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Continuous Bending of Surfaces of Rotation into Helical Surfaces

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Abstract. When a surface undergoes bending, the length of any curve on it remains unchanged. In the general case, a curve on a surface can be defined by a functional relationship between curvilinear coordinates – that is, through an intrinsic equation. The length of such a curve can be computed using the first fundamental form. Since this length remains invariant under bending, the expression of the first fundamental form also remains unchanged. This invariance forms the foundation of the theory of surface bending. Bending of surfaces occurs under certain constraints on their deformation. For ruled surfaces – whether developable or non-developable – a typical constraint is the preservation of straight-line generatrices. A clear geometric example is the bending of developable surfaces while keeping their generatrices unchanged. In the case of non-developable surfaces, a non-ruled surface can be bent into a ruled one. A classic example is the bending of a surface of revolution, such as the catenoid, into a helical surface, such as the helicoid. The helicoid is a ruled surface; however, when its pitch is gradually decreased during the bending process, the surface becomes non-ruled. By continuously reducing the pitch, one can construct a one-parameter set of intermediate surfaces, making the bending process continuous. When the pitch reaches zero, the helicoid transforms into a catenoid. This example illustrates the bending of helical surfaces into surfaces of revolution. According to Bour's theorem, when a helicoidal surface is bent into a surface of revolution, the helical lines correspond to parallels, and their orthogonal trajectories correspond to meridians. The present work explores the inverse process – the bending of a surface of revolution into a helicoidal surface. The surface of revolution is defined via the explicit equation of its meridian. Parametric equations describing the one-parameter set of intermediate surfaces are derived, and several of these surfaces are constructed. The paper also considers the continuous bending of the catenoid into the helicoid.

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1 Introduction

Helical surfaces have wide applications in engineering practice and architecture. These surfaces may be developable or non-developable. A developable helicoid (also called a torsal helicoid) can be obtained by bending a flat development, which can be constructed explicitly. The analytical description of the bending process is based on the invariance of the curvature of the edge of regression. This approach is presented in [4], where a sector of a developable helicoid is gradually deformed by reducing its pitch.

In [6], the bending of a torsal helicoid is performed according to a prescribed law of variation of the lead angle of its edge of regression, while maintaining constant curvature. The lead angle may be constant or variable. Under gradual reduction of a constant elevation angle, the pitch of the torsal helicoid decreases, eventually transforming it into a flat development. A variable lead angle is also considered, in which case the pitch also becomes variable.

The bending of non-developable helical surfaces has been studied in [2, 5] with the aim of constructing approximate developments. In [5], a helical conoid is bent into a catenoid, while in [2], an open right helicoid is bent into a one-sheeted hyperboloid of revolution. The resulting surfaces are approximated by a frustum of a cone, whose development serves as an approximate flattening (development) of the original helical surface.

The bending of a catenoid into a helical conoid is described in fundamental works on differential geometry, for example, in [8]. In that work, the bending is based on the theory of minimal surfaces, to which both the helical conoid and the catenoid belong. A characteristic feature of this bending is that the family of intermediate configurations also consists of minimal surfaces. The possibility of bending minimal surfaces, including the bending of a catenoid into a helicoid, is examined in [7].

The present article proposes a different approach that is not restricted to minimal surfaces. In [1], the theory of bendings is combined with applied mechanics, while [3] considers infinitesimal bendings of surfaces generated by the sum of two spatial curves.

This article considers the inverse problem – the bending of a surface of revolution into a helical surface. If its meridian is defined by an explicit equation of the form $z = f(\rho)$, then the surface of revolution is described by the following parametric equations:

$$X = \rho \cos \alpha; \qquad Y = \rho \sin \alpha; \qquad Z = f(\rho), \tag{1}$$

where α and ρ are the curvilinear coordinates on the surface, with ρ representing the distance from the vertical axis of revolution to a point on the surface, and α being the angular coordinate.

The length of a curve on the surface can be found using the first fundamental form of the surface:

$$dS^2 = E\, d\alpha^2 + 2F\, d\alpha\, d\rho + G\, d\rho^2, \tag{2}$$

where the coefficients E, F, and G are found through the partial derivatives of Equations (1):

$$E = \left(\frac{\partial X}{\partial \alpha}\right)^2 + \left(\frac{\partial Y}{\partial \alpha}\right)^2 + \left(\frac{\partial Z}{\partial \alpha}\right)^2; \qquad F = \frac{\partial X}{\partial \alpha} \cdot \frac{\partial X}{\partial \rho} + \frac{\partial Y}{\partial \alpha} \cdot \frac{\partial Y}{\partial \rho} + \frac{\partial Z}{\partial \alpha} \cdot \frac{\partial Z}{\partial \rho};$$
$$G = \left(\frac{\partial X}{\partial \rho}\right)^2 + \left(\frac{\partial Y}{\partial \rho}\right)^2 + \left(\frac{\partial Z}{\partial \rho}\right)^2. \tag{3}$$

