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Extension of the Formula for the Sum of Fourth Powers of Distances in Regular Polytopes

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Abstract. We extend distance–power identities for regular figures by deriving formulas for the sum of fourth powers of distances from an arbitrary point to the vertices of regular polygons and polyhedra. The method employs embeddings of these configurations into higher–dimensional Euclidean spaces and systematic use of the Pythagorean theorem. Our results unify and generalize known formulas in lower dimensions, providing a broader framework for distance relations in regular polytopes.

Key Words: Regular polytope, regular polygon, Pythagorean theorem, fourth powers of distances

MSC 2020: 51M04 (primary), 51N20

1 Introduction

In his book *Mathematical Circus* (see [4, 5, 8]), Martin Gardner presented an elegant formula relating the sum of the fourth powers and the sum of the second powers of the distances from any point to the vertices of an equilateral triangle. The formula can be stated as follows:

Theorem 1. *Given an equilateral triangle ABC with side length a and a point P in the plane containing the triangle, the following equation holds (See Figure 1):*

$$3(|PA|^4 + |PB|^4 + |PC|^4 + a^4) = (|PA|^2 + |PB|^2 + |PC|^2 + a^2)^2.$$

This theorem was later generalized by John Bentin (see [1]) to regular simplices as follows:

Theorem 2 (Bentin (1995))**.** *Let $\mathcal{A} = A_0A_1 \dots A_n$ be a regular simplex with edge length a in $\mathbb{E}^n$. Let P be any point in $\mathbb{E}^n$. (See Figure 2). Then,*

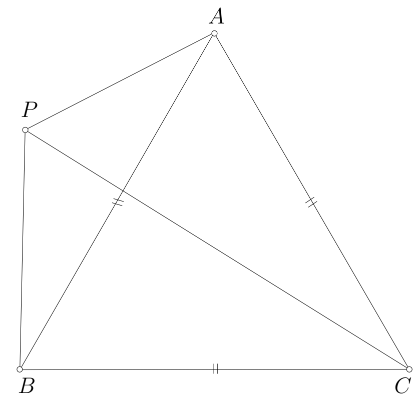

Figure 1: Equilateral triangle in plane with arbitrary point.

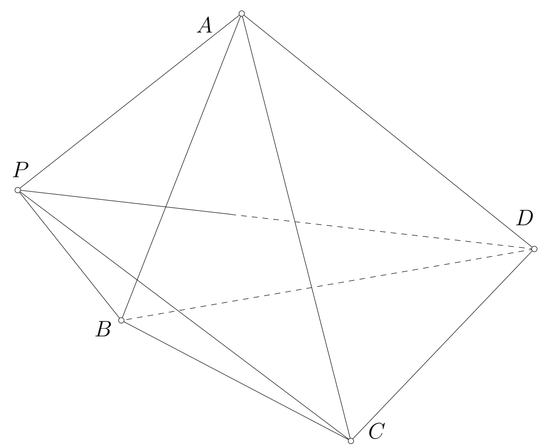

Figure 2: A regular tetrahedron in 3D space with an arbitrary point.

$$(n+1)\left(\sum_{i=0}^{n}|PA_i|^4 + a^4\right) = \left(\sum_{i=0}^{n}|PA_i|^2 + a^2\right)^2. \tag{1}$$

John Bentin also extended Theorem 1 to regular polygons in the plane, as follows (see [2]):

Theorem 3 (Bentin (1997))**.** *Consider a regular polygon with at least two vertices in the plane and a general point P. Let r be the radius of the polygon from its center to any vertex, and let s^2 and q^4 denote the averages of the squares and fourth powers of the distances from P to the vertices of the polygon, respectively. (See Figure 3). Then,*

$$q^4 + 3r^4 = (s^2 + r^2)^2. \tag{2}$$

Prior to discussing the extension proposed by Poo-Sung Park, we shall briefly review the classification of regular polytopes in Euclidean space $\mathbb{E}^n$ for $n \geq 2$.

