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Center of the Ellipse at the Base of a Cone with a Constant Base Area

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Abstract. In this paper, the locus of the center of the top surface in an inverted cone was considered geometrically when the cone was tilted. When the area of the ellipse on the top surface of the volume is constant, the locus of the center of the top surface is found to be part of an ellipsoid. The locus of the center of the ellipse is also shown for the case where the volume between the top surface and the cone is held constant and the length of the major axis of the top surface of the volume is constant. Here, the behavior of the center of the top surface of a right circular cone cut in a plane is applicable to describe the motion of conduction electrons in single-layer graphene, which is being tremendously studied recently in the field of physics, since the behavior of conduction electrons in single-layer graphene is described by a right circular cone (Dirac cone).

Key Words: ellipsoid cone, elliptic cone, center of ellipse
MSC 2020: 51N30 (primary), 51M25, 51M10

1 Introduction

When a right circular cone is cut by a plane in Euclidean space, the (edge) shape of the cross section can be a circle, an ellipse, a parabola, or a hyperbola, depending on the angle of the plane to be cut [1, 3–6]. In this study, the case in which the cross-sectional shape is an ellipse is considered. Specifically, the following positions were considered: The center of the ellipse on the upper surface when the major axis length of the ellipse on the upper surface of the right circular cone is constant, i.e., the center of the upper line segment when the right circular cone is a triangle viewed from the side; the center of the ellipse on the upper surface when the area of the upper surface of the right circular cone remains constant; and the center of the ellipse on the upper surface when the volume of the right circular cone remains constant, respectively. In particular, the position of the center of the ellipse on the top surface when the volume of the right circular cone is held constant corresponds to the behavior of the

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center of the top surface of the "liquid"[1] when the "cocktail glass" is tilted while the liquid in the "cocktail glass" is considered an upside-down right circular cone (Fig. 1). The red dot represents the center of the ellipse on the top surface. Figure 1 shows the "liquid" poured into a vertical inverted cone. The upper surface of the "liquid" corresponds to the surface when cut by a plane perpendicular to the central axis of the right circular cone (Fig. 1(a)). The shape of the upper surface is a circle. The height of the center depends on the volume of the "liquid" and the slope of the sides of the glass. The horizontal position of the center is on the central axis of the right circular cone. If the normal "down" direction changes while keeping the volume of the "liquid" constant as in case (a), i.e., the plane cutting the right circular cone is not perpendicular to the central axis of the right circular cone, the top surface becomes an ellipse. As seen in Fig. 1(b), the height and horizontal position of the center of the top surface is different from that in Fig. 1(a). The behavior of the center of the top surface of a right circular cone cut in a plane is applicable not only to the displacement of the center of the top surface of a "liquid" inverted cone, but also to describe the motion of conduction electrons in single-layer graphene, which is being tremendously studied recently in the field of physics, since the behavior of conduction electrons in single-layer graphene is described by a right circular cone (Dirac cone) [2]. First, for example, consider the case where the length of the inverted triangle to the top is constant. If the origin is the apex of an inverted cone and the slope of the cone's side is m, the locus of the midpoint $D(\tilde{x}, \tilde{y}, \tilde{z})$ of upper line segment is expressed as

$$\frac{\tilde{x}^2 + \tilde{y}^2}{(r/m)^2} + \frac{\tilde{z}^2}{(mr)^2} = 1,$$

where, r is the radius of the circle on top of the inverted right cone, and which is a part of an ellipsoid of revolution [3]. Second, for example, consider the case where the sum of the lengths of the two lower sides of an inverted triangle is constant. The locus of point D is expressed as

$$\tilde{z} = mr.$$

In this study, we consider other cases which are important for applications in physics. The above two discussions may be trivial, however, they will serve as an introduction to what follows.

