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Angles of Skew Quadrilaterals and Tetrahedra

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Abstract. Theorem 1 is to prove angle sum conditions for a skew quadrilateral to be planar. Theorem 2 is about the angle sum of a non-planar skew quadrilateral. Theorem 3 proves that a tetrahedron must have a vertex with all three angles acute.

A skew quadrilateral with pairwise equal opposite edges is called reversible. A tetrahedron that contains a reversible skew quadrilateral is reversible. An equal-angled skew quadrilateral may not be reversible. However, Theorem 4 states that if a tetrahedron contains an equal-angled skew quadrilateral, then the tetrahedron must be reversible. Our last Theorem 5 is on an angle condition of an isosceles tetrahedron.

Key Words: skew quadrilateral, quadrilateral, law of cosines, spherical law of cosines, reversible skew quadrilateral, reversible tetrahedron, isosceles tetrahedron

MSC 2020: 51M04

1 Introduction

Let A, B, C and D be distinct points in the space $\mathbb{R}^3$. The line segment AB with its endpoints is denoted by $[AB]$, the segment AB without its endpoints is denoted by (AB), and its length is denoted by $|AB|$. By joining the points A, B, C, D with line segments $[AB], [BC], [CD]$ and $[DA]$, we obtain a *skew quadrilateral*, denoted by $\mathrm{SQ}(ABCD)$, if (1) $[AB] \cap [CD] = \varnothing$ and $[DA] \cap [BC] = \varnothing$, and (2) if X, Y, Z are distinct elements from the set $\{A, B, C, D\}$, then $(XY) \cap (YZ) = \varnothing$ (see Figure 1). If $\mathrm{SQ}(ABCD)$ is *planar*, then we say that $\mathrm{SQ}(ABCD)$ is a *quadrilateral*, and denote it by $\mathrm{Q}(ABCD)$.

The angle of a skew quadrilateral $\mathrm{SQ}(ABCD)$ at D is, denoted by $\angle CDA$, is (1) the angle of the triangle $\triangle CDA$ at D if C, D, A are non-collinear, or (2) $\angle CDA = \pi$ if C, D, A are collinear in this order: (If C, D, A are collinear in this order, then $\mathrm{SQ}(ABCD)$ is the triangle $\triangle ABC$ having the point D on the edge (CA).) So, an angle of a skew quadrilateral is at most π (see Remark 1 below). If there is no confusion, we denote $\angle CDA$ by $\angle D$. A skew quadrilateral $\mathrm{SQ}(ABCD)$ is said to be *equal-angled* if $\angle A = \angle B = \angle C = \angle D$.

Skew quadrilaterals are discussed in [1]. However, no discussions on angles are given there. We will investigate angles of skew quadrilaterals and of tetrahedra.

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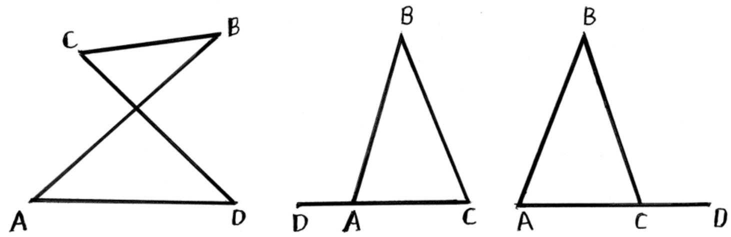

Figure 1: These are examples that are not Q($ABCD$).

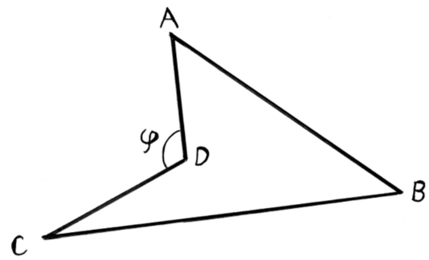

Figure 2: The marked angle φ is the angle $\sphericalangle CDA$ of (planar) Q($ABCD$) with our definition. The conventional angle $\sphericalangle CDA$ of Q($ABCD$) is the interior angle $2\pi - \varphi$.

Remark 1. We'd like to point out our angle measurements of a quadrilateral can be different from the conventional angle measurement. Suppose Q($ABCD$) is the non-convex one in Figure 2. Then the angle indicated by φ is *our* definition of the angle $\angle CDA$, while the conventional angle measurement of $\angle CDA$ is the interior angle of Q($ABCD$) at D, which is $2\pi - \varphi$ and larger than π. While a quadrilateral has the conventional angle sum of 2π, this is not true by our definition of the angle measurement. Clarification of this is Lemma 1 below. Please be noted.

Lemma 1. *The four angles of a quadrilateral can be labeled α, β, γ, δ so that $\alpha+\beta+\gamma+\delta = 2\pi$ or $\alpha = \beta + \gamma + \delta$.*

Proof. Suppose Q($ABCD$) is a quadrilateral. We will consider two cases; (Case 1), for any distinct X, Z in $\{A, B, C, D\}$, $[XZ]$ is inside or an edge of Q($ABCD$); or (Case 2) for some distinct X, Z in $\{A, B, C, D\}$, (XZ) is outside of Q($ABCD$).

