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Tangential n-gons with Commensurability Conditions

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Abstract. Each regular polygon $\mathcal{P}$ is clearly tangential, has commensurable sides and every choice of three consecutive vertices among those of $\mathcal{P}$ determines a triangle whose interior angles are pairwise commensurable.

In this article we prove that these three conditions are also sufficient for a convex polygon to be equilateral. It turns out a new characterization of convex regular polygons with an odd number of sides.

Key Words: tangential polygon, equilateral polygon, incommensurability, regular polygon

MSC 2020: 51M40 (primary), 97G40, 97D80

1 Introduction

In Euclidean geometry, a *regular polygon* is an equiangular (all angles are equal in size) and equilateral (all sides have the same length) polygon. Clearly, every regular polygon $\mathcal{P}$ is *cyclic*, that is all vertices of $\mathcal{P}$ belong to a circle, and it is *tangential*, that is all sides of $\mathcal{P}$ are tangential to a circle $\mathcal{C}$. Tangential polygons are also called *circumscribed polygons*.

Another relevant concept in geometry is the incommensurability. Recall that two magnitudes are *incommensurable* if their ratio is not given by a pair of positive integers [1]. Indeed the incommensurability is a relevant topic not only in mathematics, and its discovery represented an important moment in the development of human thought (see [4, p.472]).

In addition to the classic discovery of the incommensurability between side and diagonal of a square and side and diagonal of a regular pentagon, recent articles prove that commensurability between diagonals of a regular polygon does not occur frequently (see [1, 8, 15, 16]).

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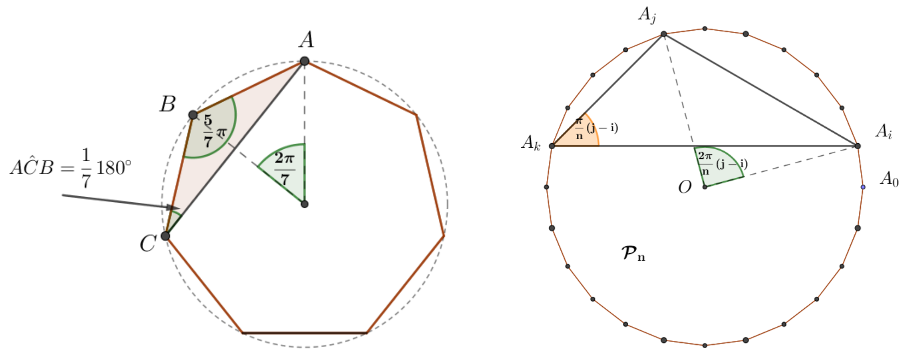

Figure 1: Left: description of the magnitude of the internal angles of the triangle ABC. Right: the interior angles of any triangle whose vertices are between $A_0, A_1, \dots, A_{n-1}, O$, are *rational in radians*.

It should also be noted that commensurability can also be referred to angular quantities (see [3, 7, 10, 12, 13]).

In the following, we will say that an angle is *rational in radians* when it is commensurable with π. Clearly, an angle is rational in radians if and only if its measure (in radians) is a rational multiple of π, that is say if and only if its measure in degrees is a rational multiple of 360°.

For example, it might be interesting to know that in every Pythagorean triangle the acute angles are incommensurable (see [2, Corollary 1], [10]), as well as it could be noted that the interior angles of any triangle whose vertices are among those of a regular polygon $\mathcal{P}_n = (A_0, A_1, \dots, A_{n-1})$ are *rational in radians*, (see Figure 1, right), so that in particular they are pairwise commensurable (see [16, Lemma 2.1], for example).

Therefore, looking at the commensurability properties of a regular polygon $\mathcal{P}_n$, we can say that:

(i) $\mathcal{P}_n$ is tangential.

(ii) Any two consecutive sides of $\mathcal{P}_n$ are commensurable.

(iii) Every triangle $\mathcal{T}$ whose vertices are three consecutive vertices of $\mathcal{P}_n$, has all interior angles pairwise commensurable.

