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Some Remarks on Arne Dür's Equation for Oblique Axonometry

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Abstract. This paper provides a derivation of A. Dür's necessary and sufficient condition for an axonometric reference system to be, up to a uniform scale factor, the parallel image of an orthonormal reference system. After that, simple formulas relating the uniform scale factor and the projection direction are given. Finally, A. Dür's condition is extended to the case of the Pohlke-Schwarz theorem.

Key Words: oblique axonometry, Pohlke theorem, A. Dür's condition

MSC 2020: 51N10 (primary), 51N20, 51N05

1 Introduction

In [2, p. 139] A. Dür stated, without proof, a generalized form of the well known Gauss' fundamental equation of orthogonal axonometry (see [1, p. 308], [5, Eq. (3)]). This condition characterizes oblique axonometry when the direction of the parallel projection onto the image plane, say ω, is expressed in the form $\mathbf{n} + \mathbf{v}$ with vectors $\mathbf{n}$, $\mathbf{v}$ such that

$$\mathbf{n} \perp \omega, \ \|\mathbf{n}\| = 1 \quad \text{and} \quad \mathbf{v} \parallel \omega. \tag{1}$$

In details, given a plane ω in the Euclidean 3-space $\mathbb{E}^3$ and a point $O \in \omega$, we choose cartesian coordinates x, y, z such that $\omega = \{(x, y, z) \in \mathbb{E}^3 \mid z = 0\}$ and $O = (0, 0, 0)$. We then identify vectors $\overrightarrow{OP} \parallel \omega$ with complex numbers: if $P = (a, b, 0) \in \omega$, we say that

$$\overrightarrow{OP} \simeq p \quad \text{with} \quad p = a + bi \in \mathbb{C}.$$

Then, the following holds:

Theorem 1 (A. Dür)**.** *Under parallel projection onto ω in the direction $\mathbf{n} + \mathbf{v}$, three non-collinear vectors $\overrightarrow{OW_1}$, $\overrightarrow{OW_2}$, $\overrightarrow{OW_3} \subset \omega$ are images of three pairwise orthogonal vectors $\overrightarrow{OQ_1}$, $\overrightarrow{OQ_2}$, $\overrightarrow{OQ_3}$ with equal norm if and only if, setting*

$$\overrightarrow{OW_k} \simeq w_k \quad \textit{and} \quad \mathbf{v} \simeq v \qquad (v, w_k \in \mathbb{C}),$$

w_1, w_2, w_3 and v satisfy the equation

$$w_1^2 + w_2^2 + w_3^2 = \frac{1}{2}\Big(|w_1|^2 + |w_2|^2 + |w_3|^2 - |w_1^2 + w_2^2 + w_3^2|\Big)v^2. \tag{2}$$

2 Main Results and Motivations

The first goal of this paper is to provide a simple derivation Theorem 1. To this end, we apply the results of [3] where Pohlke theorem is proved and the projection direction and the reference trihedron (i.e., vectors $\overrightarrow{OQ}_k$) are analytically determined.

In particular from [3] we get that the projection direction is uniquely determined up to symmetry with respect to the image plane ω. Thus, if we express the direction of the projection in the form $\mathbf{n}+\mathbf{v}$, with $\mathbf{n}$ and $\mathbf{v}$ as in (1), we know that $\mathbf{v}$ in uniquely determined, except for the sense. Taking these considerations into account, to prove Theorem 1 it will be sufficient to show that the projection directions determined by Dür's equation (2) are the same as found in [3]. See Section 3.

By means of an explicit formula for the common norm of vectors $\overrightarrow{OQ}_k$ (which is uniquely determined), we can also improve Theorem 1 somewhat. Let

$$\rho = |OQ_k| \quad (1 \le k \le 3), \tag{3}$$

an let $\mathbf{n}$, $\mathbf{v}$ as in (1). In Section 4 we get that

Claim 1. *Under projection onto ω in the direction $\mathbf{n}+\mathbf{v}$, three non-collinear vectors $\overrightarrow{OW}_1$, $\overrightarrow{OW}_2$, $\overrightarrow{OW}_3 \subset \omega$ are images of three pairwise orthogonal vectors with norm ρ if and only if*