The systematic study of regular polytopes was brought to its classical form by H. S. M. Coxeter in his influential monograph Regular Polytopes, [3]. A regular polytope in $\mathbb{E}^n$ is defined as a highly symmetric convex figure whose symmetry group is transitive on its flags. This property ensures that all facets are congruent regular polytopes of one dimension lower, and that the entire structure exhibits the highest degree of uniformity attainable in the given dimension.

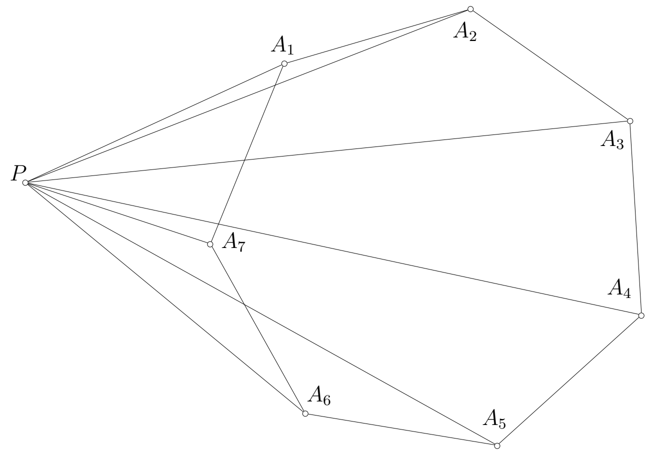

Figure 3: A regular heptagon in plane with an arbitrary point.

In two dimensions, regular polytopes reduce to the familiar family of regular polygons, of which there are infinitely many. In three dimensions, the classification yields the five Platonic solids, the tetrahedron, cube, octahedron, dodecahedron, and icosahedron. Coxeter showed that the situation changes dramatically in higher dimensions. In four-dimensional Euclidean space, there exist six convex regular polytopes: the four-dimensional analogues of the simplex, cube, and cross-polytope (often called the orthoplex or hyperoctahedron), together with three exceptional figures, the 24-cell, the 120-cell, and the 600-cell.

For every dimension $n \geq 5$, the constraints of Euclidean geometry permit only three infinite families of regular polytopes:

1. The simplex $\{3, 3, \ldots, 3\}$, the generalization of the triangle and the tetrahedron, consisting of $n + 1$ vertices.
2. The hypercube $\{4, 3, \ldots, 3\}$, the n-dimensional analogue of the cube.
3. The cross-polytope $\{3, 3, \ldots, 4\}$, also known as the orthoplex or hyperoctahedron, which is dual to the hypercube.

The Schläfli symbol, introduced by Schläfli and elaborated by Coxeter in Regular Polytopes, provides a concise notation for describing regular polytopes in $\mathbb{E}^n$, including the simplex $3, 3, \ldots, 3$, hypercube $4, 3, \ldots, 3$, and cross-polytope $3, 3, \ldots, 4$.

Thus Coxeter's classification reveals a striking pattern: infinite richness in two dimensions, finite yet varied regularity in three and four dimensions, and, beyond that, only three universal families that persist in all higher dimensions.

Poo-Sung Park further generalized Theorem 1 to three types of regular polytopes in $\mathbb{E}^n$ $(n \geq 2)$—the regular simplex, the n-cube, and the n-orthoplex (see [7]):

Theorem 4 (Poo-Sung Park (2016)). *Let $\mathcal{M}$ be an n-dimensional regular polytope (a simplex, hypercube, or orthoplex) with circumscribed radius r. If q^4 is the average of the fourth powers of the distances from a point to the vertices of $\mathcal{M}$, and s^2 is the average of the squares of those distances. (See Figure 5). Then:*

$$q^4 + \frac{4(n+1)}{n^2} r^4 = \left(s^2 + \frac{2}{n} r^2\right)^2. \tag{3}$$

In the case where $\mathcal{M}$ is an n-dimensional regular simplex, formula (3) reduces to formula (1).

