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A Family of Eight-Point Conics Associated with the Cyclic Quadrilateral

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Abstract. We consider the following configuration. Let $ABCD$ be a cyclic quadrilateral with circumcenter O, and for each vertex X, let H_X be the orthocenter of the triangle formed by the other three. Then A, B, C, D, H_A, H_B, H_C, H_D all lie on a single conic. In this paper we study a certain generalization of this fact as follows. For an arbitrary point P_D on the Euler line of $\triangle ABC$, we define corresponding points P_A, P_B, P_C on the respective Euler lines such that the ratio $P_XH_X : P_XO$ is constant for all X. We show that the four vertices A, B, C, D and the four isogonal conjugates Q_A, Q_B, Q_C, Q_D of the points P_X all lie on a single conic. This result is given distinct treatments, synthetic, projective, and algebraic. Furthermore, we situate the points P_X within the list of triangle centers.

Key Words: Euler line, isogonal conjugate, conic, cyclic quadrilateral, triangle center, orthocenter, Shinagawa coefficients

MSC 2020: 51M04 (primary), 51M05, 51M15

1 Introduction

Consider a triangle $\triangle ABC$ with orthocenter H. It is well established that if one considers a rectangular hyperbola Φ passing through the vertices of $\triangle ABC$, then H lies on Φ [1, 9]. From this fact, it immediately follows that if one considers D as the point of intersection of Φ with the circumcircle of $\triangle ABC$, the orthocenters of $\triangle ABD, \triangle ACD, \triangle BCD$ lie on Φ [6]. We state this below.

Proposition 1. *Let $ABCD$ be a cyclic quadrilateral, and by H_A denote the orthocenter of triangle $\triangle BCD$, similarly define H_B, H_C, H_D. Then, A, B, C, D, H_A, H_B, H_C, H_D all lie on a single conic.*

This paper studies a generalization of this fact, extending it to a family of points defined on the Euler line. This generalization was first stated in [8]; however the proof offered was found to contain a fatal error, leading to the paper's retraction. This paper provides three distinct geometric treatments and establishes a connection to the modern theory of triangle centers [4, 5, 9].

Notation. We work in the projective plane, denoting the line at infinity as ℓ_∞ and points on this line as ∞_X or ∞_k if we are referring to the infinity point of a line k. Moreover, for a given segment AB: M_{AB} denotes its midpoint, $|AB|$ denotes its Euclidean length. Furthermore, by $X(Y) \longmapsto X'(Y')$ we will denote a projectivity from Y to Y' in which $X \in Y$ maps to $X' \in Y'$, by $\mathcal{L}(X)$ we denote the set of lines passing through point X.

2 Main Result

2.1 Preliminaries

We recall a list of Lemmas.

Lemma 2 (Steiner's Theorem)**.** *Given two pencils of lines, P and Q, let ϕ be a projective map between them. Then:*

1. *If $\phi(PQ) = PQ$, the locus of points $x \cap \phi(x)$ for $x \in P$ forms a straight line Γ.*
2. *If $\phi(PQ) \neq PQ$, the locus of points $x \cap \phi(x)$ for $x \in P$ forms a conic Γ that passes through points P and Q.*

Moreover, the transformation

$$x(P) \longmapsto x \cap \phi(x)\ (\Gamma) \tag{1}$$

is projective.

For the proof of Lemma 2 and related theory see [2]. A direct corollary of Steiner's Theorem is the following.

Lemma 3 (Isogonal Conjugate of a Line)**.** *Given a triangle ABC and a line ℓ, the locus of the isogonal conjugates of points on ℓ with respect to $\triangle ABC$ is:*

1. *a line Γ, if ℓ passes through any of the vertices of $\triangle ABC$;*
2. *a circumconic Γ of $\triangle ABC$, otherwise.*

Moreover, the transformation

$$X(\ell) \longmapsto X'(\Gamma) \tag{2}$$

is projective, where $X \in \ell$ and X' is the isogonal conjugate of the point X with respect to triangle ABC.

Remark 4. In the case of the Euler line, this circumconic is the Jeřábek hyperbola [4], whose center is X_{125} in the ETC.

Lemma 5. *Let $\mathcal{H}$ be a rectangular hyperbola, and let A, B, C be distinct points on it, and let H denote the orthocenter of $\triangle ABC$. If $\mathcal{H}$ intersects the circumcircle of $\triangle ABC$ for the fourth time at D, then the midpoint of HD is the center of $\mathcal{H}$.*

Proof. Let O be the circumcenter of $\triangle ABC$. By ∞_D denote the isogonal conjugate of D with respect to $\triangle ABC$. Let the line $O\infty_D$ intersect the circumcircle of $\triangle ABC$ at K and L, and by ∞_K and ∞_L we denote the isogonal conjugates of K and L with respect to $\triangle ABC$ respectively. Then, we have

$$-1 = (O, \infty_D; K, L) = (H, D; \infty_K, \infty_L), \tag{3}$$

which implies that the asymptotes of $\mathcal{H}$ intersect each other on HD. By symmetry, the intersection point must be in the midpoint – which is the center of $\mathcal{H}$. □