For the surface of revolution given by (1), the first fundamental form (2) takes the form:

$$dS^2 = [1 + f'^2(\rho)]\, d\rho^2 + \rho^2\, d\alpha^2. \tag{4}$$

The middle coefficient $F = 0$, which indicates that the coordinate net formed by parallels and meridians is orthogonal. If an intrinsic equation is specified in the form $\alpha = \alpha(\rho)$ or $\rho = \rho(\alpha)$, then a corresponding curve is defined on the surface (1), and its length can be determined using expression (4). However, during the bending process, the lengths of such curves remain unchanged. This implies that we must find a new set of parametric equations for the surface (1), according to which it transforms into a helical surface, while ensuring that the first fundamental form (4) remains unchanged.

2 Association of the Helical Surface with an Orthogonal Net Formed by Families of Coordinate Lines

The helical surface, into which the surface of revolution (1) will be bent, is represented in the form:

$$X = w \cos t; \qquad Y = w \sin t; \qquad Z = \varphi + ht, \tag{5}$$

where

- t and w are the independent variables of the surface;
- h is the helical parameter (a constant);
- $\varphi = \varphi(w)$ is the axial section of the helical surface (a function to be found).

When $h = 0$, the helical surface (5) transforms into a surface of revolution. It is necessary to find such a dependence $\varphi = \varphi(w)$ so that, when $h = 0$ – that is, when it is bent into a surface of revolution – it describes the same curve (meridian) as the function $f = f(\rho)$.

According to Formulas (3), we find the coefficients E, F, and G:

$$E = 1 + \varphi'^2; \qquad F = h\varphi'; \qquad G = w^2 + h^2. \tag{6}$$

The first fundamental form, according to (2), is written as:

$$dS^2 = (1 + \varphi'^2)\, dw^2 + 2h\varphi'\, dw\, dt + (w^2 + h^2)\, dt^2. \tag{7}$$

From (7), it is clear that the coordinate line net is not orthogonal, since $F \neq 0$. As previously noted, when bending the helical surface into a surface of revolution, the helical lines coincide with the parallels, and their orthogonal trajectories – with the meridians. Therefore, it follows that we need to transition to a rectangular (orthogonal) net of coordinate lines. One family of these lines is the family of helical lines, and the other is the family of their orthogonal trajectories. To find the second family, it is necessary to solve the following differential equation:

$$F\, dw + G\, dt = 0. \tag{8}$$

After the substitution of the values of the coefficients F and G from (6) into Equation (8), and taking into account that the variables are w and t, we obtain:

$$t = -h \int \frac{\varphi'}{w^2 + h^2}\, dw + \alpha, \tag{9}$$

where α is the constant of integration.

The constant of integration α can be assigned different numerical values, each corresponding to a specific line on the surface. These lines will be perpendicular to the family of helical lines. Therefore, the constant α can be considered as a new independent variable. In this case, we can write:

$$dt = d\alpha - \frac{h\varphi'}{w^2 + h^2}\, dw. \tag{10}$$

After the substitution of dt from (10) into the line element (7) and performing the simplification, we obtain:

$$dS^2 = \left(1 + \frac{w^2\varphi'^2}{w^2 + h^2}\right) dw^2 + (w^2 + h^2)\, d\alpha^2. \tag{11}$$

After this substitution, the middle coefficient F disappears, meaning that the coordinate line net is orthogonal. In this sense, the first fundamental form (11) is similar to the analogous form (4) for the surface of revolution. In both forms, the variable α plays the same role – it is the angle of rotation of the meridian for the surface of revolution and the corresponding curve for the helical surface. Further, it is necessary to transform the fundamental form (11) into the form of (4). This means that in the fundamental form (11) one needs to transition from the variable w to the variable ρ so that the first fundamental form remains unchanged for both surfaces.

The next task is to reduce the line element (11) to the form of (4), that is, to change variables in the line element (11) from w and t to α and ρ such that it remains invariant.