$$w_1^2 + w_2^2 + w_3^2 = \rho^2 v^2 \tag{4}$$

and

$$|w_1|^2 + |w_2|^2 + |w_3|^2 - |w_1^2 + w_2^2 + w_3^2| = 2\rho^2. \tag{5}$$

Finally, using an argument from [4], in Section 5 we extend Theorem 1 to the more general case of Pohlke-Schwarz theorem. In details, given three non-collinear vectors $\overrightarrow{OW}_1$, $\overrightarrow{OW}_2$, $\overrightarrow{OW}_3$ in ω, with $W_k = (x_k, y_k, 0)$, and three non-coplanar vectors $\overrightarrow{OT}_1$, $\overrightarrow{OT}_2$, $\overrightarrow{OT}_3$, with $T_k = (\tilde{x}_k, \tilde{y}_k, \tilde{z}_k)$, we consider the matrices

$$\mathcal{W} = \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathcal{T} = \begin{pmatrix} \tilde{x}_1 & \tilde{x}_2 & \tilde{x}_3 \\ \tilde{y}_1 & \tilde{y}_2 & \tilde{y}_3 \\ \tilde{z}_1 & \tilde{z}_2 & \tilde{z}_3 \end{pmatrix}. \tag{6}$$

Furthermore, we define

$$\mathcal{U} = \mathcal{W}\mathcal{T}^{-1} = \begin{pmatrix} x_1' & x_2' & x_3' \\ y_1' & y_2' & y_3' \\ 0 & 0 & 0 \end{pmatrix}, \tag{7}$$

and then we indicate with $\overrightarrow{OU}_1$, $\overrightarrow{OU}_2$, $\overrightarrow{OU}_3$ the non-collinear vectors of ω corresponding to the columns of the matrix $\mathcal{U}$, that is, we set

$$U_k = (x_k', y_k', 0) \quad (1 \le k \le 3). \tag{8}$$

With $\mathbf{n}$, $\mathbf{v}$ as in (1), the following holds.

Claim 2. *Under parallel projection onto ω in the direction $\mathbf{n}+\mathbf{v}$, three non-collinear vectors $\overrightarrow{OW}_1$, $\overrightarrow{OW}_2$, $\overrightarrow{OW}_3 \subset \omega$ are images of three non-coplanar vectors $\overrightarrow{OZ}_1$, $\overrightarrow{OZ}_2$, $\overrightarrow{OZ}_3$ that show reciprocal ratios and angles like $\overrightarrow{OT}_1$, $\overrightarrow{OT}_2$, $\overrightarrow{OT}_3$ if and only if, setting*

$$\overrightarrow{OU}_k \simeq u_k \quad \text{and} \quad \mathbf{v} \simeq v \qquad (u_k, v \in \mathbb{C}),$$

u_1, u_2, u_3 and v satisfy the equation

$$u_1^2 + u_2^2 + u_3^2 = \frac{1}{2}\Big(|u_1|^2 + |u_2|^2 + |u_3|^2 - |u_1^2 + u_2^2 + u_3^2|\Big)v^2. \tag{9}$$

3 A Proof of Theorem 1

As remarked in the previous section it will be enough to prove that Dür's Equation (2) gives the same projection directions as found in [3].

Let $W_k = (x_k, y_k, 0)$, for $1 \le k \le 3$. As in [3], we consider the row vectors

$$A_1 = (x_1, x_2, x_3) \quad \text{and} \quad A_2 = (y_1, y_2, y_3), \tag{10}$$

which are linearly independent because we assume $\overrightarrow{OW}_1$, $\overrightarrow{OW}_2$, $\overrightarrow{OW}_3$ non-collinear.

It is then easy to see that

$$|w_1|^2 + |w_2|^2 + |w_3|^2 = \|A_1\|^2 + \|A_2\|^2, \tag{11}$$

$$w_1^2 + w_2^2 + w_3^2 = \|A_1\|^2 - \|A_2\|^2 + 2(A_1 \cdot A_2)i. \tag{12}$$

This means that (2) is equivalent to

$$v^2 = \frac{2\Big[\|A_1\|^2 - \|A_2\|^2 + 2(A_1 \cdot A_2)i\Big]}{\|A_1\|^2 + \|A_2\|^2 - \sqrt{(\|A_1\|^2 - \|A_2\|^2)^2 + 4(A_1 \cdot A_2)^2}}. \tag{13}$$

To prove (13), according to [3, Defs. (3.6), (3.21)] we define the quantities:

$$\gamma \doteqdot \arccos\left(\frac{A_1 \cdot A_2}{\|A_1\|\|A_2\|}\right), \quad \lambda \doteqdot \frac{\|A_1\|}{\|A_2\|}, \tag{14}$$

$$\eta \doteqdot \frac{\lambda^2 + 1 + \sqrt{(\lambda^2+1)^2 - 4\lambda^2 \sin^2\gamma}}{2\lambda^2 \sin^2\gamma}.^1 \tag{15}$$