2 Center of the Ellipse at the Base of a Cone with a Constant Base Area

As shown in Fig. 2, consider a right circular cone in three dimensions. The upper figure in Fig. 2 shows the top view, and the lower figure in Fig. 2 shows the side of the cone. Consider two lines with slopes of $\pm m$ ($m > 0$) passing through the origin. The surface generated by the rotation of these two lines (cone generating line) around the y-axis is the slant of the right circular cone. The hatched area in the upper figure of Fig. 2 is the top surface of the right circular cone with the radius length r.

Let's find the coordinates of the center of the ellipse if the ellipse in the cross section of the cone moves while the area of this hatched area remains constant ($= \pi r^2$). Let the major

[1] In this study, a liquid is defined as a substance which moves downward under the force of gravity and changes shape to match the container. The physical concepts of such as surface tension, wettability, and viscosity are ignored.

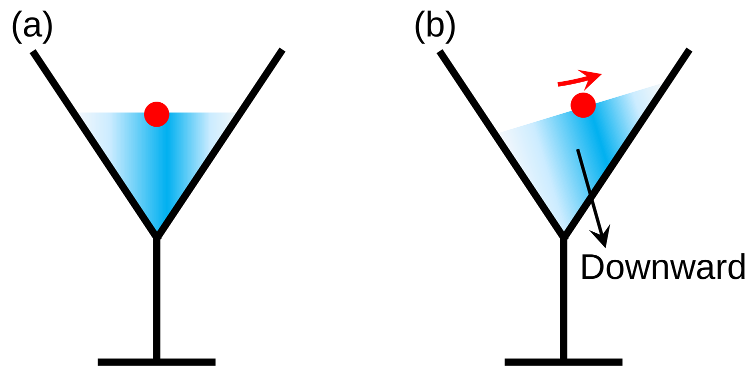

Figure 1: "Liquid" in a "cocktail glass" when standing upright (a) and when tilted (b). The red point indicates the center of the top surface's ellipse.

axis length of the ellipse be $2a$ and the minor axis length be $2b$. The coordinates of the top (A) and bottom (B) edges of the ellipse on the top surface are expressed as

$$(x_1, mx_1) \qquad (x_1 > 0) \tag{1}$$

and

$$(x_2, -mx_2) \qquad (x_2 < 0), \tag{2}$$

respectively. Let the center of the ellipse be point D.

Theorem 1. *The locus of point D is expressed as*

$$\left[\frac{\tilde{z}^2}{m^2} + m^2(\tilde{x}^2 + \tilde{y}^2)\right]\left[\frac{\tilde{z}^2}{m^2} - (\tilde{x}^2 + \tilde{y}^2)\right] = r^4, \tag{3}$$

in three-dimensional coordinates.

Proof. Now, consider the lower side view of Fig. 2, i.e., from symmetry, let the coordinate of D be (X, Y) in two-dimensional coordinates, where X and Y are expressed as

$$X = \frac{x_1 + x_2}{2} \tag{4}$$

and

$$Y = \frac{mx_1 - mx_2}{2}, \tag{5}$$

respectively. Therefore, in the lower view in Fig. 2, the length $|AB|$ is expressed as

$$\begin{aligned} |AB| &= \sqrt{(x_1 - x_2)^2 + (mx_1 + mx_2)^2} \\ &= 2a. \end{aligned} \tag{6}$$

