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A Look at Archimedes' Book of Lemmas

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Abstract. In this work, the fifteen Lemmas of *Archimedes* are revisited, new proofs are given to Lemmas, and sometimes more than one proof is given. Generalizations of several Lemmas are added as *Remarks* that extend the scope of several basic lemmas and opens the way for researchers and college and high school students to look for new generalizations and proofs. As examples; in *Lemma 2* it is proved that if D is a point of a semicircle with diameter AB and the tangents at B and D meet at T, then AT bisects the perpendicular DE to AB and in Case (II) of *Remark 2(a)* it is proved that the conclusion of *Lemma 2* is also true for semi-ellipses and also as the point D moves from A to B, the locus of the midpoint of DE is a semi-ellipse. The relation with trisecting angles is discussed in *Lemma 8* and in *Remark 8(a)* it is shown that, by using *Lemma 8* and dynamic geometry, any angle can be trisected. Also trisected angles appeared in *Lemma 15*. In *Lemma 12* it is proved that if TP and TQ are tangents to a circle, AB a diameter, and PB and QA meet at R, then TR produced to meet AB at M, then $TM \perp AB$ and in *Remark 12(a)* this result of *Lemma 12* is extended to the case where diameter AB intersects PQ and the locus of R and locus of M as diameter AB changes position are determined. Visual diagrams are drawn as squares in Lemmas 4, 7, and 14, that make the proofs of these lemmas more feasible without computations. In *Remark 15(a)* the converse of *Lemma 15* is proved that shows that the conclusion that EM is equal to the radius of the semicircle Ω in *Lemma 15* is true only if the side AC is a side of a regular pentagon. Also in this remark a new method to inscribe a regular pentagon in a circle by compass and straight edge is proved.

Key Words: arbelos, salinon, trisecting angles, dynamic geometry

MSC 2020: 51M04 (primary), 01A20, 01A30

1 Introduction

The *Book of Lemmas* was first introduced in Arabic by *Thābit ibn Qurra* who lived in Baghdad (826 or 836-901) during the time of Abbasid Caliphate and he attributed the work to

Archimedes. The Arabic manuscript was first translated into Latin in 1661 under the name *Liber Assumptorum* by *Abraham Ecchellensis* and *Giovanni A. Borelli* [2]. *T.L. Heath* translated the Latin work of *Johan Ludvig Heiberg's* into English in [3]. An Arabic manuscript of *Thābit ibn Qurra's* is recently discovered in Suleymanye Manuscript Library in Istanbul and translated into English by *Emre Coşkun* [1].

The scope of this article is to see how these lemmas are related and to generalize some of these lemmas that show new concepts that will open the way for researchers and college and high school students and teachers to find new proofs and new generalizations that can be also to other conic sections, as in *Remark 2(a)* where the results of *Lemma 2* are also true for semi-ellipses, or to a higher dimension. Also it is good for students to know the *Euclid's Propositions* applied in the proofs of lemmas and remarks and to know the concept of *Dynamic Geometry*, as it is described in *Remark 8(a)* to trisect any given angle, that will help students to discover new generalizations and to solve some problems studying moving points on circular arcs or curves similar to some remarks as *Remark 3(a).* High school teachers can play an important role in helping students in this direction.

So, in this work we revisit the fifteen lemmas, we give sometimes more than one proof using elementary Euclidean Geometry and in the proofs we refer to the known *Propositions in Euclid's Elements* that are not mentioned in the proofs that appeared in [1] and for this a link is provided in [4]. Some visual diagrams are drawn to make the proofs more feasible without computations as in *Lemmas 4, 7,* and *14.* Several remarks for lemmas are stated as generalizations that show new concepts for these lemmas and opens the way for high school students and college students to find new proofs and new ideas.

Lemma 1 is the only *Lemma* that is used to prove other *Lemmas*, mainly to prove *Lemma 5 (Arbelos 2)* and *Lemma 6.* A second proof to *Lemma 2.* is a result of the proof of *Remark 2(a)* and for example four proofs are given for *Lemma 13. Remark 3(a)* is a generalization of *Lemma 3* so that P can be any point of any circular arc APB with base AB of a circle Ω and it is proved that if the circular arc with center P and passing through A meets Ω at Q and the line AB at D, and if the lines PD and QD meet Ω at C and G, respectively. Then $BD = BQ = BC$ and $GD = GC = GA$. The proof of *Lemma 4* can be seen easily from the drawn square $CDEF$ in Figure 4. In *Lemma 6*, it is proved that the quadrilaterals $CDJF$ and $RINP$ are squares. To prove *Lemma 7*, two squares are drawn in Figure 7 that show clearly that the area of the circle Ω is double the area of the circle Γ.

