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Poisson Structures Compatible with the Metric of the Three-Dimensional Walker Manifold

Mahamane Saminou Ali[1], Ibrahima Hamidine[2], Issa Ousmane Toudou[3], Mouhamadou Hassirou[2]

[1] *Université d'Agadez, Agadez, Niger*
mahamanesaminou@gmail.com

[2] *Université Abdou Moumouni, Niamey, Niger*
ibrahima.hamidine@uam.edu.ne,, mouhamadouhassirou@gamil.com

[3] *Université Boubakar Ba de Tillabéri, Tillabéri, Niger*
outoud26@yahoo.fr

Abstract. The object of the present paper is to construct a characterization of a three-dimensional Walker-Poisson manifolds with Walker metric; in other words, study the compatibility between Walker and Poisson structures. Some examples are given.

Key Words: Poisson manifold, Poisson structure, Walker manifold, Walker metric

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1 Introduction

Poisson geometry and Walker geometry are interesting subjects in differential geometry. The two notions of Poisson and Walker manifold have various applications in mathematics, theoretical physics and physics. Consequently, they have motivated several studies.

A Poisson manifold is just a smooth manifold M endowed with a Poisson bracket which is a Lie bracket, on the space of smooth function $\mathcal{C}^\infty(M)$, satisfying the Jacobi identity. It's also equivalent to the data of a bi-vector field π such that $[\pi, \pi] = 0$, where $[,]$ is the Schouten-Nijenhuis bracket. Poisson manifolds were introduced by André Lichnorowicz in 1977 [8], but the notion of Poisson bracket were first introduced by Siméon Dennis Poisson in his work on mathematical mechanic [9]. They play a crucial role in physic mathematics, namely in classic mechanic and quantization. This motived several study of such manifold (see [12]).

A Walker manifold is also a smooth manifold, endowed with a Walker metric which is a class of semi-Riemannian metric. It's a specific type of manifold characterized by neutral metric and a parallel distribution of totally null planes. More specifically, a Walker n-manifold is a semi-Riemannian manifold which admits a field of parallel nul r-planes with $r \leq \frac{n}{2}$. Their canonical form were investigate by Walker in the early 1950 [11]. This geometry was named after Arthur Goffrey Walker who studied it extensively and it's applications. Walker geometry have many applications namely in cosmology, relativity, string theory, ...

The study of Poisson and pseudo-Riemannian structure, in the same manifold, was first done by I. Vaisman [10] with the introduction of the notion of contravariant connection. This contravariant connection was more generalized by R. L. Fernades [6] and used by M. Boucetta [3, 4] to introduce the notion of Riemann-Poisson manifold. This notion is based on the notion of compatibility between the Poisson structure and pseudo-Riemannian one. This allow more characterization of Poisson manifold and the introduction of Poisson manifold with lightlike Kaehler foliation [1, 7]. In finite dimension several study of Riemann-Poisson manifolds are done [2].

In this paper, we will discuss the relation between the Walker metric and the Poisson structures in three-dimensional manifolds. The starting point of this work is the search of a notion of compatibility between a Poisson structure and a Walker metric. Given a Poisson manifold endowed with a Walker metric (M, π, g_f^*), it is natural to look for a notion of compatibility between the Poisson tensor π and the metric g_f^*. The first idea is to assume that $\mathcal{D}\pi = 0$, where $\mathcal{D}$ is the Levi-Civita contravariant connection associated with g_f^*.

Our paper is organized as follows. Section 1 introduces the topic. In Section 2, we recall some basics notions on Walker manifolds and Poisson manifold. In Section 3, we study three-dimensional Walker-Poisson manifolds in a Walker 3-manifold. In Section 4, we prove Theorem 2, which characterises Poisson and Liouville-Poisson vector fields. We end the paper with some example of a vector field which is both Poisson and Liouville.

2 Preliminaries

2.1 Some Basic Facts on Riemann-Poisson Manifolds

Let (M, π) be a Poisson manifold endowed with her Poisson tensor π. It's well known that this tensor induce a morphism $\sharp \colon T^*M \to TM$ such that $\beta(\sharp(\alpha)) = \pi(\alpha, \beta)$ for all $\alpha, \beta \in \Omega^1(M)$. Thus, a contravariant connection $\mathcal{D}$ on M is defined as

$$\mathcal{D} \colon \Omega^1(M) \times \Omega^1(M) \to \Omega^1(M),$$

satisfying the following equalities

$$\begin{aligned} \mathcal{D}_{f\alpha}\beta &= f\mathcal{D}_\alpha\beta \\ \mathcal{D}_\alpha(f\beta) &= f\mathcal{D}_\alpha\beta + \sharp(\alpha)(f)\beta, \end{aligned}$$

for all $f \in \mathcal{C}^\infty(M)$ and $\alpha, \beta \in \Omega^1(M)$.

Let g be a pseudo-Riemannian manifold on M. If ∇ is the Levi-Civita connection associated te g, the the first compatibility between the Poisson structure π and the Riemannian structure g was introduced by I. Vaisman [10] which is equivalent to $\nabla\pi = 0$. This compatibility induced that the Poisson manifold (M, π) is regular. However, the most important Poisson structure are not regular, and this condition proves to be insufficient for the study of pseudo-Riemannian Poisson manifolds. Hence, M. Boucetta introduce a notion of compatibility using the contravariant Levi-Civita connection [2]. Indeed, the Riemannian metric

g induces, on T^*M, a Riemannian metric that can be noted g^*. This metric have it's Levi-Civita associated contravariant connection $\mathcal{D}$. This contravariant connection is characterized by the Koszul type formula:

$$2g^*(D_\alpha\beta,\gamma) = \pi_\sharp(\alpha)\cdot g^*(\beta,\gamma) + \pi_\sharp(\beta)\cdot g^*(\alpha,\gamma) - \pi_\sharp(\gamma)\cdot g^*(\alpha,\beta) + g^*([\alpha,\beta]_\pi,\gamma) + g^*([\gamma,\alpha]_\pi,\beta) + g^*([\gamma,\beta]_\pi,\alpha).$$