Table 1: Detailed list of regular polytopes in $\mathbb{E}^n$ $(n \geq 2)$

n	Schläfli	Name	Vertices	Dual	Remarks
2	$\{p\}$	Regular polygon	p	Self-dual	$p \geq 3$
3	$\{3,3\}$	Tetrahedron (3-simplex)	4	Self-dual	
	$\{4,3\}$	Cube	8	$\{3,4\}$	
	$\{3,4\}$	Octahedron	6	$\{4,3\}$	
	$\{5,3\}$	Dodecahedron	20	$\{3,5\}$	
	$\{3,5\}$	Icosahedron	12	$\{5,3\}$	
4	$\{3,3,3\}$	4-simplex (5-cell)	5	Self-dual	
	$\{4,3,3\}$	4-cube (tesseract)	16	$\{3,3,4\}$	
	$\{3,3,4\}$	16-cell (orthoplex)	8	$\{4,3,3\}$	
	$\{3,4,3\}$	24-cell	24	Self-dual	Exceptional in 4D
	$\{5,3,3\}$	120-cell	600	$\{3,3,5\}$	Exceptional in 4D
	$\{3,3,5\}$	600-cell	120	$\{5,3,3\}$	Exceptional in 4D
$n \geq 5$	$\{\underbrace{3,3,\ldots,3}_{n-1}\}$	n-simplex	$n+1$	Self-dual	Group A_n
	$\{4,\underbrace{3,\ldots,3}_{n-2}\}$	n-cube (hypercube)	2^n	$\{3,\ldots,4\}$	Group B_n
	$\{\underbrace{3,\ldots,3}_{n-2},4\}$	n-orthoplex (cross-polytope)	$2n$	$\{4,3,\ldots,3\}$	Group B_n

This paper aims to extend Theorems 2 and 4 to cases where P lies in $\mathbb{E}^m$ $(m \geq n)$, and the regular polytope (a simplex, hypercube, or orthoplex) lies in $\mathbb{E}^n \subset \mathbb{E}^m$, where $\mathbb{E}^n$ is viewed as an n-dimensional subspace of $\mathbb{E}^m$. We extend Theorem 2 similarly as follows:

Theorem 5 (Extension of Theorem 2)**.** *In m-dimensional space $\mathbb{E}^m$, consider the subspace $\mathcal{P} = \mathbb{E}^n$ $(n \leq m)$ containing the n-simplex $\mathcal{A} = A_0A_1\ldots A_n$ with edge length a. Let P be any point in $\mathbb{E}^m$. (See Figure 6). Then,*

$$(n+1)\Big(\sum_{i=0}^{n}|PA_i|^4 + a^4 + 2h^2a^2\Big) = \Big(\sum_{i=0}^{n}|PA_i|^2 + a^2\Big)^2,$$

where h is the distance from P to the n-plane $\mathcal{P}$.

When P lies in $\mathcal{P}$, we have $h = 0$, which reduces the equation to Theorem 2.

When $m = 3$ and $n = 2$, Theorem 5 reduces to following Theorem 6. Here, the equilateral triangle ABC lies in a plane $\mathcal{P}$ in space, and P is any point in space. The following theorem states this result:

Theorem 6 (A generalization of Theorem 1 to three-dimensional space)**.** *In three-dimensional space, let ABC be an equilateral triangle with side length a, lying in a plane $\mathcal{P}$ (see Figure 4). Let P be any point in space. Then,*

$$3(|PA|^4 + |PB|^4 + |PC|^4 + a^4 + 2h^2a^2) = (|PA|^2 + |PB|^2 + |PC|^2 + a^2)^2,$$

where h is the distance from P to the plane $\mathcal{P}$.