For *Lemma 10*, two proofs are given and it is generalized in *Remark 10(a)* by constructing two new circles, Γ with diameter OT that passes through A and B and Φ with diameter OB where O is the center of the circle Ω and TA and TB are tangents to Ω from T as shown in Figure 10(I). It is proved that if E is any point of Γ and the line EA produced meets Ω at D, then the line BD that meets Φ at H is parallel to the line TE and the line EH passes through O and is perpendicular to both TE and BD and EH bisects BD. *Lemma 12* is generalized in the *Remark 12(a)* to the case where the diameter AB intersects PQ. The converse of *Remark 15* is proved in *Remark 15(a)* that shows that the result of *Remark 15* is true only if the side AC is a side of a regular pentagon.

2 Archimedes' Book of Lemmas

Lemma 1. *If two circles touch at A externally or internally, and if CD, EF be parallel diameters in them, then FAD and EAC are straight lines in the external case and FDA and*

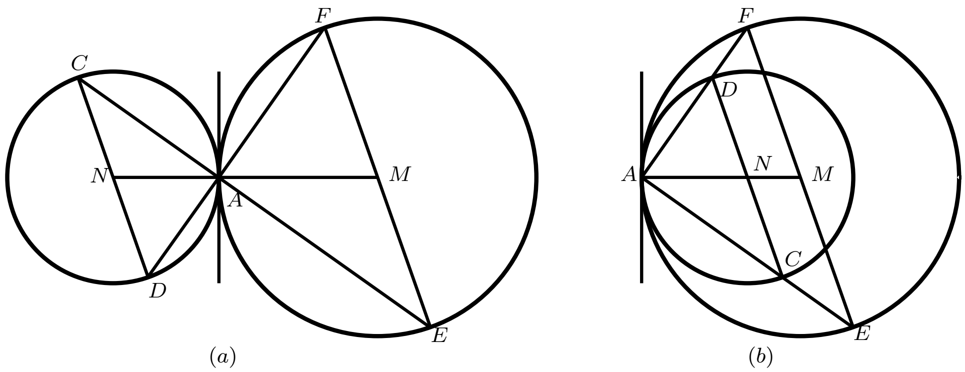

Figure 1: Illustrating the proof of Lemma 1

ECA are straight lines in the internal case as seen in Figure 1.

Proof. Let the circles Ω and Γ touch externally or internally at A as in Figure 1 and let their centers be N, M, respectively. Join AF, AD, AM and AN. Since AM and AN are lines joining the centers of the two circles to the point of contact, it follows by *Euclid's Proposition III-12* that MAN in case (a) and MNA in case (b) are straight lines. We want to prove that FAD in case (a) and FDA in case (b) are straight lines. Suppose that FAD in case (a) and FDA in case (b) are not straight lines. So, let FA produced in case (a) and FA in case (b) meet the ray $\overrightarrow{ND}$ at D'. Since, by *Euclid's Proposition I-29*, the two lines MAN and FAD' meeting at A and falling on the parallel lines MF and ND' make the alternate angles and opposite angles of $\triangle MFA$ and $\triangle ND'A$ equal one another and hence these two triangles are equiangular (i.e. similar). Thus by *Euclid's Proposition VI-4* we get $\frac{MF}{ND'} = \frac{MA}{NA}$. But $MF = MA$. Therefore $ND' = NA = ND$. So, $D' = D$ and FAD in case (a) and FDA in case (b) are straight lines and similarly EAC in case (a) and ECA in case (b) are also straight lines as required. □

Note that Lemma 1 is used in proving Lemmas 5 and 6.