3 Reduction of the Surface of Revolution and the Helical Surface to a Common First Fundamental Form

Let us start by equating the right-hand sides of Equations (4) and (11): $\rho^2 = w^2 + h^2$. From this, we obtain:

$$\rho = \sqrt{w^2 + h^2}; \qquad d\rho = \frac{w\, dw}{\sqrt{w^2 + h^2}}. \tag{12}$$

Next, we equate the left-hand sides of Equations (4) and (11), using the value of the differential $d\rho$ from (12):

$$(1 + f'^2)\frac{w^2\, dw^2}{w^2 + h^2} = \left(1 + \frac{w^2\varphi'^2}{w^2 + h^2}\right) dw^2. \tag{13}$$

We solve Equation (13) with respect to ϕ in order to subsequently find the dependence $\varphi = \varphi(w)$:

$$\varphi' = \frac{d\varphi}{dw} = \frac{\sqrt{w^2 f'^2 - h^2}}{w}, \tag{14}$$

where

- φ' is the sought-after function describing the axial section curve of the helical surface;
- f' is the derivative of the explicit curve equation (the meridian of the given surface of revolution).

For further solution, we need a specific function $f = f(\rho)$, that is, the explicit meridian equation for the surface of revolution.

4 Examples

Let us consider the bending of certain surfaces of revolution into helical surfaces, including some well-known examples of such bendings.

Example 1

We start with a classical example. As a surface of revolution, we take the catenoid, whose meridian is a catenary curve. Its explicit equation and derivative have the following form:

$$f = a\,\mathrm{Arccosh}\left(\frac{\rho}{a}\right); \qquad f' = \frac{a}{\sqrt{\rho^2 - a^2}}. \tag{15}$$

To substitute (15) into (14), it is necessary to change variables to the variable w according to (12):

$$f' = \frac{a}{\sqrt{\rho^2 - a^2}} = \frac{a}{\sqrt{w^2 + h^2 - a^2}}. \tag{16}$$

After the substitution, we obtain:

$$\frac{d\varphi}{dw} = \frac{\sqrt{w^2 f'^2 - h^2}}{w} = \frac{1}{w}\sqrt{\frac{a^2 w^2}{w^2 + h^2 - a^2} - h^2}. \tag{17}$$

Let us return in (17) to the variable ρ. According to (12), we write:

$$w = \sqrt{\rho^2 - h^2}; \qquad dw = \frac{\rho\, d\rho}{\sqrt{\rho^2 - h^2}}. \tag{18}$$

Substituting expressions (18) into (17) and performing simplifications and transformations, we finally obtain:

$$\frac{d\varphi}{d\rho} = \frac{\rho^2}{\rho^2 - h^2}\sqrt{\frac{a^2 - h^2}{\rho^2 - a^2}}. \tag{19}$$

In the same way, we find the expression for the angle t. For this, we substitute (17) into (10) and, after simplification, we obtain:

$$dt = d\alpha - \frac{h}{w}\sqrt{\frac{a^2 - h^2}{(w^2 + h^2 - a^2)(w^2 + h^2)}}\, dw. \tag{20}$$

Let us transform (20) to the variable ρ. Substituting expressions (18) into (20) and simplifying, we obtain:

$$\frac{dt}{d\rho} = d\alpha - \frac{h}{\rho^2 - h^2}\sqrt{\frac{a^2 - h^2}{\rho^2 - a^2}}. \tag{21}$$

Equations (5) involve the independent variables w and t, as well as the dependency $\varphi = \varphi(w)$. After transforming the first fundamental form of the helical surface (5) from the variables w and t to the surface of revolution (1) with independent variables ρ and α, the following relations were obtained:

- according to (18):

$$w = \sqrt{\rho^2 - h^2}; \tag{22}$$

- according to (19):

$$\varphi = \int \frac{\rho^2}{\rho^2 - h^2} \sqrt{\frac{a^2 - h^2}{\rho^2 - a^2}}\, d\rho; \tag{23}$$

- according to (21):

$$t = \alpha - \int \frac{h}{\rho^2 - h^2} \sqrt{\frac{a^2 - h^2}{\rho^2 - a^2}}\, d\rho \tag{24}$$

Substituting (22), (23), and (24) into the equations of the helicoidal surface (5), we obtain the following parametric equations:

$$\begin{aligned} X &= \sqrt{\rho^2 - h^2} \cos\left(\alpha - h \int \frac{1}{\rho^2 - h^2} \sqrt{\frac{a^2 - h^2}{\rho^2 - a^2}}\, d\rho\right); \\ Y &= \sqrt{\rho^2 - h^2} \sin\left(\alpha - h \int \frac{1}{\rho^2 - h^2} \sqrt{\frac{a^2 - h^2}{\rho^2 - a^2}}\, d\rho\right); \\ Z &= \int \frac{\rho^2}{\rho^2 - h^2} \sqrt{\frac{a^2 - h^2}{\rho^2 - a^2}}\, d\rho - h^2 \int \frac{1}{\rho^2 - h^2} \sqrt{\frac{a^2 - h^2}{\rho^2 - a^2}}\, d\rho + h\alpha. \end{aligned} \tag{25}$$

Let us find the first fundamental form of the surface (25) using Formulas (2), (3):

$$dS^2 = \left[\frac{\rho^2}{\rho^2 - a^2}\right] d\rho^2 + \rho^2\, d\alpha^2. \tag{26}$$

If we substitute the derivative expression from (15) into the first fundamental form (4), it will exactly coincide with (26). The constant h does not appear in it, although it affects the shape of the surface. Therefore, this constant is a bending parameter.