Then, by [3, Formulas (3.10), (4.6)], the direction of the parallel projection $\pi\colon \mathbb{E}^3 \to \omega$ such that $\pi(\overrightarrow{OQ}_k) = \overrightarrow{OW}_k$ ($1 \le k \le 3$, $\overrightarrow{OQ}_k$ pairwise orthogonal and with equal norm) is given by the vector $\overrightarrow{OU}$, with $U = (-\alpha, -\beta, 1)$ and

$$(\alpha, \beta) \doteqdot \pm\Big(\sqrt{\eta\lambda^2 - 1}, \operatorname{sgn}(\cos\gamma)\sqrt{\eta - 1}\Big).^2 \tag{16}$$

So, taking into account (1), it will be sufficient to show that the right-hand side of (13) is equal to $(\alpha + \beta i)^2$. To begin with, noting (15), we can rewrite η as

$$\eta = \frac{2}{\lambda^2 + 1 - \sqrt{(\lambda^2-1)^2 + 4\lambda^2\cos^2\gamma}}. \tag{17}$$

Then, from (16) and (17), we immediately have

$$\alpha^2 - \beta^2 = \eta(\lambda^2 - 1) = \frac{2(\lambda^2 - 1)}{(\lambda^2+1) - \sqrt{(\lambda^2-1)^2 + 4\lambda^2\cos^2\gamma}}. \tag{18}$$

[1]Note that η is well defined because $\gamma \neq 0, \pi$.

[2]Here, $\operatorname{sgn}(t) = 1$ if $t \ge 0$, $\operatorname{sgn}(t) = -1$ otherwise.

Still from (16) and (17), for $\alpha^2\beta^2$ we find the expression

$$\begin{aligned}\alpha^2\beta^2 &= (\eta\lambda^2-1)(\eta-1)\\ &= \left(\frac{\lambda^2-1+\sqrt{(\lambda^2-1)^2+4\lambda^2\cos^2\gamma}}{\lambda^2+1-\sqrt{(\lambda^2-1)^2+4\lambda^2\cos^2\gamma}}\right)\left(\frac{-(\lambda^2-1)+\sqrt{(\lambda^2-1)^2+4\lambda^2\cos^2\gamma}}{\lambda^2+1-\sqrt{(\lambda^2-1)^2+4\lambda^2\cos^2\gamma}}\right)\\ &= \frac{4\lambda^2\cos^2\gamma}{\left(\lambda^2+1-\sqrt{(\lambda^2-1)^2+4\lambda^2\cos^2\gamma}\right)^2}.\end{aligned} \tag{19}$$

This means that

$$\alpha\beta = \operatorname{sgn}(\cos\gamma)\sqrt{\eta\lambda^2-1}\sqrt{\eta-1} = \frac{2\lambda\cos\gamma}{\lambda^2+1-\sqrt{(\lambda^2-1)^2+4\lambda^2\cos^2\gamma}}. \tag{20}$$

Then, from (18) and (20), we obtain

$$(\alpha+\beta i)^2 = \frac{2[\lambda^2-1+2(\lambda\cos\gamma)i]}{\lambda^2+1-\sqrt{(\lambda^2-1)^2+4\lambda^2\cos^2\gamma}}. \tag{21}$$

Finally, noting the definitions of (14) and substituting into the right hand side of (21) we obtain the right hand side of (13). Dür's equation (2) is therefore verified.

4 Proof of Claim 1

It will be enough to show that (4), (5) give the same value of ρ and the same projection directions as found in [3]. By [3, Formula (1.4)] and the definition of γ, we have

$$\rho = \frac{\sqrt{2}\|A_1\|\|A_2\|\sin\gamma}{\sqrt{\|A_1\|^2+\|A_2\|^2+\sqrt{(\|A_1\|^2+\|A_2\|^2)^2-4\|A_1\|^2\|A_2\|^2\sin^2\gamma}}}. \tag{22}$$

Then, an easy calculation gives

$$\begin{aligned}\rho^2 &= \frac{\|A_1\|^2+\|A_2\|^2-\sqrt{(\|A_1\|^2+\|A_2\|^2)^2-4\|A_1\|^2\|A_2\|^2\sin^2\gamma}}{2}\\ &= \frac{\|A_1\|^2+\|A_2\|^2-\sqrt{(\|A_1\|^2-\|A_2\|^2)^2+4\|A_1\|^2\|A_2\|^2\cos^2\gamma}}{2}.\end{aligned} \tag{23}$$

Taking (11), (12) into account, from (23) we get (5). Also noting (13), we have

$$\rho^2 v^2 = \|A_1\|^2-\|A_2\|^2+2(A_1\cdot A_2)i = w_1^2+w_2^2+w_3^2. \tag{24}$$

So, Equation (4) is proved.