Lemma 2. *Let AB be the diameter of the semicircle Ω, and let the tangents to it at B and at any other point D on it meet in T. If now DE be drawn perpendicular to AB, and if AT, DE meet at F, then $DF = FE$.*

Proof. Let AD and BT produced meet at K. Since $\angle ADB$ is a right angle by *Euclid's Proposition III-31* and $TD = TB$ by *equal tangents theorem*, we have $\angle TBD = \angle TDB$ by *Euclid's Proposition I-5*. But each of the two sums $\angle TDK + \angle TDB$ and $\angle TKD + \angle TBD$ is a right angle and hence $\angle TDK = \angle TKD$. Thus $TK = TD = TB$ by *Euclid's Proposition I-6*. Since we have $ED \parallel BK$, we have $\triangle ADF$ and $\triangle AKT$ are equiangular (i.e. similar) and also $\triangle AFE$ and $\triangle ATB$ are equiangular. Hence by *Euclid's Proposition VI-4*, we have

$$\frac{DF}{TK} = \frac{AF}{AT} = \frac{FE}{TB} \text{ but } TK = TB. \text{ Therefore}$$
$$DF = FE \text{ as required.} \quad \square$$

Note that the next *Remark 2(a)* leads to a new proof and a generalization of *Lemma 2* where the semicircle Ω is replaced by a semi-ellipse.

Remark 2(a). Case(I). Let AB be the diameter of the semicircle Ω, and let the tangents to it at A and B meet the tangent at any other point D in T and C as shown in Figure 2(a)(I).

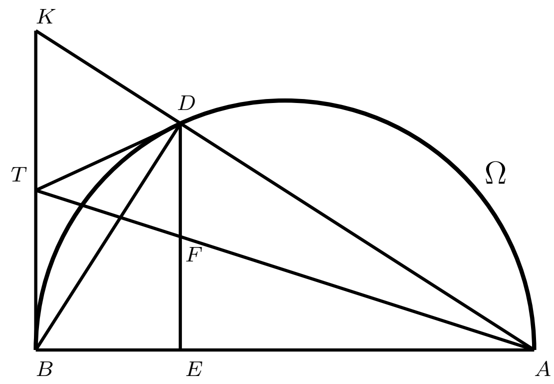

Figure 2: Illustrating the proof of Lemma 2

If AT and BC meet at F and DF produced meets AB at E, then DE is perpendicular to AB and $DF = FE$ and as the point D moves on the semicircle Ω from A to B, the point F moves from A to B on the semi-ellipse Γ.

Proof. Since the diameter AB is perpendicular to the tangents TB, CA by *Euclid's Proposition III-18,* it follows that the sum of the interior angles $\angle TBA$ and CAB is two right angles and hence $TB \parallel CA$ by *Euclid's Proposition I-28.* Also by *equal tangents theorem,* we have $TB = TD$ and $CA = CD$. Therefore $\frac{CD}{TD} = \frac{CA}{TB}$. So, since TB is parallel to CA and AT and BC are cutting these parallel lines, it follows that the alternate angles $\angle BTF$ and $\angle CAF$ are equal and also $\angle TBF = \angle ACF$. Thus the triangles $\triangle TBF$ and $\triangle ACF$ are equiangular (i.e. similar) and hence by *Euclid's Proposition VI-4,* we have

$$\frac{AC}{TB} = \frac{AF}{TF} = \frac{CF}{BF}. \quad \text{So, } \frac{CD}{TD} = \frac{AC}{TB} = \frac{AF}{TF} \text{ and } DF \parallel CA \text{ by } \textit{Euclid's Proposition VI-2.}$$

Therefore, $EF \parallel CA$ and the opposite angle in triangles $\triangle TDF$ and $\triangle TCA$ and also in $\triangle BFE$ and $\triangle BCA$ are equal by *Euclid's Proposition I-29* and hence $\triangle TDF$ and $\triangle TCA$ are equiangular and also $\triangle BFE$ and $\triangle BCA$ are equiangular and $\angle FEB$ is a right angle as wanted.

Thus by *Euclid's Proposition VI-4,* we have

$$\frac{CA}{DF} = \frac{TA}{TF} = \frac{AF}{TF} + 1 \text{ and } \frac{CA}{EF} = \frac{CF}{BF} + 1. \quad \text{But } \frac{AF}{TF} = \frac{CF}{BF}. \quad \text{So, } \frac{CA}{DF} = \frac{CA}{EF}$$

and hence $DF = EF$ as required.

So, we conclude from all of above that DE is perpendicular to AB and $DF = FE$ as required.

Note that we conclude from Remark 2(a) that if DE is perpendicular to AB, then DE is bisected by AT and by BC as wanted in *Lemma 2* and in fact this part of *Remark 2(a)* and *Lemma 2* are equivalent.

