Journal for Geometry and Graphics
Volume 29 (2025), No. 1, 23–32

Developability Conditions and Position Vectors of Ruled Surfaces According to the Darboux Frame in $\mathbf{E}^3$

Soukaina Ouarab

Department of Mathematics and Computer Science, Laboratory of LAVETE, Équipe de Mathématiques et Applications, Faculty of Science and Technology of Settat, Hassan First University of Settat, Morocco
soukaina.ouarab.sma@gmail.com

Abstract. In this paper, we investigate the developability conditions of ruled surfaces in Euclidean 3-space according to the Darboux frame of an arbitrary regular surface along their common base curve. Moreover, the position vectors of such surfaces are found.

Key Words: ruled surface, developable surface, position vector, Darboux frame, Euclidean 3-space

MSC 2020: 53M04 (primary), 53A05

1 Introduction

In the classical differential geometry [2, 9], a surface is said to be ruled if it is generated by moving a straight line continuously along a curve. These lines are said to be rulings, and each curve that intersects all the rulings is called a base curve. In the theory of surfaces, ruled surfaces are one of the most interesting topics to be studied because of their great impact to many applications in several areas, such as mathematical physics, kinematics and Computer Aided Geometric Design (CAGD).

Ruled surfaces which can be transformed into the plane without any deformation and distortion, with vanishing Gaussian curvature, are called developable surfaces. They form a relatively small subset that contains cylinders, cones, and the tangent surfaces [1, 3, 5]. That notion of developability is one of the most important properties of ruled surfaces.

Study of ruled surfaces according to Darboux frame was and still is the focus of interest for many researchers. In [4], the authors studied ruled surface according to the Darboux frame in Minkowski space, they gave sufficient and necessary condition for the well-known ruled surface to be developable. In [10], the authors presented characterizations of ruled surface according to Darboux frame along the base curve by using the relationship between

ISSN 1433-8157/ © 2025 by the author(s), licensed under CC BY SA 4.0.

Frenet and Darboux frame. In [8], we made a comparative study between a ruled surface defined with linear combination of Darboux frame vectors and an arbitrary regular surface along their common curve in Euclidean 3-space. We have also conducted two more studies on ruled surface according to the Darboux frame, where in [6] we have introduced for the first time the notion of partner ruled surfaces and studied their simultaneous developability. Furthermore, in [7], we introduced the notion of Smarandache ruled surfaces in Euclidean 3-space.

In the present paper, we find sufficient and necessary condition for ruled surfaces generated by Darboux frame to be developable. Moreover, we present their position vectors by the means the Darboux frame curvatures. Finally, we give an example to visualize that kind of surfaces in $\mathbb{R}^3$.

2 Preliminaries

In the Euclidean 3-space E^3, we consider the usual metric given by

$$\langle , \rangle = \mathrm{d}x_1 + \mathrm{d}x_2 + \mathrm{d}x_3,$$

where (x_1, x_2, x_3) is a rectangular coordinate system of E^3.

Let $\Psi\colon (s,v) \in I \times \mathbb{R} \mapsto \gamma(s) + v\overrightarrow{X}(s)$, be a ruled surface in E^3, where I is an open interval of $\mathbb{R}$, $\gamma(s)$ is a curve and $\overrightarrow{X}(s)$ is a non-vanishing vector field along $\gamma(s)$.

The unit normal $\overrightarrow{N}$ on the ruled surface Ψ at a regular point is

$$\overrightarrow{N} = \frac{\Psi_s \wedge \Psi_v}{\|\Psi_s \wedge \Psi_v\|} = \frac{(\gamma' + v\overrightarrow{X}') \wedge \overrightarrow{X}}{\|(\gamma' + v\overrightarrow{X}') \wedge \overrightarrow{X}\|},$$

where $\Psi_s = \frac{\partial \Psi}{\partial s}$ and $\Psi_v = \frac{\partial \Psi}{\partial v}$.

The first I and the second II fundamental forms of ruled surface Ψ at a regular point, are defined respectively by

$$I(\Psi_s\,\mathrm{d}s + \Psi_v\,\mathrm{d}v) = E\,\mathrm{d}s^2 + 2F\,\mathrm{d}s\,\mathrm{d}v + G\,\mathrm{d}v^2,$$
$$II(\Psi_s\,\mathrm{d}s + \Psi_v\,\mathrm{d}v) = e\,\mathrm{d}s^2 + 2f\,\mathrm{d}s\,\mathrm{d}v + g\,\mathrm{d}v^2,$$

where

$$E = \|\Psi_s\|^2, \quad F = \langle \Psi_s, \Psi_v \rangle, \quad G = \|\Psi_v\|^2,$$
$$e = \langle \Psi_{ss}, \overrightarrow{N} \rangle, \quad f = \langle \Psi_{vs}, \overrightarrow{N} \rangle, \quad g = \langle \Psi_{vv}, \overrightarrow{N} \rangle = 0.$$

The Gaussian curvature K of the ruled surface Ψ at a regular point is given by

$$K = -\frac{f^2}{EG - F^2}.$$

Theorem 1 ([4])**.** *A developable surface is a ruled surface with vanishing Gaussian curvature.*

Definition 2 ([2])**.** A tangent surface is a ruled surface whose ruling direction is the unit tangent vector of its base curve.

