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On Min/Max Angle Conditions for Simplices in Arbitrary Space Dimension

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Abstract. We investigate a hierachical relation between minimum and maximum angle conditions widely used in the interpolation theory and finite element analysis over simplicial partitions. We prove that the minimum angle condition implies the maximum angle condition in arbitrary space dimension.

Key Words: simplex, dihedral angle, finite element method, angle conditions, interpolation.

MSC 2020: 65N30 (primary), 65M60, 65D05

1 Introduction

It is well-known that the sum of angles in a triangle is constant and equal to π. Therefore, it is easy to prove that the existence of a positive lower bound on angles of all triangles in all triangulations used (minimum angle condition in $2d$) implies the existence of an upper bound (less than π) on angles of all triangles in all triangulations used (maximum angle condition in $2d$). Both angle conditions guarantee the optimal interpolation properties in appropriate Sobolev norms and, therefore, are widely used in the finite element analysis, see e.g. [1, 3, 7, 9, 12, 14]. However, the situations with tetrahedra and simplices is quite different. Thus, the sum of dihedral angles in a tetrahedron is not constant anymore and varies between 2π and 3π. Similar situation appears in higher dimensions. See [6]. Therefore, the implication between minimum and maximum angle conditions in higher dimensions is much less trivial than in the case of triangulations. Simplicial meshes are often used for problems in higher dimensions $d \geq 4$ which appear e.g. in financial mathematics, statistical physics, etc., see [4, 13].

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2 Denotations and Definitions

Let $\Omega \subset \mathbf{R}^d$ be a bounded domain for $d \in \{2, 3, \dots\}$. Assume that $\overline{\Omega}$ is *polytopic.* By this we mean that $\overline{\Omega}$ is the closure of a domain whose boundary $\partial\overline{\Omega}$ is contained in a finite number of $(d-1)$-dimensional hyperplanes.

Let A_0, A_1, ..., A_d be points in $\mathbf{R}^d, d \in \{1, 2, \dots\}$, not lying in one hyperplane. Then the convex hull of these points

$$S = \mathrm{conv}\{A_0, A_1, \dots, A_d\}$$

is called a *d-simplex* or shortly a *simplex.* The points A_0, A_1, ..., A_d are called the *vertices* of S, and the symbol F_j stands for the facet opposite to the vertex A_j.

Next we define a simplicial partition of a bounded closed polytopic domain $\overline{\Omega} \subset \mathbf{R}^d$ as follows. We subdivide $\overline{\Omega}$ into a finite number of d-simplices, so that their union is $\overline{\Omega}$, any two distinct simplices have disjoint interiors, and any facet of any simplex is either a facet of another simplex from the partition or belongs to the boundary $\partial\overline{\Omega}$. The set of such simplices will be called *simplicial partition* and denoted by $\mathcal{T}_h$.

For a given partition $\mathcal{T}_h$ the *discretization parameter* h stands for the maximum length of all edges in the partition, i.e.,

$$h = \max_{S \in \mathcal{T}_h} h_S,$$

where

$$h_S = \mathrm{diam}\, S.$$

Definition 1. An infinite sequence $\mathcal{F} = \{\mathcal{T}_h\}_{h \to 0}$ of simplicial partitions of $\overline{\Omega}$ is called a *family of simplicial partitions* if for every $\varepsilon > 0$ there exists $\mathcal{T}_h \in \mathcal{F}$ with $h < \varepsilon$.

Definition 2. The dihedral angles of a d-simplex are defined as the complementary angles of the angles between the outward unit normals to the corresponding facets and can thus be calculated by means of the inner product

$$\cos \alpha = -n_i \cdot n_j, \tag{1}$$

where n_i and n_j, $i \neq j$, are outward unit normals of the facets F_i and F_j, respectively. Similarly we can define dihedral angles for lower-dimensional simplices. For $d = 3$, the dihedral angle is the standard interior angle between tetrahedral faces, and for $d = 2$, it is the usual angle between sides of a triangle.

In the following definition we present a generalization of the concept of the minimum angle condition proposed by Zlámal [14] to any dimension, see [11].