Next, to prove that the locus of the point F is the semi-ellipse Γ as D moves from A to B on the semicircle Ω with center M, let N be the midpoint of Ω, L the midpoint of NM, and F_1 and F_2 are points of AB such that $LF_1 = LF_2 = AM$. Then we prove that the semi-ellipse Γ, with semi-major axis AM, semi-minor axis LM, and the two foci points F_1 and F_2, passes through the midpoint F of DE as D moves from B to A on Ω. So, let $AM = 2m$ and $ME = 2c$. Then $LM = m, LF_1 = 2m = AM = DM$ and by Pythagorean theorem applied to the right triangles $\triangle LMF_1$, $\triangle DEM$, $\triangle FEF_1$, and $\triangle FEF_2$ we get,

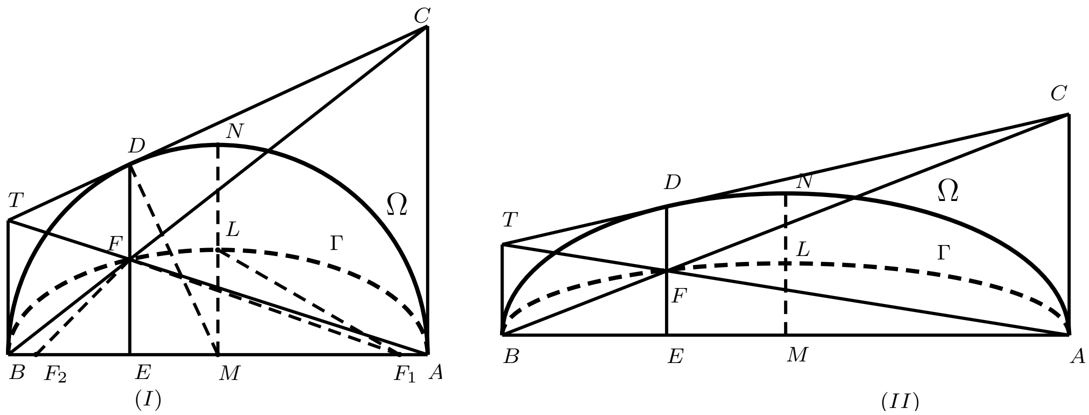

Figure 2(a): Illustrating the proof of Remark 2(a)

$(MF_1)^2 = 3m^2$, $EF_1 = m\sqrt{3}+2c$, $EF_2 = m\sqrt{3}-2c$, $DE = 2\sqrt{m^2-c^2}$, and $FE = \sqrt{m^2-c^2}$. Thus $(FF_1)^2 = (FE)^2 + (EF_1)^2 = (4m^2+3c^2) + 4\sqrt{3}mc$, $(FF_2)^2 = (FE)^2 + (EF_2)^2 = (4m^2+3c^2) - 4\sqrt{3}mc$, and hence $(FF_1)(FF_2) = \sqrt{(4m^2+3c^2)^2 - 48m^2c^2} = 4m^2 - 3c^2$ and $(FF_1)^2 + (FF_2)^2 = 8m^2 + 6c^2$. Therefore $(FF_1 + FF_2)^2 = (FF_1)^2 + (FF_2)^2 + 2(FF_1)(FF_2) = 16m^2$ and hence $FF_1 + FF_2 = 4m = AB$, the major axis of the semi-ellipse Γ. Thus we conclude that the semi-ellipse Γ passes through the midpoint F of DE as D moves on Ω from B to A as required. □

Note that we get the same result by using the algebraic equation $x^2 + y^2 = a^2, y > 0$ of the semicircle Ω, when the origin of the coordinate system is the midpoint of AB, the x axis is the line AB, and radius of Ω is $a = \frac{AB}{2}$. So, if the coordinates of $D = (x, y)$, then the coordinates of $F = (x, \frac{y}{2})$. So, when $D = N = (0, a)$, we have $F = L = (0, \frac{a}{2})$ and it is clear that the equation of the semi-ellipse Γ, with semi-major axis a and semi-minor axis $\frac{a}{2}$, is $x^2 + 4y^2 = a^2, y > 0$ for if $x = c$, then $D = (c, \sqrt{a^2-c^2})$ is a point of Ω and $F = (c, \frac{\sqrt{a^2-c^2}}{2})$ is a point of Γ as required.

