d(P, \Omega_{A^*B^*D^*}) = \frac{|\overrightarrow{A^*P} \cdot \overrightarrow{n_{A^*B^*D^*}}|}{|\overrightarrow{n_{A^*B^*D^*}}|} = \frac{2abckML}{(L+kM)\sqrt{(bcL)^2 + (acL)^2 + 4(abM)^2}}.$$

Suppose $a = b = c = 1$ and $k = 2$. Then $M = \sqrt{6}$, and $L = \sqrt{18}$. Then by Lemma 7, $\nabla ABCD$ is weakly reversible, and we have

$$d(P, \Omega_{A^*B^*C^*}) = \tfrac{4\sqrt{6}\sqrt{18}}{(\sqrt{6}+\sqrt{18})\sqrt{84}} \neq \tfrac{4\sqrt{6}\sqrt{18}}{(\sqrt{6}+\sqrt{18})\sqrt{80}} = d(P, \Omega_{A^*B^*D^*}).$$

Therefore, the in-center of a weakly reversible tetrahedron may not be the in-center of its ex-center-tetrahedron.

On the other hand, by noting that any isosceles tetrahedron $\nabla ABCD$ can be embedded in $\mathbb{R}^3$ by letting $A = (-a, -b, c)$, $B = (a, b, c)$, $C = (-a, b, 0)$, $D = (a, -b, 0)$ for some $a, b, c > 0$. So let $k = 1$ in Lemma 7. Then $\nabla ABCD$ becomes an isosceles tetrahedron and $L = M$, $A^* = (a, b, 0)$, $B^* = (-a, -b, 0)$, $C^* = (a, -b - c)$, $D^* = (-a, b, c)$, and $P = \left(0, 0, \frac{c}{2}\right)$ is the in-center of $\nabla ABCD$ as well as $\nabla A^*B^*C^*D^*$. We can see this in an alternate way. Suppose $\nabla ABCD$ is isosceles. Then the in-center P is the centroid of $\nabla ABCD$, and we know that $\nabla ABCD$ and $\nabla A^*B^*C^*D^*$ are twins (see [5, Theorem 3]) so that P is also the in-center of $\nabla A^*B^*C^*D^*$. Hence, A, B, C, D are the ex-centers of $\nabla A^*B^*C^*D^*$. (Please see [5, Definition 3], [4, Definition 4], or [1] for the definition of "twin".)

Example 3. Let $\nabla ABCD$ be the reversible tetrahedron in Lemma 7. By calculating the determinants, we have

$$\overrightarrow{AB} \cdot (\overrightarrow{AC} \times \overrightarrow{AD}) = \begin{vmatrix} 2a & 2b & 0 \\ -ka + a & kb + b & -c \\ ka + a & -kb + b & -c \end{vmatrix} = -8abck,$$

and

$$\overrightarrow{A^*B^*} \cdot (\overrightarrow{A^*C^*} \times \overrightarrow{A^*D^*}) = \begin{vmatrix} -\frac{2akM}{L} & -\frac{2bkM}{L} & 0 \\ \frac{aL}{M} - \frac{akM}{L} & -\frac{bL}{M} - \frac{bkM}{L} & -c \\ -\frac{aL}{M} - \frac{akM}{L} & \frac{bL}{M} - \frac{bkM}{L} & -c \end{vmatrix} = -8abck.$$

Therefore, $\mathcal{V} = \frac{8abck}{6} = \mathcal{V}^*$ as we expected from Theorem 2.

Also, from Lemma 7, we can calculate that

$$r = \frac{2abck}{L + kM}, \quad r_{A^*} = r_{B^*} = \frac{2abck}{L}, \quad \text{and} \quad r_{C^*} = r_{D^*} = \frac{2abc}{M}.$$

Hence, $\frac{1}{r_{A^*}} + \frac{1}{r_{B^*}} + \frac{1}{r_{C^*}} + \frac{1}{r_{D^*}} = \frac{L}{abck} + \frac{M}{abc} = \frac{L+kM}{abck} = \frac{2}{r}$ as expected from Corollary 1.

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