Expressions (23) and (24) can be integrated:

$$\varphi = h \operatorname{Arctg}\left(\frac{h}{\rho} \sqrt{\frac{\rho^2 - a^2}{a^2 - h^2}}\right) + \sqrt{a^2 - h^2} \ln\left(\rho + \sqrt{\rho^2 - a^2}\right), \tag{27}$$

$$t = \alpha - \int \frac{h}{\rho^2 - h^2} \sqrt{\frac{a^2 - h^2}{\rho^2 - a^2}}\, d\rho = \alpha - \frac{1}{h} \operatorname{Arctg}\left(\frac{h}{\rho} \sqrt{\frac{\rho^2 - a^2}{a^2 - h^2}}\right). \tag{28}$$

In Figure 1, according to Equations (25) and taking into account (27) and (28), several intermediate surfaces during the bending of a catenoid with constant $a = 2$ are constructed.

In expressions (27) and (28), square roots are present, which imposes certain constraints on the pitch h and the range of the parameter ρ. The radius ρ can take values $\rho \geq a$. This is expected, since for $\rho < a$, the catenoid surface does not exist. There is also a restriction on the pitch h: $a > h > 0$. In general, when $h = 0$, Equations (25) should describe the catenoid surface, which they do quite accurately for $h = 0.001$ (Figure 1a). As the pitch h increases, the smallest parallel of the catenoid with radius a transforms into a helical line, which, in the limiting case, becomes a straight line – the axis of the surface (Figure 1d). This occurs at $h = a = 2$, which is approximated quite accurately at $h = 1.999$.