Theorem 3 ([2])**.** *A tangent surface is developable.*

Let $c(s) = \varphi(u(s), v(s))$ be a unit speed curve that lies on a regular surface $\varphi = \varphi(u, v)$. Along c we define a frame, called Darboux frame, and denoted $\{\vec{T}(s), \vec{g}(s), \vec{n}(s)\}$, where $\vec{T}(s) = c'(s)$ is the unit tangent, $\vec{n}(s) = \frac{\varphi_s \wedge \varphi_v}{\|\varphi_s \wedge \varphi_v\|}(u(s), v(s))$ is the unit normal vector on the surface along $c(s)$ and $\vec{g}(s) = \vec{n}(s) \wedge \vec{T}(s)$ is the unit vector defined by the cross product of $\vec{n}$ and $\vec{T}$. The derivative formulas of Darboux frame (also called "Darboux formulas"), are expressed as:

$$\begin{pmatrix} \vec{T}'(s) \\ \vec{g}'(s) \\ \vec{n}'(s) \end{pmatrix} = \begin{pmatrix} 0 & \rho_g(s) & \rho_n(s) \\ -\rho_g(s) & 0 & \theta_g(s) \\ -\rho_n(s) & -\theta_g(s) & 0 \end{pmatrix} \begin{pmatrix} \vec{T}(s) \\ \vec{g}(s) \\ \vec{n}(s) \end{pmatrix}, \tag{1}$$

where ρ_n is the normal curvature, ρ_g is the geodesic curvature and θ_g is the geodesic torsion of the curve c on the surface φ.

Definition 4. For a curve c lying on a regular surface φ, we have the following definitions:

1. c is an asymptotic line on φ if and only if its normal curvature ρ_n vanishes.
2. c is a geodesic curve on φ if and only if its geodesic curvature ρ_g vanishes.
3. c is a principal line on φ if and only if its geodesic torsion θ_g vanishes.

3 Developability Conditions and Position Vectors of Ruled Surfaces According to the Darboux Frame in $\mathbf{E}^3$

In this main section, we investigate two fundamental theorems: In the first one, we find the position vector of a $\vec{T}$-ruled surface (as a natural developable surface) generated by the Darboux frame, by the means of the position vector of its base curve and also according to the relationship between the Darboux curvatures ρ_g, ρ_n, θ_g. In the second one, we give the sufficient and necessary condition for a $\vec{g}$-ruled surface and $\vec{n}$-ruled surface generated by the Darboux frame to be developable. Finally, we present their position vectors.

Theorem 5. *The position vector of $\vec{T}$-developable ruled surface according to Darboux frame $\{\vec{T}, \vec{g}, \vec{n}, \rho_n, \rho_g, \theta_g\}$ of an arbitrary regular surface φ is given by*

$$\begin{aligned} {}^0\Psi(s, v) = {} & \Big(\langle \gamma(s), \vec{T}(s)\rangle + v\Big)\vec{T}(s) \\ & - \Big(\int \Big[\langle \gamma(s), \vec{T}(s)\rangle \rho_g(s) - \langle \gamma(s), \vec{n}(s)\rangle \theta_g(s)\Big]\,\mathrm{d}s + c_0\Big)\vec{g}(s) \\ & - \Big(\int \Big[\langle \gamma(s), \vec{T}(s)\rangle \rho_n(s) + \langle \gamma(s), \vec{g}(s)\rangle \theta_g(s)\Big]\,\mathrm{d}s + d_0\Big)\vec{n}(s), \end{aligned}$$

where $\gamma(s)$ is the base curve and c_0, d_0 are arbitrary constants.

Proof. Let $\gamma\colon s \in I \subset \mathbb{R} \mapsto \gamma(s)$ be a C^2-differentiable unit speed curve lying on a regular surface $\varphi = \varphi(u, v)$, whose Darboux frame is given by $\{\vec{T}(s), \vec{g}(s), \vec{n}(s)\}$. The position vector of the curve $\gamma(s) = \varphi(u(s), v(s))$ on the surface φ according to its Darboux frame is expressed as follows

$$\gamma(s) = \gamma_0(s)\vec{T}(s) + \gamma_1(s)\vec{g}(s) + \gamma_2(s)\vec{n}(s), \tag{2}$$

where γ_0, γ_1, γ_2 are the C^2-differentiable functions of $s \in I$, defined respectively by the following scalar products

$$\begin{cases} \gamma_0(s) = \langle \gamma(s), \vec{T}(s)\rangle, \\ \gamma_1(s) = \langle \gamma(s), \vec{g}(s)\rangle, \\ \gamma_2(s) = \langle \gamma(s), \vec{n}(s)\rangle. \end{cases}$$