Since $\mathcal{D}$ has vanished torsion, one can obtain (see [2])

$$0 = -[\pi,\pi]_S(\alpha,\beta,\gamma) = \mathcal{D}\pi(\alpha,\beta,\gamma) + \mathcal{D}\pi(\beta,\gamma,\alpha) + \mathcal{D}\pi(\gamma,\alpha,\beta),$$
$$\mathcal{D}\pi(\gamma,\alpha,\beta) = -d\gamma(\pi(\alpha),\pi(\beta)) - \pi(\mathcal{D}_\alpha\gamma,\beta) - \pi(\alpha,\mathcal{D}_\beta\gamma),$$
$$\pi(\mathcal{D}_\alpha\beta) - \pi(\mathcal{D}_\beta\alpha) = [\pi(\alpha),\pi(\beta)].$$

Definition 1. Let (M,g^*,π) be a manifold endowed with a pseudo-Riemannian metric g^* and a Poisson tensor π. (M,g^*,π) is said to be a pseudo-Riemann-Poisson manifold, if

$$\mathcal{D}_\alpha\pi(\beta,\gamma) \coloneqq \pi_\sharp(\alpha)\cdot\pi(\beta,\gamma) - \pi(\mathcal{D}_\alpha\beta,\gamma) - \pi(\beta,\mathcal{D}_\alpha\gamma) = 0, \tag{1}$$

where $\alpha,\beta,\gamma \in \Omega^1(M)$.

This class of manifolds are characterized by the following theorem.

Proposition 1 ([3])**.** *Let (M,π,g^*) be a Poisson manifold endowed with a pseudo-Riemannian metric on T^*M. Let $\mathcal{D}$ be the Levi-Civita contravariant connection associated with the couple (π,g^*). Then the following assertions are equivalent.*

1. *The triplet (M,π,g^*) is a pseudo-Riemann-Poisson manifold.*
2. *For all $\alpha,\beta \in \Omega^1(M)$ and all $f \in \mathcal{C}^\infty(M)$,*

$$\pi(D_\alpha df,\beta) + \pi(\alpha,\mathcal{D}_\beta df) = 0.$$

3. *For all $\alpha,\beta,\gamma \in \Omega^1(M)$,*

$$d\gamma(\pi(\alpha),\pi(\beta)) + \pi(\mathcal{D}_\alpha\gamma,\beta,) + \pi(\alpha,\mathcal{D}_\beta\gamma) = 0.$$

2.2 Three Dimensional Walker Manifold

In this section, we recall some basics notions on Walker manifolds taken from [11]. The metric of three-dimensional Walker manifold (M,g_f) with coordinates (x,y,z) is expressed as

$$g_f = \epsilon dy^2 + f dz^2 + dx\otimes dz + dz\otimes dx$$

and its matrix form as

$$g_f = \begin{pmatrix} 0 & 0 & 1\\ 0 & \epsilon & 0\\ 1 & 0 & f\end{pmatrix}, \qquad \text{with its inverse} \qquad g_f^{-1} = \begin{pmatrix} -f & 0 & 1\\ 0 & \epsilon & 0\\ 1 & 0 & 0\end{pmatrix} \tag{2}$$

for some smooth function $f(x,y,z)$, where $\epsilon = \pm 1$. A Walker 3-manifold noted M, is a pseudo-Riemannian manifold, which admits a parallel null vector field U. Notice that when $\epsilon = 1$ and $\epsilon = -1$ the Walker manifold has signature $(2,1)$ and $(1,2)$ respectively, and therefore is Lorentzian in both cases.

3 Poisson Structures on 3-dimensional Walker Manifolds

Let (M, g_f) be a 3-dimensional pseudo-Riemannian manifold endowed with Walker metric g_f:

$$g_f = \epsilon dy^2 + f dz^2 + dx \otimes dz + dz \otimes dx \tag{3}$$

Let π be a Poisson tensor given in local coordinates (x, y, z) by:

$$\pi = \pi_{12}\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} + \pi_{13}\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial z} + \pi_{23}\frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z}, \tag{4}$$

where

$$\pi_{12} = \pi(dx, dy), \quad \pi_{13} = \pi(dx, dz), \quad \pi_{23} = \pi(dy, dz)$$

are differentiable functions on $\mathbb{R}^3$. We therefore have

$$\begin{cases} \sharp_\Pi(dx) = \pi_{12}\frac{\partial}{\partial y} + \pi_{13}\frac{\partial}{\partial z}, \\ \sharp_\Pi(dy) = -\pi_{12}\frac{\partial}{\partial x} + \pi_{23}\frac{\partial}{\partial z}, \\ \sharp_\Pi(dz) = -\pi_{13}\frac{\partial}{\partial x} - \pi_{23}\frac{\partial}{\partial y}, \end{cases}$$

and the bracket $[.,.]_\pi$ is defined by

$$\begin{cases} [dx, dy]_\pi = d\pi_{12} = \frac{\partial \pi_{12}}{\partial x}dx + \frac{\partial \pi_{12}}{\partial y}dy + \frac{\partial \pi_{12}}{\partial z}dz \\ [dx, dz]_\pi = d\pi_{13} = \frac{\partial \pi_{13}}{\partial x}dx + \frac{\partial \pi_{13}}{\partial y}dy + \frac{\partial \pi_{13}}{\partial z}dz \\ [dy, dz]_\pi = d\pi_{23} = \frac{\partial \pi_{23}}{\partial x}dx + \frac{\partial \pi_{23}}{\partial y}dy + \frac{\partial \pi_{23}}{\partial z}dz. \end{cases} \tag{5}$$