(Case 1) Suppose for any distinct points X, Z in $\{A, B, C, D\}$, $[XZ]$ is inside or an edge of Q($ABCD$). Then the union of $\triangle DBC$ and $\triangle DAB$ with their interior is Q($ABCD$) with its interior. (In this case, the conventional angle measurement and ours are the same.) Then

$$\begin{aligned}\angle CDA + \angle DAB + \angle ABC + \angle BCD \\ = (\angle CDB + \angle BDA) + \angle DAB + (\angle ABD + \angle DBC) + \angle BCD \\ = (\angle CDB + \angle DBC + \angle BCD) + (\angle BDA + \angle DAB + \angle ABD) = 2\pi.\end{aligned}$$

(Case 2) Suppose for some distinct X, Z in $\{A, B, C, D\}$, (XZ) is outside of Q($ABCD$). Suppose (AC) is outside of Q($ABCD$). Then D and B are on the same side of $\overline{AC}$. (This is the case of Figure 2.) Then $\angle BDA > \frac{\pi}{2}$ and $\angle CDB > \frac{\pi}{2}$. Since $(\angle DAB + \angle ABD + \angle BDA) = \pi$ and $(\angle DBC + \angle BCD + \angle CDB) = \pi$ are the angle sums of $\triangle DAB$ and $\triangle DBC$, we have

$$\begin{aligned}\angle DAB + \angle ABC + \angle BCD = \angle DAB + (\angle ABD + \angle DBC) + \angle BCD \\ = (\angle DAB + \angle ABD) + (\angle DBC + \angle BCD) = (\pi - \angle BDA) + (\pi - \angle CDB) \\ = 2\pi - (\angle BDA + \angle CDB) = \sphericalangle CDA.\end{aligned}$$

Therefore, the four angles of a quadrilateral can be labeled α, β, γ, δ so that $\alpha+\beta+\gamma+\delta = 2\pi$, or $\alpha = \beta+\gamma+\delta$. $\square$

Theorem 1 is the converse of Lemma 1. Theorem 2 is to show that a non-planar skew quadrilateral has the four-angle sum being less than 2π and the three-angle sum being not equal to the fourth angle.

If edges $[AC]$ and $[BD]$ are added to a non-planar skew quadrilateral SQ$(ABCD)$, the resulting solid is a *tetrahedron*, denoted by $\nabla ABCD$, and $\nabla ABCD$ is said to be *generated* by SQ$(ABCD)$. Equivalently, a tetrahedron $\nabla ABCD$ is said to *contain* or to *have* three skew quadrilaterals SQ$(ABCD)$, SQ$(ACBD)$, and SQ$(ACDB)$ (see Figure 3).

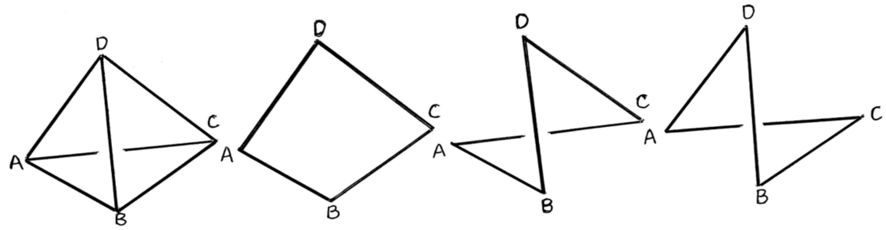

Figure 3: The figure on the left is a tetrahedron $\nabla ABCD$. Then the remaining three figures are the skew quadrilaterals SQ$(ABCD)$, SQ$(ABDC)$, and SQ$(ACBD)$ that are contained in $\nabla ABCD$. Any one of SQ$(ABCD)$, SQ$(ABDC)$, and SQ$(ACBD)$ can generate $\nabla ABCD$.

An angle of a triangular face of a tetrahedron is said to be an *angle* of the tetrahedron. Hence, a tetrahedron $\nabla ABCD$ has twelve angles, and three angles $\sphericalangle ADB$, $\sphericalangle ADC$, $\sphericalangle BDC$ are said to be *angles* of $\nabla ABCD$ at D. Theorem 3 will show that a tetrahedron must have a vertex with three acute angles. (An angle θ is *acute* if $0 < \theta < \frac{\pi}{2}$, and *obtuse* if $\theta \geq \frac{\pi}{2}$.)

If $|AB| = |CD|$ and $|AD| = |BC|$, then SQ$(ABCD)$ is said to be *reversible*. A reversible quadrilateral is a parallelogram.

A tetrahedron that contains a reversible skew quadrilateral is said to be *reversible*. A tetrahedron $\nabla ABCD$ is said to be *isosceles* if $|AB| = |CD|$, $|AC| = |BD|$ and $|AD| = |BC|$. An isosceles tetrahedron has four congruent triangular faces.