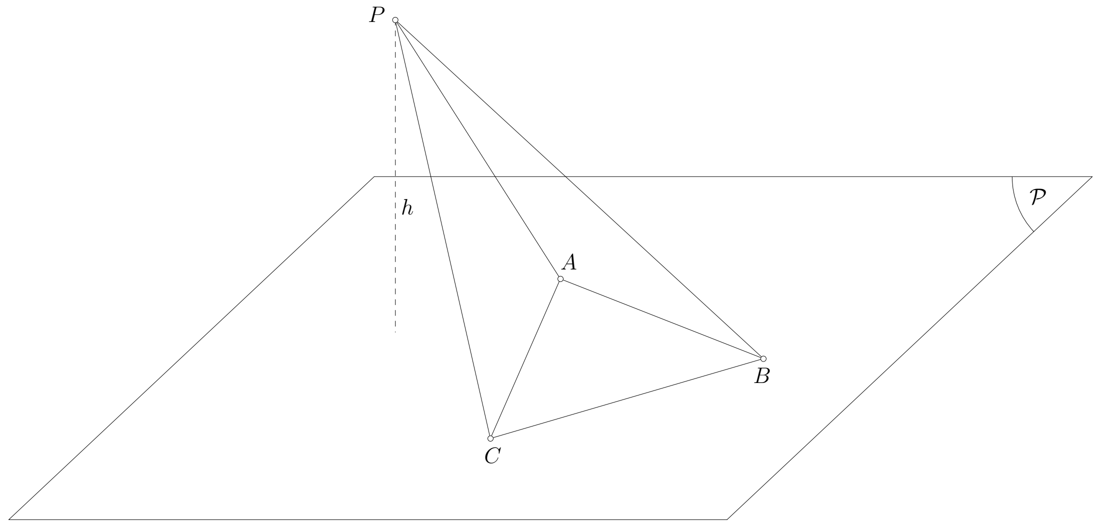

Figure 4: An equilateral triangle in plane $\mathcal{P}$ embedded in 3D space with an arbitrary point in 3D space.

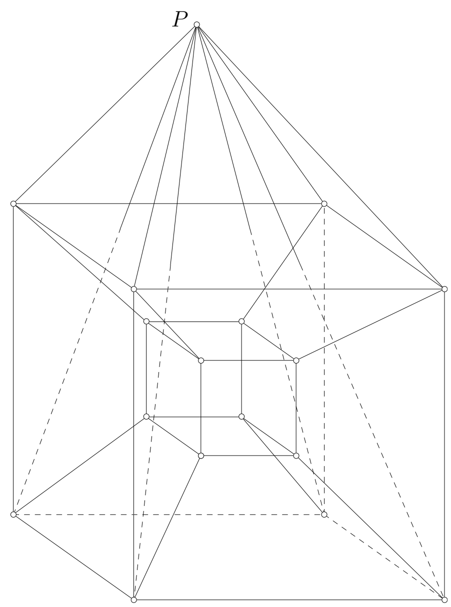

Figure 5: A tesseract in 4D space with an arbitrary point.

Theorem 7 (Extension of Theorem 4)**.** *In m-dimensional space $\mathbb{E}^m$, consider the subspace $\mathcal{P} = \mathbb{E}^n$ $(n \leq m)$ containing the regular polytope (a simplex, hypercube, or orthoplex) $\mathcal{M} = A_1 \ldots A_k$. Let r be the distance from center of $\mathcal{M}$ to any its vertex (which is known as its circumradius). Let P be any point in $\mathbb{E}^m$. Then*

$$\frac{1}{k}\left(\sum_{i=1}^{k} PA_i^4\right) + \frac{4(n+1)}{n^2}r^4 + \frac{4h^2r^2}{n} = \left(\frac{1}{k}\left(\sum_{i=1}^{k} PA_i^2\right) + \frac{2}{n}r^2\right)^2, \tag{4}$$

where h is the distance from P to the n-plane $\mathcal{P}$. (See Figure 5).

When P lies on $\mathcal{P}$, we get $h = 0$, Theorem 7 provides another version of Theorem 4.