Case (II). Let AB be the major axis of the semi-ellipse Ω, and let the tangents to it at A and B meet the tangent at any other point D of Ω at C and T as shown in Figure 2(a)(II). Then BC and AT intersect at the midpoint F of the perpendicular DE to AB and the point F moves from A to B on the semi-ellipse Γ as D moves on the semi-ellipse Ω from A to B.

Proof. Let the coordinate system be as in Case (I), the semi-major axis of the semi-ellipse Ω be equal to a and its semi-minor axis be equal to b. Then the algebraic equation of the semi-ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, a > b > 0, y > 0$ and the slope of the tangent TC at $D = (x, y)$ is equal $\frac{-b^2x}{a^2y}$. Thus if $D = (c, d)$, then $d = \frac{b}{a}\sqrt{a^2-c^2}$ and the equation of the tangent line CT is given by $\frac{y-d}{x-c} = -\frac{b^2c}{a^2d}$ and hence the coordinates of the points C and T are given by $C = \left(a, \frac{a^2d^2+b^2c^2-acb^2}{a^2d}\right)$ and $T = \left(-a, \frac{a^2d^2+b^2c^2+acb^2}{a^2d}\right)$. So, if $DE \perp AB$, then the midpoint of DE is the point $F = (c, \frac{d}{2}) = (c, \frac{b}{2a}\sqrt{a^2-c^2})$. We want to show that the lines BC and AT passes through the point F as D moved from A to B. Since the slope of BC is equal $\frac{AC}{AB} = \left(\frac{a^2d^2+b^2c^2-acb^2}{2a^3d}\right)$ and $B = (-a, 0)$, it follows that equation of BC is $y = \left(\frac{(x+a)(a^2d^2+b^2c^2-acb^2)}{2a^3d}\right)$. But $F = (c, \frac{d}{2})$. So, we substitute $x = c, y = \frac{d}{2}$ in the equation of BC and we get $a^3d^2 = (a+c)(a^2d^2 + b^2c^2 - acb^2)$ which is equivalent to $a^2d^2c + (a+c)(b^2c^2 - acb^2) = 0$. But $d^2 = \frac{b^2}{a^2}(a^2-c^2)$. Therefore $b^2c(a^2-c^2) + b^2c(a+c)(c-a) =$

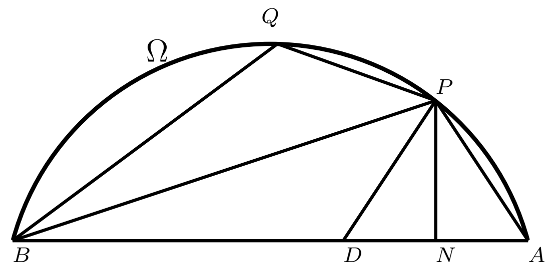

Figure 3: Illustrating the proof of Lemma 3

$b^2c(a^2-c^2+c^2-a^2)=0$ as wanted. Thus we conclude that BC passes through F and similarly AT passes through F as required.

Next, we prove that the locus of the point F is the semi-ellipse Γ as D moves from A to B on the semi-ellipse Ω. Since the algebraic equation of the semi-ellipse Ω is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, y>0$. So, if the coordinates of $D=(x,y)$, then the coordinates of $F=(x,\frac{y}{2})$. So, when $D=(0,b)$, we have $F=(0,\frac{b}{2})$. Therefore the equation of the semi-ellipse Γ, with semi-major axis a and semi-minor axis $\frac{b}{2}$, is $\frac{x^2}{a^2}+\frac{4y^2}{b^2}=1$, $y>0$. Thus if $x=c$, then $D=(c,\frac{b}{a}\sqrt{a^2-c^2})$ is a point of Ω and $F=(c,\frac{b}{2a}\sqrt{a^2-c^2})$ satisfies the equation of Γ and Γ is the locus of F as required. □

Lemma 3. *Let P be any point on a segment of a circle Ω whose base is AB, and let PN be perpendicular to AB. Take D on AB so that $AN=ND$. If PQ is an arc equal to the arc PA and BQ joined, then BQ, BD shall be equal.*