By differentiating (2) with respect to s and using Darboux frame formulas (1), we get

$$\begin{aligned}\gamma'(s) = \quad & [\gamma_0'(s) - \gamma_1(s)\rho_g(s) - \gamma_2(s)\rho_n(s)]\vec{T}(s)\\ & + [\gamma_1'(s) + \gamma_0(s)\rho_g(s) - \gamma_2(s)\theta_g(s)]\vec{g}(s)\\ & + [\gamma_2'(s) + \gamma_0(s)\rho_n(s) + \gamma_1(s)\theta_g(s)]\vec{n}(s).\end{aligned}$$

By replacing $\gamma'(s)$ with $\vec{T}(s)$, we get

$$\begin{aligned}\vec{T}(s) = \quad & [\gamma_0'(s) - \gamma_1(s)\rho_g(s) - \gamma_2(s)\rho_n(s)]\vec{T}(s)\\ & + [\gamma_1'(s) + \gamma_0(s)\rho_g(s) - \gamma_2(s)\theta_g(s)]\vec{g}(s)\\ & + [\gamma_2'(s) + \gamma_0(s)\rho_n(s) + \gamma_1(s)\theta_g(s)]\vec{n}(s).\end{aligned}$$

Then, by identification, we get

$$\begin{cases}\gamma_0'(s) - \gamma_1(s)\rho_g(s) - \gamma_2(s)\rho_n(s) = 1,\\ \gamma_1'(s) + \gamma_0(s)\rho_g(s) - \gamma_2(s)\theta_g(s) = 0,\\ \gamma_2'(s) + \gamma_0(s)\rho_n(s) + \gamma_1(s)\theta_g(s) = 0.\end{cases} \tag{3}$$

Let us consider the $\vec{T}$-ruled surface generated by the Darboux frame of the curve $\gamma(s)$ on the regular surface φ:

$$^{0}\Psi(s,v) = (\gamma_0(s)\vec{T}(s) + \gamma_1(s)\vec{g}(s) + \gamma_2(s)\vec{n}(s)) + v\vec{T}(s). \tag{4}$$

It is clear that $^{0}\Psi\,(s,v)$ is developable because it is a tangent surface. Then, by using the properties (3), we get the position vector of such a developable surface as follows

$$\begin{aligned}^{0}\Psi(s,v) = (\gamma_0(s)+v)\vec{T}(s) &- \left(\int\Big[\gamma_0(s)\rho_g(s) - \gamma_2(s)\theta_g(s)\Big]\,\mathrm{d}s + c_0\right)\vec{g}(s)\\ &- \left(\int\Big[\gamma_0(s)\rho_n(s) + \gamma_1(s)\theta_g(s)\Big]\,\mathrm{d}s + d_0\right)\vec{n}(s). \quad \square\end{aligned}$$

Theorem 6. *The $\vec{g}$-ruled surface and $\vec{n}$-ruled surface according to Darboux frame $\{\vec{T}, \vec{g}, \vec{n}, \rho_n, \rho_g, \theta_g\}$ of an arbitrary regular surface φ are developable if and only if their base curve is a principle line on the surface φ. Moreover, their position vectors are given respectively by*

$$\begin{aligned}^{1}\Psi(s,v) = & \langle\gamma(s),\vec{T}(s)\rangle\vec{T}(s)\\ & - \left(\int\Big[\langle\gamma(s),\vec{T}(s)\rangle\rho_g(s) - \langle\gamma(s),\vec{n}(s)\rangle\theta_g(s)\Big]\,\mathrm{d}s + c_1 + v\right)\vec{g}(s)\\ & - \left(\int\Big[\langle\gamma(s),\vec{T}(s)\rangle\rho_n(s) + \langle\gamma(s),\vec{g}(s)\rangle\theta_g(s)\Big]\,\mathrm{d}s + d_1\right)\vec{n}(s),\end{aligned}$$

$$\begin{aligned}^{2}\Psi(s,v) = & \langle\gamma(s),\vec{T}(s)\rangle\vec{T}(s)\\ & - \left(\int\Big[\langle\gamma(s),\vec{T}(s)\rangle\rho_g(s) - \langle\gamma(s),\vec{n}(s)\rangle\theta_g(s)\Big]\,\mathrm{d}s + c_2\right)\vec{g}(s)\\ & - \left(\int\Big[\langle\gamma(s),\vec{T}(s)\rangle\rho_n(s) + \langle\gamma(s),\vec{g}(s)\rangle\theta_g(s)\Big]\,\mathrm{d}s + d_2 + v\right)\vec{n}(s),\end{aligned}$$

where $\gamma(s)$ is the base curve and c_1, c_2, d_1, d_2 are arbitrary constants.

Proof. Now, let us search under which conditions our second ruled surface ${}^1\Psi(s,v)$ could be developable.