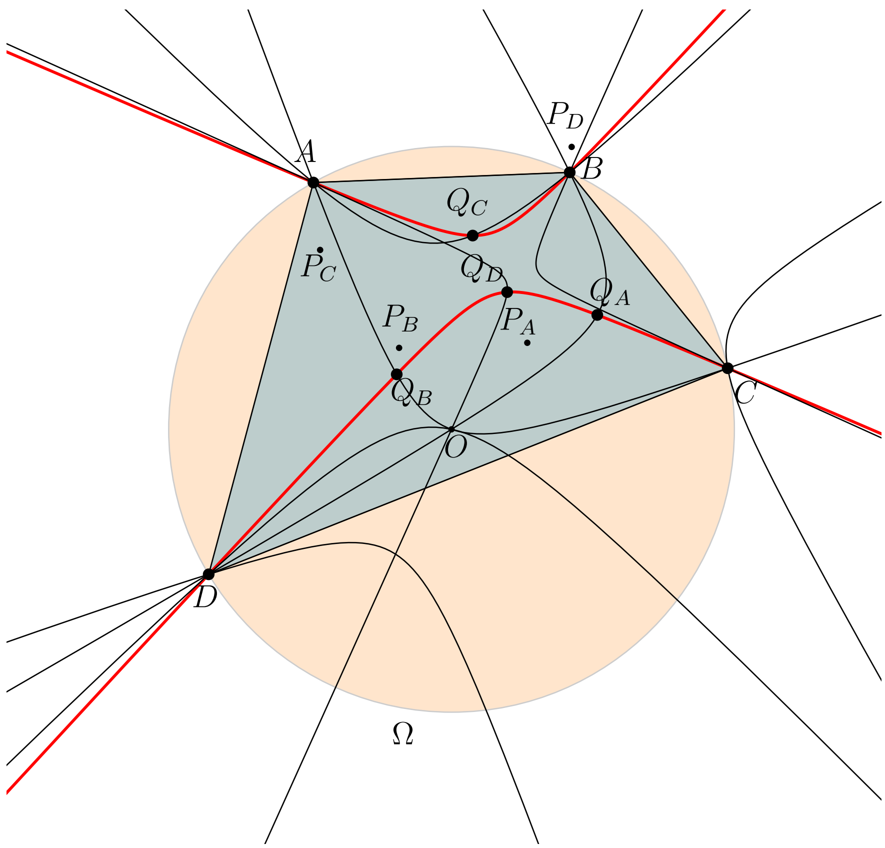

Figure 1: A, B, C, D, Q_A, Q_B, Q_C, Q_D lie on a conic.

Now we state the main result of this paper (Fig. 1).

Theorem 6. *Let A, B, C, D be distinct ordered points lying on a circle, denoted by Ω with center O. By H_D, H_B, H_C, H_A denote the orthocenters of $\triangle ABC$, $\triangle ACD$, $\triangle ABD$, $\triangle BCD$, and let P_D be any point on the Euler line of $\triangle ABC$. Let P_A be the point on the Euler line of $\triangle BCD$ such that*

$$\frac{P_A H_A}{P_A O} = \frac{P_D H_D}{P_D O}, \tag{4}$$

where H_A is the orthocenter of $\triangle BCD$. Define P_B, P_C similarly on the Euler lines of $\triangle ACD$ and $\triangle ABD$. Let Q_D be the isogonal conjugate of P_D with respect to $\triangle ABC$, and define Q_A, Q_B, Q_C analogously in their respective triangles. Then the points

$$A,\ B,\ C,\ D,\ Q_A,\ Q_B,\ Q_C,\ Q_D \tag{5}$$

all lie on a single conic.

Without loss of generality, we will prove that the points A, B, C, D, Q_A, and Q_D lie on a single conic. By symmetry, an analogous argument can be constructed for the remaining pairs of Q points. Since five points, no three collinear, uniquely determine a conic, this is sufficient to establish that all eight points lie on a single conic.

Furthermore, we assume that no sub-triangle is equilateral. For an equilateral triangle, the Euler line degenerates to a single point where the orthocenter and circumcenter coincide. In such a case, Theorem 6 reduces to that of Proposition 1.

2.2 More (or Less) Synthetic Treatments

We give the first proof, using projective and synthetic tools.

Remark 7 (Degenerated cases). In all the proofs presented in this paper, whenever we refer to Lemma 3 and Lemma 5, we assume that the locus of points described in these lemmas is a conic. If the locus happens to be a line, such cases can be handled separately by an argument of continuity in geometry, since degenerate cases are finitely many.

Proof. By ∞_A and ∞_D denote the isogonal conjugates of points A and D with respect to triangles DBC and ABC, respectively. Since the quadrilateral $ABCD$ is cyclic, the isogonal conjugates ∞_A and ∞_D lie on the line at infinity.

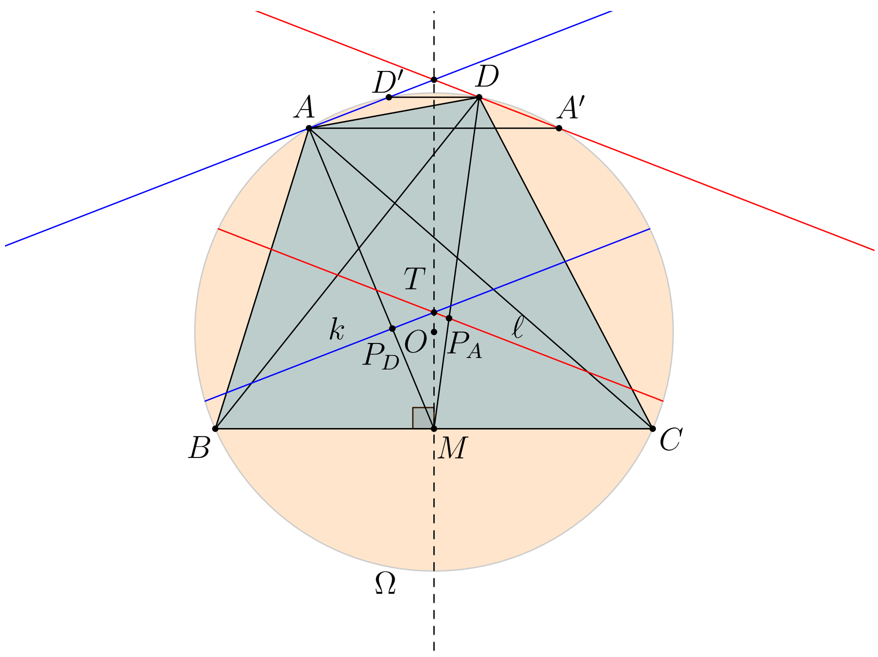

Figure 2: Point T lies on the perpendicular bisector of the segment BC.