Finally, we note that Theorem 3 can also be extended to three-dimensional space as follows:

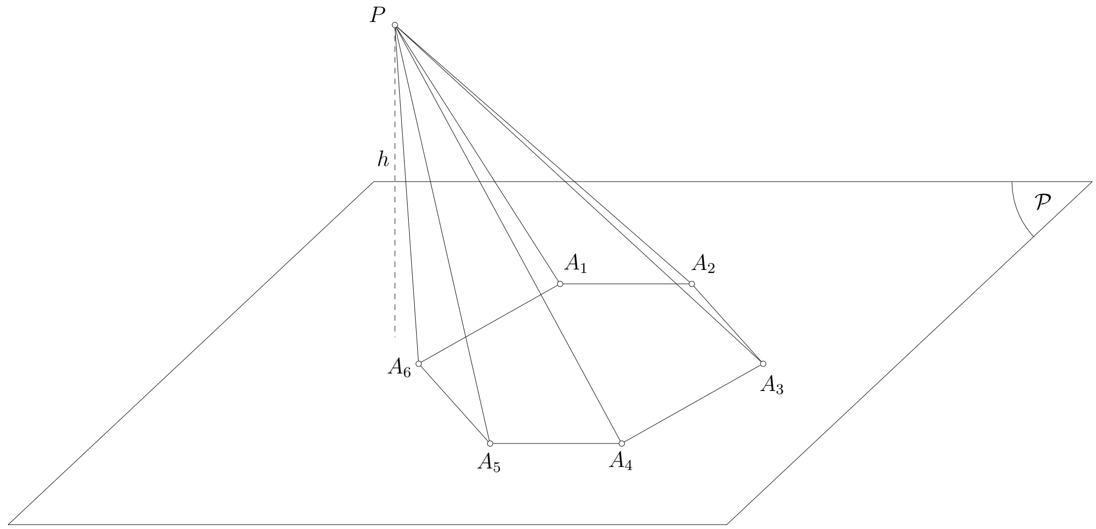

Figure 6: Regular hexagon in plane $\mathcal{P}$ embedded in 3D space with an arbitrary point in 3D space.

Theorem 8 (Extension of Theorem 3)**.** *Consider a regular polygon $\mathcal{A} = A_1 \dots A_k$ $(k \geq 1)$ lying in a plane $\mathcal{P}$ within three-dimensional space $\mathbb{E}^3$, and P is an arbitrary point in $\mathbb{E}^3$. Let r be the distance from center of $\mathcal{A}$ to any its vertex (which is known as its circumradius). (See Figure 6). Then,*

$$\frac{1}{k}\Big(\sum_{i=1}^{k} PA_i^4\Big) + 3r^4 + 2h^2r^2 = \Big(\frac{1}{k}\Big(\sum_{i=1}^{k} PA_i^2\Big) + r^2\Big)^2, \tag{5}$$

where h is the distance from P to the plane $\mathcal{P}$.

2 Proof of the Theorems

Proof of Theorem 5. When $n \leq m$, the subspace $\mathcal{P} = \mathbb{E}^n$ can be regarded as a linear subspace of $\mathbb{E}^m$, effectively serving as an n-dimensional plane within $\mathbb{E}^m$. Let H denote the orthogonal projection of P onto $\mathcal{P}$. By Theorem 2, we have:

$$(n+1)\Big(\sum_{i=0}^{n} |HA_i|^4 + a^4\Big) = \Big(\sum_{i=0}^{n} |HA_i|^2 + a^2\Big)^2. \tag{6}$$

Since H is the orthogonal projection of P onto $\mathcal{P}$, the distance from P to $\mathcal{P}$ is $|PH| = h$. By the Pythagorean theorem, we have:

$$|PA_i|^2 = |HA_i|^2 + h^2, \quad (i = 0, \dots, n). \tag{7}$$

Substituting (7) into (6), we obtain:

$$(n+1)\Big(\sum_{i=0}^{n}(|PA_i|^2 - h^2)^2 + a^4\Big) = \Big(\sum_{i=0}^{n}(|PA_i|^2 - h^2) + a^2\Big)^2. \tag{8}$$