From Eqs. (4) and (5),

$$mx_1 + mx_2 = 2mX \tag{7}$$

and

$$x_1 - x_2 = \frac{2Y}{m} \tag{8}$$

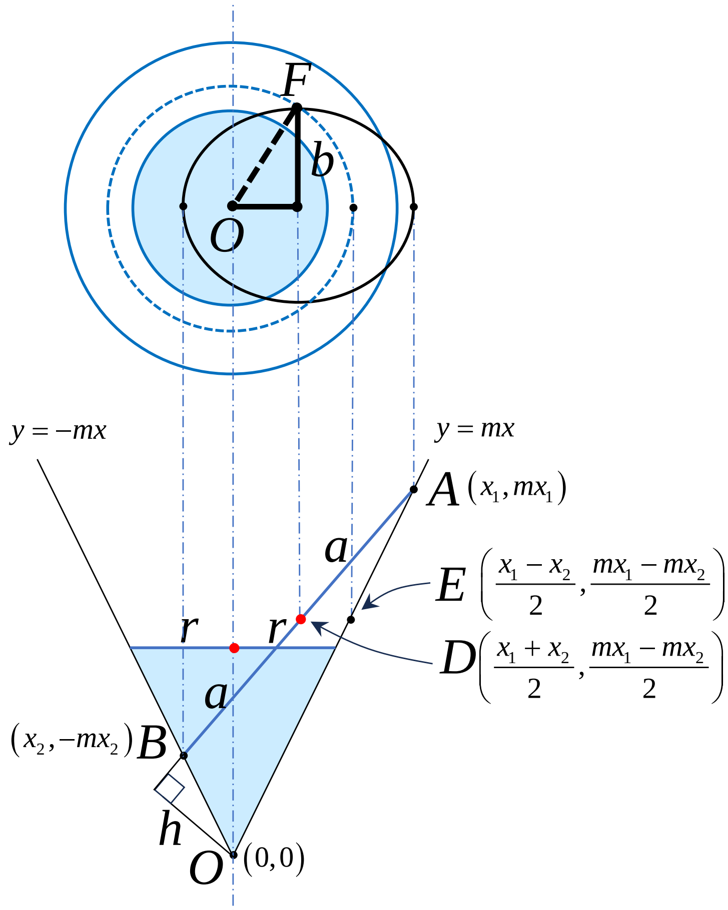

Figure 2: The locus of the center of the top surface and the definition of each point. The drawing emphasizes the changes in the upper surface.

are obtained. Substituting Eqs. (2) and (8) into Eq. (6), we obtain the following relationship;

$$\sqrt{\frac{Y^2}{m^2} + m^2X^2} = a. \tag{9}$$

In the upper view of Fig. 2, F is the point where the minor axis of the ellipse intersects the slope of the right circular cone. The y-coordinate of point F is the same as that of point D. Therefore, from the theorem of three squares in triangle ODF,

$$\sqrt{\left(\frac{x_1+x_2}{2}\right)^2 + \left(\frac{mx_1-mx_2}{2}\right)^2 + b^2} = |OF| = |OE| = \sqrt{\left(\frac{x_1-x_2}{2}\right)^2 + \left(\frac{mx_1-mx_2}{2}\right)^2} \tag{10}$$

This equation can be rearranged to

$$b = \sqrt{-x_1x_2}. \tag{11}$$

Substituting Eqs. (2) and (8) into Eq. (11), we obtain the following relationship;

$$\sqrt{\frac{Y^2}{m^2} - X^2} = b. \tag{12}$$

Since the area of the ellipse on the top surface is constant;

$$\pi r^2 = \pi ab, \tag{13}$$

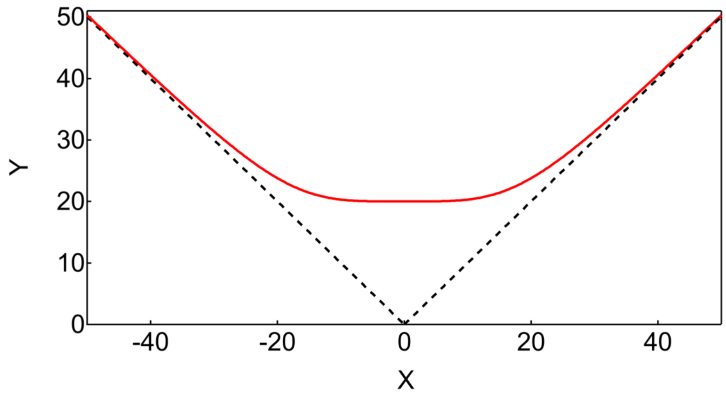

Figure 3: The locus drawn by point D when $m = 1$ and $r = 20$ in the case where the top surface area is constant.

from Eqs. (9) and (12), we obtain

$$r^2 = ab = \sqrt{\tfrac{Y^2}{m^2} + m^2X^2}\sqrt{\tfrac{Y^2}{m^2} - X^2}. \tag{14}$$

Thus, considered in three-dimensional coordinates, Eq. (3) is obtained. □

As the absolute value of X increases, Y also increases. The curve drawn by point D is of infinite length.