Let g_f^* be the metric on the cotangent bundle defined by $g_f^*(\alpha, \beta) = g_f(\sharp_g\alpha, \sharp_g\beta)$. Let $\mathcal{D}$ be the contravariant Levi-Civita connection associated to g^*. We call it the Levi-Civita contravariant connection associated with the couple (π, g_f^*). It's characterised by the Koszul type formula:

$$\begin{aligned} 2g_f^*(\mathcal{D}_\alpha\beta, \gamma) = {} & \sharp_\pi(\alpha)g_f^*(\beta, \gamma) + \sharp_\pi(\beta)g_f^*(\alpha, \gamma) - \sharp_\pi(\gamma)g_f^*(\alpha, \beta) \\ & + g_f^*([\alpha, \beta]_\pi, \gamma) + g_f^*([\gamma, \alpha]_\pi, \beta) + g_f^*([\gamma, \beta]_\pi, \alpha), \end{aligned} \tag{6}$$

where α, β, $\gamma \in \Omega^1(M)$, the Lie bracket $[\,,]_\pi$ is given by

$$[\alpha, \beta]_\pi = \mathcal{L}_{\sharp_\pi(\alpha)}\beta - \mathcal{L}_{\sharp_\pi(\beta)}\alpha - d\pi(\alpha, \beta),$$

where $\mathcal{L}_{\sharp_\pi(\alpha)}$ denotes the Lie derivative along a differentiable vector field $\sharp_\pi(\alpha)$ and $\sharp_\pi : T^*M \to TM$ denotes the bundle map given by $\beta\big(\sharp_\pi(\alpha)\big) = \pi(\alpha, \beta)$.

$\mathcal{D}$ has vanishing torsion, i.e.,

$$\mathcal{D}_\alpha\beta - \mathcal{D}_\beta\alpha = [\alpha, \beta]_\pi \tag{7}$$

$\mathcal{D}$ is compatible with g_f^* i.e.

$$\sharp_\pi(\alpha)g_f^*(\beta, \gamma) = g_f^*(\mathcal{D}_\alpha\beta, \gamma) + g(\beta, \mathcal{D}_\alpha\gamma). \tag{8}$$

Using (6), we calculate the non-zero Christoffel symbols of a Walker cometric g_f^* of $\mathcal{D}$ as:

$$\Gamma_{11}^1 = \frac{\partial \pi_{13}}{\partial z}, \qquad \Gamma_{11}^2 = \epsilon \frac{\partial \pi_{12}}{\partial z}, \tag{9}$$

$$\Gamma_{12}^1 = \frac{1}{2}\Big(-\frac{\partial \pi_{12}}{\partial x} + f\frac{\partial \pi_{12}}{\partial z} + \epsilon\frac{\partial \pi_{13}}{\partial y} + \frac{\partial \pi_{13}}{\partial z}\Big), \qquad \Gamma_{12}^3 = -\frac{\partial \pi_{12}}{\partial z}, \tag{10}$$

$$\Gamma_{13}^1 = \frac{1}{2} f\frac{\partial \pi_{13}}{\partial z}, \qquad \Gamma_{13}^3 = -\frac{\partial \pi_{13}}{\partial z}, \tag{11}$$

$$\Gamma_{13}^2 = \frac{1}{2}\Big(\epsilon\frac{\partial \pi_{12}}{\partial x} + \epsilon f\frac{\partial \pi_{12}}{\partial z} - \frac{\partial \pi_{13}}{\partial y} - \epsilon\frac{\partial \pi_{23}}{\partial z}\Big), \tag{12}$$

$$\Gamma_{21}^1 = \frac{1}{2}\Big(\frac{\partial \pi_{12}}{\partial x} + f\frac{\partial \pi_{12}}{\partial z} + \epsilon\frac{\partial \pi_{13}}{\partial y} + \frac{\partial \pi_{23}}{\partial z}\Big), \qquad \Gamma_{21}^2 = \frac{\partial \pi_{12}}{\partial y}, \tag{13}$$

$$\Gamma_{22}^1 = \epsilon\Big(f\frac{\partial \pi_{12}}{\partial y} + \frac{\partial \pi_{23}}{\partial y}\Big), \qquad \Gamma_{22}^3 = -\epsilon\frac{\partial \pi_{12}}{\partial y} \tag{14}$$

$$\Gamma_{23}^1 = \frac{1}{2}\Big(f\frac{\partial \pi_{12}}{\partial x} + f^2\frac{\partial \pi_{12}}{\partial z} + \epsilon f\frac{\partial \pi_{13}}{\partial y} + \frac{\partial \pi_{23}}{\partial x}\Big), \tag{15}$$

$$\Gamma_{23}^2 = -\frac{\partial \pi_{23}}{\partial y}, \qquad \Gamma_{23}^3 = -\Gamma_{21}^1, \tag{16}$$

$$\Gamma_{31}^2 = \frac{1}{2}\Big(\epsilon\frac{\partial \pi_{12}}{\partial x} + \epsilon f\frac{\partial \pi_{12}}{\partial z} + \frac{\partial \pi_{13}}{\partial y} - \epsilon\frac{\partial \pi_{23}}{\partial z}\Big), \tag{17}$$

$$\Gamma_{32}^1 = \frac{1}{2} f\Big(\frac{\partial \pi_{12}}{\partial x} + f\frac{\partial \pi_{12}}{\partial z} + \epsilon\frac{\partial \pi_{13}}{\partial y} - \frac{\partial \pi_{13}}{\partial z}\Big), \qquad \Gamma_{32}^3 = -\frac{1}{f}\Gamma_{32}^1, \tag{18}$$

$$\Gamma_{33}^1 = \frac{1}{2}\Big(f^2\frac{\partial \pi_{13}}{\partial z} - \pi_{13}\frac{\partial f}{\partial x} - \pi_{23}\frac{\partial f}{\partial z}\Big). \tag{19}$$