Let line k be the line passing through the points P_D and ∞_D, and let line ℓ be the line passing through the points P_A and ∞_A. Denote T as the intersection of k and ℓ. Note that since ∞_A and ∞_D are fixed and independent of the specific choice of P_A and P_D, and $P_AP_D \parallel H_AH_D$, the locus of the point T is a fixed straight line passing through O. We will show that this line is the perpendicular bisector of the side BC – denoted by m (Fig. 2).

Now consider the case $P_A = H_A$, which implies $P_D = H_D$. It suffices to demonstrate that T lies on the perpendicular bisector of the side BC. Consider A', D', H'_A and H'_D – reflections of A, D, H_A and H_D, respectively, in the perpendicular bisector of BC. We have $AD' \parallel k$

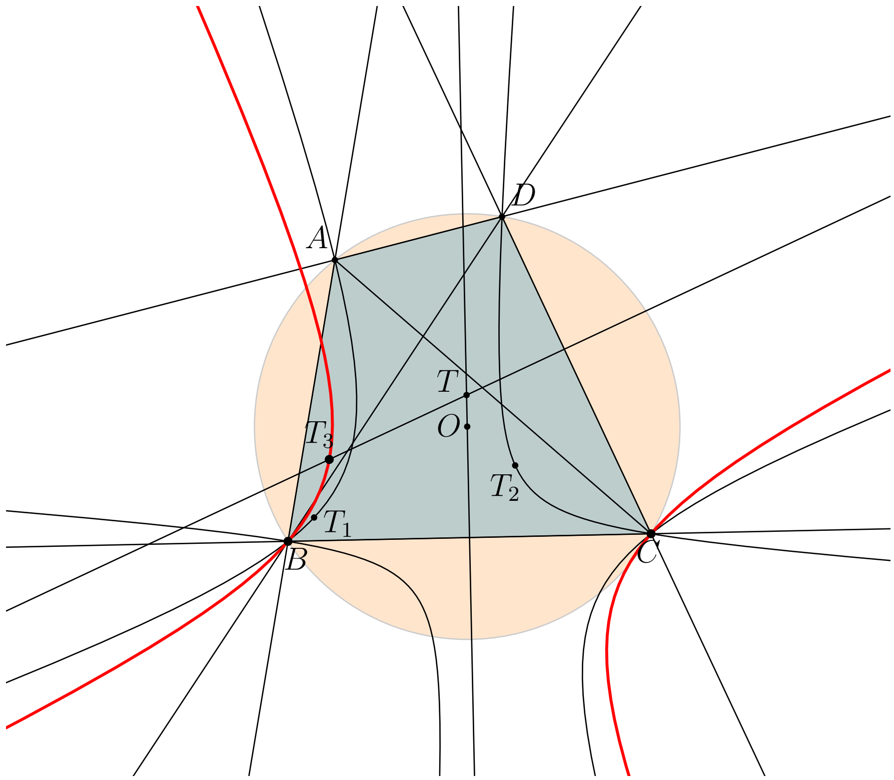

Figure 3: TT_3 goes through the point ∞_D.

and $DA' \parallel \ell$ from the definition and that AD' and DA' are reflections of each other in our bisector. Moreover,

$$|AH_D| = 2|OM_{BC}| = |DH_A| = |D'H'_A|. \tag{6}$$

Now, from $AH_D \parallel D'H'_A$, we find that $AD'H_DH'_A$ is a parallelogram, which gives $AD' \parallel H_DH'_A \implies \ell = H_DH'_A$. Similarly we get that $k = H_AH'_D$, and our claim follows.

Let T_1 and T_2 denote the isogonal conjugates of the point T with respect to $\triangle ABC$ and $\triangle DBC$, respectively. By Lemma 3, the line ℓ, under isogonal conjugation with respect to $\triangle DBC$, maps to a circumconic of $\triangle DBC$. Since the point ∞_A lies on ℓ and is the isogonal conjugate of point A with respect to $\triangle DBC$, it follows that A also lies on this conic. Furthermore, by the definition of point T_2 and the fact that point P_A lies on ℓ, both T_2 and Q_A lie on this conic.

By a similar argument, conjugating the line k with respect to $\triangle ABC$ yields that the points A, B, C, D, T_1, and Q_D lie on a single conic. Therefore, to establish that A, B, C, D, Q_A, and Q_D lie on a single conic, it suffices to prove that points A, B, C, D, T_1, and T_2 lie on a single conic.

Now let T_3 denote the isogonal conjugate of T_2 with respect to $\triangle ABC$. We will show now that line TT_3 goes through point ∞_D (Fig. 3).