For example, when the angle of the oblique side of the inverted cone is 45 degrees, i.e., when $m = 1$, then Eq. (14) is

$$r^2 = \sqrt{Y^2 + X^2}\sqrt{Y^2 - X^2}, \tag{15}$$

thus,

$$Y = (X^4 + r^4)^{\frac{1}{4}} \qquad (Y > 0). \tag{16}$$

Furthermore, for example, when $r = 20$, the locus of D is plotted as a solid curve in Fig. 3. The dashed lines in the figure are the lines with a slope of ± 1, corresponding to the position of the side of the inverted cone. The top surface of the inverted cone can be drawn as a tangent to the curve. It should be noted that this is different from the behavior of the center of the "liquid" surface when a 20 deep "liquid" is poured into an inverted cone with sides which have a 45 degree tilt.

Next, let's find the locus of the center D of the ellipse when the perimeter of the upper surface is constant. Now, consider the lower side view of Fig. 2. Then, let the coordinate of D be (X, Y), where X and Y are expressed as

$$X = \frac{x_1 + x_2}{2} \tag{17}$$

and

$$Y = \frac{mx_1 - mx_2}{2}, \tag{18}$$

respectively. From Eq. (9), the major axis a of the ellipse on the top surface is described as

$$a = \sqrt{\tfrac{Y^2}{m^2} + m^2X^2}. \tag{19}$$

Also, from Eq. (12), the minor axis b of the ellipse on the top surface is described as

$$b = \sqrt{\tfrac{Y^2}{m^2} - X^2}. \tag{20}$$

Therefore, using the complete elliptic integral of the second kind ($E(k) = \int_0^{\frac{\pi}{2}} \sqrt{1 - k^2 \sin^2\theta}\, \mathrm{d}\theta$), the length L of the perimeter of ellipse on the top surface is expressed as

$$L = 4aE(e), \tag{21}$$

where e is the eccentricity and is expressed as

$$e = \sqrt{1 - \tfrac{b^2}{a^2}}. \tag{22}$$

Then, Eq. (21) is re-expressed as

$$L = 4aE\Big(\sqrt{1 - \tfrac{b^2}{a^2}}\Big) = 4a \int_0^{\frac{\pi}{2}} \sqrt{1 - \Big(1 - \frac{b^2}{a^2}\Big)\sin^2\theta}\, \mathrm{d}\theta. \tag{23}$$

If the top upper segment is horizontal,

$$L_0 = 2\pi r \tag{24}$$

holds. Since the perimeter of the top surface is constant, from Eqs. (23) and (24), we obtain

$$\begin{aligned}
2\pi r &= 4\sqrt{m^2X^2 + \tfrac{Y^2}{m^2}}\,E\Big(\sqrt{1 - \tfrac{b^2}{a^2}}\Big) = 4\sqrt{m^2X^2 + \tfrac{Y^2}{m^2}} \int_0^{\frac{\pi}{2}} \sqrt{1 - \Big(1 - \tfrac{b^2}{a^2}\Big)\sin^2\theta}\, \mathrm{d}\theta \\
&= 4X\sqrt{m^2 + \tfrac{1}{m^2}\Big(\tfrac{Y}{X}\Big)^2}\,E\left(\sqrt{\tfrac{1+m^2}{m^2+\frac{1}{m^2}(\frac{Y}{X})^2}}\right) \\
&= 4X\sqrt{m^2 + \tfrac{1}{m^2}\Big(\tfrac{Y}{X}\Big)^2} \int_0^{\frac{\pi}{2}} \sqrt{1 - \tfrac{1+m^2}{m^2+\frac{1}{m^2}\left(\frac{Y}{X}\right)^2}\sin^2\theta}\, \mathrm{d}\theta \\
&= 4X \int_0^{\frac{\pi}{2}} \sqrt{m^2 + \tfrac{1}{m^2}\Big(\tfrac{Y}{X}\Big)^2 - \Big\{m^2 + \tfrac{1}{m^2}\Big(\tfrac{Y}{X}\Big)\Big\}\tfrac{1+m^2}{m^2+\frac{1}{m^2}\left(\frac{Y}{X}\right)^2}\sin^2\theta}\, \mathrm{d}\theta \\
&= 4X \int_0^{\frac{\pi}{2}} \sqrt{m^2 + \tfrac{1}{m^2}\Big(\tfrac{Y}{X}\Big)^2 - (1+m^2)\sin^2\theta}\, \mathrm{d}\theta.
\end{aligned} \tag{25}$$