We now expand both sides of (8) as follows:

$$(n+1)\Big(\sum_{i=0}^{n}(|PA_i|^2-h^2)^2+a^4\Big)=(n+1)\Big(\sum_{i=0}^{n}|PA_i|^4-2h^2\sum_{i=0}^{n}|PA_i|^2+(n+1)h^4+a^4\Big)$$
$$=(n+1)\Big(\sum_{i=0}^{n}|PA_i|^4+a^4\Big)-2(n+1)h^2\sum_{i=0}^{n}|PA_i|^2+(n+1)^2h^4. \quad (9)$$

On the other hand:

$$\Big(\sum_{i=0}^{n}(|PA_i|^2-h^2)+a^2\Big)^2=\Big(\sum_{i=0}^{n}|PA_i|^2+a^2-(n+1)h^2\Big)^2$$
$$=\Big(\sum_{i=0}^{n}|PA_i|^2+a^2\Big)^2-2(n+1)h^2\Big(\sum_{i=0}^{n}|PA_i|^2+a^2\Big)+(n+1)^2h^4. \quad (10)$$

From (8), (9), and (10), we deduce that:

$$(n+1)\Big(\sum_{i=0}^{n}|PA_i|^4+a^4+2h^2a^2\Big)=\Big(\sum_{i=0}^{n}|PA_i|^2+a^2\Big)^2. \quad (11)$$

This completes the proof of Theorem 5. □

Proof of Theorem 7. Let H be the orthogonal projection of P onto $\mathcal{P}$, where the distance from P to $\mathcal{P}$ is $|PH|=h$. By the Pythagorean theorem, we have:

$$|PA_i|^2=|HA_i|^2+h^2, \quad (i=1\dots k). \quad (12)$$

Since H lies on $\mathbb{E}^n$, also by Theorem 4, we have:

$$\frac{1}{k}\Big(\sum_{i=1}^{k}|HA_i|^4\Big)+\frac{4(n+1)}{n^2}r^4=\Big(\frac{1}{k}\Big(\sum_{i=1}^{k}|PA_i|^2\Big)+\frac{2}{n}r^2\Big)^2. \quad (13)$$

Using (12) to transform both sides of (13), we obtain:

$$\frac{1}{k}\Big(\sum_{i=1}^{k}|HA_i|^4\Big)+\frac{4(n+1)}{n^2}r^4=\frac{1}{k}\sum_{i=1}^{k}\Big(|PA_i|^2-h^2\Big)^2+\frac{4(n+1)}{n^2}r^4$$
$$=\frac{1}{k}\Big(\sum_{i=1}^{k}|PA_i|^4\Big)-2h^2\cdot\frac{1}{k}\Big(\sum_{i=1}^{k}|PA_i|^2\Big)+h^4+\frac{4(n+1)}{n^2}r^4. \quad (14)$$

Meanwhile:

$$\Big(\frac{1}{k}\Big(\sum_{i=1}^{k}|HA_i|^2\Big)+\frac{2}{n}r^2\Big)^2=\Big(\frac{1}{k}\Big(\sum_{i=1}^{k}|PA_i|^2\Big)-h^2+\frac{2}{n}r^2\Big)^2$$
$$=\Big(\frac{1}{k}\Big(\sum_{i=1}^{k}|PA_i|^2\Big)+\frac{2}{n}r^2\Big)^2-2h^2\Big(\frac{1}{k}\Big(\sum_{i=1}^{k}|PA_i|^2\Big)+\frac{2}{n}r^2\Big)+h^4$$
$$=\Big(\frac{1}{k}\Big(\sum_{i=1}^{k}|PA_i|^2\Big)+\frac{2}{n}r^2\Big)^2-2h^2\cdot\frac{1}{k}\Big(\sum_{i=1}^{k}|PA_i|^2\Big)+h^4-\frac{4h^2r^2}{n}. \quad (15)$$

Combining (13), (14), and (15), we get:

$$\frac{1}{k}\Big(\sum_{i=1}^{k}|PA_i|^4\Big)+\frac{4(n+1)}{n^2}r^4+\frac{4h^2r^2}{n}=\Big(\frac{1}{k}\Big(\sum_{i=1}^{k}|PA_i|^2\Big)+\frac{2}{n}r^2\Big)^2. \tag{16}$$