Proof. Join PA, PD, PB, and PQ, as shown in Figure 3. Since PN is the perpendicular bisector of AD, it follows that $\triangle PNA \cong \triangle PND$ by *Euclid's proposition I-4* (i.e. SAS rule) and hence $\angle PDA=\angle PAB$. But $\angle PDA+\angle PDB$ is equal to two right angles and $\angle PAB+\angle PQB$ is equal to two right angles by *Euclid's proposition III-22.* Therefore, $\angle PDB=\angle PQB$. But $PA=PQ$ and hence $\angle PAQ=\angle PQA$. Also by *Euclid's proposition III-21,* we have $\angle PBA=\angle PQA$ as angles in the same segment with base PA and $\angle PBQ=\angle PAQ$ as angles in the same segment with base PQ. Thus $\angle PBA=\angle PBQ$. So, $\triangle PDB \cong \triangle PQB$ by *Euclid's proposition I-26* (i.e. ASA rule) and hence $QB=DB$ as required. □

Note that in *Lemma 3* and also as it appeared in [1], for the point Q to be a point of the arc PB of Ω and the point D to be between A and B of AB, it is necessary that P is a point between A and the midpoint M of Ω and the angle $\angle AMB$ is greater or equal to a right angle.

In the next *Remark 3(a)* we extend *Lemma 3* so that P can be any point of any circular arc APB with base AB of a circle Ω, Q a point of Ω, and D is a point of the line AB.

Remark 3(a). Let AB be the base of a circular arc AMB, with midpoint M, of a circle Ω with center O and P a point of the segment AMB and MO produced meet Ω at L. Let the circular arc with center P and passing through A meets Ω at Q and the line AB at D. Also let the lines PD and QD meet Ω at C and G, respectively. Then $BD=BQ=BC$ and $GD=GC=GA$.

Proof. We consider the following two cases:

Case (I). The segment AMB of Ω with base AB and $\angle AMB$ is greater than or equal to a right angle.

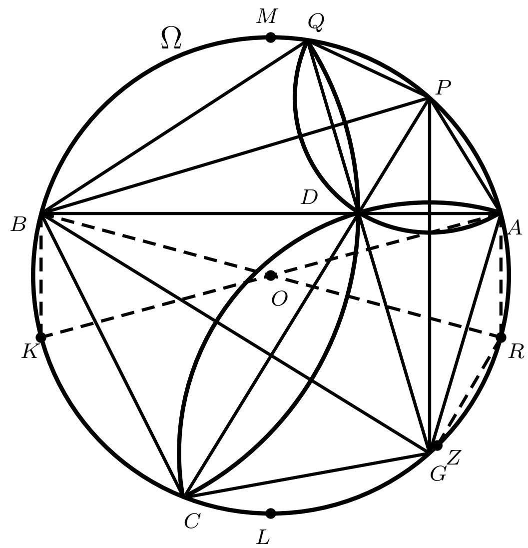

Figure 3(I(i)): Illustrating the proof of I(i)

I(i). Let P be a point between A and M and we refer to Figure 3(I(i)). Then as P approaches M, the point Q approaches B and D approaches B on AB. Also C and G are points of the major arc of Ω with base AB. So, it follows from Lemma 3 that $BD = BQ$. Since $\angle BDC = \angle PDA$ and $PA = PD$, we have $\angle BDC = \angle PDA = \angle PAD = \angle PAB$. But $\angle PAB = \angle PCB = \angle DCB$ as angles in the same segment with base PB. Therefore $\angle BDC = \angle DCB$. Thus $BD = BC$ and we conclude that $BD = BQ = BC$ as required.

Similarly, since $PQ = PD$, we have $\angle PQD = \angle PDQ$. But $\angle PQG = \angle GCP$ as angles in the same segment with base PG and $\angle PDQ = \angle GDC$. Therefore $\angle PQD = \angle GCD$ and hence $\angle GCD = \angle GDC$. Thus $GD = GC$. Also, by Lemma 3 we have $BD = BQ$ and hence $\angle BQD = \angle BDQ$. But $\angle BDQ = \angle GDA$ and $\angle BQG = \angle BAG$ as angles in the same segment with base BG. Thus $\angle GDA = \angle GAD$ and hence $GD = GA$. So we conclude that $GC = GD = GA$ as required.