Having calculated the Christoffel symbols, we get the following possibly non-vanishing components of the Levi-Civita contravariant connection on (M, g_f^*):

$$\mathcal{D}_{dx}dx = \Gamma_{11}^1 dx + \Gamma_{11}^2 dy, \qquad \mathcal{D}_{dx}dy = \Gamma_{12}^1 dx + \Gamma_{12}^3 dz \tag{20}$$

$$\mathcal{D}_{dx}dz = \Gamma_{13}^1 dx + \Gamma_{13}^2 dy + \Gamma_{13}^3 dz, \tag{21}$$

$$\mathcal{D}_{dy}dx = \Gamma_{21}^1 dx + \Gamma_{21}^2 dy, \qquad \mathcal{D}_{dy}dy = \Gamma_{22}^1 dx + \Gamma_{22}^3 dz, \tag{22}$$

$$\mathcal{D}_{dy}dz = \Gamma_{23}^1 dx + \Gamma_{23}^2 dy + \Gamma_{23}^3 dz, \tag{23}$$

$$\mathcal{D}_{dz}dx = \Gamma_{31}^2 dy, \ \mathcal{D}_{dz}dy = \Gamma_{32}^1 dx + \Gamma_{32}^3 dz, \ \mathcal{D}_{dz}dz = \Gamma_{33}^1 dx. \tag{24}$$

Definition 2. With the notations above, the triple (M, π, g_f^*) is called a Walker-Poisson manifold if, for any $\alpha, \beta, \gamma \in \Omega^1(M)$,

$$\mathcal{D}\pi(\alpha,\beta,\gamma) := \sharp_\pi(\alpha)\cdot\pi(\beta,\gamma) - \pi(\mathcal{D}_\alpha\beta,\gamma) - \pi(\beta,\mathcal{D}_\alpha\gamma) = 0. \tag{25}$$

Finally, we quote the following result as a characterization of a three-dimensional Poisson manifold with Walker metric:

Theorem 1. *A Poisson tensor* $\pi = \pi_{12}\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y} + \pi_{13}\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial z} + \pi_{23}\frac{\partial}{\partial y}\wedge\frac{\partial}{\partial z}$ *is compatible with* *the Walker cometric g_f^* iff:*

$$\left\{\begin{aligned}
0 &= \pi_{12}\Big(\frac{\partial\pi_{12}}{\partial y} - \frac{\partial\pi_{13}}{\partial z}\Big) + 2\pi_{13}\frac{\partial\pi_{12}}{\partial z}\\
0 &= \pi_{12}\Big(\epsilon\frac{\partial\pi_{12}}{\partial x} + \epsilon f\frac{\partial\pi_{12}}{\partial z} - 3\frac{\partial\pi_{13}}{\partial y} - \epsilon\frac{\partial\pi_{23}}{\partial z}\Big)\\
&\quad + 2\epsilon\pi_{23}\frac{\partial\pi_{12}}{\partial z} - 2\pi_{13}\frac{\partial\pi_{13}}{\partial z},\\
-\pi_{12}\Big(\pi_{12}\frac{\partial f}{\partial y} + \pi_{13}\frac{\partial f}{\partial z}\Big) &= \pi_{13}\Big(\frac{\partial\pi_{12}}{\partial x} - f\frac{\partial\pi_{12}}{\partial z} - \epsilon\frac{\partial\pi_{13}}{\partial y} + \frac{\partial\pi_{23}}{\partial z}\Big)\\
&\quad + 2(f\pi_{12} + \pi_{23})\frac{\partial\pi_{13}}{\partial z} + 2\pi_{12}\frac{\partial\pi_{23}}{\partial y},\\
0 &= \pi_{12}\Big(\frac{\partial\pi_{12}}{\partial x} + \frac{\epsilon}{2}\frac{\partial\pi_{13}}{\partial y} + \frac{1}{2}\frac{\partial\pi_{23}}{\partial z}\Big) - 2\epsilon\pi_{13}\frac{\partial\pi_{12}}{\partial y} + f\frac{\partial\pi_{12}}{\partial z},\\
0 &= \pi_{12}\Big(\frac{\partial\pi_{13}}{\partial x} - \frac{\partial\pi_{23}}{\partial y}\Big) - \pi_{13}\frac{\partial\pi_{12}}{\partial x} - \pi_{23}\frac{\partial\pi_{13}}{\partial z},\\
-\pi_{12}\Big(\pi_{12}\frac{\partial f}{\partial x} - \pi_{23}\frac{\partial f}{\partial z}\Big) &= (f\pi_{12} + 2\pi_{23})\frac{\partial\pi_{12}}{\partial x} - 2\epsilon f\pi_{13}\frac{\partial\pi_{12}}{\partial y} + (f^2\pi_{12} + f\pi_{23})\frac{\partial\pi_{12}}{\partial z}\\
&\quad + \epsilon(f\pi_{12} + \pi_{23})\frac{\partial\pi_{13}}{\partial y} - 2\pi_{12}\frac{\partial\pi_{23}}{\partial x} + (3\pi_{23} - f\pi_{12})\frac{\partial\pi_{23}}{\partial z},\\
-\pi_{23}\Big(\pi_{12}\frac{\partial f}{\partial y} + \pi_{13}\frac{\partial f}{\partial z}\Big) &= f\pi_{13}\Big(\frac{\partial\pi_{12}}{\partial x} + f\frac{\partial\pi_{12}}{\partial z} + \epsilon\frac{\partial\pi_{13}}{\partial y} - \frac{\partial\pi_{23}}{\partial z}\Big)\\
&\quad - 2f\pi_{23}\frac{\partial\pi_{12}}{\partial y} - 2\pi_{12}\frac{\partial\pi_{13}}{\partial x},\\
(\epsilon\pi_{12}\pi_{23} + \pi_{13}^2 - f\pi_{13}^2)\frac{\partial f}{\partial z} &= (\epsilon\pi_{12}^2 - \pi_{13}^2)\frac{\partial f}{\partial x} + \pi_{12}\pi_{13}(f-1)\frac{\partial f}{\partial y}\\
&\quad + \pi_{23}\Big(\epsilon\frac{\partial\pi_{12}}{\partial x} + \epsilon f\frac{\partial\pi_{12}}{\partial z} + 3\frac{\partial\pi_{13}}{\partial y}\Big) + 2f\pi_{13}\Big(\frac{\partial\pi_{13}}{\partial x} + (f-1)\frac{\partial\pi_{13}}{\partial z}\Big)\\
&\quad - 2\epsilon\pi_{12}\frac{\partial\pi_{23}}{\partial x} - (2+2f)\frac{\partial\pi_{23}}{\partial z},\\
f\pi_{12}\Big(\pi_{12}\frac{\partial f}{\partial y} + \pi_{13}\frac{\partial f}{\partial z}\Big) &= f\pi_{13}\Big(\epsilon\frac{\partial\pi_{13}}{\partial y} + \frac{\partial\pi_{12}}{\partial x} + f\frac{\partial\pi_{12}}{\partial z} - \frac{\partial\pi_{23}}{\partial z}\Big) - 2(f\pi_{12} + \pi_{23})\frac{\partial\pi_{13}}{\partial x}\\
&\quad - 2f(f\pi_{12} + \pi_{23})\frac{\partial\pi_{13}}{\partial z} + 2\pi_{13}\frac{\partial\pi_{23}}{\partial x} + 2\pi_{23}\frac{\partial\pi_{23}}{\partial y}.
\end{aligned}\right. \tag{26}$$