This completes the proof of Theorem 7. □

Proof of Theorem 8. Let H denote the orthogonal projection of P onto the plane $\mathcal{P}$. According to Theorem 3, we have

$$\frac{1}{k}\Big(\sum_{i=1}^{k}|HA_i|^4\Big)+3r^4=\Big(\frac{1}{k}\Big(\sum_{i=1}^{k}|HA_i|^4\Big)+r^2\Big)^2. \tag{17}$$

Noting that $|PA_i|^2=|HA_i|^2+h^2$, for $i=1,\dots,n$ (by the Pythagorean theorem), we can transform both sides of Equation (17) as follows:

$$\begin{aligned}\frac{1}{k}\Big(\sum_{i=1}^{k}|HA_i|^4\Big)+3r^4&=\frac{1}{k}\sum_{i=1}^{k}(|PA_i|^2-h^2)^2+3r^4\\&=\frac{1}{k}\sum_{i=1}^{k}(|PA_i|^4-2h^2|PA_i|^2+h^4)+3r^4\\&=\frac{1}{k}\Big(\sum_{i=1}^{k}|PA_i|^4\Big)-2h^2\cdot\frac{1}{k}\Big(\sum_{i=1}^{k}|PA_i|^2\Big)+h^4+3r^4.\end{aligned} \tag{18}$$

Meanwhile,

$$\begin{aligned}\Big(\frac{1}{k}\sum_{i=1}^{k}|HA_i|^2+r^2\Big)^2&=\Big(\frac{1}{k}\sum_{i=1}^{k}(|PA_i|^2-h^2\Big)+r^2)^2\\&=\Big(\frac{1}{k}\Big(\sum_{i=1}^{k}|PA_i|^2\Big)+r^2-h^2\Big)^2\\&=\Big(\frac{1}{k}\Big(\sum_{i=1}^{k}|PA_i|^2\Big)+r^2\Big)^2-2h^2\cdot\Big(\frac{1}{k}\Big(\sum_{i=1}^{k}|PA_i|^2\Big)+r^2\Big)+h^4\\&=\Big(\frac{1}{k}\Big(\sum_{i=1}^{k}|PA_i|^2\Big)+r^2\Big)^2-2h^2\cdot\frac{1}{k}\Big(\sum_{i=1}^{k}|PA_i|^2\Big)+h^4-2h^2r^2.\end{aligned} \tag{19}$$

From Equations (17), (18), and (19), we deduce that:

$$\frac{1}{k}\Big(\sum_{i=1}^{k}PA_i^4\Big)+3r^4+2h^2r^2=\Big(\frac{1}{k}\Big(\sum_{i=1}^{k}PA_i^2\Big)+r^2\Big)^2. \tag{20}$$

This completes the proof of Theorem 8. □

3 Applications

We present the following theorem as an application of Theorem 7.

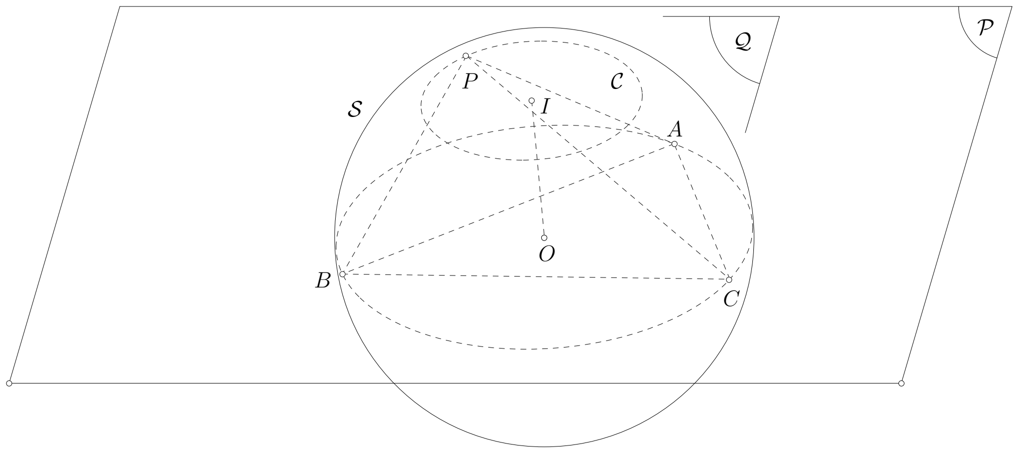

Figure 7: Sum of fourth powers of distance remains constant in equilateral triangle embedded in 3D space.