Note that as P approaches M, the points Q, D, and C approaches B, and G approaches L and PG is the perpendicular bisector of AD. So, at $P = M$ we have, $B = D = Q = C$ and $G = L$ and hence $BD = BQ = BC = 0$ and $GA = LA = LB = GD = GC$ as required. Also, as P approaches A, the points Q and D approaches A, the point G approaches R where RA is perpendicular to AB, and the point C approaches Z where $RA = RZ$, RZ is perpendicular to BZ, and $BZ = BA$. So, at $P = A$ we have, $A = D = Q$, $G = R$,and $C = Z$ and hence $PA = PQ = PD = 0$, $BD = BQ = BC = BA$, and $RZ = RD = RA$ as required.

Also it is clear that each point of D, P, B, G is the orthocenter of triangle formed by the other three points.

We can also conclude from Figure 3(I(i)) that if L is the midpoint of the major circular arc Ω with base AB and the diameters AK and BR meet Ω at K and R, respectively, and the point G between R and L is the center of the circular arc ADC (i.e. $GA = GD = GC$), then B and P are centers of the circular arcs CDQ and QDA, respectively. First we prove that $BC = BD = BQ$. Since $GA = GD = GC$, it follows as in the proof of Lemma 3 that BG bisects the angles DBC and DGC and hence the triangles GBD and GBC are congruent. Thus $BC = BD$. Also from the isosceles triangle GAD we have the equal base angles $\angle GDA = \angle GAD$ by *Euclid's Proposition I-5* and since $\angle GDA = \angle BDQ$ and $\angle GAD = \angle GAB = \angle BQG = \angle BQD$ as equal angles in the same segment with base BG

by *Euclid's Proposition III-21,* it follows that $\angle BQD = \angle BDQ$ and hence $BD = BQ$. Thus $BC = BD = BQ$ as required.

Similarly, from the isosceles triangle CGD we have $\angle GDC = \angle GCD$, $\angle CDG = \angle QDP$, and $\angle PCG = \angle GQP$ as equal angles in the same segment with base PG, Thus $\angle PDQ = \angle PQD$ and hence $PD = PQ$. Also, from the isosceles triangle CBD we have $\angle BDC = \angle BCD$, $\angle CDB = \angle PDA$, and $\angle BCP = \angle BAP$ as equal angles in the same segment with base PB. So, $\angle PDA = \angle PAD$ and hence $PA = PD$. Thus $PA = PD = PQ$ as required.

I(ii). Let P be a point between M and B and we refer to Figure 3(I(ii)). Note that as P moves from M to B on the arc MB, the point C moves from B to A on the arc BMA and if $P = C$, then DP is tangent to Ω at P, D moves on the ray AB from B to a point H so that $BH = BA$, G moves from L on the minor arc LB to a point K so that AK is a diameter of Ω and PG is the perpendicular bisector of AD. The point Q moves from B to the point N where HK produced meets Ω and if $G = Q$, then DG is tangent to Ω at G.

First, we assume that $PA = PD = PQ$ and prove that $BD = BQ = BC$ and $GA = GD = GC$. From the isosceles triangles PDQ and PAD, we get $\angle PDQ = \angle PQD$ and $\angle PDA = \angle PAD$. But $\angle PCB = \angle PAB = \angle PQB$ as equal angles in the same segment with base PB and $\angle PAD = \angle PDA$ as base angles of the isosceles triangle PAD. Thus $\angle PDB = \angle PQB$, $\angle PDQ = \angle PQD$, and $\angle BDC = \angle BCD$ and hence $\angle BDQ = \angle BQD$ and $\angle BDC = \angle BCD$. Therefore the triangles BDQ and BDC are an isosceles triangles and $BD = BQ = BC$ as required.

Next, we prove that $GA = GD = GC$. Since $PQ = PD$, we have $\angle PDQ = \angle PQD$. But $\angle PQD = \angle PQG = \angle PCG$ as angles in the same segment with base PG. Therefore $\angle PDG = \angle PCG$ and hence $GD = GC$. Since $GD = GC$ and $BD = BC$, it follows that $\angle GDC = \angle GCD$ and $\angle BDC = \angle BCD$ and hence $\angle GDB = \angle GCB$. But $\angle GCB = \angle GAB$ as angles in the same segment with base BG. Therefore $\angle GDA = \angle GDB = \angle GAB = \angle GAD$ and hence $GA = GD$. Thus $GA = GD = GC$ as required.