Differentiating the second line of (4) with respect to s and v, respectively, and using Darboux frame formulas (1), we get

$$\begin{cases} {}^1\Psi_s = [\gamma_0' - \gamma_2\rho_n - (\gamma_1+v)\rho_g]\overrightarrow{T} + [\gamma_1' + \gamma_0\rho_g - \gamma_2\theta_g]\overrightarrow{g} + [\gamma_2' + \gamma_0\rho_n + (\gamma_1+v)\theta_g]\overrightarrow{n}, \\ {}^1\Psi_v = \overrightarrow{g}. \end{cases} \tag{5}$$

By determining the cross product of both vectors ${}^1\Psi_s$ and ${}^1\Psi_v$, we get the normal vector on the ruled surface ${}^1\Psi(s,v)$:

$${}^1\Psi_s \wedge {}^1\Psi_v = -[\gamma_2' + \gamma_0\rho_n + (\gamma_1+v)\theta_g]\overrightarrow{T} + [\gamma_0' - \gamma_2\rho_n - (\gamma_1+v)\rho_g]\overrightarrow{n},$$

so under regularity condition, the unit normal vector takes the following form:

$$\frac{{}^1\Psi_s \wedge {}^1\Psi_v}{\|{}^1\Psi_s \wedge {}^1\Psi_v\|} = \frac{-[\gamma_2' + \gamma_0\rho_n + (\gamma_1+v)\theta_g]\overrightarrow{T} + [\gamma_0' - \gamma_2\rho_n - (\gamma_1+v)\rho_g]\overrightarrow{n}}{\sqrt{[\gamma_2' + \gamma_0\rho_n + (\gamma_1+v)\theta_g]^2 + [\gamma_0' - \gamma_2\rho_n - (\gamma_1+v)\rho_g]^2}}. \tag{6}$$

From (5), we get the components of the first fundamental form of the ruled surface ${}^1\Psi(s,v)$, at regular points, as follows

$$\begin{cases} {}^1E = [\gamma_0' - \gamma_2\rho_n - (\gamma_1+v)\rho_g]^2 + [\gamma_1' + \gamma_0\rho_g - \gamma_2\theta_g]^2 \\ \qquad + [\gamma_2' + \gamma_0\rho_n + (\gamma_1+v)\theta_g]^2 \\ {}^1F = \gamma_1' + \gamma_0\rho_g - \gamma_2\theta_g \\ {}^1G = 1. \end{cases} \tag{7}$$

On the other hand, differentiating ${}^1\Psi_s$ and ${}^1\Psi_v$ with respect to s and v, respectively and using Darboux formulas (1), we get

$$\begin{cases} {}^1\Psi_{ss} = [\gamma_0'' - 2(\gamma_1'\rho_g + \gamma_2'\rho_n) - \gamma_0(\rho_g^2+\rho_n^2) - \gamma_1(\rho_g' + \theta_g\rho_n) + \gamma_2(\theta_g\rho_g - \rho_n')]\overrightarrow{T} \\ \qquad - v(\rho_g' + \rho_n\theta_g)\overrightarrow{T} \\ \quad + [\gamma_1'' + 2(\gamma_0'\rho_g - \gamma_2'\theta_g) + \gamma_0(\rho_g' - \rho_n\theta_g) - \gamma_1(\rho_g^2+\theta_g^2) - \gamma_2(\theta_g' + \rho_n\rho_g)]\overrightarrow{g} \\ \qquad - v(\rho_g^2+\theta_g^2)\overrightarrow{g} \\ \quad + [\gamma_2'' + 2(\gamma_0'\rho_n + \gamma_1'\theta_g) + \gamma_0(\rho_n' + \rho_g\theta_g) + \gamma_1(\theta_g' - \rho_g\rho_n) - \gamma_2(\rho_n^2+\theta_g^2)]\overrightarrow{n} \\ \qquad + v(\theta_g' - \rho_n\rho_g)\overrightarrow{n}, \\ {}^1\Psi_{vs} = -\rho_g\overrightarrow{T} + \theta_g\overrightarrow{n}, \\ {}^1\Psi_{vv} = 0. \end{cases} \tag{8}$$

Hence, from (6) and (8), we get the components of the second fundamental form of ${}^1\Psi(s,v)$

at regular points:

$$\begin{cases} {}^1e = \dfrac{-[\gamma_2' + \gamma_0\rho_n + (\gamma_1 + v)\theta_g][\gamma_0'' - 2(\gamma_1'\rho_g + \gamma_2'\rho_n) - \gamma_0(\rho_g^2 + \rho_n^2)]}{\sqrt{[\gamma_2' + \gamma_0\rho_n + (\gamma_1 + v)\theta_g]^2 + [\gamma_0' - \gamma_2\rho_n - (\gamma_1 + v)\rho_g]^2}} \\ \qquad - \dfrac{[\gamma_2' + \gamma_0\rho_n + (\gamma_1 + v)\theta_g][-\gamma_1(\rho_g' + \theta_g\rho_n) + \gamma_2(\theta_g\rho_g - \rho_n') - v(\rho_g' + \rho_n\theta_g)]}{\sqrt{[\gamma_2' + \gamma_0\rho_n + (\gamma_1 + v)\theta_g]^2 + [\gamma_0' - \gamma_2\rho_n - (\gamma_1 + v)\rho_g]^2}} \\ \qquad + \dfrac{[\gamma_0' - \gamma_2\rho_n - (\gamma_1 + v)\rho_g][\gamma_2'' + 2(\gamma_0'\rho_n + \gamma_1'\theta_g) + \gamma_0(\rho_n' + \rho_g\theta_g)]}{\sqrt{[\gamma_2' + \gamma_0\rho_n + (\gamma_1 + v)\theta_g]^2 + [\gamma_0' - \gamma_2\rho_n - (\gamma_1 + v)\rho_g]^2}} \\ \qquad + \dfrac{[\gamma_0' - \gamma_2\rho_n - (\gamma_1 + v)\rho_g][\gamma_1(\theta_g' - \rho_g\rho_n) - \gamma_2(\rho_n^2 + \theta_g^2) + v(\theta_g' - \rho_n\rho_g)]}{\sqrt{[\gamma_2' + \gamma_0\rho_n + (\gamma_1 + v)\theta_g]^2 + [\gamma_0' - \gamma_2\rho_n - (\gamma_1 + v)\rho_g]^2}}, \\ {}^1f = \dfrac{\rho_g(\gamma_2' + \gamma_0\rho_n) + \theta_g(\gamma_0' - \gamma_2\rho_n)}{\sqrt{[\gamma_2' + \gamma_0\rho_n + (\gamma_1 + v)\theta_g]^2 + [\gamma_0' - \gamma_2\rho_n - (\gamma_1 + v)\rho_g]^2}}, \\ {}^1g = 0. \end{cases} \tag{9}$$

From (7) and (9), we get the Gaussian curvature of ruled surface ${}^1\Psi(s,v)$, at regular points as follows

$${}^1K = -\left[\frac{\rho_g(\gamma_2' + \gamma_0\rho_n) + \theta_g(\gamma_0' - \gamma_2\rho_n)}{[\gamma_2' + \gamma_0\rho_n + (\gamma_1 + v)\theta_g]^2 + [\gamma_0' - \gamma_2\rho_n - (\gamma_1 + v)\rho_g]^2}\right]^2. \tag{10}$$

We deduce that the second ruled surface ${}^1\Psi(s,v)$ is developable if and only if $\rho_g(\gamma_2' + \gamma_0\rho_n) + \theta_g(\gamma_0' - \gamma_2\rho_n) = 0$. Then from (3) this equation becomes $\theta_g = 0$, which means that the ruled surface ${}^1\Psi(s,v)$ is developable if and only if the curve $\gamma(s)$ is a principal line on the surface φ.

Now, according to the properties (3), we conclude that the position vector of the ruled surface ${}^1\Psi(s,v)$ is expressed as follows

$$\begin{aligned} {}^1\Psi(s,v) = \gamma_0(s)\vec{T}(s) - \left(\int [\gamma_0(s)\rho_g(s) - \gamma_2(s)\theta_g(s)]\,\mathrm{d}s + c_1 + v\right)\vec{g}(s) \\ - \left(\int [\gamma_0(s)\rho_n(s) + \gamma_1(s)\theta_g(s)]\,\mathrm{d}s + d_1\right)\vec{n}(s). \end{aligned}$$

Now, let us consider the third ruled surface ${}^2\Psi(s,v)$:

Differentiating the third line of (4) with respect to s and v, respectively and using Darboux frame formulas (1), we get

$$\begin{cases} {}^2\Psi_s = [\gamma_0' - \gamma_1\rho_g - \rho_n(\gamma_2 + v)]\vec{T} + [\gamma_1' + \gamma_0\rho_g - \theta_g(\gamma_2 + v)]\vec{g} + [\gamma_2' + \gamma_0\rho_n + \gamma_1\theta_g]\vec{n}, \\ {}^2\Psi_v = \vec{n}. \end{cases} \tag{11}$$

By determining the cross product of both vectors ${}^2\Psi_s$ and ${}^2\Psi_v$, we get the normal vector on the ruled surface ${}^2\Psi(s,v)$:

$${}^2\Psi_s \wedge {}^2\Psi_v = [\gamma_1' + \gamma_0\rho_g - \theta_g(\gamma_2 + v)]\vec{T} - [\gamma_0' - \gamma_1\rho_g - \rho_n(\gamma_2 + v)]\vec{g}.$$

Then, under regularity condition, the unit normal vector is:

$$\frac{{}^2\Psi_s \wedge {}^2\Psi_v}{\|{}^2\Psi_s \wedge {}^2\Psi_v\|} = \frac{[\gamma_1' + \gamma_0\rho_g - \theta_g(\gamma_2 + v)]\vec{T} - [\gamma_0' - \gamma_1\rho_g - \rho_n(\gamma_2 + v)]\vec{g}}{\sqrt{[\gamma_1' + \gamma_0\rho_g - \theta_g(\gamma_2 + v)]^2 + [\gamma_0' - \gamma_1\rho_g - \rho_n(\gamma_2 + v)]^2}}. \tag{12}$$