It is reasonable to ask whether the three above properties (i), (ii) and (iii), besides being necessary conditions, also express together a sufficient condition for a convex polygon to be regular. We will see that this is not generally true, as the example of rhombus in Figure 1 shows. On the other hand we will prove that if a polygon satisfies the conditions (i), (ii) and (iii), then it is equilateral. In particular, such polygons are regular when n is odd (see Theorem 5).

To complete the investigation we will see that the conditions (i), (ii) and (iii) are independent (see Examples 2, 4, and 3), so that two of these conditions are not sufficient for a polygon to be equilateral.

The article is suitable for a broad audience.

2 Recalls and Preliminary Results

The following result, which is not widely known in the literature (see [3, p.228]), will be relevant for our investigation.

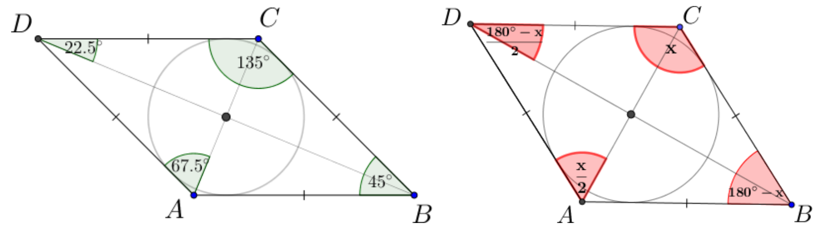

Figure 2: Left: a rhombus with the property (iii). Right: when x is not rational in radians, then the properties (iii) is not satisfied.

Proposition 1. *Every triangle all of whose sides have rational length and all of whose angles are rational in radians must be equilateral.*

If a triangle has all sides pairwise commensurable, then it is similar to one that has sides with rational length. Moreover,

Remark 1. If a triangle has all interior angles pairwise commensurable, then they are rational in radians (see [14, Lemma (c), p. 47]).

These last considerations, allow us to state the above Proposition 1 in terms of commensurability.

Remark 2. Every triangle all of whose sides are pairwise commensurable and all of whose angles are pairwise commensurable must be equilateral (and equiangular).

We also note that

Remark 3. If two inner angles of a triangle are rational in radians, then the third is rational in radians too. Therefore in the above remark and in the next results referred to triangles, the condition "all of whose inner angles are rational in radians (or in degrees), can be replaced by the condition "in which at least two angles are rational in radians" (resp. in degrees).

In the case of quadrilaterals, the situation differs significantly. Indeed, one can readily construct quadrilaterals satisfying conditions (i), (ii) and (iii) but that are not equiangular, (see Figure 2).

Example 1. Clearly, any rhombus satisfies both conditions (i) and (ii); moreover, if its interior angles are rational in radians, then it also satisfies condition (iii) (see for example Figure 2, left).

Another result showing restrictions on triangles with commensurability conditions is due to Evans and Isaacs (see [7]):

Theorem 1 (Evans-Isaacs)**.** *If Δ is a non-isosceles triangle with two or more rational sides and with all angles rational (measured in degrees), then Δ has angles $30°$, $60°$ and $90°$.*

Remark 4. Note that if a triangle $\mathcal{T}$ has two commensurable sides and its interior angles are rational in radians, then it is similar to a triangle Δ which satisfies the hypothesis of the Evans-Isaacs' theorem. Then, by Remark 3, Evans-Isaacs' result can be stated as follows:

Theorem 2. *If $\mathcal{T}$ is a non-isosceles triangle with two or more commensurable sides and with two or more angles rational in radians, then $\mathcal{T}$ has angles $30°, 60°$ and $90°$. In particular the hypotenuse is twice the length of a leg and it is incommensurable with the other one.*

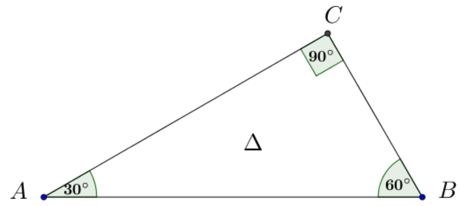

Figure 3: The hypothenuse of a right triangle of the type "30-60-90" is twice the shorter leg and it is incommensurable with the other one.