Therefore, when the perimeter of the ellipse on the top surface is constant, the locus of the center D of the ellipse is expressed as

$$\begin{aligned}
2\pi r &= 4X\sqrt{m^2 + \tfrac{1}{m^2}\Big(\tfrac{Y}{X}\Big)^2}\,E\left(\sqrt{\tfrac{1+m^2}{m^2+\frac{1}{m^2}\left(\frac{Y}{X}\right)^2}}\right) \\
&= 4X \int_0^{\frac{\pi}{2}} \sqrt{m^2 + \tfrac{1}{m^2}\Big(\tfrac{Y}{X}\Big)^2 - (1+m^2)\sin^2\theta}\, \mathrm{d}\theta,
\end{aligned} \tag{26}$$

where $E(k)$ $(= \int_0^{\frac{\pi}{2}} \sqrt{1 - k^2\sin^2\theta}\, \mathrm{d}\theta)$ is the complete elliptic integral of the second kind. Thus, considered in three-dimensional coordinates, Eq. (26) becomes

$$\begin{aligned}
2\pi r &= 4\sqrt{(\tilde{x}^2 + \tilde{y}^2)m^2 + \tfrac{\tilde{z}^2}{m^2}}\,E\left(\sqrt{\tfrac{1+m^2}{m^2+\frac{1}{m^2}\frac{\tilde{z}^2}{\tilde{x}^2+\tilde{y}^2}}}\right) \\
&= 4\sqrt{\tilde{x}^2 + \tilde{y}^2} \int_0^{\frac{\pi}{2}} \sqrt{m^2 + \tfrac{1}{m^2}\tfrac{\tilde{z}^2}{\tilde{x}^2+\tilde{y}^2} - (1+m^2)\sin^2\theta}\, \mathrm{d}\theta
\end{aligned} \tag{27}$$

Here, since Eq. (25) cannot be solved analytically, using Ramanujan's approximation of the elliptic circumference (first formula) [7]

$$L_{\mathrm{Ram}} = \pi\Big\{3(a+b) - \sqrt{(a+3b)(3a+b)}\Big\}, \tag{28}$$

Eq. (25) is expressed approximately as

$$\begin{aligned}2\pi r = \pi\Big\{3(a+b) - \sqrt{(a+3b)(3a+b)}\Big\} &= 3\pi\Big(\sqrt{\tfrac{Y^2}{m^2}+m^2X^2}+\sqrt{\tfrac{Y^2}{m^2}-X^2}\Big)\\ &- \pi\sqrt{\Big(\sqrt{\tfrac{Y^2}{m^2}+m^2X^2}+3\sqrt{\tfrac{Y^2}{m^2}-X^2}\Big)\Big(3\sqrt{\tfrac{Y^2}{m^2}+m^2X^2}+\sqrt{\tfrac{Y^2}{m^2}-X^2}\Big)},\end{aligned} \tag{29}$$

then,

$$\begin{aligned}2r &= 3\Big(\sqrt{\tfrac{Y^2}{m^2}+m^2X^2}+\sqrt{\tfrac{Y^2}{m^2}-X^2}\Big)\\ &- \sqrt{\Big(\sqrt{\tfrac{Y^2}{m^2}+m^2X^2}+3\sqrt{\tfrac{Y^2}{m^2}-X^2}\Big)\Big(3\sqrt{\tfrac{Y^2}{m^2}+m^2X^2}+\sqrt{\tfrac{Y^2}{m^2}-X^2}\Big)}.\end{aligned} \tag{30}$$

As the absolute value of X increases, Y also increases. The curve drawn by point D is of infinite length.