Example 1. Note that if SQ$(ABCD)$ is a reversible skew quadrilateral, then $\sphericalangle DAB = \sphericalangle BCD$ and $\sphericalangle ABC = \sphericalangle CDA$. However, the converse is not true. We will construct an equal-angled SQ$(ABCD)$ such that $|BC| \neq |DA|$. Let $\nabla ABCD$ be a tetrahedron such that $|AB| = |BC| = |CD| = 2, |BD| = |AC| = \sqrt{2}$, and $|DA| = 1$. (The volume,[1] of $\nabla ABCD$ is $\frac{\sqrt{2}}{6}$.) Hence, $\triangle ABD \cong \triangle DCA$ and $\triangle ABC \cong \triangle BCD$ so that $\sphericalangle DAB = \sphericalangle ADC$ and $\sphericalangle ABC = \sphericalangle BCD$. From the law of cosines, we have

$$2 = |BD|^2 = |AB|^2 + |DA|^2 - 2|AB||DA| \cos\angle DAB = 5 - 4\cos\angle DAB \quad \text{and}$$
$$2 = |AC|^2 = |AB|^2 + |BC|^2 - 2|AB||BC| \cos\angle ABC = 8 - 8\cos\angle ABC.$$

Hence, $\cos\angle DAB = \frac{3}{4} = \cos\angle ABC$. Therefore, $\angle DAB = \angle ABC = \angle BCD = \angle CDA = \cos^{-1}\frac{3}{4}$ so that SQ$(ABCD)$ is *equal-angled*. Since $|BC| = 2$ and $|DA| = 1$,

[1]If (a and α), (b and β), (c and γ) are pairs of opposite edges of a tetrahedron of volume V, then

$$\begin{aligned}144V^2 =& a^2\alpha^2(-a^2-\alpha^2+b^2+\beta^2+c^2+\gamma^2)+b^2\beta^2(a^2+\alpha^2-b^2-\beta^2+c^2+\gamma^2)\\ &+c^2\gamma^2(a^2+\alpha^2+b^2+\beta^2-c^2-\gamma^2)-(a^2b^2c^2+a^2\beta^2\gamma^2+a^2b^2\gamma^2+\alpha^2\beta^2c^2).\end{aligned}$$

A positive righthand side of this equation is a necessary condition for the existence of such tetrahedron.

SQ($ABCD$) is *not reversible*. SQ($ACBD$) is not reversible, either. However, SQ($ACDB$) *is reversible* since $|AC| = |DB| = \sqrt{2}$ and $|CD| = |BA| = 2$, and therefore, the *tetrahedron* $\nabla ABCD$ is *reversible.*

Even though an equal-angled skew quadrilateral may not be reversible as we saw in the above example, Theorem 4 is to show that if a tetrahedron contains an equal-angled skew quadrilateral, then the tetrahedron must be reversible. The converse of Theorem 4 is not true (see Example 3 below).

Lastly, Theorem 5 is to show that a tetrahedron is isosceles if, and only if, it contains two equal-angled skew quadrilaterals.

2 Angle Sum of a Skew Quadrilateral

The sum of four angles of a skew quadrilateral can be made very close to 0 as well as 2π (see Lemma 2). We will prove that for a non-planar skew quadrilateral, the sum of four angles is less than 2π, and the sum of three angles is not equal to the fourth angle.

Lemma 2. *Let* $0 < \theta < \frac{\pi}{2}$. *Then there is a skew quadrilateral* SQ($ABCD$) *such that* $\sphericalangle A = \sphericalangle B = \sphericalangle C = \sphericalangle D = \theta$.

Proof. Let Q($ABCD_1$) be a quadrilateral such that $|AB| = |BC| = |CD_1| = |D_1A|$ and $\sphericalangle AD_1C = \sphericalangle ABC = \theta$. So $\triangle AD_1C$ and $\triangle ABC$ are congruent isosceles triangle. Since $\sphericalangle BAD_1 = (\pi - \theta) > \frac{\pi}{2} > \theta$, we can rotate $\triangle AD_1C$ about the edge $[AC]$ so that $\sphericalangle BAD_1$ become θ, and relabeling this D_1 by D. Then we obtain $\sphericalangle BAD = \sphericalangle BCD = \theta$. Then SQ($ABCD$) is a skew quadrilateral such that $\sphericalangle A = \sphericalangle B = \sphericalangle C = \sphericalangle D = \theta$. □

The interior angle between the triangular faces $\triangle ABC$ and $\triangle ADC$ at the edge $[AC]$ of a tetrahedron $\nabla ABCD$ is said to be the *dihedral angle* at the edge $[AC]$, and it is denoted by $\sphericalangle AC$. The next lemma is called *spherical law of cosines* and it is well-known.

Lemma 3. *(Spherical Law of Cosines) Let* $\sphericalangle BDC = \alpha$, $\sphericalangle ADC = \beta$, $\sphericalangle ADB = \gamma$ *and* $\sphericalangle AD = \theta$ *for a tetrahedron* $\nabla ABCD$. *Then*

$$\cos \sphericalangle AD = \frac{\cos\alpha - \cos\beta\cos\gamma}{\sin\beta\sin\gamma} \quad \text{and} \quad \cos \sphericalangle \alpha = \frac{\cos\sphericalangle AD + \cos\sphericalangle BD \cos\sphericalangle CD}{\sin\sphericalangle BD \sin\sphericalangle CD}.$$

Additional tools to prove Theorem 1 are the next three lemmas.

Lemma 4. *Let* $\nabla ABCD$ *be a tetrahedron. Then* $\sphericalangle DAB < \sphericalangle DAC + \sphericalangle CAB$.