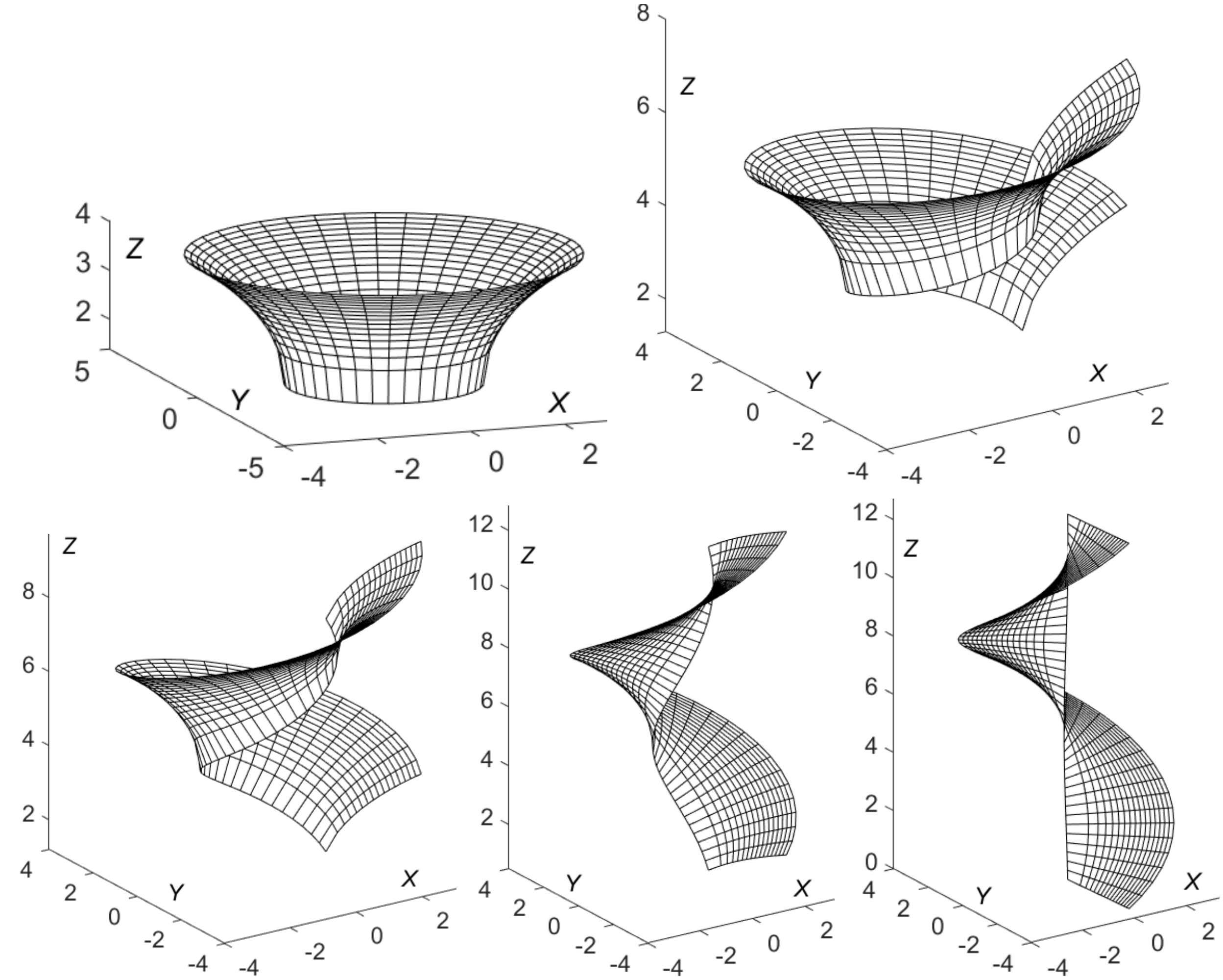

Figure 1: Intermediate surfaces as a result of bending the catenoid into a helicoidal conoid, obtained for different values of the parameter h: a) $h = 0.001$; b) $h = 0.5$; c) $h = 1$; d) $h = 1.75$; e) $h = 1.999$

Example 2

Let us replace the catenoid with a similar surface by introducing an additional constant b into the meridian (15). Then Equation (15) and its derivative take the following form:

$$f = b\,\mathrm{Arccosh}\left(\frac{\rho}{a}\right); \qquad f' = \frac{b}{\sqrt{\rho^2 - a^2}}. \tag{29}$$

When $a = b$, the modified surface of revolution becomes a catenoid.

As a result of performing the mathematical operations according to the developed algorithm, expressions for the integrals were obtained and then integrated:

$$\varphi = \int \frac{\rho}{\rho^2 - h^2}\sqrt{\frac{b^2(\rho^2 - h^2)}{\rho^2 - a^2} - h^2}\,d\rho = h\,\mathrm{Arctg}\left(\frac{h\sqrt{\rho^2 - a^2}}{\sqrt{(a^2 - b^2)h^2 + (b^2 - h^2)\rho^2}}\right)$$
$$+ \sqrt{b^2 - h^2}\,\ln\left[\sqrt{b^2 - a^2}\left(\sqrt{(b^2 - a^2)(\rho^2 - a^2)} + \sqrt{(a^2 - b^2)h^2 + (b^2 - h^2)\rho^2}\right)\right]. \tag{30}$$