Remark 5. If $\mathcal{T} = ABC$ is an isosceles triangle with two or more commensurable sides, say $l_1 = AB$ and $l_2 = AC$, and with all angles rational in radians, then l_1 and l_2 must be congruent. Indeed, if $l_1 \neq l_2$, then the third side BC must be congruent either to AB or AC, and hence all sides of $\mathcal{T}$ are pairwise commensurable. It follows that $\mathcal{T}$ is equilateral by Proposition 1. This contradiction shows that $l_1 = l_2$.

Corollary 1. *Let $\mathcal{T} = (ABC)$ be a triangle with two commensurable sides AB and BC, and two or more angles rational in radians. If $|AB| \neq |BC|$ then $\mathcal{T}$ has angles $30°$, $60°$ and $90°$.*

By Theorem 2, and Remark 5 it follows that:

Corollary 2. *Let $\mathcal{T}$ be a triangle in which at least two sides are commensurable and two or more angles are rational in radians. If $\mathcal{T}$ has an obtuse angle α, then $\mathcal{T}$ is isosceles and α is the vertex angle of $\mathcal{T}$.*

We conclude this section with a technical result on convex n-gons.

Proposition 2. *Let $n > 2$ be a positive integer. Then any convex n-gon $\mathcal{P}_n$ has at most three interior acute angles.*

Proof. For a convex n-gon the sum of external angles is 2π. Therefore, four inner acute angles cause four external angles greater than $\pi/2$, so that their sum would exceed 2π. □

3 Tangential Polygons with Commensurability Conditions

Tangential polygons have very interesting properties (see [5, 9, 11] and references therein). Starting from quadrilaterals, it is well-known that they are tangential if and only if the sum of the lengths of the opposite side coincides. What about tangential n-gons, for $n > 4$?

In this fashion we highlight the following nice and "recent" results:

Theorem 3 (Djukić et al. [6, p. 561])**.** *In a tangential polygon with an even number of sides, the sum of the odd numbered sides' lengths is equal to the sum of the even numbered sides' lengths.*

Theorem 4 (De Villiers, [5, Theorem 2])**.** *A circumscribed polygon has all sides equal, if and only if, the alternate angles are equal. (For n odd, it becomes regular).*

In the same article the author shows interesting examples of tangential (circumscribed) equilateral polygons.

Another remark that will be useful to recall in the context of tangential polygons, is the following:

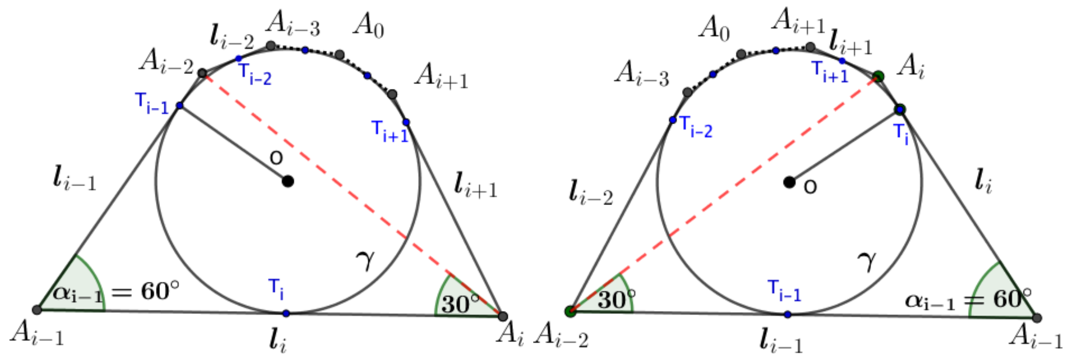

Figure 4: Tangential Polygons: Left, $l_{i-1} < l_i$; right, $l_{i-1} > l_i$.