Definition 3. A family $\mathcal{F} = \{\mathcal{T}_h\}_{h \to 0}$ of partitions of a polytope $\overline{\Omega} \subset \mathbf{R}^d$ into d-simplices is said to satisfy the *d-dimensional minimum angle condition* if there exists a constant $\alpha_0 > 0$ such that for any $\mathcal{T}_h \in \mathcal{F}$ and any simplex $S \in \mathcal{T}_h$ and any subsimplex $S' \subseteq S$ with vertex set contained in the vertex set of S, the minimum dihedral angle in S' is not less than α_0.

Further, we present a generalization of the concept of maximum angle condition proposed in [12] to any dimension, see also [8].

Definition 4. A family $\mathcal{F} = \{\mathcal{T}_h\}_{h\to 0}$ of partitions of a polytope $\overline{\Omega} \subset \mathbf{R}^d$ into d-simplices is said to satisfy the *d-dimensional maximum angle condition* if there exists a constant $\gamma_0 < \pi$ such that for $\mathcal{T}_h \in \mathcal{F}$ and any simplex $S \in \mathcal{T}_h$ and any subsimplex $S' \subseteq S$ with vertex set contained in the vertex set of S, the maximum dihedral angle in S' is less than or equal to γ_0.

In what follows, we will use the definition for the d-dimensional sine of angles in $\mathbf{R}^d$ introduced in [5]. In terms of the simplex S, for any of its vertices A_i, the d-dimensional sine of the angle of S at A_i, denoted by $\hat{A}_i$, is defined as follows (see (3) in [5, p. 72]):

$$\sin_d(\hat{A}_i | A_0 A_1 \dots A_d) = \frac{d^{d-1}\,(\mathrm{meas}_d\, S)^{d-1}}{(d-1)!\,\prod_{j=0, j\neq i}^{d} \mathrm{meas}_{d-1} F_j}. \tag{2}$$

where the symbol $\prod$ stands for the standard product according to the involved indices.

The symbol sin without any subindex will stand for the usual sinus function in what follows.

Remark 1. For $d = 2$, $\sin_2(\hat{A}_i | A_0 A_1 A_2)$ is the standard sine of the angle $\hat{A}_i$ in the triangle $A_0 A_1 A_2$, due to the following well-known formula, e.g. for $i = 0$, we get from (2) that

$$\mathrm{meas}_2(A_0 A_1 A_2) = \frac{1}{2} |A_0 A_1|\, |A_0 A_2| \sin \hat{A}_0. \tag{3}$$

In fact, one can similarly define a sine for any k-dimensional (vertex) angle of any k-dimensional facet of S for $k \in \{2, \dots, d\}$ (which will include the case of vertices of S itself). Namely, let us denote the k-dimensional facet of S spanned by the $k+1$ (distinct) vertices $A_{i_0}, A_{i_1}, \dots, A_{i_k}$ by $F_{i_0, i_1, \dots, i_k}$. Then for any index $i_\ell \in \{i_0, \dots, i_k\}$ we set

$$\sin_k(\hat{A}_{i_\ell} | A_{i_0} A_{i_1} \dots A_{i_k}) = \frac{k^{k-1}\,(\mathrm{meas}_k\, F_{i_0, i_1, \dots, i_k})^{k-1}}{(k-1)!\,\prod_{i_j \in \{\{i_0, \dots, i_k\} \setminus \{i_\ell\}\}} \mathrm{meas}_{k-1} F^{i_j}_{i_0, i_1, \dots, i_k}}, \tag{4}$$

where $F^{i_j}_{i_0, i_1, \dots, i_k}$ denotes $(k-1)$-dimensional facet of $F_{i_0, i_1, \dots, i_k}$ (which is clearly itself a $(k-1)$-dimensional simplex) lying against the vertex A_{i_j}.

Remark 2. Notice that in the above denotation we have $S \equiv F_{0,1,\dots,d}$ and $F_j \equiv F^j_{0,1,\dots,d}$ for $j = 0,\ 1,\ \dots,\ d$.