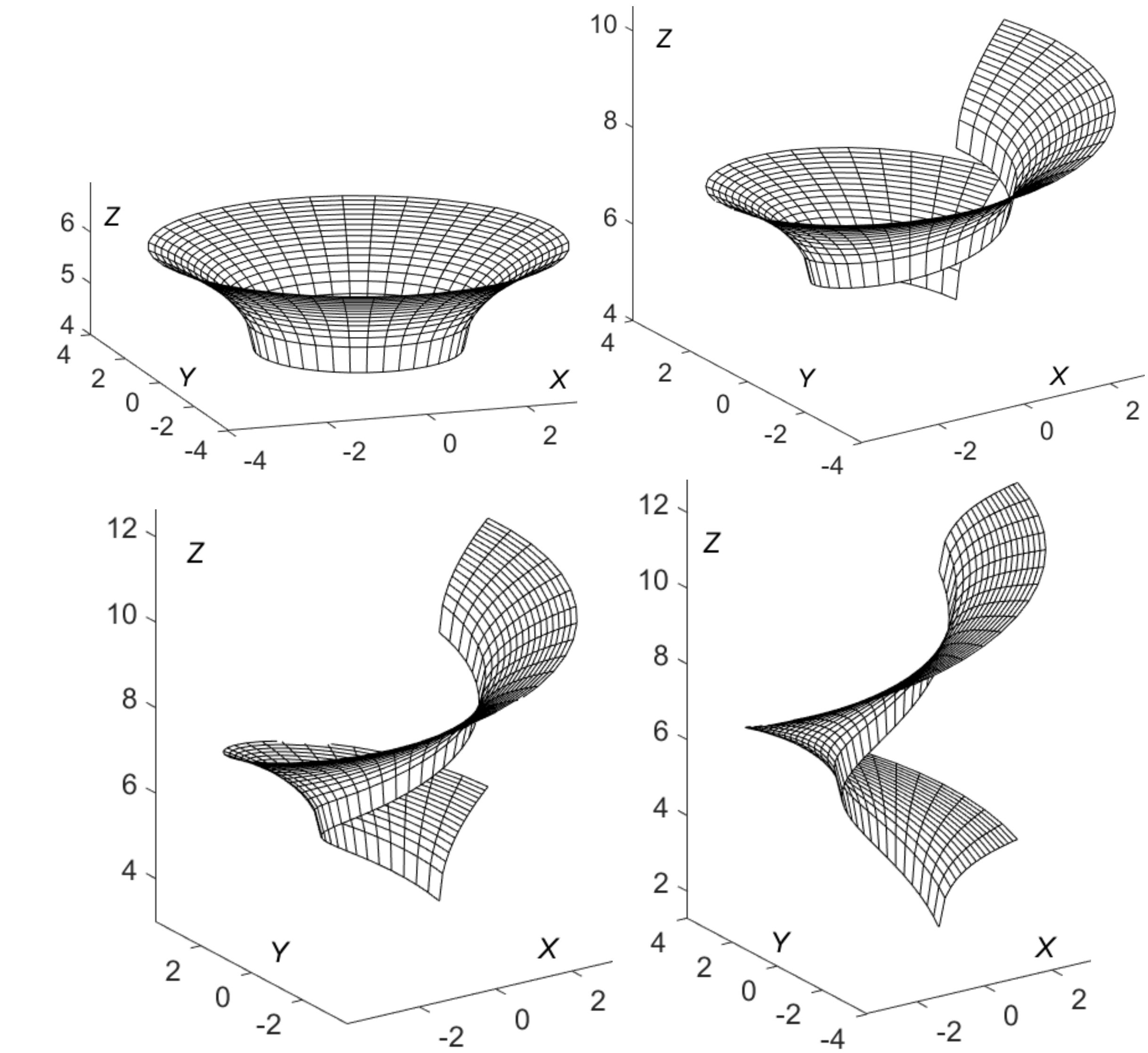

Figure 2: Intermediate surfaces resulting from the bending of the modified catenoid surface for various values of the parameter h: a) $h = 0.001$; b) $h = 0.5$; c) $h = 1$; d) $h = 1.499$

$$\begin{aligned} t &= \alpha - \int \frac{h}{\rho(\rho^2 - h^2)} \sqrt{\frac{b^2(\rho^2 - h^2)}{\rho^2 - a^2} - h^2}\, d\rho \\ &= \alpha - \frac{\sqrt{a^2 - b^2}}{a} \operatorname{Arctg}\left(\frac{a}{h\sqrt{a^2 - b^2}} \sqrt{\frac{b^2(\rho^2 - h^2)}{\rho^2 - a^2} - h^2} \right) - \operatorname{Arctg}\left(\frac{1}{h} \sqrt{\frac{b^2(\rho^2 - h^2)}{\rho^2 - a^2} - h^2} \right). \end{aligned} \tag{31}$$

Substituting these expressions into the parametric Equations (25) defines a set of surfaces – bendings of the surface of revolution of the modified catenoid.

This case also involves certain constraints: $b > h > 0$, $a > b$, $\rho \geq a$. Figure 2 shows selected surfaces from this set for $a = 2$, $b = 1.5$.

In this case, it is not possible to bend the surface of revolution in such a way that its smallest parallel transforms into the axis of the helicoidal surface.

Example 3

The meridian of the surface of revolution is a straight-line segment inclined at an angle β to the horizontal axis, meaning that the surface of revolution is a cone. The equation of the meridian and its derivative have the following form:

$$f = \rho \operatorname{tg} \beta; \qquad f' = \operatorname{tg} \beta. \tag{32}$$

As a result of further operations according to the developed algorithm, the following expressions were obtained:

$$\varphi = \int \frac{\rho\sqrt{\rho^2 \sin^2 \beta - h^2}}{(\rho^2 - h^2)\cos \beta}\, d\rho = \frac{\sqrt{\rho^2 \sin^2 \beta - h^2}}{\cos \beta} - h \operatorname{Arctg}\left(\frac{\sqrt{\rho^2 \sin^2 \beta - h^2}}{h \cos \beta}\right), \tag{33}$$

$$\begin{aligned} t &= \alpha - h \int \frac{\sqrt{\rho^2 \sin^2 \beta - h^2}}{\rho(\rho^2 - h^2)\cos \beta}\, d\rho \\ &= \alpha - \frac{1}{\cos \beta} \operatorname{Arctg}\left(\frac{\sqrt{\rho^2 \sin^2 \beta - h^2}}{h}\right) - \operatorname{Arctg}\left(\frac{\sqrt{\rho^2 \sin^2 \beta - h^2}}{h \cos \beta}\right). \end{aligned} \tag{34}$$