Remark 6. Let $\mathcal{C}$ be a circle and let A be a point outside it. Denote by T' and T'' the points of tangency of the two tangents drawn from A to $\mathcal{C}$. Then the magnitude of the angle $\widehat{T'AT''}$ decreases as the distance from point A to the curve $\mathcal{C}$ increases.

We are now in a position to show our main theorem.

Theorem 5. *Let $\mathcal{P}_n$ be an n-gon satisfying the following conditions:*

(i) $\mathcal{P}_n$ is tangential.

(ii) Any two consecutive sides of $\mathcal{P}_n$ are commensurable.

(iii) Every triangle $\mathcal{T}$ whose vertices are three consecutive vertices of $\mathcal{P}_n$, has all interior angles pairwise commensurable.

Then $\mathcal{P}_n$ is equilateral. In particular, if n is odd, then $\mathcal{P}_n$ is a regular polygon.

Proof. By Remark 2, we may suppose $n > 3$. Let $\mathcal{P}_n$ be a tangential polygon, and denote by T_i all tangential points (see Figure 4). By contradiction, suppose that the statement is false. Then $\mathcal{P}_n$ has at least one pair of consecutive non-congruent sides. Without loss of generality, we may assume that $l_{n-1} = A_{n-2}A_{n-1}$ and $l_n = A_{n-1}A_0$ are two non-congruent sides of $\mathcal{P}_n$.

First, we prove that for every pair of consecutive sides of different length, say l_{i-1} and l_i, the following conditions hold:

(j) The triangle $\mathcal{T}_{i-1} = A_{i-2}A_{i-1}A_i$ is 30-60-90 and the angle $\alpha_{i-1} = 60°$.

(jj) One of the following cases appears:

- (jj') $l_{i-1} < l_i$, α_{i-2} is obtuse and $30° < \alpha_i < 60°$ (see Figure 4, left).
- (jj") $l_{i-1} > l_i$, α_i is obtuse and $30° < \alpha_{i-2} < 60°$ (see Figure 4, right).

(jjj) The sides l_{i-2} and l_{i-1} and l_i and l_{i+1} have the same length: $l_{i-2} = l_{i-1}$ and $l_i = l_{i+1}$.

- Proof of (j). It follows by Corollary 1.
- Proof of (jj). By (j), A_iA_{i-2} is the longer leg of the right triangle $\mathcal{T}_{i-1}$, thus two possibilities appear:
  - (jj') If $l_{i-1} < l_i$, then the vertex of the right angle of $\mathcal{T}_{i-1}$ is A_{i-2} (Figure 4, left). Clearly, the vertex A_{i-2} is distinct from the tangential point T_{i-1} and therefore $\alpha_{i-2} = A_{i-3}\widehat{A_{i-2}}A_{i-1}$ is obtuse, and $\alpha_i = A_{i+1}\widehat{A_i}A_{i-1} > 30°$. So that: $A_{i-1}T_i = A_{i-1}T_{i-1} < A_{i-1}A_{i-2} = \frac{1}{2}A_{i-1}A_i$. Thus, $A_{i-1}T_i < A_iT_i$, and hence, by Remark 6, $\alpha_i = A_{i+1}\widehat{A_i}A_{i-1} < \alpha_{i-1} = 60°$.
  - (jj") If $l_i < l_{i-1}$, the vertices of the right angle of $\mathcal{T}_{i-1}$ is A_i (Figure 4, right) and the same (specular) argument as above shows that α_i is obtuse and $30° < \alpha_{i-2} < 60°$.

  The (jj) is proved.