Considered in three-dimensional coordinates, Eq. (30) becomes

$$\begin{aligned}2r &= 3\Big(\sqrt{\tfrac{\tilde z^2}{m^2}+m^2(\tilde x^2+\tilde y^2)}+\sqrt{\tfrac{\tilde z^2}{m^2}-(\tilde x^2+\tilde y^2)}\Big)\\ &- \sqrt{\Big(\sqrt{\tfrac{\tilde z^2}{m^2}+m^2(\tilde x^2+\tilde y^2)}+3\sqrt{\tfrac{\tilde z^2}{m^2}-(\tilde x^2+\tilde y^2)}\Big)\Big(3\sqrt{\tfrac{\tilde z^2}{m^2}+m^2(\tilde x^2+\tilde y^2)}+\sqrt{\tfrac{\tilde z^2}{m^2}-(\tilde x^2+\tilde y^2)}\Big)}.\end{aligned} \tag{31}$$

For example, when the angle of the oblique side of the inverted cone is 45 degrees, i.e., when $m = 1$, Eq. (25) becomes

$$\begin{aligned}2\pi r &= 4X\sqrt{1+\left(\tfrac{Y}{X}\right)^2}\,E\left(\sqrt{\tfrac{2}{1+\left(\frac{Y}{X}\right)^2}}\right)\\ &= 4X\int_0^{\frac{\pi}{2}}\sqrt{1+\left(\tfrac{Y}{X}\right)^2-2\sin^2\theta}\,\mathrm{d}\theta.\end{aligned} \tag{32}$$

And Eq. (30) becomes

$$\begin{aligned}2r &= 3(\sqrt{Y^2+X^2}+\sqrt{Y^2-X^2})\\ &\quad - \sqrt{(\sqrt{Y^2+X^2}+3\sqrt{Y^2-X^2})(3\sqrt{Y^2+X^2}+\sqrt{Y^2-X^2})}\\ &= 3(\sqrt{Y^2+X^2}+\sqrt{Y^2-X^2})\\ &\quad - \sqrt{3(Y^2+X^2)+\sqrt{Y^4-X^4}+9\sqrt{Y^4-X^4}+3(Y^2-X^2)}\\ &= 3(\sqrt{Y^2+X^2}+\sqrt{Y^2-X^2}) - \sqrt{6Y^2+10\sqrt{Y^4-X^4}}.\end{aligned} \tag{33}$$

Furthermore, for example, when $r = 20$, the locus of D is plotted as a solid curve in Fig. 4. The dashed lines in the figure are the lines with a slope of ± 1, corresponding to the position of the side of the inverted cone. The top surface of the inverted cone can be drawn as a tangent to the curve. It should be noted that this is different from the behavior of the

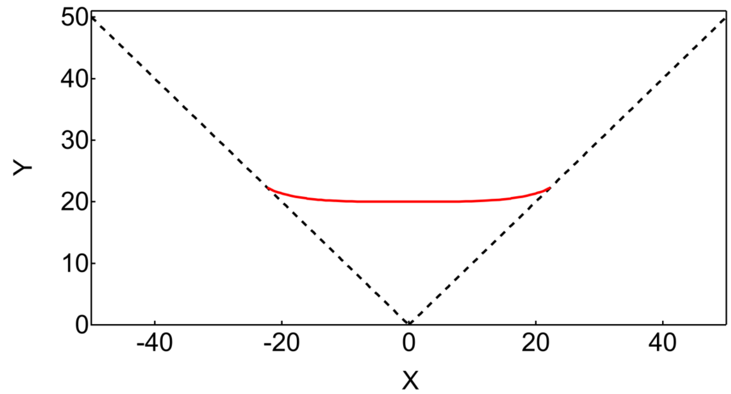

Figure 4: The locus drawn by point D when $m = 1$ and $r = 20$ in the case where the perimeter of the ellipse on the top surface is constant.