5 Proof of Claim 2

This statement follows from the explicit determination, given in [4, § 2], of the projection direction in the case of Pohlke-Schwarz's theorem. In fact, it has been shown that in the case of Pohlke-Schwarz theorem the projection direction is the same as in the Pohlke theorem, but with the vectors $\overrightarrow{OU}_1$, $\overrightarrow{OU}_2$, $\overrightarrow{OU}_3$ instead of $\overrightarrow{OW}_1$, $\overrightarrow{OW}_2$, $\overrightarrow{OW}_3$.

More precisely, let $\mathcal{V}$ be the column vector[3] with elements given by the components of the projection direction, that is, the vector $\mathbf{n}+\mathbf{v}$. Besides, let $\mathcal{Z}$ be the matrix with the columns given by the coordinates of the points Z_1, Z_2, Z_3.[4] Then, taking into account (6), it is immediate to note that the following facts are equivalent:

(i) *Under projection in the direction* $\mathbf{n}+\mathbf{v}$, $\overrightarrow{OW}_1$, $\overrightarrow{OW}_2$, $\overrightarrow{OW}_3$ *are the images of three vectors* $\overrightarrow{OZ}_1$, $\overrightarrow{OZ}_2$, $\overrightarrow{OZ}_3$ *that show reciprocal ratios and angles like* $\overrightarrow{OT}_1$, $\overrightarrow{OT}_2$, $\overrightarrow{OT}_3$*;*

(ii) $\mathcal{Z}-\mathcal{W}=\mathcal{V}\Lambda$ *and* $\mathcal{Z}=\mathcal{Q}\mathcal{T}$*, with* $\mathcal{Q}$ *a nonzero multiple of an orthogonal matrix and* $\Lambda=(\lambda_1,\lambda_2,\lambda_3)$ *a suitable row vector.*[3]

To proceed, recalling (7), we rewrite the equations of *(ii)* simply as

$$\mathcal{Q}-\mathcal{U}=(\mathcal{Z}-\mathcal{W})\mathcal{T}^{-1}=(\mathcal{V}\Lambda)\mathcal{T}^{-1}=\mathcal{V}(\Lambda\mathcal{T}^{-1})=\mathcal{V}\widetilde{\Lambda}, \tag{25}$$

where $\widetilde{\Lambda}=\Lambda\mathcal{T}^{-1}$ is still a row vector. Therefore, *(ii)* is equivalent to requiring

$$\mathcal{Q}-\mathcal{U}=\mathcal{V}\widetilde{\Lambda}, \tag{26}$$

with $\mathcal{Q}$ a nonzero multiple of an orthogonal matrix and $\widetilde{\Lambda}=(\tilde{\lambda}_1,\tilde{\lambda}_2,\tilde{\lambda}_3)$ a suitable row vector. Noting that the column vector $\mathcal{V}$ in (26) has remained the same, we deduce that the direction of the projection has not changed. In over words, by indicating with Q_1, Q_2, Q_3 the points corresponding to the columns of $\mathcal{Q}$ we obtain that *(ii)* is equivalent to the following:

(iii) *Under projection in the direction* $\mathbf{n}+\mathbf{v}$*, the non-collinear vectors* $\overrightarrow{OU}_1$, $\overrightarrow{OU}_2$, $\overrightarrow{OU}_3$ *are the images of three pairwise orthogonal vectors* $\overrightarrow{OQ}_1$, $\overrightarrow{OQ}_2$, $\overrightarrow{OQ}_3$ *with equal norm.*

The last statement is exactly the first part of Theorem 1, with the vectors $\overrightarrow{OU}_k$ instead of $\overrightarrow{OW}_k$. This means that *(iii)* holds if and only if v satisfies Dür's Equation (2) with the complex numbers u_1, u_2, u_3 instead of w_1, w_2, w_3. That is, Equation (9).

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[3] That is, a 3×1 matrix. Similarly, we consider Λ a 1×3 matrix.

[4] As already done in (6) for the matrices $\mathcal{W}$, $\mathcal{T}$ and in (7), (8) for $\mathcal{U}$.