Proof. Firstly, from (25) and (20)-(22) a straightforward calculation shows that the vanishing of $\mathcal{D}\pi$ is equivalent to

$$\left\{\begin{aligned}
\pi_{12}\frac{\partial\pi_{12}}{\partial y} + \pi_{13}\frac{\partial\pi_{12}}{\partial z} &= \pi_{12}\Gamma_{11}^1 + \pi_{13}\Gamma_{12}^3\\
\pi_{12}\frac{\partial\pi_{13}}{\partial y} + \pi_{13}\frac{\partial\pi_{13}}{\partial z} &= \pi_{13}\Gamma_{11}^1 + \pi_{23}\Gamma_{11}^2 + \pi_{12}\Gamma_{13}^2 + \pi_{13}\Gamma_{13}^3\\
\pi_{12}\frac{\partial\pi_{23}}{\partial y} + \pi_{13}\frac{\partial\pi_{23}}{\partial z} &= \pi_{13}\Gamma_{12}^1 + \pi_{23}\Gamma_{12}^3 - \pi_{12}\Gamma_{13}^1 + \pi_{23}\Gamma_{13}^3\\
-\pi_{12}\frac{\partial\pi_{12}}{\partial x} + \pi_{23}\frac{\partial\pi_{12}}{\partial z} &= \pi_{12}\Gamma_{12}^1 - \pi_{23}\Gamma_{12}^3 + \pi_{13}\Gamma_{22}^3\\
-\pi_{12}\frac{\partial_{13}}{\partial x} + \pi_{23}\frac{\partial\pi_{13}}{\partial z} &= \pi_{13}\Gamma_{12}^1 + \pi_{12}\Gamma_{23}^2 + \pi_{13}\Gamma_{23}^3\\
-\pi_{12}\frac{\partial\pi_{23}}{\partial x} + \pi_{23}\frac{\partial\pi_{23}}{\partial z} &= \pi_{13}\Gamma_{22}^1 - \pi_{12}\Gamma_{23}^1 + \pi_{23}\Gamma_{23}^3\\
-\pi_{13}\frac{\partial\pi_{12}}{\partial x} - \pi_{23}\frac{\partial\pi_{12}}{\partial y} &= -\pi_{23}\Gamma_{31}^3 + \pi_{13}\Gamma_{32}^3\\
-\pi_{13}\frac{\partial\pi_{13}}{\partial x} - \pi_{23}\frac{\partial\pi_{13}}{\partial y} &= \pi_{23}\Gamma_{31}^2 + \pi_{12}\Gamma_{33}^2 + \pi_{13}\Gamma_{33}^3\\
-\pi_{13}\frac{\partial\pi_{23}}{\partial x} - \pi_{23}\frac{\partial\pi_{23}}{\partial y} &= \pi_{13}\Gamma_{32}^1 - \pi_{12}\Gamma_{33}^1 + \pi_{23}\Gamma_{33}^3.
\end{aligned}\right. \tag{27}$$

Now, by (9)–(19) it is easy to see that (27) is equivalent to (26). □

Here is an example of how to use it as an application.

Example 1. Let π be a Poisson tensor given by:

$$\pi = 3xy\partial_x \wedge \partial_y + xz\ln(x^2)\partial_x \wedge \partial_z + 2yz\partial_y \wedge \partial_z,$$

with $(x, y, z) \in (\mathbb{R}^*)^3$. The non-zero Christoffel symbols are given by the following expressions

$$\Gamma^1_{11} = x\ln(x^2),\ \Gamma^1_{12} = \frac{1}{2}(-3y + x\ln(x^2)),\ \Gamma^1_{13} = \frac{1}{2}xf\ln(x^2),\ \Gamma^2_{13} = \frac{1}{2}\epsilon y,$$
$$\Gamma^3_{13} = -x\ln(x^2),\ \Gamma^1_{21} = \frac{5}{2}y,\ \Gamma^2_{21} = 3x,\ \Gamma^1_{22} = \epsilon(3xf + 2z),\ \Gamma^3_{22} = -3\epsilon x,$$
$$\Gamma^1_{23} = \frac{3}{2}yf,\ \Gamma^2_{23} = -2z,\ \Gamma^3_{23} = -\frac{5}{2}y,\ \Gamma^2_{31} = \frac{1}{2}\epsilon y,\ \Gamma^1_{32} = \frac{1}{2}(3y - x\ln(x^2))f,$$
$$\Gamma^3_{32} = \frac{1}{2}(x\ln(x^2) - y),\ \Gamma^1_{33} = \frac{1}{2}\Big(x\ln(x^2)f^2 - xz\ln(x^2)\frac{\partial f}{\partial x} - 2yz\frac{\partial f}{\partial z}\Big).$$

Our goal in this example is to find a function f for which π and g_f are compatible.