Now, from Lemma 3 it follows that T_2 moves along a conic passing through D, B, and C, as the isogonal conjugate of $T \in m$ with respect to $\triangle DBC$. Let us denote this conic by Γ_1. We then have the following maps:

$$T(m) \longmapsto T_2(\Gamma_1), \tag{7}$$

$$T_2(\Gamma_1) \longmapsto CT_2(\mathcal{L}(C)) \longmapsto CT_3(\mathcal{L}(C)), \tag{8}$$

$$T_2(\Gamma_1) \longmapsto BT_2(\mathcal{L}(B)) \longmapsto BT_3(\mathcal{L}(B)). \tag{9}$$

From Equations (8) and (9), we obtain the following projective map:

$$CT_3(\mathcal{L}(C)) \longmapsto BT_3(\mathcal{L}(B)). \tag{10}$$

By Lemma 2, we conclude that T_3 lies on a conic passing through points B and C, which we denote by Γ_2. Thus, we have the following projective map:

$$BT_3(\mathcal{L}(B)) \longmapsto T_3(\Gamma_2). \tag{11}$$

Note that since $D \in \Gamma_1$, it follows that $\infty_D \in \Gamma_2$. Therefore, we have the following projective map:

$$T_3(\Gamma_2) \longmapsto \infty_D T_3(\mathcal{L}(\infty_D)) \longmapsto m \cap \infty_D T_3(m). \tag{12}$$

By composing the above projective maps, we find that:

$$T(m) \longmapsto m \cap \infty_D T_3(m). \tag{13}$$

We will show that this map is the identity mapping. It is enough to show this in three cases, as a projective map is a Möbius transformation.

1. $T = \infty_m \implies T_2 \in \Omega \implies T_3 \in \ell_\infty \implies T, T_3, \infty_D$ are collinear. Therefore, the map (13) is the identity mapping.
2. Let $T = O$. By Lemma 3, we obtain that the conic passing through A, B, C, H_D, and H_A is the isogonal conjugate of TT_3 with respect to $\triangle ABC$. Furthermore, by Proposition 1, we know that A, B, C, H_D, H_A, and D lie on a single conic. Hence $\infty_D \in TT_3$, which proves the collinearity of T, T_3, and ∞_D. Therefore, the map (13) is the identity mapping.
3. $T = M \implies T_2 = D \implies T_3 = \infty_D$. Thus $\infty_D T_3$ is tangent to Γ_2. Therefore by Lemma 8 (proved below) we obtain that this tangent line DT_3 passes through point $M = T$. Therefore, the map (13) is the identity mapping.

We have shown that map (13) is the identity mapping in three cases. Therefore, it is the identity mapping in general. It follows directly that T, T_3, and ∞_D are collinear.

Now we take the isogonal conjugates of points on the line TT_3 with respect to $\triangle ABC$. Then, by Lemma 3 and by taking the isogonal conjugates of the points T, T_3, and ∞_D with respect to $\triangle ABC$, we obtain that points

$$A,\ B,\ C,\ D,\ T_1,\ T_2 \tag{14}$$

lie on a single conic. Which completes our proof. □

Lemma 8. *Let A, B, C, D be four distinct points on a circle Ω. Denote ℓ to be the perpendicular bisector of BC. Denote $\mathcal{H}_1$ to be the isogonal conjugate of ℓ with respect to $\triangle BCD$ and $\mathcal{H}_2$ to be the isogonal conjugate of $\mathcal{H}_1$ in with respect to $\triangle ABC$. Then, $\mathcal{H}_1$ and $\mathcal{H}_2$ are rectangular hyperbolas, and the midpoint of BC is their center (Fig. 4).*

Remark 9. To finish the proof of Theorem 6, we only really need the part that M_{BC} is the center of $\mathcal{H}_2$, but for completeness, we include the other result too.

Proof. First, it is clear that $\mathcal{H}_1$ is a rectangular hyperbola – the intersections of ℓ with Ω are antipodal on Ω, hence it follows that the asymptotes of $\mathcal{H}_1$ are perpendicular. Let $\mathcal{H}_1$ intersect Ω for the fourth time at D', and let H be the orthocenter of $\triangle BCD$. Notice that

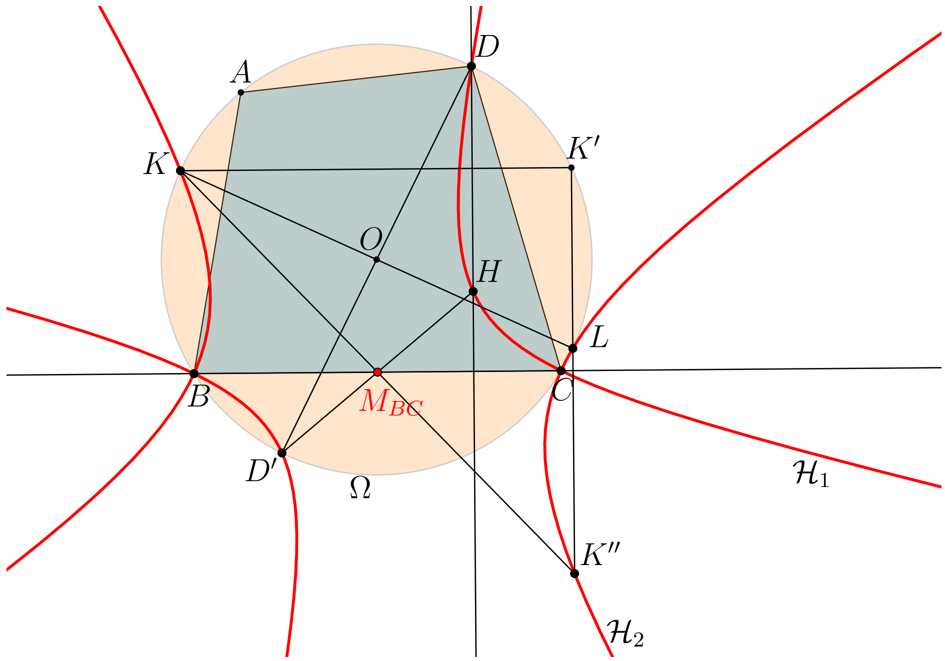

Figure 4: M_{BC} is the center of the hyperbolas.