Proof. Let $\sphericalangle DAC = \alpha$, $\sphericalangle CAB = \beta$, and $\sphericalangle DAB = \theta$. We have to show that $\theta < \alpha + \beta$. If $\alpha + \beta \geq \pi$, then $\alpha + \beta > \theta$ since $0 < \theta < \pi$. We assume that $\alpha + \beta < \pi$. By Lemma 3, we have $\cos \angle AC = \frac{\cos\theta - \cos\alpha\cdot\cos\beta}{\sin\alpha\cdot\sin\beta}$. Hence,

$$\begin{aligned}\frac{\cos\theta - \cos\alpha\cdot\cos\beta}{\sin\alpha\cdot\sin\beta} &= \cos\angle AC > -1 \\ &= \frac{\cos\alpha\cdot\cos\beta - \sin\alpha\cdot\sin\beta - \cos\alpha\cdot\cos\beta}{\sin\alpha\cdot\sin\beta} = \frac{\cos(\alpha+\beta) - \cos\alpha\cdot\cos\beta}{\sin\alpha\cdot\sin\beta}.\end{aligned}$$

This implies that $\cos\theta > \cos(\alpha+\beta)$ since $\sin\alpha\cdot\sin\beta > 0$. Since $0 < \alpha+\beta < \pi$, and since the cosine function is decreasing on the interval $[0,\pi]$, $\cos\theta > \cos(\alpha+\beta)$ implies that $\theta < \alpha + \beta$ or $\sphericalangle DAB < \sphericalangle DAC + \sphericalangle CAB$. □

Lemma 5. *The four-angle sum of a non-planar skew quadrilateral is less than 2π.*

Proof. Suppose SQ($ABCD$) is non-planar. Hence, SQ($ABCD$) generates $\nabla ABCD$. By Lemma 4, we have $\angle DAB < \angle DAC + \angle CAB$ and $\angle BCD < \angle BCA + \angle ACD$. Then

$$\begin{aligned}\angle A + \angle B + \angle C + \angle D &= \angle DAB + \angle ABC + \angle BCD + \angle CDA \\ &< (\angle DAC + \angle CAB) + \angle ABC + (\angle BCA + \angle ACD) + \angle CDA \\ &= (\angle CDA + \angle DAC + \angle ACD) + (\angle ABC + \angle BCA + \angle CAB).\end{aligned}$$

Since $\angle CDA + \angle DAC + \angle ACD$ and $\angle ABC + \angle BCA + \angle CAB$ are the angle sums of triangles ΔACD and ΔABC, respectively, we have

$$\angle A + \angle B + \angle C + \angle D < (\angle CDA + \angle DAC + \angle ACD) + (\angle ABC + \angle BCA + \angle CAB) = 2\pi.$$

This proves the lemma. □

Lemma 6. *If the sum of some three angles of a skew quadrilateral is equal to the remaining fourth angle, then the skew quadrilateral is a (planar) quadrilateral.*

Proof. Suppose SQ($ABCD$) is a skew quadrilateral such that $\sphericalangle A + \sphericalangle B + \sphericalangle C = \sphericalangle D$. Let $\sphericalangle DAB = \alpha$, $\sphericalangle ABD = \beta_1$, $\sphericalangle DBC = \beta_2$, $\sphericalangle BCD = \gamma$, $\sphericalangle ADB = \delta_1$, $\sphericalangle CDB = \delta_2$, and $\sphericalangle CDA = \theta$. The dihedral angle $\sphericalangle BD$ of $\nabla ABCD$ at the edge BD is give by

$$\cos \sphericalangle BD = \frac{\cos\theta - \cos\delta_1 \cos\delta_2}{\sin\delta_1 \sin\delta_2}.$$

We will show that $\sphericalangle BD = \pi$ so that $\triangle ABD$ and $\triangle CBD$ are on the same plane. Since $\sphericalangle A + \sphericalangle B + \sphericalangle C = \sphericalangle D$ and $\sphericalangle B = \beta_1 + \beta_2$, we have $\alpha + \beta_1 + \beta_2 + \gamma = \theta$.

We have $\delta_1 = \pi - (\beta_1 + \alpha)$ and $\delta_2 = \pi - (\beta_2 + \gamma)$ from ΔADB and ΔCDB. Hence,

$$\begin{aligned}\cos\delta_1 &= \cos[\pi - (\beta_1+\alpha)] = -\cos(\beta_1+\alpha), &\quad \cos\delta_2 &= \cos[\pi - (\beta_2+\gamma)] = -\cos(\beta_2+\gamma),\\ \sin\delta_1 &= \sin[\pi - (\beta_1+\alpha)] = \sin(\beta_1+\alpha), \quad \text{and} &\quad \sin\delta_2 &= \sin[\pi - (\beta_2+\gamma)] = \sin(\beta_2+\gamma),\end{aligned}$$

Thus, $\cos\theta - \cos\delta_1\cos\delta_2 = \cos[(\beta_1+\alpha) + (\beta_2+\gamma)] - \cos(\beta_1+\alpha)\cos(\beta_2+\gamma) = -\sin(\beta_1 + \alpha)\sin(\beta_2+\gamma) = -\sin\delta_1\sin\delta_2$. Therefore,

$$\cos \sphericalangle BD = \frac{\cos\theta - \cos\delta_1\cos\delta_2}{\sin\delta_1\sin\delta_2} = \frac{-\sin\delta_1\sin\delta_2}{\sin\delta_1\sin\delta_2} = -1.$$

This implies that $\sphericalangle BD = \pi$. Hence, $\triangle ABD$ and $\triangle CBD$ are on the same plane so that SQ($ABCD$) is a quadrilateral. □

Now we have our first theorem.