There are constraints on the range of the pitch h: $\rho \sin \beta > h > 0$. For the initial value $\rho = 2$ and angle $\beta = \pi/3$, the pitch h can vary within the interval $1.733 > h > 0$. During the bending process, the cone is transformed into a developable helicoid, whose pitch increases up to its maximum value.

At the maximum value, the smaller base of the frustum of a cone (shown with a bold line) becomes the edge of regression. The lead angle β of this edge (Figure 3d) is equal to the inclination angle of the cone's generatrices (Figure 3a). It should be noted that this type of bending of a developable surface differs from the traditional case, in which the straight generatrices maintain their direction along the surface and the length of the edge of regression remains unchanged during the bending process. In this case, however, the length of the edge of regression does change. For a cone, the edge of regression is a point (the apex of the cone). As the bending begins, it becomes a helical line whose pitch gradually increases up to $h = \rho \sin \beta$.

Example 4

The meridian is a parabola, meaning that the surface is a paraboloid of revolution. The equation and derivative of the meridian are given by:

$$f = a\rho^2; \qquad f' = 2a\rho. \tag{35}$$

The parametric equations describing the set of bent surfaces take the form:

$$\begin{aligned} X &= \sqrt{\rho^2 - h^2} \cos\left[\alpha - h \int \frac{\sqrt{4a^2\rho^2(\rho^2 - h^2) - h^2}}{\rho(\rho^2 - h^2)}\, d\rho\right]; \\ Y &= \sqrt{\rho^2 - h^2} \sin\left[\alpha - h \int \frac{\sqrt{4a^2\rho^2(\rho^2 - h^2) - h^2}}{\rho(\rho^2 - h^2)}\, d\rho\right]; \\ Z &= \int \frac{\rho\sqrt{4a^2\rho^2(\rho^2 - h^2) - h^2}}{\rho^2 - h^2}\, d\rho - h^2 \int \frac{\sqrt{4a^2\rho^2(\rho^2 - h^2) - h^2}}{\rho(\rho^2 - h^2)}\, d\rho + h\alpha. \end{aligned} \tag{36}$$