From (11), we get the components of the first fundamental form of the ruled surface ${}^2\Psi(s,v)$:

$$\begin{cases} {}^2E = [\gamma_0' - \gamma_1\rho_g - \rho_n(\gamma_2+v)]^2 + [\gamma_1' + \gamma_0\rho_g - \theta_g(\gamma_2+v)]^2 + [\gamma_2' + \gamma_0\rho_n + \gamma_1\theta_g]^2, \\ {}^2F = \gamma_2' + \gamma_0\rho_n + \gamma_1\theta_g, \\ {}^2G = 1. \end{cases} \tag{13}$$

On the other hand, differentiating ${}^2\Psi_s$ and ${}^2\Psi_v$ with respect to s and v, respectively and using the Darboux formulas (1), we get

$$\begin{cases} {}^2\Psi_{ss} = [\gamma_0'' - 2(\gamma_1'\rho_g + \gamma_2'\rho_n) - \gamma_0(\rho_g^2+\rho_n^2) - \gamma_1(\rho_g' + \rho_n\theta_g) + \gamma_2(\rho_g\theta_g - \rho_n')]\vec{T} \\ \qquad\quad + v(\rho_g\theta_g - \rho_n')\vec{T} \\ \qquad + [\gamma_1'' + 2(\gamma_0'\rho_g - \gamma_2'\theta_g) + \gamma_0(\rho_g' - \rho_n\theta_g) - \gamma_1(\rho_g^2+\theta_g^2) - \gamma_2(\theta_g' + \rho_g\rho_n)]\vec{g} \\ \qquad\quad - v(\theta_g' + \rho_g\rho_n)\vec{g} \\ \qquad + [\gamma_2'' + 2(\gamma_0'\rho_n + \gamma_1'\theta_g) + \gamma_0(\rho_n' + \rho_g\theta_g) + \gamma_1(\theta_g' - \rho_g\rho_n) - \gamma_2(\rho_n^2+\theta_g^2)]\vec{n} \\ \qquad\quad - v(\rho_n^2+\theta_g^2)\vec{n}, \\ {}^2\Psi_{vs} = -\rho_n\vec{T} - \theta_g\vec{g}, \\ {}^2\Psi_{vv} = 0. \end{cases} \tag{14}$$

Hence, from (12) and (14), we get the components of the second fundamental form of ${}^2\Psi(s,v)$:

$$\begin{cases} {}^2e = \dfrac{[\gamma_1' + \gamma_0\rho_g - \theta_g(\gamma_2+v)][\gamma_0'' - 2(\gamma_1'\rho_g + \gamma_2'\rho_n) - \gamma_0(\rho_g^2+\rho_n^2)]}{\sqrt{[\gamma_1' + \gamma_0\rho_g - \theta_g(\gamma_2+v)]^2 + [\gamma_0' - \gamma_1\rho_g - \rho_n(\gamma_2+v)]^2}} \\ \qquad \dfrac{[\gamma_1' + \gamma_0\rho_g - \theta_g(\gamma_2+v)][-\gamma_1(\rho_g' + \rho_n\theta_g) + \gamma_2(\rho_g\theta_g - \rho_n') + v(\rho_g\theta_g - \rho_n')]}{\sqrt{[\gamma_1' + \gamma_0\rho_g - \theta_g(\gamma_2+v)]^2 + [\gamma_0' - \gamma_1\rho_g - \rho_n(\gamma_2+v)]^2}} \\ \qquad - \dfrac{[\gamma_0' - \gamma_1\rho_g - \rho_n(\gamma_2+v)][\gamma_1'' + 2(\gamma_0'\rho_g - \gamma_2'\theta_g) + \gamma_0(\rho_g' - \rho_n\theta_g)]}{\sqrt{[\gamma_1' + \gamma_0\rho_g - \theta_g(\gamma_2+v)]^2 + [\gamma_0' - \gamma_1\rho_g - \rho_n(\gamma_2+v)]^2}} \\ \qquad + \dfrac{[\gamma_0' - \gamma_1\rho_g - \rho_n(\gamma_2+v)][-\gamma_1(\rho_g^2+\theta_g^2) + \gamma_2(\theta_g' + \rho_g\rho_n) + v(\theta_g' + \rho_g\rho_n)]}{\sqrt{[\gamma_1' + \gamma_0\rho_g - \theta_g(\gamma_2+v)]^2 + [\gamma_0' - \gamma_1\rho_g - \rho_n(\gamma_2+v)]^2}}, \\ {}^2f = \dfrac{\theta_g(\gamma_0' - \gamma_1\rho_g) - \rho_n(\gamma_1' + \gamma_0\rho_g)}{\sqrt{[\gamma_1' + \gamma_0\rho_g - \theta_g(\gamma_2+v)]^2 + [\gamma_0' - \gamma_1\rho_g - \rho_n(\gamma_2+v)]^2}}, \\ {}^2g = 0. \end{cases} \tag{15}$$