D' is the antipode of D on Ω, as $\infty_{D'}$ – the isogonal conjugate of D' with respect to $\triangle BCD$ is the point at infinity of ℓ, and as $\ell \perp BC \implies D\infty_D \perp BC \implies H \in D\infty_D$, we must have DD' passing through the center of Ω (in the exceptional case where $D \in \ell$, its isogonal conjugate will be a line. For our lemma to make sense, in such case we assume that the isogonal conjugate of D with respect to $\triangle BCD$ is the whole line BC, and $\mathcal{H}_1$ degenerates to two perpendicular lines – namely BC and ℓ). From Lemma 5, the center of $\mathcal{H}_1$ is the midpoint of HD', but it is clearly also the midpoint of BC.

Now, as $\mathcal{H}_1$ is a conic passing through B and C, $\mathcal{H}_2$, being its isogonal conjugate with respect to $\triangle ABC$, must also be a conic passing through B and C (in the exceptional case where $A \in \mathcal{H}_1$, we again assume that $\mathcal{H}_2$ degenerates to two perpendicular lines – the same as before). Isogonal conjugates of D and D' with respect to $\triangle ABC$ lie on $\mathcal{H}_2$ and are at infinity. Because $AD \perp AD'$ (as DD' is the diameter of Ω), we get that $\mathcal{H}_2$ is also a rectangular hyperbola. Now label the points at infinity of $\mathcal{H}_1$ as ∞_K and ∞_L, and let K and L be their isogonal conjugates with respect to $\triangle ABC$, respectively. It is clear that K and L lie on Ω as well as on $\mathcal{H}_2$. K and L are also the antipodes of each other on Ω, because

$$\sphericalangle KAL = \sphericalangle \infty_K A \infty_L = 90^\circ. \tag{15}$$

Let K' be the reflection of K in ℓ. Now $KK' \parallel BC$ and $\sphericalangle KK'L = 90^\circ \implies K'L \perp BC$. Let K'' be the reflection of K' in BC. It follows that K'' is the orthocenter of $\triangle BCL$, and from Lemma 5, the midpoint of KK'' is the center of $\mathcal{H}_2$. As BC passes through the midpoint of $K'K''$ and $BC \parallel KK'$, we get that the midpoint of KK'' lies on BC. □

A more synthetic proof of the main result can be seen below.

Proof. Consider the isogonal conjugate of the circumconic $ABCDQ_D$ with respect to $\triangle ABC$. It will be a line, denoted k, through P_D and ∞_D, where ∞_D is the isogonal conjugate of D with respect to $\triangle ABC$. Similarly define the line ℓ through P_A and ∞_A, where ∞_A is the isogonal conjugate of A with respect to $\triangle BCD$.

Repeating the same treatment as at the start of the first proof of Theorem 6, we get that k and ℓ are reflections of each other in the perpendicular bisector of BC.

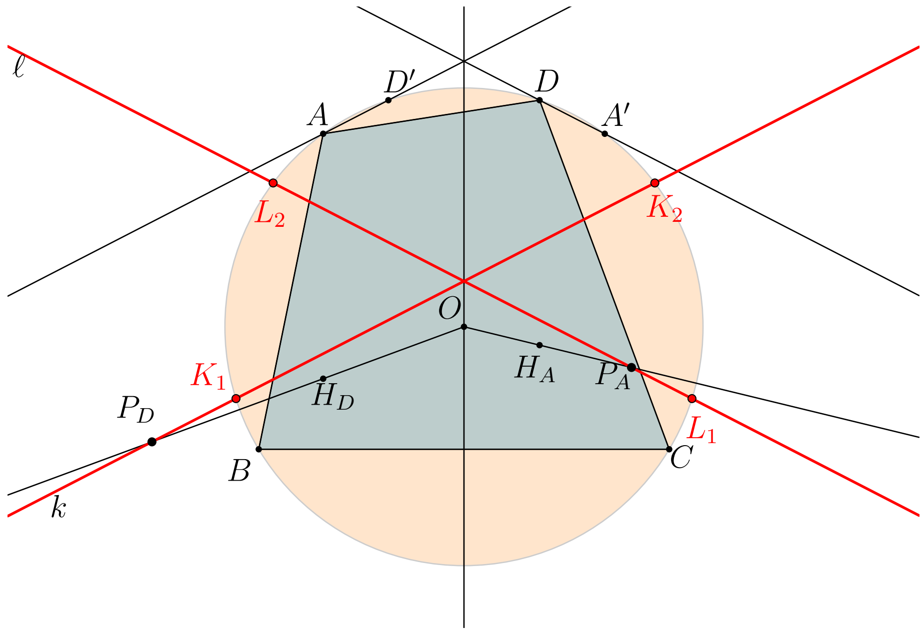

Figure 5: k and ℓ are reflections of each other in the perpendicular bisector of BC.

We have to prove that the isogonal conjugate of ℓ with respect to $\triangle BCD$ and the isogonal conjugate of k with respect to $\triangle ABC$ are the same conic. To do this, we first assume that the conics are hyperbolas, that is, lines k and ℓ intersect the circle on $ABCD$, each at two points.