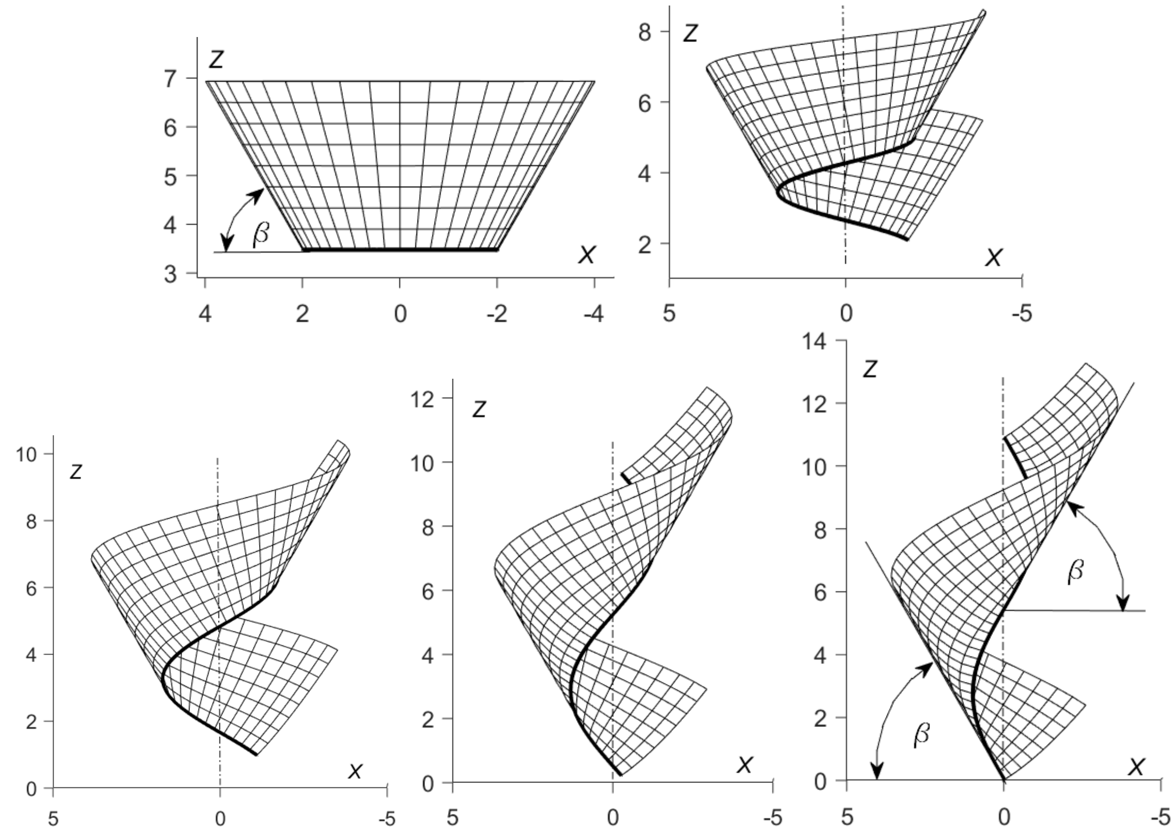

Figure 3: Intermediate surfaces resulting from the bending of a frustum of a cone, obtained for different values of the parameter h: a) $h = 0.001$; b) $h = 0.5$; c) $h = 1$; d) $h = 1.5$; e) $h = 1.732$

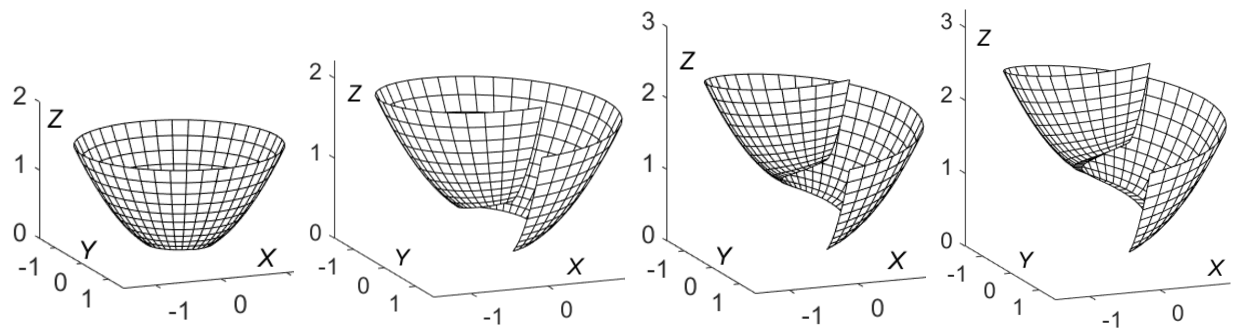

Figure 4: Intermediate surfaces resulting from the bending of a paraboloid of revolution, obtained for different values of the parameter h: a) $h = 0.001$; b) $h = 0.1$; c) $h = 0.2$; d) $h = 0.24$

The integrals appearing in Equations (36) can be integrated, but their expressions are omitted due to their cumbersome form. There is also a constraint on bending the paraboloid into a helicoidal surface, namely $\rho > h$. This means that a paraboloid of revolution can be bent into a helicoidal surface only if its lower part is “cut off”. Figure 4 shows several intermediate configurations of the surface for $a = 0.5$. In this case, there is also a limitation on the maximum value of the pitch h.

For this case, the maximum value of the pitch h is $h = 0.24$.

5 Conclusion

Any surface of revolution can be bent into a helicoidal surface. In this transformation, the parallels of the surface of revolution become helical lines, and the meridians become lines orthogonal to them. The coordinate net formed by these two families of curves remains orthogonal throughout the bending process. This process of bending can be intuitively visualized as a stretching of a turn (or loop) of the surface of revolution along its axis. In the resulting helicoidal surface, the pitch increases from zero to its maximum value. These limits depend on the shape of the original surface of revolution. In the classical case of bending a catenoid into a helicoidal conoid, the maximum pitch is equal to the length of the smallest parallel of the catenoid. For a frustum of a cone, the maximum pitch is determined by the inclination angle of its straight generatrix with respect to the horizontal plane. At this maximum pitch, the smaller base of the cone becomes the edge of regression of the developable helicoid into which the cone is bent. In this case, the lead angle of the edge of regression equals the angle of the generatrix of the original frustum of a cone. For other surfaces of revolution, the maximum pitch of the corresponding helicoidal surface depends on the surface shape and can be determined analytically from the mathematical description of the bending process.

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