From (13) and (15), we get the Gaussian curvature of the ruled surface ${}^2\Psi(s,v)$ at regular points as follows

$${}^2K = -\left[\frac{\theta_g(\gamma_0' - \gamma_1\rho_g) - \rho_n(\gamma_1' + \gamma_0\rho_g)}{[\gamma_1' + \gamma_0\rho_g - \theta_g(\gamma_2+v)]^2 + [\gamma_0' - \gamma_1\rho_g - \rho_n(\gamma_2+v)]^{22}}\right]^2.$$

We deduce that the ruled surface ${}^2\Psi(s,v)$ is developable if and only if $\gamma_0'\theta_g - \gamma_1'\rho_n - \rho_g(\gamma_0\rho_n + \gamma_1\theta_g) = 0$. Then from the properties (3), this equation becomes $\theta_g = 0$, which means that the ruled surface ${}^2\Psi(s,v)$ is developable if and only if the curve $\gamma(s)$ is a principle line on the surface φ.

On the other hand, according to the properties (3), we conclude that the position vector of such a ruled surface is

$$
\begin{aligned}
{}^2\Psi(s,v) = \gamma_0(s)\vec{T}(s) &- \Big(\int [\gamma_0(s)\rho_g(s) - \gamma_2(s)\theta_g(s)]\,\mathrm{d}s + c_2\Big)\vec{g}(s) \\
&- \Big(\int [\gamma_0(s)\rho_n(s) + \gamma_1(s)\theta_g(s)]\,\mathrm{d}s + d_2 + v\Big)\vec{n}(s). \quad \square
\end{aligned}
$$

In the following, we present an example of that type of ruled surface generated by Darboux frame, and visualize them in $\mathbb{R}^3$.

Example. Let consider the unit speed curve $\gamma(s) = \big(\frac{1}{\sqrt{3}}\cos(s); 1 - \sin(s); -\frac{\sqrt{2}}{\sqrt{3}}\cos(s)\big)$ that lies on the regular surface

$$\varphi(u,v) = \Big(\tfrac{1}{\sqrt{3}}\cos(u) + v\sin(u); 1 - \sin(u) + v\sqrt{3}\cos(u); -\tfrac{\sqrt{2}}{\sqrt{3}}\cos(u) + v\sqrt{3}\Big).$$

It is clear that $\gamma(s) = \varphi(s,0)$, then, Darboux frame vectors of $\gamma(s)$ on φ are:

$$\vec{T} = \begin{pmatrix} -\frac{1}{\sqrt{3}}\sin(s) \\ -\cos(s) \\ \frac{\sqrt{2}}{\sqrt{3}}\sin(s) \end{pmatrix}, \quad \vec{g} = \frac{1}{P}\begin{pmatrix} \frac{2}{3}\sin^3(s) + \frac{\sqrt{2}}{\sqrt{3}}\sin^2(s) \\ \sqrt{2}\sin(s)\cos(s) + \frac{2}{\sqrt{3}}\sin^2(s)\cos(s) \\ \frac{1}{\sqrt{3}}(1 + \frac{\sqrt{2}}{\sqrt{3}}\sin(s))(1 + 2\cos^2(s)) \end{pmatrix},$$

$$\vec{n} = \frac{1}{P}\begin{pmatrix} -\sqrt{3}\cos(s) - \sqrt{2}\sin(s)\cos(s) \\ \frac{\sqrt{2}}{\sqrt{3}}\sin^2(s) + \sin(s) \\ 0 \end{pmatrix},$$

where $P = \sqrt{\frac{1}{3}(3 + 2\sqrt{6}\sin(s) + 2\sin^2(s))(2\cos^2(s) + 1)}$.

The following figures show us the three ruled surfaces according to that Darboux frame: Figure 1 is the $\vec{T}$-ruled surface, Figure 2 is the $\vec{g}$-ruled surface and Figure 3 is the $\vec{n}$-ruled surface.

Conclusion

We showed that according to the Darboux frame $\{\vec{T}, \vec{g}, \vec{n}\}$ of an arbitrary regular surface φ along the base curve c, the $\vec{T}$-ruled surface is naturally developable. Furthermore, $\vec{g}$-ruled surfaces and $\vec{n}$-ruled surfaces whose base curves are principle lines on the reference surface, are also developable. On the other hand, the position vectors of the last three ruled surfaces are given by the means of the Darboux frame curvatures.