Lemma 1. *The (generalized)* $\sin_d$ *of any angle in* S *is always positive and not greater than* 1*:*

$$0 < \sin_d(\hat{A}_i | A_0, A_1, \dots, A_d) \le 1 \quad \forall\ i = 0, 1, \dots, d. \tag{5}$$

Proof. From the definition (2) we see that the generalized sinus is always positive. Without loss of generality we may assume that $i = 0$. According to [5, p. 76], we find that

$$\sin_d(\hat{A}_0 | A_0 A_1 \dots A_d) \le \sin_{d-1}(\hat{A}_0 | A_0 A_1 \dots A_{d-1}) \le \dots \le \sin_2(\hat{A}_0 | A_0 A_1 A_2) \le 1. \tag{6}$$

Hence, (5) holds. □

3 The Minimum Angle Condition is Stronger than the Maximum Angle Condition

In [2], the following definition on the families of simplicial partitions is introduced.

Definition 5. A family $\mathcal{F} = \{\mathcal{T}_h\}_{h\to 0}$ of partitions of a polytope $\overline{\Omega} \subset \mathbf{R}^d$ into d-simplices is said to satisfy the generalized Zlámal condition if there exists a constant $C > 0$ such that for any $\mathcal{T}_h \in \mathcal{F}$ and any $S = \mathrm{conv}\{A_0, \dots, A_d\} \in \mathcal{T}_h$ we have

$$\sin_d(\hat{A}_i|A_0A_1 \dots A_d) \geq C > 0 \qquad \forall\, i \in \{0, 1, \dots, d\}, \tag{7}$$

where $\sin_d$ is defined in (2).

Theorem 1. *The d-dimensional minimum angle condition and the generalized Zlámal condition are equivalent.*

For the proof see [2].

Using the above statements we will prove the following theorem in any dimension, generalizing the result obtained in [10] for tetrahedra, but by a different argument.

Theorem 2. *The d-dimensional minimum angle condition implies the d-dimensional maximum angle condition.*

Proof. For a d-simplex $S = A_0A_1 \dots A_d$, the following relations between sinuses of different angles hold

$$\sin_d(\hat{A}_0|A_0A_1 \dots A_d) = \sin_{d-1}(\hat{A}_0|A_0A_1 \dots A_{d-2}A_{d-1}) \prod_{j=1}^{d-1} \sin \beta_{jd}, \tag{8}$$

where β_{jd} is the dihedral angle between the facet omitting A_j and the facet omitting A_d.

Due to Theorem 1, the left-hand side in (8) is bounded by constant α_0 from below. Further, we note that all sinuses involved are positive and not greater than one, therefore any $\sin \beta_{jd} \geq \alpha_0$, i.e., any dihedral angle involved is bounded from π. As this argument is valid for any vertex, the estimate is valid for any dihedral angle of S. Expanding $\sin_{d-1}$, then $\sin_{d-2}$, etc. in (8), and using relevant analogues of (8) within (lower-dimensional) facets of S, we, similarly to the above argumentation, can prove that all other dihedral angles between lower-dimensional facets of S are bounded from above. □

Remark 3. The converse statement is not true in any dimension $d \geq 2$. We can illustrate this by considering the d-simplex with the following vertices

$$\begin{aligned} A_0 &= (0, 0, 0, \dots, 0, 0),\\ A_1 &= (1, 0, 0, \dots, 0, 0),\\ A_2 &= (1, 1, 0, \dots, 0, 0),\\ &\vdots\\ A_{d-1} &= (1, 1, 1, \dots, 1, 0),\\ A_d &= (1, 1, 1, \dots, 1, \varepsilon), \end{aligned}$$

where $\varepsilon > 0$. It can be easily checked that all angles of this simplex are not greater than $\frac{\pi}{2}$, i.e., the d-dimensional maximum angle condition is satisfied, but the d-dimensional minimum angle condition does not hold, since by (6)

$$\sin_d(\hat{A}_0|A_0A_1 \dots A_d) \leq \sin_2(\hat{A}_0|A_0A_{d-1}A_d) \to 0$$

as $\varepsilon \to 0$.

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