Theorem 9. *In the m-dimensional space $\mathbb{E}^m$, consider a subspace $\mathcal{P} = \mathbb{E}^n$ $(n \leq m)$ containing an n-dimensional regular polytope $\mathcal{M}$, which could be a simplex, hypercube, or orthoplex. Let O be the center of $\mathcal{M}$. Suppose $\mathcal{S}$ is a hypersphere centered at O within $\mathbb{E}^m$, making $\mathcal{S}$ an $(m-1)$-sphere. Cutting $\mathcal{S}$ by an n-plane parallel to $\mathcal{P}$ results in an n-dimensional sphere $\mathcal{S}^n$. For any point P on $\mathcal{S}^n$, the sum of the fourth powers of the distances from P to the vertices of $\mathcal{M}$ remains constant.*

Proof. Let $\{A_i\}_{i\in I}$ denote the set of vertices of $\mathcal{M}$. Since $\mathcal{M}$ is a regular polytope, its center O is also the centroid of $\mathcal{M}$ (see [3]). When P lies on $\mathcal{S}$, the distance $|OP|$ is constant. By the Generalized Parallel Axis Theorem (GPAT) from [6], it follows that $\sum_{i\in I} |PA_i|^2$ is constant. Moreover, when P lies on an n-plane parallel to $\mathcal{P}$, the distance h from P to $\mathcal{P}$ remains unchanged. Combining these two facts, when P lies on $\mathcal{S}^n$, both $\sum_{i\in I} |PA_i|^2$ and the distance h from P to $\mathcal{P}$ are constant. Applying Theorem 7, we conclude that the sum of the fourth powers of the distances from P to the vertices of $\mathcal{M}$ is constant. □

A simpler example of Theorem 9 in three-dimensional space, involving an equilateral triangle, is as follows:

Theorem 10 (Corollary of Theorem 9)**.** *Let ABC be an equilateral triangle with circumcenter O in three-dimensional space. Consider a sphere $\mathcal{S}$ centered at O. Cutting $\mathcal{S}$ with a plane $\mathcal{Q}$ parallel to the plane containing the triangle ABC results in a circle $\mathcal{C}$. For any point P on $\mathcal{C}$, the sum $|PA|^4 + |PB|^4 + |PC|^4$ remains constant. (See Figure 7).*

This theorem extends the problem where, when P lies on the circumcircle of the equilateral triangle ABC, the sum $|PA|^4 + |PB|^4 + |PC|^4$ is always constant.

4 Is Further Generalization Possible?

We observe that when equilateral triangles or regular polygons are placed in three-dimensional space, Theorems 1 and 2 transform into Theorems 6 and 8, respectively. On the other hand, when $\mathcal{M}$ is an n-dimensional regular polytope $(n \geq 2)$, such as a simplex, hypercube, or orthoplex, and it is placed in a higher-dimensional space $\mathbb{E}^m$ $(m \geq n)$, Theorems 2 and 4 generalize into Theorems 5 and 7. We all know that in $\mathbb{E}^n$ with $n \geq 5$, there are only three types of regular polytopes: the simplex, hypercube, and orthoplex. Therefore, the general theorem, Theorem 5, will be sufficient for $n \geq 5$. When $n = 2$, the general theorem, Theorem 7, is adequately comprehensive. However, for $n = 3$, there are five types of regular polytopes, but Theorem 5 only addresses three of them. Similarly, when $n = 4$, Theorem 5 also covers only three types of regular polytopes. So what about the 24-cell, 60-cell, and 600-cell?

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