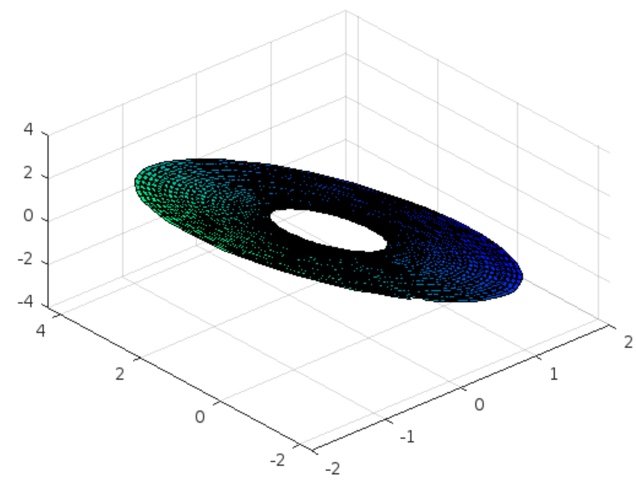

Figure 1: $\overrightarrow{T}$-ruled surface

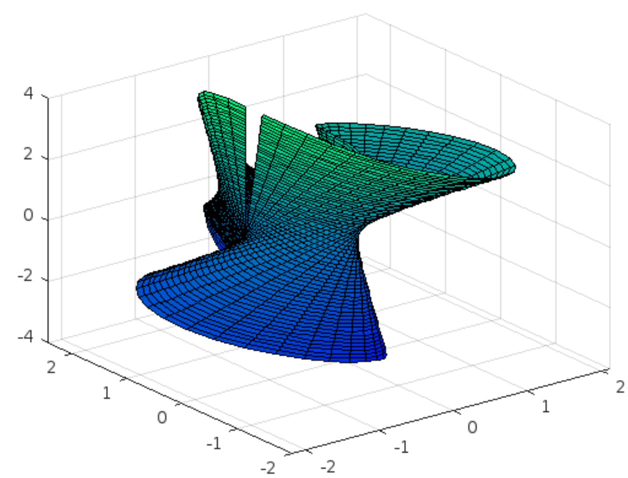

Figure 2: $\overrightarrow{g}$-ruled surface

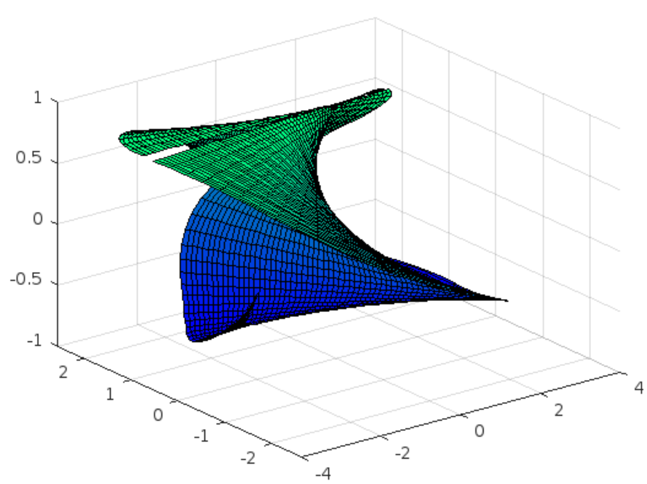

Figure 3: $\overrightarrow{n}$-ruled surface

References

[1] P. Alegre, K. Arslan, A. Carriazo, C. Murathan, and G. Öztürk: *Some special types of developable ruled surface*. Hacettepe Journal of Mathematics and Statistics **39**(3), 319–325, 2010.

[2] M. P. Do Carmo: *Differential Geometry of Curves and Surfaces*. Prentice Hall, Englewood Cliffs, NJ, 1976.

[3] G. Hu, H. Cao, J. Wu, and G. Wei: *Construction of developable surfaces using generalized C-Bézier bases with shape parameters*. Computational and Applied Mathematics **39**(3), 2020. doi: 10.1007/s40314-020-01185-9.

[4] S. Kiziltuğ and A. Çakmak: *Developable Ruled Surfaces with Darboux Frame in Minkowski 3-Space*. Life Science Journal **10**(4), 2013.

[5] T. G. Nelson, T. K. Zimmerman, S. P. Magleby, R. J. Lang, and L. L. Howell: *Developable mechanisms on developable surfaces*. Science Robotics **4**(27), 2019. doi: 10.1126/scirobotics.aau5171.

[6] S. Ouarab: *Simultaneous Developability of Partner Ruled Surfaces according to Darboux Frame in E^3*. Abstract and Applied Analysis **2021**, 1–9, 2021. doi: 10.1155/2021/3151501.

[7] S. Ouarab: *Smarandache Ruled Surfaces according to Darboux Frame in E^3*. Journal of Mathematics **2021**, 1–10, 2021. doi: 10.1155/2021/9912624.

[8] S. Ouarab, A. O. Chahdi, and M. Izid: *Ruled surface generated by a curve lying on a regular surface and its characterizations*. Journal for Geometry and Graphics **24**(2), 257–267, 2020.

[9] D. J. Struik: *Lectures on Classical Differential Geometry*. Addison-Wesley, Dover, 2nd ed., 1988.

[10] G. Y. Şentürk and S. Yüce: *Characteristic properties of the ruled surface with Darboux frame in E^3*. Kuwait Journal of Science & Engineering **42**(2), 14–33, 2015.

Received February 6, 2025; final form April 10, 2025.