center of the "liquid" surface when a 20 deep "liquid" is poured into an inverted cone with sides that have a 45 degree tilt. Equations (32) and (33) become

$$\begin{aligned} 10\pi &= X\sqrt{1+\left(\tfrac{Y}{X}\right)^2}\,E\left(\sqrt{\frac{2}{1+\left(\tfrac{Y}{X}\right)^2}}\right) \\ &= X\int_0^{\frac{\pi}{2}}\sqrt{1+\left(\tfrac{Y}{X}\right)^2-2\sin^2\theta}\,\mathrm{d}\theta \end{aligned} \tag{34}$$

and

$$40 = 3(\sqrt{Y^2+X^2}+\sqrt{Y^2-X^2})-\sqrt{6Y^2+10\sqrt{Y^4-X^4}}, \tag{35}$$

respectively. It is then drawn as in Fig. 4. The circle with a perimeter of 40π becomes an ellipse as the glass tilts and gradually thins, becoming a line segment of constant length 20π (i.e., 40π in round-trip length) when it touches the oblique edge. Thus, the coordinate of the center of the line segment is $(X, Y) = (5\sqrt{2}\pi, 5\sqrt{2}\pi)$.

Next, let's find the locus of the center D of the ellipse when the volume of the cone is constant. The diagram is shown in Fig. 2. Find the coordinates of the center of the ellipse if the ellipse in the cross section of the cone moves while the volume indicate by hatched area of lower in Fig. 2 remains constant. h is the length of the perpendicular line from the origin O to the line AB, which is the height of the elliptic cone.

Given that the height of the original cone is mr, the volume of the original cone V_0 is expressed as

$$V_0 = \frac{1}{3}\pi m r^3. \tag{36}$$

Now, consider elliptic cones. Consider the line passing through point AB. From Fig. 2, the slope of the line is

$$\frac{mx_1+mx_2}{x_1-x_2} = m\frac{x_1+x_2}{x_1-x_2}. \tag{37}$$

Since the line passes through point (x_1, mx_1), Eq. (37) is expressed as

$$y - mx_1 = m\frac{x_1+x_2}{x_1-x_2}(x-x_1). \tag{38}$$

Thus, the line is expressed as

$$m(x_1+x_2)x-(x_1-x_2)y-2mx_1x_2 = 0. \tag{39}$$

Since $m > 0$, $x_1 > 0$, and $x_2 < 0$, thus,

$$h = \frac{|-2mx_1x_2|}{\sqrt{m^2(x_1+x_2)^2+(x_1-x_2)^2}} = \frac{-2mx_1x_2}{\sqrt{m^2(x_1+x_2)^2+(x_1-x_2)^2}}. \tag{40}$$

From Eqs. (4), (5), (11), and (12), Eq. (40) can be expressed in terms of the coordinates of the center of the ellipse (X, Y) as

$$h = \frac{\frac{Y^2}{m^2} - X^2}{\sqrt{X^2 + \frac{Y^2}{m^4}}}. \tag{41}$$

From Eqs. (14) and (41), therefore, the volume of the elliptic cone V is expressed as

$$\begin{aligned} V &= \frac{1}{3}\pi abh \\ &= \frac{1}{3}\pi\sqrt{\tfrac{Y^2}{m^2} + m^2X^2}\sqrt{\tfrac{Y^2}{m^2} - X^2}\,\frac{\frac{Y^2}{m^2} - X^2}{\sqrt{X^2 + \frac{Y^2}{m^4}}} \\ &= \frac{1}{3}\pi m\left(\tfrac{Y^2}{m^2} - X^2\right)^{\frac{3}{2}} \end{aligned} \tag{42}$$

If the volume is constant, $V_0 = V$ holds, then, from Eqs. (36) and (42),

$$\frac{1}{3}\pi m r^3 = \frac{1}{3}\pi m\left(\tfrac{Y^2}{m^2} - X^2\right)^{\frac{3}{2}}. \tag{43}$$

This equation can be rearranged to

$$-\frac{X^2}{r^2} + \frac{Y^2}{m^2r^2} = 1, \tag{44}$$

which represents a hyperbolic curve.