Theorem 1. *A skew quadrilateral is a (planar) quadrilateral if, and only if, the sum of four angles is 2π or the sum of three angles is equal to the fourth angle.*

Proof. By Lemmas 1 and 6, we only have to prove that if the sum of four angles is 2π, then the skew quadrilateral is quadrilateral. Suppose a skew quadrilateral has the four-angle sum is 2π. Then it cannot be a non-planar skew quadrilateral by Lemma 5, i.e., it has to be a quadrilateral. This proves the theorem. □

Corollary 1. *Suppose* SQ$(ABCD)$ *is non-planar and equal-angled. Then* $\sphericalangle A = \sphericalangle B = \sphericalangle C = \sphericalangle D < \frac{\pi}{2}$.

Proof. This is because $\sphericalangle A + \sphericalangle B + \sphericalangle C + \sphericalangle D < 2\pi$. □

Corollary 2. *Support* SQ$(ABCD)$ *is a skew quadrilateral such that* $\sphericalangle A = \sphericalangle B = \sphericalangle C = \sphericalangle D = \frac{\pi}{2}$. *Then* SQ$(ABCD)$ *is a rectangle.*

Proof. Since $\sphericalangle A + \sphericalangle B + \sphericalangle C + \sphericalangle D = 2\pi$, SQ$(ABCD)$ must be a quadrilateral having each angle being $\frac{\pi}{2}$ so that it is Q$(ABCD)$, and Q$(ABCD)$ is a rectangle. □

Theorem 2. *For a non-planar skew quadrilateral, the sum of four angles is less than* 2π*, and the sum of three angles is not equal to the fourth angle.*

Proof. By Lemma 5, the sum of four angles of a skew quadrilateral is less than 2π and the sum of three angle cannot be equal to the fourth angle by Lemma 6. □

Conjecture. The sum of three angles of a non-planar skew quadrilateral is larger than the fourth angle.

3 Angles at a Vertex of a Tetrahedron

A tetrahedron may have four obtuse triangular faces, but Theorem 3 shows that four obtuse angles cannot be distributed to four distinct vertices.

Theorem 3. *A tetrahedron must have a vertex with all three angles acute.*

Proof. On the contrary to the theorem, suppose $\nabla ABCD$ does not have a vertex with three acute angles. A triangle has at most one obtuse angle. Hence, $\nabla ABCD$ is a tetrahedron having four obtuse angles, one obtuse angle at each vertex. First, we establish a simple way to denote twelve angles of $\nabla ABCD$. Since the angle $\sphericalangle ABC$ is on the plane ABC (not containing the point D) at the vertex B, we denote $\sphericalangle ABC$ by D_B. In a similar way, we can denote the twelve angles of $\nabla ABCD$ by A_B, A_C, A_D; B_A, B_C, B_D; C_A, C_B, C_D; D_A, D_B, D_C. (Note that there are NO angles of the forms A_A, B_B, C_C, D_D).

Without loss of generality, we assume that A_B is an obtuse angle of $\nabla ABCD$. We have to select the sets of all possible four obtuse angles of $\nabla ABCD$ on each face at each vertex. Keeping in mind that one obtuse angle is on a face at each vertex, a set of four obtuse angles is of the form $\{A_B, B_X, C_Y, D_Z\}$, where X, Y, Z are distinct elements in the set $\{A, C, D\}$. This enables us to find these sets to be

$$\{A_B, B_A, C_D, D_C\}, \quad \{A_B, B_C, C_D, D_A\}, \quad \text{and} \quad \{A_B, B_D, C_A, D_C\}.$$

Case 1: Suppose $\nabla ABCD$ has obtuse angles A_B, B_A, C_D, D_C. Since $C_D = \angle ADB$, $A_B = \angle DBC$, $D_C = \angle BCA$, and $B_A = \angle CAD$, the set of obtuse angles $\{A_B, B_A, C_D, D_C\}$ are the angles of the non-planar skew quadrilateral SQ$(ADBC)$. But this is impossible by Theorem 2.

Case 2: Suppose $\nabla ABCD$ has obtuse angles A_B, B_C, C_D, D_A. (These are not angles of a skew quadrilateral contained in $\nabla ABCD$.)

$A_B \geq \frac{\pi}{2}$ implies that $|CD| > |BC|$ and $|CD| > |BD|$ from $\triangle BCD$.
$B_C \geq \frac{\pi}{2}$ implies that $|AD| > |AC|$ and $|AD| > |CD|$ from $\triangle ACD$.
$D_A \geq \frac{\pi}{2}$ implies that $|BC| > |AB|$ and $|BC| > |AC|$ from $\triangle ABC$.
$C_D \geq \frac{\pi}{2}$ implies that $|AB| > |AD|$ and $|AB| > |BD|$ from $\triangle ABD$.
Hence, $|BC| > |AB| > |AD| > |CD| > |BC|$. This is a contradiction.

Case 3: Suppose $\nabla ABCD$ has obtuse angles A_B, B_D, C_A, D_C. This is similar to Case 2.

$A_B \geq \frac{\pi}{2}$ implies that $|CD| > |BC|$ and $|CD| > |BD|$ from $\triangle BCD$.
$B_D \geq \frac{\pi}{2}$ implies that $|AC| > |AD|$ and $|AC| > |CD|$ from $\triangle ACD$.
$D_C \geq \frac{\pi}{2}$ implies that $|AB| > |AC|$ and $|AB| > |BC|$ from $\triangle ABC$.
$C_A \geq \frac{\pi}{2}$ implies that $|BD| > |AB|$ and $|BD| > |AD|$ from $\triangle ABD$.