As the aforementioned isogonal conjugates of k and ℓ share points A, B, C, D, it is sufficient to prove that they have the same points at infinity. Let k intersect Ω at K_1 and K_2, and let ℓ intersect Ω at L_1 and L_2 (Fig. 5).

As k and ℓ are reflections of each other in the perpendicular bisector of BC, we can see that $K_1K_2L_1L_2$ is a trapezoid with its bases parallel to BC. Without loss of generality, assume that K_1 and L_2 lie on the same side of the aforementioned bisector. Now because arcs $A'K_2$ and AL_2 have the same length and $L_1L_2 \parallel DA'$, we get that $DK_2 \parallel AL_1$. Now from the trapezoid, the line DK_2 is clearly the isogonal conjugate of the line DL_2 in the angle BDC, and AL_1 is the isogonal conjugate of the line AK_1 in the angle BAC, hence, from $DK_2 \parallel AL_1$, the isogonal conjugate of L_2 in BCD is the same point at infinity as the isogonal conjugate of K_1 in ABC, which proves our claim.

Now there are clearly infinitely many instances of P_A, so that k and ℓ have intersections with Ω (when P_A is inside the circle), hence, we have proven our Theorem for infinitely many cases. To get from this to the Theorem for all the cases, we can use a *polynomial moving points* argument. When P_A moves on OH_A, Q_A, being its isogonal conjugate, by Lemma 2, moves on a conic Γ_1, and the map

$$P_A(OH_A) \longmapsto Q_A(\Gamma_1) \tag{16}$$

is projective. Similarly Q_D also moves on a conic Γ_2, and the map

$$P_A(OH_A) \longmapsto P_D(OH_D) \longmapsto Q_D(\Gamma_2) \tag{17}$$

is also projective. Now we get that the map

$$Q_A(\Gamma_1) \longmapsto P_A(OH_A) \longmapsto P_D(OH_D) \longmapsto Q_D(\Gamma_2) \tag{18}$$

is projective, since projective maps are bijective. Therefore we can express coordinates of the point Q_D, as a polynomial of coordinates of the point Q_A, which lies on a fixed conic. Thus, to prove that A, B, C, D, Q_A, Q_D lie on a single conic it is sufficient to check some finite number of cases, but we have infinitely many. □

2.3 A More Algebraic Approach

As it turns out this result also comes about purely projectively, using techniques well known in algebraic geometry. To proceed with the proof, consider the space of all conics considered as homogeneous forms of degree 2. Conics passing through any four non-collinear points P_1, P_2, P_3, P_4 form a line in that space; let us call this line $C(P_1, P_2, P_3, P_4) \cong \mathbb{P}^1$. We will need the following Lemma.

Lemma 10. *Let ABC be a triangle, ℓ be a line different from the line in infinity that passes through at most one vertex of the triangle, X be a point in the plane not lying on any of the triangle's sides such that its isogonal conjugate does not lie on ℓ. Then there exists a projectivity from ℓ to $C(A, B, C, X)$ that maps each point possessing an isogonal conjugate to a conic passing through the isogonal conjugate, which moreover satisfies the two following conditions:*

1. *if X lies on the circumcircle of ABC, the image of the point in infinity is the circumcircle of ABC,*
2. *if ℓ passes through the circumcenter O of $\triangle ABC$ then the image of O is a rectangular hyperbola (perhaps one reducing to two perpendicular lines).*

Proof. First consider the case when ℓ does not pass through any of the triangle's vertices. Take the conic Γ and the parametrisation $\gamma\colon \ell \to \Gamma$ from Lemma 2; had X lied on Γ, it would be the image of some point X' in γ. It is easy to see X' must not lie on any of the triangle's sides, meaning that X is the isogonal conjugate of X', a contradiction. The parametrisation γ induces a map in the reverse direction: from the space of homogeneous forms of degree 2 on $\mathbb{P}^2$ restricted to Γ to the space $H^4(2)$ of homogeneous forms of degree 4 on ℓ. In particular, we have a map ϕ from $C(A, B, C, X)$ to $H^4(2)$. 0 does not lie in the image of ϕ, as X does not lie on Γ – hence ϕ is a well-defined projectivity from $C(A, B, C, X)$ to $\operatorname{Im} \phi$. Now, three of the roots of any form in the image of ϕ correspond to the preimages of A, B, C in the aforementioned parametrisation; we can compute the fourth root by Viete's formulas, inducing a projective map ψ from $\operatorname{Im} \phi$ to ℓ. Both ψ and ϕ are projectivities – denote the inverse of their composition by f. By our construction, it is a map from ℓ to $C(A, B, C, X)$ that sends every point to the conic passing through its image in γ; thus f is the sought after projectivity.

In the case where ℓ passes through one of the triangle's vertices (say A), the reasoning is very similar except we directly restrict conics to the line Γ given by Lemma 2.2. We know one of the resulting form's roots will be A, and we extract the second one by Vieta's formulas as above.

The last two claims are easy to prove synthetically except two nuances; the first proposition has a slight problem when ℓ is parallel to one of the triangle's sides, and the second – when ABC is right-angled. These issues can be addressed manually or with an argument by continuity. □

Now we can give another proof of the main result of this paper.

Proof. We will assume the isogonal conjugate of A with respect to $\triangle BCD$ does not lie on OH_A and similarly for D; otherwise, it is not hard to see the conic degenerates to a rectangular hyperbola through A, B, C, D.