Considered in three-dimensional coordinates, Eq. (44) becomes

$$-\frac{\tilde{x}^2 + \tilde{y}^2}{r^2} + \frac{\tilde{z}^2}{m^2r^2} = 1. \tag{45}$$

Equation (44) expresses the hyperbolic curve. As the absolute value of X increases, Y also increases. The curve drawn by point D is of infinite length.

For example, when the angle of the oblique side of the inverted cone is 45 degrees, i.e., when $m = 1$, then Eq. (44) is

$$-\frac{X^2}{r^2} + \frac{Y^2}{r^2} = 1, \tag{46}$$

thus,

$$Y = \sqrt{X^2 + r^2} \quad (Y > 0). \tag{47}$$

Furthermore, for example, when $r = 20$, the locus of D is plotted as a solid curve in Fig. 5. The dashed lines in the figure are the lines with a slope of ± 1, corresponding to the position of the side of the inverted cone. The top surface of the inverted cone can be drawn as a tangent to the curve. It corresponds to the position of the center of the top surface when the inverted cone is viewed from the side.

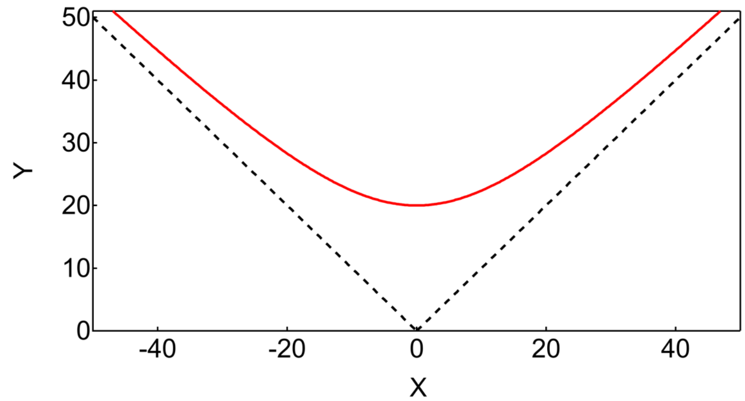

Figure 5: The locus drawn by point D when $m = 1$ and $r = 20$ in the case where the volume is constant.

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References

[1] T. M. Apostol and M. A. Mnatsakanian: *New Horizons in Geometry*. Mathematical Association of America, 2013.

[2] A. H. Castro Neto, F. Guinea, N. M. R. Peres, K. S. Novoselov, and A. K. Geim: *The electronic properties of graphene*. Reviews of Modern Physics **81**(1), 109–162, 2009. doi: 10.1103/revmodphys.81.109.

[3] G. Glaeser, H. Stachel, and B. Odehnal: *The Universe of Conics. From the ancient Greeks to 21st century developments*. Springer, Berlin, Heidelberg, 2 ed., 2024. ISBN 978-3-662-70305-2. doi: 10.1007/978-3-662-70306-9.

[4] A. G. Horváth: *Affine Construction of Conic Sections from Their Five Points*. Journal for Geometry and Graphics **28**, 029, 2024.

[5] A. Kobiera: *Ellipse, Hyperbola and Their Conjunction*, 2019. arXiv: 1805.02111.

[6] M. H. Protter and P. E. Protter: *Calculus*. Jones & Bartlett Learning, 4th ed., 1988.

[7] J.-F. Tian and Z.-H. Yang: *Absolute monotonicity of the accuracy of Ramanujan approximations for perimeter of an ellipse*. Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas **117**(3), 2023. doi: 10.1007/s13398-023-01465-z.

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