Hence, $|AB| > |AC| > |CD| > |BD| > |AB|$. This is a contradiction.

From these three cases, we can see that $\nabla ABCD$ cannot have four obtuse angles on each face. Therefore, a tetrahedron must have a vertex with all three angles there being acute. □

Example 2. We construct a tetrahedron having four non-acute angles at three vertices. Let $\nabla ABCD$ be a tetrahedron such that $|AB| = 4$, $|BC| = 3$, $|AC| = 5$, and $|DA| = 10$, $|DB| = 13$, $|DC| = 12$. Then the volume V of $\nabla ABCD$ is given by $144V^2 = 13832$ and the tetrahedron exists.

$$|AB|^2 + |BC|^2 = 25 = |AC|^2, \quad |DA|^2 + |AB|^2 = 116 < 169 = |DB|^2$$
$$|DC|^2 + |CB|^2 = 153 < 169 = |DB|^2, \quad \text{and} \quad |DA|^2 + |AC|^2 = 125 < 144 = |DC|^2.$$

Therefore, $\sphericalangle ABC = \frac{\pi}{2}$ and $\sphericalangle BAD$, $\sphericalangle BCD$, $\sphericalangle CAD > \frac{\pi}{2}$ so that $\nabla ABCD$ is a tetrahedron having one right angle at the vertex B, two obtuse angles at A, and one obtuse angle at C. Three angles at D are acute angles.

Corollary 3. *A face of an isosceles tetrahedron has to be an acute triangle.*

Proof. Let $\nabla ABCD$ be an isosceles tetrahedron. Since four faces are congruent, it suffices to prove that $\triangle ABC$ is an acute triangle. Let $\alpha = \sphericalangle CAB, \beta = \sphericalangle ABC,$ and $\gamma = \sphericalangle BCA$. Since $\nabla ABCD$ is isosceles, each vertex has angles α, β, γ. By Theorem 3, α, β, γ must be acute. Hence, $\triangle ABC$ is an acute triangle. □

Corollary 3 is not new, but we could not find any references. Two alternate proofs of Corollary 3 will be given in Remark 2 at the end.

4 A Tetrahedron Having an Equal-Angled Skew Quadrilateral

The next theorem was motivated by Example 1 in the introduction.

Theorem 4. *If a tetrahedron contains an equal-angled skew quadrilateral, then the tetrahedron must be reversible.*

Proof. Suppose SQ($ABCD$) is an equal-angled (non-planar) skew quadrilateral contained in a tetrahedron $\nabla ABCD$. Let $\sphericalangle DAB = \sphericalangle ABC = \sphericalangle BCD = \sphericalangle CDA = \theta > 0$.

Let $|AB| = a, |CD| = x$ and $|BC| = b, |DA| = y$. Then by the law of cosines, we have
(1) $|BD|^2 = a^2 + y^2 - 2ay\cos\theta = b^2 + x^2 - 2bx\cos\theta$ and
(2) $|AC|^2 = a^2 + b^2 - 2ab\cos\theta = x^2 + y^2 - 2xy\cos\theta$.

$$|BD|^2 = a^2 + y^2 - 2ay\cos\theta = b^2 + x^2 - 2bx\cos\theta \quad \text{and} \tag{1}$$
$$|AC|^2 = a^2 + b^2 - 2ab\cos\theta = x^2 + y^2 - 2xy\cos\theta. \tag{2}$$

We will consider three cases when ($ay - bx \neq 0$ and $ab - xy \neq 0$), or when $ay - bx = 0$, or when $ab - xy = 0$.

Case 1: Suppose $ay - bx \neq 0$ and $ab - xy \neq 0$. From Equations (1) and (2), we have $\frac{a^2+y^2-b^2-x^2}{2(ay-bx)} = \cos\theta = \frac{a^2+b^2-x^2-y^2}{2(ab-xy)}$.

$$(a^2 + y^2 - b^2 - x^2)(ab - xy) = (a^2 + b^2 - x^2 - y^2)(ay - bx).$$

Multiplications gives us

$$a^3b + aby^2 - ab^3 - abx^2 - a^2xy - xy^3 + b^2xy + x^3y \\ = a^3y + ab^2y - ax^2y - ay^3 - a^2bx - b^3x + bx^3 + bxy^2.$$

Factoring terms $(a - x)$ and $(b - y)$ gives us

$$-(a - x)(b^3 - y^3 + b^2y - by^2) + (b - y)(a^3 - x^3 + a^2x - ax^2) = 0.$$

And, finally, we have:

$$[(a + x)^2 - (b + y)^2](a - x)(b - y) = 0.$$

Hence, we have $a = x$, or $b = y$, or $a + x = b + y$ since a, b, x, $y > 0$.