The system (26) has been demonstrated to yield the desired function f as a solution to the equation

$$Af^3 + Bf^2 + cf + D = 0,$$

where

$$\begin{aligned}
A &= -x^2z^2 - 2x^2z\ln(x^2) - x^2z(z-3)\Big(\ln(x^2)\Big)^2\\
B &= 3\epsilon xy^2z + 2x^2z^2 + 4yz - 12y - 9\epsilon xy^2\\
&\quad + \Big(-13x^2z^2 - \tfrac{34}{3}xz^2 + 40x^2z - 16xz^3 - X^4z(z-3)\Big)\ln(x^2)\\
C &= -x^2z^2 - 4xz^3 + 20\epsilon y^2z^2 + 36\epsilon xy^4z - \frac{2}{3}z^2(x + 4z^2)\ln(x^2)\\
&\quad - \frac{12x^2z^4 + 10z^4}{9x^2}\Big(\ln(x^2)\Big)^2\\
D &= \frac{1}{27x}\Big[-54xy^2z^3 - 72\epsilon y^2z^2 + z^4(36x + 8)\Big(\ln(x^2)\Big)^2\Big].
\end{aligned}$$

Recall that a strict 3-dimensional Walker manifolds are 3-dimensional Walker manifolds whose parallel null vector field is spanned by ∂_x. Hence in a strict Walker 3-dimensional manifolds, f is independent of the parameter x, i.e., $f = f(y, z)$ [11].

Corollary 1. *Let (M, g^*_f) be a strict 3-dimensional Walker manifold and π a Poisson tensor*

$$\pi = c_1\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} + c_2\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial z} + c_3\frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z},$$

where c_i are the real constant. Then π is compatible with g^*_f if and only if f is constant.

Example 2. Consider the following three-dimensional tensor:

$$\pi = \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} + \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial z} + \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z}.$$

According to Theorem 1, a bivector field π on $\mathbb{R}^3$ is a Poisson tensor with respect to the Walker metric. If we choose $f(x, y, z) = const$, we can easily verify that (26) are fulfilled.

4 Characterization of Poisson and Liouville-Poisson Vector Fields

In this section we will propose a characterisation of Poisson vector fields and Liouville vector fields associated with the Poisson tensor defined on a Walker-Poisson manifold of dimension 3. To do this, we will recall their definitions.

Definition 3. Let (M, g_f, π) be a Walker-Poisson manifold of dimension 3, where g_f denotes the Walker metric and $\pi = \pi_{12}\partial_x \wedge \partial_y + \pi_{13}\partial_x \wedge \partial_z + \pi_{23}\partial_y \wedge \partial_z$ a Poisson tensor. Assume that the functions f and π_{ij} $(1 \le i < j \le 3)$ are of class $\mathcal{C}^\infty$. Let X be a vector field on TM. It is said that

1) X is a Poisson field if the Schouten bracket $[X, \pi] = 0$,
2) X is a Liouville field if the Schouten bracket $[X, \pi] = \pi$.

Remark 1. The analogous notion for Liouville vector fields on unimodular exact Poisson manifolds appears in [5].

Our result in this section, which gives a characterization of the Poisson and Liouville vector fields, is the following theorem

Theorem 2. *Let M be a three-dimensional Walker-Poisson manifold.*

1) If X is a Poisson vector field then, the following equations are satisfied:

$$\begin{cases} X_1\frac{\partial\pi_{12}}{\partial x} + X_2\frac{\partial\pi_{12}}{\partial y} + X_3\frac{\partial\pi_{12}}{\partial z} = \pi_{12}\Big(\frac{\partial X_1}{\partial x} + \frac{\partial X_2}{\partial y}\Big) - \pi_{23}\frac{\partial X_1}{\partial z} + \pi_{13}\frac{\partial X_2}{\partial z} \\ X_1\frac{\partial\pi_{13}}{\partial x} + X_2\frac{\partial\pi_{13}}{\partial y} + X_3\frac{\partial\pi_{13}}{\partial z} = \pi_{12}\frac{\partial X_3}{\partial y} + \pi_{13}\Big(\frac{\partial X_1}{\partial x} + \frac{\partial X_3}{\partial z}\Big) + \pi_{23}\frac{\partial X_1}{\partial y} \\ X_1\frac{\partial\pi_{23}}{\partial x} + X_2\frac{\partial\pi_{23}}{\partial y} + X_3\frac{\partial\pi_{23}}{\partial z} = -\pi_{12}\frac{\partial X_3}{\partial x} + \pi_{13}\frac{\partial X_2}{\partial x} + \pi_{23}\Big(\frac{\partial X_2}{\partial y} + \frac{\partial X_3}{\partial z}\Big). \end{cases} \tag{28}$$

2) If X is a Liouville vector field then, the following equations are satisfied:

$$\begin{cases} X_1\frac{\partial\pi_{12}}{\partial x} + X_2\frac{\partial\pi_{12}}{\partial y} + X_3\frac{\partial\pi_{12}}{\partial z} = \pi_{12}\Big(1 + \frac{\partial X_1}{\partial x} + \frac{\partial X_2}{\partial y}\Big) - \pi_{23}\frac{\partial X_1}{\partial z} + \pi_{13}\frac{\partial X_2}{\partial z} \\ X_1\frac{\partial\pi_{13}}{\partial x} + X_2\frac{\partial\pi_{13}}{\partial y} + X_3\frac{\partial\pi_{13}}{\partial z} = \pi_{12}\frac{\partial X_3}{\partial y} + \pi_{13}\Big(1 + \frac{\partial X_1}{\partial x} + \frac{\partial X_3}{\partial z}\Big) + \pi_{23}\frac{\partial X_1}{\partial y} \\ X_1\frac{\partial\pi_{23}}{\partial x} + X_2\frac{\partial\pi_{23}}{\partial y} + X_3\frac{\partial\pi_{23}}{\partial z} = -\pi_{12}\frac{\partial X_3}{\partial x} + \pi_{13}\frac{\partial X_2}{\partial x} + \pi_{23}\Big(1 + \frac{\partial X_2}{\partial y} + \frac{\partial X_3}{\partial z}\Big). \end{cases} \tag{29}$$