By $\infty_{H_AH_D}$ denote the point at infinity on line H_AH_D. We will use our previous Lemma for the line OH_A, the triangle BCD and the point A, as well as the line OH_D, the triangle ABC and the point D; call the obtained projectivities ϕ_A and ϕ_D. Consider the following diagram:

$$\begin{array}{ccc} OH_A & \xrightarrow{\phi_A} & C(A,B,C,D) \\ \Big\downarrow{\scriptstyle \pi_{\infty_{H_AH_D}}} & & {\scriptstyle \phi_D}\Big\uparrow \\ \mathcal{L}(\infty_{H_AH_D}) & \xrightarrow{\pi_{OH_D}} & OH_D \end{array}$$

where $\pi_{\infty_{H_AH_D}}$ is a map which assigns each point X on OH_A the line $\infty_{H_AH_D}$ and π_{OH_D} is a map which assigns each line through $\infty_{H_AH_D}$ its intersection with OH_D. If the diagram commutes, then applying the relevant maps to P_A will get us that the conic passing through its isogonal conjugate is the same as the conic passing through the isogonal conjugate of P_D, ending the proof. But all maps on the diagram are projectivities; hence it suffices to prove that $\phi_A(X) = \phi_D(\pi_{OH_D}(\pi_{\infty_{H_AH_D}}(X))$ for three cases of X.

1. For X at infinity, $\pi_{\infty_{H_AH_D}}(X)$ is at infinity as well; by Lemma 10, both conics are the circumcircle.
2. For X at H_A, $\phi_A(X)$ passes through A, B, C, D, O; there is exactly one such conic, and it must also be equal to $\phi_D(\pi_{\infty_{H_AH_D}}(X))$, ending the proof for this case.
3. For X at O, $\phi_A(X)$ is a rectangular hyperbola by Lemma 10. The same follows for $\phi_D(\pi_{\infty_{H_AH_D}}(X))$; both conics pass through A, B, C, D, implying they are one and the same. □

3 Connection to the Encyclopedia of Triangle Centers

3.1 Centers with Constant Shinagawa Coefficients

Theorem 6 holds for any point P_D on the Euler line, which in turn establishes a ratio $P_DH_D : P_DO = \lambda_0$. Since this condition is satisfied by triangle centers such as the orthocenter, it is natural to investigate the cases where P_A, P_B, P_C, P_D correspond to other triangle centers. To this end, we consult the Encyclopedia of Triangle Centers (henceforth ETC) [4].

Let X_n denote the n-th Kimberling center cataloged in the ETC. We consider the subset of these centers that lie on the Euler line. The homogeneous barycentric coordinates of such points, relative to the affine frame defined by triangle ABC, are conveniently expressed using Shinagawa coefficients [4]. We use the convention where $A = (1 : 0 : 0)$, $B = (0 : 1 : 0)$, $C = (0 : 0 : 1)$, and the Conway symbols for $\triangle ABC$ with side lengths a, b, c are given by $S = 2 \cdot \text{Area}(\triangle ABC)$ and $S_A = \frac{1}{2}(b^2 + c^2 - a^2)$, with S_B and S_C defined cyclically [9].

Definition 11 (Shinagawa coefficients)**.** Let X_n be a triangle center on the Euler line. Its homogeneous barycentric coordinates are expressed as $(f(a, b, c) : f(b, c, a) : f(c, a, b))$, where f is the triangle center function of X_n [4]. The *Shinagawa coefficients* of X_n is the pair of functions $(G(a, b, c), H(a, b, c))$ that satisfy the relation:

$$f(a, b, c) = G(a, b, c) \cdot S^2 + H(a, b, c) \cdot S_BS_C, \tag{19}$$

and analogous cyclic conditions hold.

Remark 12. Note that Shinagawa coefficients are homogeneous.

It is often the case that the Shinagawa coefficients $(G(a,b,c), H(a,b,c))$ are constants for all possible a, b, c. For example, the coefficients of the orthocenter, X_4, are $(0,1)$ (see [5] for more examples).

In this context, we claim that our points P_A, P_B, P_C, P_D are always centers with constant Shinagawa coefficients. To establish this, we first present the following lemma from [7], where lowercase letters of points denote their positional vectors.

Lemma 13. *Let ABC be a triangle on the unit circle with center O and orthocenter H. Let X be a center with constant Shinagawa coefficients (u,v). Then there exists a unique scalar $\lambda \in \mathbb{R} \cup \{\infty\}$ such that $x = \lambda h$, where*

$$\lambda = \frac{u+v}{3u+v}. \tag{20}$$

Proposition 14. *The points P_A, P_B, P_C, P_D satisfy the condition in (4) if and only if they correspond to the same triangle center with constant Shinagawa coefficients in their respective triangles.*

Proof. Assume the condition in (4) holds. This gives a constant λ_0 such that

$$\frac{p_D - h_D}{p_D} = \lambda_0, \tag{21}$$

which implies $p_D = \lambda h_D$ for $\lambda = (1-\lambda_0)^{-1}$. The condition ensures this same λ applies to all four points, so $p_X = \lambda h_X$ for $X \in \{A,B,C,D\}$. By Lemma 13, this single λ determines a unique ratio of Shinagawa coefficients (u,v):

$$\frac{u}{v} = \frac{1-\lambda}{3\lambda - 1}. \tag{22}$$

Since this ratio is the same for all four points, they must correspond to the same center with constant Shinagawa coefficients.