Subcase 1.1: Suppose $a = x$. From (1), we have:

$$y^2 - 2ay\cos\theta = b^2 - 2ab\cos\theta.$$

This factors as:

$$(b - y)\,[2a\cos\theta - (b + y)] = 0.$$

Hence, $b = y$ or $2a\cos\theta - (b + y) = 0$.
If $b = y$, then $a = x$ and $b = y$. In this case, SQ$(ABCD)$ is reversible.
So, suppose $2a\cos\theta - (b + y) = 0$. Then $\cos\theta = \frac{b+y}{2a}$.
Then from (1), we have:

$$|BD|^2 = a^2 + y^2 - 2ay\cos\theta = a^2 + y^2 - \cancel{2a} \cdot y \cdot \frac{b + y}{\cancel{2a}} = a^2 - by. \tag{3}$$

And from (2), we have:

$$|AC|^2 = a^2 + b^2 - 2ab\cos\theta = a^2 + b^2 - \cancel{2a} \cdot b \cdot \frac{b + y}{\cancel{2a}} = a^2 - by. \tag{4}$$

Note that $1 > \cos\theta = \frac{b+y}{2a} \geq \frac{2\sqrt{bc}}{2a} = \frac{\sqrt{bc}}{a}$, so that $1 > \frac{\sqrt{bc}}{a}$ or $a^2 - by > 0$.
Hence, we have $|AC|^2 = |BD|^2$ from (3) and (4). Thus, $|AC| = |DB|$.
Since we also have $|AB| = a = x = |CD|$, SQ$(ACDB)$ is reversible.
Therefore, if $a = x$, then either SQ$(ABCD)$ or SQ$(ACDB)$ is reversible.

Subcase 1.2: Suppose $b = y$. This is similar to Subcase 1.1, and we can show that either SQ$(ABCD)$ or SQ$(ACBD)$ is reversible.

Subcase 1.3: Suppose $a + x = b + y$. Then $a - y = b - x$. By (1), we have

$$|BD|^2 = a^2 + y^2 - 2ay\cos\theta = (a - y)^2 + 2ay - 2ay\cos\theta \\ = (b - x)^2 + 2ay - 2ay\cos\theta = b^2 + x^2 - 2bx + 2ay - 2ay\cos\theta.$$

By (1) again, we also have $|BD|^2 = b^2 + x^2 - 2bx\cos\theta$.

Hence, $b^2 + x^2 - 2bx + 2ay - 2ay\cos\theta = b^2 + x^2 - 2bx\cos\theta$.
This simplifies to: $(1-\cos\theta)(bx - ay) = 0$.
Since $\cos\theta \neq 1$, we must have

$$bx = ay. \tag{5}$$

Similarly, $a + x = b + y$ implies $a - b = y - x$. By (2), we have

$$\begin{aligned}|AC|^2 &= a^2 + b^2 - 2ab\cos\theta = (a-b)^2 + 2ab - 2ab\cos\theta \\ &= (y-x)^2 + 2ab - 2ab\cos\theta = x^2 + y^2 - 2xy + 2ab - 2ab\cos\theta.\end{aligned}$$

Since we also have

$$|AC|^2 = x^2 + y^2 - 2xy\cos\theta$$

by (2),

$$x^2 + y^2 - 2xy + 2ab - 2ab\cos\theta = x^2 + y^2 - 2xy\cos\theta.$$

This simplifies to $(1-\cos\theta)(ab - xy) = 0$. Since $\cos\theta \neq 1$, we have

$$ab = xy. \tag{6}$$

From (5) and (6), we have $ab^2x = axy^2$ or $ax(b^2 - y^2) = 0$, so $b = y$. By (6), $a = x$. That is, we have shown that $|AB| = |CD|$ and $|BC| = |AD|$, so that SQ($ABCD$) is reversible. Therefore, in Case 1, we have shown that $\nabla ABCD$ is reversible.

Case 2: Suppose $ay - bx = 0$. Then $ay = bx$.
By (1), we have $a^2 + y^2 - 2ay\cos\theta = b^2 + x^2 - 2bx\cos\theta$. Since $ay = bx$, we have $a^2 + y^2 = b^2 + x^2$ so that $(a+y)^2 - 2ay = (b+x)^2 - 2bx$.
Again, by $ay = bx$, we must have $(a+y)^2 = (b+x)^2$ $a + y = b + x$.
Hence, $a - b = x - y$.
By (2), we have

$$\begin{aligned}|AC|^2 &= a^2 + b^2 - 2ab\cos\theta = (a-b)^2 + 2ab - 2ab\cos\theta \\ &= (x-y)^2 + 2ab - 2xy\cos\theta = x^2 + y^2 - 2xy + 2ab - 2ab\cos\theta.\end{aligned}$$

By (2), we also have $|AC|^2 = x^2 + y^2 - 2xy\cos\theta$. Hence, we have

$$x^2 + y^2 - 2xy + 2ab - 2ab\cos\theta = x^2 + y^2 - 2xy\cos\theta$$

so that

$$ab(1-\cos\theta) = xy(1-\cos\theta).$$

Since $\cos\theta \neq 1$, we have $ab = xy$.
Thus, $ay = bx$ and $ab = xy$. These give $a^2by = bx^2y$, so $a = x$ and thus $b = y$. That is, $|AB| = |CD|$ and $|BC| = |AD|$. Therefore, $\nabla ABCD$ is reversible.

Case 3: Suppose $ab - xy = 0$. This is similar to Case 2. As in Case 2, this implies $b = y$. This is, $\nabla ABCD$ is reversible.