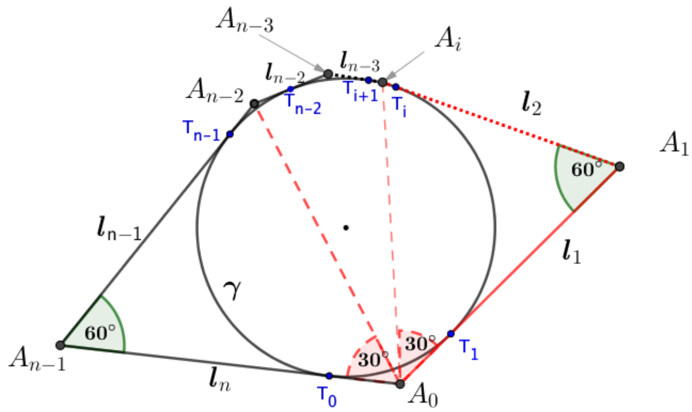

Figure 5: A polygon satisfying the conditions (i)-(iii), has no interior acute angles (when $n > 3$).

- Proof of (jjj). By hypothesis the sides l_{i-2} and l_{i-1} of the triangle $(A_{i-3}A_{i-2}A_{i-1})$ are commensurable. By contradiction suppose that $l_{i-2} \neq l_{i-1}$, then by (jj) the angle between them, α_{i-2}, cannot be 60°, 30°, or 90°. Therefore, by Theorem 5 the triangle $(A_{i-3}A_{i-2}A_{i-1})$ must be isosceles. It follows by Remark 5 that $l_{i-2} = l_{i-1}$.
  Similarly it can be proved that $l_i = l_{i+1}$. The condition (jjj) is proved.

Now, assume that $l_{n-1} < l_n$. By condition (jj') the triangle $\mathcal{T} = A_{n-1}A_0A_{n-2}$ is of the type 30-60-90 (choose $n = i$ in Figure 4, left). In particular,

$$\alpha_{n-1} = 60^\circ, \ \alpha_{n-2} > 90^\circ \quad \text{and} \quad 30^\circ < \alpha_0 < 60^\circ. \tag{1}$$

By (jjj) we have

$$l_{n-2} = l_{n-1} \quad \text{and} \quad l_n = l_1. \tag{2}$$

If $l_1 \neq l_2$, then by (j), the triangle $A_0A_1A_2$ must be 30-60-90 and $\alpha_1 = 60^\circ$ (choose $i = 2$ in Figure 4, right). By (1) $\alpha_0 < 60^\circ$, thus we have $\widehat{A_1A_0A_2} = 30^\circ$. It follows that

$$60^\circ \leq \widehat{A_1A_0A_2} + \widehat{A_{n-2}A_0A_{n-1}} \leq \alpha_0 < 60^\circ \quad \text{(see Figure 5)}.$$

This contradiction shows that

$$l_1 = l_2. \tag{3}$$

Now, if $n = 4$, then by Equation (2) we have: $l_4 = l_1 = l_2 = l_3$. That is a contradiction to the assumption $l_{n-1} < l_n$.

Thus we may suppose that $\mathcal{P}_n$ has more than 4 sides.

Clearly, a polygon that is not equilateral must possess more than one pair of consecutive non-congruent sides. Therefore we may consider a pair (l_h, l_{h+1}) of non-congruent sides with $h \neq n-1$.

On the other hand, assuming the convention that $l_{n+1} = l_1$, by Equations (3) and (2), $h \neq 1, n-2, n$, so that $1 < h < n-2$. Applying the condition (j) to the pair of sides (l_h, l_{h+1}), we have $\alpha_h = 60^\circ$ and either $\alpha_{h-1} < 60^\circ$ or $\alpha_{h+1} < 60^\circ$. It turns out that $\mathcal{P}_n$ has at least four acute angles: α_0, α_{n-1}, α_h and one between α_{h-1} and α_{h+1}. This contradicts Proposition 2. A similar contradiction can be obtained if we assume that $l_{n-1} > l_n$.

Therefore $l_{n-1} = l_n$, so that $\mathcal{P}_n$ is equilateral.