Proof. 1) Let $X = X_1\partial_x + X_2\partial_y + X_3\partial_z$ be a Poisson vector field in the three-dimensional Walker-Poisson manifold. In that case, X is the solution of the following equation

$$[X, \pi] = 0, \tag{30}$$

where $\pi = \pi_{12}\partial_x \wedge \partial_y + \pi_{13}\partial_x \wedge \partial_z + \pi_{23}\partial_y \wedge \partial_z$ is the Poisson bivector field. Then, one has

$$\begin{aligned} [X, \pi] = {} & [X_1\partial_x, \pi_{12}\partial_x \wedge \partial_y] + [X_1\partial_x, \pi_{13}\partial_x \wedge \partial_z] + [X_1\partial_x, \pi_{23}\partial_y \wedge \partial_z] \\ & + [X_2\partial_y, \pi_{12}\partial_x \wedge \partial_y] + [X_2\partial_y, \pi_{13}\partial_x \wedge \partial_z] + [X_2\partial_y, \pi_{23}\partial_y \wedge \partial_z] \\ & + [X_3\partial_z, \pi_{12}\partial_x \wedge \partial_y] + [X_3\partial_z, \pi_{13}\partial_x \wedge \partial_z] + [X_3\partial_z, \pi_{23}\partial_y \wedge \partial_z]. \end{aligned}$$

By calculating the Schouten bracket, we find

$$\begin{aligned} [X_1\partial_x, \pi_{12}\partial_x \wedge \partial_y] &= X_1\frac{\partial\pi_{12}}{\partial x}\partial_x \wedge \partial_y + \pi_{12}[X_1\partial_x, \partial_x] \wedge \partial_y + \pi_{12}\partial_x \wedge [X_1\partial_x, \partial_y] \\ &= X_1\frac{\partial\pi_{12}}{\partial x}\partial_x \wedge \partial_y - \pi_{12}\frac{\partial X_1}{\partial x}\partial_x \wedge \partial_y + \pi_{12}\partial_x \wedge \Big(-\frac{\partial X_1}{\partial y}\Big)\partial_x \\ &= \Big(X_1\frac{\partial\pi_{12}}{\partial x} - \pi_{12}\frac{\partial X_1}{\partial x}\Big)\partial x \wedge \partial_y. \end{aligned} \tag{31}$$

Likewise we get

$$[X_1\partial_x, \pi_{13}\partial_x \wedge \partial_z] = \Big(X_1\frac{\partial \pi_{13}}{\partial x} - \pi_{13}\frac{\partial X_1}{\partial x}\Big)\partial x \wedge \partial_z \tag{32}$$

$$[X_1\partial_x, \pi_{23}\partial_y \wedge \partial_z] = X_1\frac{\partial \pi_{23}}{\partial x}\partial_y \wedge \partial_z - \pi_{23}\frac{\partial X_1}{\partial y}\partial_x \wedge \partial_z + \pi_{23}\frac{\partial X_1}{\partial z}\partial_x \wedge \partial_y, \tag{33}$$

Indeed by (31)–(33), on has

$$[X_1\partial_x, \pi] = \Big(X_1\frac{\partial \pi_{12}}{\partial x} - \pi_{12}\frac{\partial X_1}{\partial x} + \pi_{23}\frac{\partial X_1}{\partial z}\Big)\partial_x \wedge \partial_y + \Big(X_1\frac{\partial \pi_{13}}{\partial x} - \pi_{13}\frac{\partial X_1}{\partial x} - \pi_{23}\frac{\partial X_1}{\partial y}\Big)\partial_x \wedge \partial_z + X_1\frac{\partial \pi_{23}}{\partial x}\partial_y \wedge \partial_z. \tag{34}$$

As the same of (34), we have

$$\begin{aligned}[X_2\partial_y, \pi] &= [X_2\partial_y, \pi_{12}\partial_x \wedge \partial_y] + [X_2\partial_y, \pi_{13}\partial_x \wedge \partial_z] + [X_2\partial_y, \pi_{23}\partial_y \wedge \partial_z] \\ &= \Big(X_2\frac{\partial \pi_{12}}{\partial y} - \pi_{12}\frac{\partial X_2}{\partial y} - \pi_{13}\frac{\partial X_2}{\partial z}\Big)\partial_x \wedge \partial_y + X_2\frac{\partial \pi_{13}}{\partial y}\partial_x \wedge \partial_z \\ &\quad + \Big(X_2\frac{\partial \pi_{23}}{\partial y} - \pi_{13}\frac{\partial X_2}{\partial x} - \pi_{23}\frac{\partial X_2}{\partial y}\Big)\partial_y \wedge \partial_z,\end{aligned} \tag{35}$$

and

$$\begin{aligned}[X_3\partial_z, \pi] &= [X_3\partial_z, \pi_{12}\partial_x \wedge \partial_y] + [X_3\partial_z, \pi_{13}\partial_x \wedge \partial_z] + [X_3\partial_z, \pi_{23}\partial_y \wedge \partial_z] \\ &= X_3\frac{\partial \pi_{12}}{\partial z}\partial_x \wedge \partial_y + \Big(X_3\frac{\partial \pi_{13}}{\partial z} - \pi_{12}\frac{\partial X_3}{\partial y} - \pi_{13}\frac{\partial X_3}{\partial z}\Big)\partial_x \wedge \partial_z \\ &\quad + \Big(X_3\frac{\partial \pi_{23}}{\partial z} + \pi_{12}\frac{\partial X_3}{\partial x} - \pi_{23}\frac{\partial X_3}{\partial z}\Big)\partial_y \wedge \partial_z.\end{aligned} \tag{36}$$