Therefore, from Cases 1-3, we have shown that the tetrahedron $\nabla ABCD$ is reversible. □

Example 3. The converse of Theorem 4 is not true. We construct a reversible tetrahedron that does not contain an equal-angled skew quadrilateral. Let $A = (1, 0, 0)$, $B = (0, 0, 2)$, $C = (-1, 0, 0)$, and $D = (0, 2, 0)$. Then $|AB| = |BC| = |CD| = |DA| = \sqrt{5}$ so that SQ($ABCD$) is reversible so that $\nabla ABCD$ is also reversible. We can calculate that

$\cos\sphericalangle DAB = \cos\sphericalangle BCD = \frac{1}{5}$ and $\cos\sphericalangle ABC = \cos\sphericalangle CDA = \frac{3}{5}$ with the law of cosine. Hence, SQ($ABCD$) is not equal-angled. Since $\cos\sphericalangle BAC = \frac{1}{\sqrt{5}}$ and $\cos\sphericalangle CDB = \frac{2}{\sqrt{10}}$, SQ($ABDC$) is not equal-angled. Since $\cos\sphericalangle CAD = \frac{1}{\sqrt{5}}$ and $\cos\sphericalangle ADB = \frac{2}{\sqrt{10}}$, SQ($ACBD$) is not equal-angled, either. Thus, $\nabla ABCD$ is reversible, but it does not contain an equal-angled skew quadrilateral.

5 Isosceles Tetrahedra

Angles of isosceles tetrahedra were investigated in [2]. We give one additional theorem related to angles of an isosceles tetrahedron.

Lemma 7. *Suppose a tetrahedron $\nabla ABCD$ is isosceles. Then the all three skew quadrilaterals* SQ($ABCD$), SQ($ACBD$), *and* SQ($ACDB$) *are equal-angled.*

Proof. The tetrahedron $\nabla ABCD$ has four congruent triangular faces $\triangle ABC \equiv \triangle BAD \equiv \triangle DCB \equiv \triangle CDA$. Hence, $\sphericalangle ABC = \sphericalangle BAD = \sphericalangle DCB = \sphericalangle CDA$ (these are angles of SQ($ABCD$)), $\sphericalangle BCA = \sphericalangle ADB = \sphericalangle CBD = \sphericalangle DAC$ (these are angles of SQ($ACBD$)), and $\sphericalangle BAC = \sphericalangle ABD = \sphericalangle CDB = \sphericalangle DCA$ (these are angles of SQ($ACDB$)). (See Figure 3.) Therefore, this proves the lemma. □

A tetrahedron is isosceles if and only if it contains two reversible skew quadrilaterals. Theorem 5 resembles to this statement.

Theorem 5. *A tetrahedron is isosceles if, and only if, it contains two equal angled skew quadrilaterals.*

Proof. Suppose SQ($ABCD$) and SQ($ACDB$) are equal-angled in $\nabla ABCD$. Then $\sphericalangle ABC = \sphericalangle BAD = \sphericalangle DCB = \sphericalangle CDA$ and $\sphericalangle BAC = \sphericalangle ABD = \sphericalangle CDB = \sphericalangle DCA$. Since $\triangle ABC$ and $\triangle BAD$ share a common edge. So $\triangle ABC \equiv \triangle BAD$. Similarly, we can prove $\triangle BAD \equiv \triangle DCB$ and $\triangle DCB \equiv \triangle CDA$. Therefore, $\triangle ABC \equiv \triangle BAD \equiv \triangle DCB \equiv \triangle CDA$. That is, four faces of $\nabla ABCD$ are congruent, and $\nabla ABCD$ is an isosceles tetrahedron.

The converse is shown in Lemma 7. Hence, this proves the theorem. □

Remark 2. We give two alternate proofs of Corollary 3.

Proof 1. Let $\nabla ABCD$ be an isosceles tetrahedron. Then by Lemma 7, skew quadrilateral SQ($ABCD$), SQ($ACBD$) and SQ($ACDB$) have equal angles, say α, β, γ respectively. By Corollary 1, we have $0 < \alpha, \beta, \gamma < \frac{\pi}{2}$. But, $\alpha = \sphericalangle ABC$, $\beta = \sphericalangle ACB$, $\gamma = \sphericalangle BAC$, and these are angles of $\triangle ABC$. This proves Corollary 3. □

Proof 2. A tetrahedron T is isosceles if, and only if, the parallelepiped that circumscribe is a rectangular box (see [1, Theorem 291, page 94]). So let $a \times b \times c$ the dimension of the rectangular box that contains T for some $a, b, c > 0$. Then the three edges of a face of T are given by $\sqrt{a^2+b^2}$, $\sqrt{b^2+c^2}$ and $\sqrt{c^2+a^2}$. Let θ be the angle between $\sqrt{a^2+b^2}$ and $\sqrt{b^2+c^2}$. Then by the law of cosines,

$$\cos\theta = \frac{(a^2+b^2)+(b^2+c^2)-(c^2+a^2)}{2\sqrt{a^2+b^2}\sqrt{b^2+c^2}} = \frac{b^2}{\sqrt{a^2+b^2}\sqrt{b^2+c^2}} > 0.$$

Hence, θ is acute. Similarly, other angles of the triangular face can be shown to be acute. This proves Corollary 3. □

References

[1] N. Altshiller-Court: *Modern Pure Solid Geometry.* The Macmillan Co., New York, 1935.

[2] H. Katsuura: *Angles of Isosceles Tetrahedra.* J. Geom. Graph. **25**(2), 241–249, 2021.

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