Conversely, assume P_A, P_B, P_C, P_D are the same center with constant Shinagawa coefficients (u,v). By Lemma 13, these coefficients determine a unique scalar λ via (22). This implies $p_X = \lambda h_X$ for each point, which is equivalent to the ratio condition in (4). □

Following Proposition 14 the isogonal conjugates of centers with constant Shinagawa coefficients are exactly the family of points that satisfy the main result of this paper, meaning it can be rephrased as follows.

Theorem 15. *Assume X is a triangle center with constant Shinagawa coefficients (u,v). Let $ABCD$ be a cyclic quadrilateral, by X_A, X_B, X_C, X_D denote the X points of $\triangle BCD$, $\triangle ACD$, $\triangle ABD$ and $\triangle ABC$, respectively. Furthermore by Y_A denote the isogonal conjugate of X_A with respect to $\triangle BCD$, similarly define Y_B, Y_C, Y_D. Then A, B, C, D, Y_A, Y_B, Y_C, Y_D lie on a conic, denoted by $\Phi(u,v)$.*

X_n	X'_n	X_n	X'_n	X_n	X'_n
X_2	X_6	X_3	X_4	X_5	X_{54}
X_{20}	X_{64}	X_{30}	X_{74}	X_{381}	X_{3431}
X_{140}	X_{1173}	X_{376}	X_{3426}	X_{547}	X_{57714}
X_{382}	X_{11270}	X_{546}	X_{57713}	X_{550}	X_{16835}
X_{548}	X_{57715}	X_{549}	X_{14483}	X_{1656}	X_{13472}
X_{631}	X_{3527}	X_{632}	X_{57730}	X_{3091}	X_{14528}
X_{1657}	X_{13452}	X_{3090}	X_{43908}	X_{3523}	X_{52518}
X_{3146}	X_{3532}	X_{3522}	X_{22334}	X_{3534}	X_{11738}
X_{3524}	X_{3531}	X_{3529}	X_{43719}	X_{3830}	X_{20421}
X_{3543}	X_{43713}	X_{3628}	X_{34567}	X_{5071}	X_{44731}
X_{5054}	X_{14491}	X_{5059}	X_{43691}	X_{12100}	X_{14487}
X_{8703}	X_{13603}	X_{10304}	X_{14490}	X_{46853}	X_{46851}
X_{33703}	X_{44763}	X_{33923}	X_{46848}	X_{61138}	X_{61137}

Table 1: Pairs (X_n, X'_n), where X_n is a center with constant Shinagawa coefficients.

3.2 A Catalog of Centers that Satisfy the Main Result

We now identify all cataloged pairs of centers X_n and their isogonal conjugates X'_n that satisfy Theorem 15. See Table 1.

Remark 16. Note that for $n \leq 61371$, there exist 721 centers with constant Shinagawa coefficients; however, most of them do not have a cataloged isogonal conjugate pair [4, 5, 7] (as of 20th of July 2025).

3.3 Equation of the Eight-Point Conic Family

We conclude our study by providing the explicit equation for the conic $\Phi(u, v)$ from Theorem 15. We first recall two results concerning barycentric coordinates [9].

Lemma 17. *Let $D = (d : e : f)$ and $X = (p : q : r)$ be two distinct points, with homogeneous barycentric coordinates given relative to the affine frame defined by $\triangle ABC$. The conic passing through the five points A, B, C, D, X has the equation:*

$$\sum_{\text{cyc}} frxy(ep - dq) = 0. \tag{23}$$

Lemma 18 (Isogonal conjugate coordinates)**.** *Let a point P have homogeneous barycentric coordinates $(x : y : z)$ relative to the affine frame defined by $\triangle ABC$. Its isogonal conjugate with respect to $\triangle ABC$ is the point Q given by:*

$$Q = \left(\frac{a^2}{x} : \frac{b^2}{y} : \frac{c^2}{z}\right), \tag{24}$$

where a, b, c are the lengths of sides BC, AC, AB respectively.

Proposition 19. *In the affine frame of $\triangle ABC$, the equation of the conic passing through A, B, C, D, Y_A, Y_B, Y_C, Y_D, denoted $\Phi(u, v)$ where (u, v) are the Shinagawa coefficients of X_i, is given by*

$$\sum_{\text{cyc}} \frac{c^2 rxy}{uS^2 + vS_A S_B}\left(\frac{b^2 p}{uS^2 + vS_A S_C} - \frac{a^2 q}{uS^2 + vS_B S_C}\right) = 0, \tag{25}$$

where $D = (p : q : r)$ *are the homogeneous barycentric coordinates of the fourth vertex.*

Proof. By Theorem 6, all eight of our points lie on a single conic, which in turn is uniquely determined by the five points A, B, C, D, and Y_D. To find its equation, we can, without loss of generality, work within the affine frame of $\triangle ABC$.

By Proposition 14, Y_D is the isogonal conjugate of the center X_D with constant Shinagawa coefficients (u, v). Applying Lemma 18, the coordinates of Y_D in the frame of $\triangle ABC$ are:

$$Y_D = \left(\frac{a^2}{uS^2 + vS_BS_C} : \frac{b^2}{uS^2 + vS_AS_C} : \frac{c^2}{uS^2 + vS_AS_B}\right). \tag{26}$$

Substituting the coordinates of $D = (p : q : r)$ and Y_D into the five-point conic formula (Lemma 17) directly yields the equation for $\Phi(u, v)$ as stated. □

Remark 20. A computational verification that all eight points lie on $\Phi(u, v)$ is also possible. The procedure involves performing an affine transformation [3] to reframe the coordinates of each point Y_i from its native triangle into the affine frame of $\triangle ABC$. One can then confirm by direct substitution that these transformed coordinates are zeros of Equation (25).

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