The last part of the statement follows from Theorem 4. □

Remark 7. We note that the condition (iii) of the Theorem 5 cannot be replaced by the following natural condition:

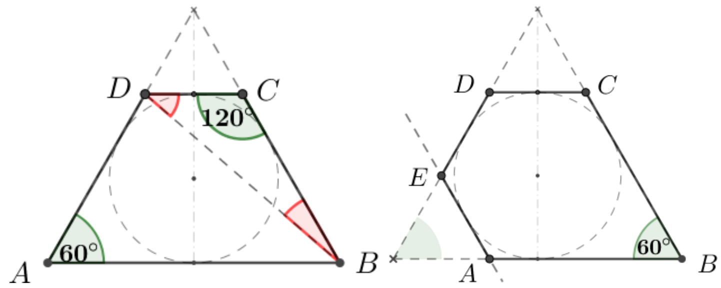

Figure 6: Tangential polygons with commensurable sides. Here red angles are not commensurable with $\widehat{DCB}$.

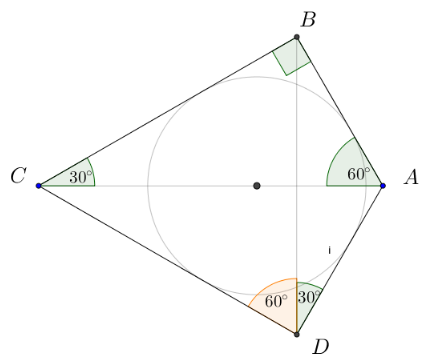

Figure 7: Construction of a Kite with a pair of non-commensurable sides and satisfying (iii).

(iii') *Each pair of consecutive interior angles of* $\mathcal{P}_n$ *are commensurable.*

Indeed the polygons shown in the Figure 6 are tangential, have the sides pairwise commensurable and the angles rational in radians. On the other hand, the angle $\widehat{DCB}$ is obtuse and $DC \neq CB$. Then, by Corollary 2 both the angles $\widehat{CBD}$ and $\widehat{BDC}$ are not rational in radians, and hence they are not commensurable with $\widehat{DCB} = 120°$. In particular (iii) is not satisfied.

The following three examples show that the conditions (i), (ii) and (iii) of the Theorem 5 are independent.

Example 2. As we have seen, it is not hard to find examples of polygons satisfying (i) and (ii), but not (iii) (see polygons shown in Figure 6).

Example 3. To find examples of polygons satisfying conditions (i) and (iii), but not (ii), it is sufficient to consider *kites* constructed as in Figure 7.

Example 4. To find examples of polygons satisfying (ii) and (iii), but not (i), we will consider a *special* trapezoid $\mathcal{Q} = (ABCD)$ constructed as in Figure 8. It turns out that $\mathcal{Q}$ satisfies both the conditions (ii) and (iii). In particular, $\mathcal{Q}$ is not equilateral, so that it cannot satisfy also the condition (i) of the Theorem 5.

Conclusions

In this article, we have explored the geometric structure of tangential polygons under conditions of commensurability imposed on their sides and angles. Our results show that such constraints lead, in a surprisingly natural way, to strong regularity properties.

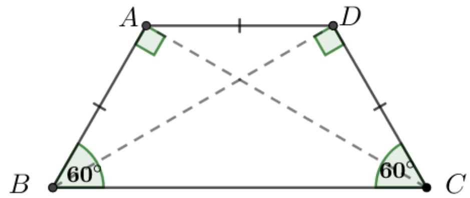

Figure 8: Special isosceles trapezoid. They are not tangenzial.

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These findings highlight an elegant interplay between metric and angular conditions, from which interesting geometric properties arise.

From a theoretical standpoint, in line with researches as in [9] the results contribute to a deeper understanding of the rigid structures underlying classical polygonal configurations.

Finally an open question which is connected with Example 3.

Question. Do there exist n-gons (with $n > 4$) satisfying conditions (i) and (iii) of Theorem 5, which possess at least one pair of consecutive sides that are not commensurable?

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