From the equations (34)–(36), we have

$$\begin{aligned}[X, \pi] = \Big(&X_1\frac{\partial \pi_{12}}{\partial x} + X_2\frac{\partial \pi_{12}}{\partial y} + X_3\frac{\partial \pi_{12}}{\partial z} - \pi_{12}\Big(\frac{\partial X_1}{\partial x} + \frac{\partial X_2}{\partial y}\Big) + \pi_{23}\frac{\partial X_1}{\partial z} \\ &- \pi_{13}\frac{\partial X_2}{\partial z}\Big)\partial_x \wedge \partial_y + \Big(X_1\frac{\partial \pi_{13}}{\partial x} + X_2\frac{\partial \pi_{13}}{\partial y} + X_3\frac{\partial \pi_{13}}{\partial z} - \pi_{12}\frac{\partial X_3}{\partial y} \\ &- \pi_{13}\Big(\frac{\partial X_1}{\partial x} + \frac{\partial X_3}{\partial z}\Big) - \pi_{23}\frac{\partial X_1}{\partial y}\Big)\partial_x \wedge \partial_z + \Big(X_1\frac{\partial \pi_{23}}{\partial x} + X_2\frac{\partial \pi_{23}}{\partial y} \\ &+ X_3\frac{\partial \pi_{23}}{\partial z} + \pi_{12}\frac{\partial X_3}{\partial x} - \pi_{13}\frac{\partial X_2}{\partial x} - \pi_{23}\Big(\frac{\partial X_2}{\partial y} + \frac{\partial X_3}{\partial z}\Big)\Big)\partial_y \wedge \partial_z.\end{aligned} \tag{37}$$

Since X is a Poisson vector field, one has $[X, \pi] = 0$, Equation (37) gives the system (28).

2) In order to prove (29), we set $[X, \pi] = \pi$ and the result comes from 1).

□

Corollary 2. *Consider the following Poisson tensor with respect to the Walker metric on* $\mathbb{R}^3$*:*

$$\pi = \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} + \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial z} + \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z}.$$

Let $X = X_1\partial_x + X_2\partial_y + X_3\partial_z$ *be a vector field on the three-dimensional real vector space* $\mathbb{R}^3$*.*

1) If X is a Poisson vector field then, the following equations are satisfied:

$$\operatorname{div}(X)\vec{\varepsilon} = \overrightarrow{\operatorname{grad}}(h), \quad \textit{with} \quad h = X_1 - X_2 + X_3, \tag{38}$$

2) If X is a Liouville vector field then, the following equations are satisfied:

$$(\operatorname{div}(X) + 1)\vec{\varepsilon} = \overrightarrow{\operatorname{grad}}(h), \quad \textit{with} \quad h = X_1 - X_2 + X_3, \tag{39}$$

where $\vec{\varepsilon} = (1, -1, 1)$, $\operatorname{div}(X)$ denote the divergence of X and $\overrightarrow{\operatorname{grad}}(h)$ the gradient vector field of h.

Furthermore, in the case that X is a Poisson or Liouville field, if and only if $h = X_1 - X_2 + X_3$ is a solution of the following differential system:

$$\frac{\partial h}{\partial z} = \frac{\partial h}{\partial x} = -\frac{\partial h}{\partial y}. \tag{40}$$

Proof. It is evident that, in consideration of the system (28) of Theorem 2, the following equation can be established

$$\begin{cases} \frac{\partial X_1}{\partial x} - \frac{\partial X_1}{\partial z} + \frac{\partial X_2}{\partial y} + \frac{\partial X_2}{\partial z} = 0 \\ \frac{\partial X_1}{\partial x} + \frac{\partial X_1}{\partial y} + \frac{\partial X_3}{\partial y} + \frac{\partial X_3}{\partial z} = 0 \\ \frac{\partial X_2}{\partial x} + \frac{\partial X_2}{\partial y} - \frac{\partial X_3}{\partial x} + \frac{\partial X_3}{\partial z} = 0 \end{cases} \Longleftrightarrow \begin{cases} \frac{\partial X_1}{\partial x} + \frac{\partial X_2}{\partial y} + \frac{\partial X_3}{\partial z} - \frac{\partial X_3}{\partial z} - \frac{\partial X_1}{\partial z} + \frac{\partial X_2}{\partial z} = 0 \\ \frac{\partial X_1}{\partial x} + \frac{\partial X_1}{\partial y} + \frac{\partial X_3}{\partial y} + \frac{\partial X_3}{\partial z} + \frac{\partial X_2}{\partial y} - \frac{\partial X_2}{\partial y} = 0 \\ \frac{\partial X_1}{\partial x} - \frac{\partial X_1}{\partial x} + \frac{\partial X_2}{\partial x} + \frac{\partial X_2}{\partial y} - \frac{\partial X_3}{\partial x} + \frac{\partial X_3}{\partial z} = 0 \end{cases} \tag{41}$$

Then the system (41) becomes

$$\begin{cases} \operatorname{div}(X) - \frac{\partial h}{\partial z} = 0 \\ \operatorname{div}(X) + \frac{\partial h}{\partial y} = 0 \\ \operatorname{div}(X) - \frac{\partial h}{\partial x} = 0 \end{cases} \Longleftrightarrow \begin{cases} \operatorname{div}(X) = \frac{\partial h}{\partial z} \\ -\operatorname{div}(X) = \frac{\partial h}{\partial y} \\ \operatorname{div}(X) = \frac{\partial h}{\partial x} \end{cases}, \tag{42}$$

or, equivalently

$$\operatorname{div}(X)\vec{\varepsilon} = \overrightarrow{\operatorname{grad}}(h),$$

which represents the equality (38).

From system (29) and (41), we obtain

$$\begin{cases} \operatorname{div}(X) + 1 - \frac{\partial h}{\partial z} = 0 \\ \operatorname{div}(X) + 1 + \frac{\partial h}{\partial y} = 0 \\ \operatorname{div}(X) + 1 - \frac{\partial h}{\partial x} = 0 \end{cases} \Longleftrightarrow (\operatorname{div}(X) + 1)\vec{\varepsilon} = \overrightarrow{\operatorname{grad}}(h), \tag{43}$$

from which we have (39).

Moreover, using (42) and (43), we get

$$\frac{\partial h}{\partial z} = \frac{\partial h}{\partial x} = -\frac{\partial h}{\partial y},$$

which implies (40). □

Example 3. Let $X = x\partial_x + y\partial_y + z\partial_z$ be a vector field in a Poisson-Walker manifold equipped with a Poisson tensor as in Corollary 2. Then X is a Poisson vector field, and the associated function $h = x - y + z$ satisfies (40).

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