Similarly, as in **I(i)** we can prove that if G is a point between L and K of the major arc ALB and $GA = GC = GD$, then $BD = BQ = BC$ and $PA = PQ = PD$. Since $GD = GA$, we have $\angle GDB = \angle GAB$. But $\angle GAB = \angle GQB$. Thus $\angle GDB = \angle GQB$ and hence $BD = BQ$, Also, since $GD = GA$, we have $\angle GDB = \angle GAB$. But $\angle GAB = \angle GCB$ as angles of the same segment with base BG. Therefore $\angle GDB = \angle GCB$ and since $GC = GD$, we have $\angle GDC = \angle GCB$. Thus $\angle BDC = \angle BCD$ and hence $BD = BC$ and $BD = BQ = BC$ as required.

Also, since $BD = BC$, we have $\angle BDP = \angle BDC = \angle BCD = \angle BCP$. But $\angle BCP = \angle BAP$ as angles in the same segment with base PB. Thus $\angle BDP = \angle BAP$ and hence $\angle PDA = \angle PAD$ and $PA = PD$. So, since $GA = GD$ and $PA = PD$, the triangles GAP and GDP are congruent and hence $\angle PGA = \angle PGD$. But $\angle PGA = \angle PQA$ as angles of the same segment with base PA and $\angle PGD = \angle PAQ$ for the sum of the two angles PGD and PGQ is two right angles as the two opposite angles PGQ and PAQ of the cyclic Quadrilateral $PGQA$ by *Euclid's proposition III-22.* Therefore $\angle PQA = \angle PAQ$ and hence $PA = PQ$ and $PA = PQ = PD$ as required.

Case (II). The segment ALB of Ω with base AB, midpoint L, and $\angle ALB$ is less than a right angle. We proved in **I(i)** and **I(ii)** and as shown in Figures 3(I(i)) and 3(I(ii)), that if the point G is between R and K and $GA = GD = GC$, then $BD = BQ = BC$ and $PA = PD = PQ$. So, to complete the proof for P as a point of the major circular arc ALB, we consider P between A and R and P between K and B.

II(i). Let P be a point between A and R and $PA = PQ = PD$ and we refer to Figure 3(II(i)).

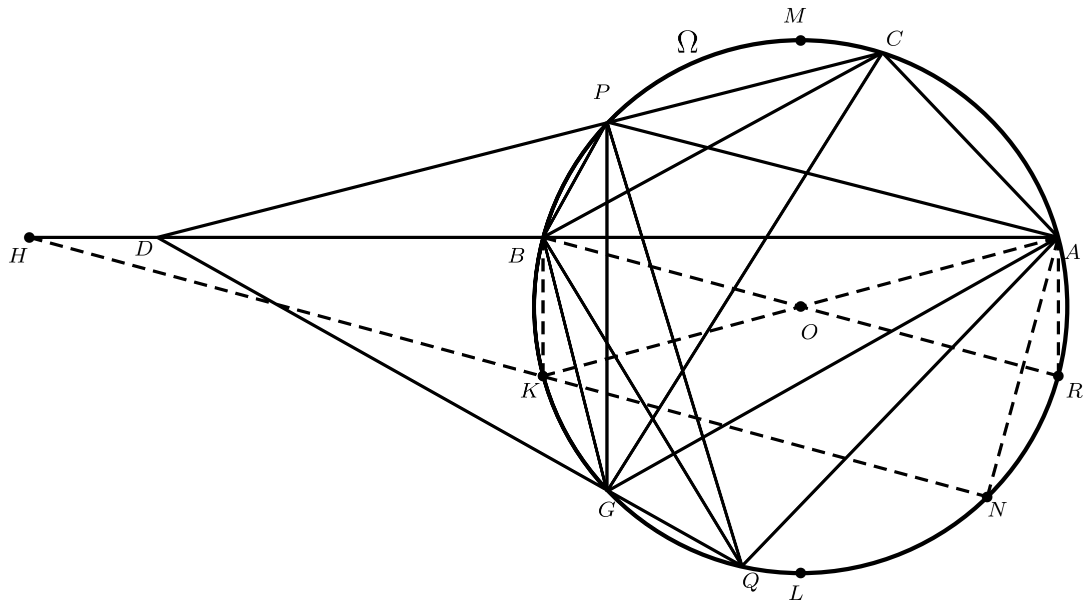

Figure 3(I(ii)): Illustrating the proof of I(ii)

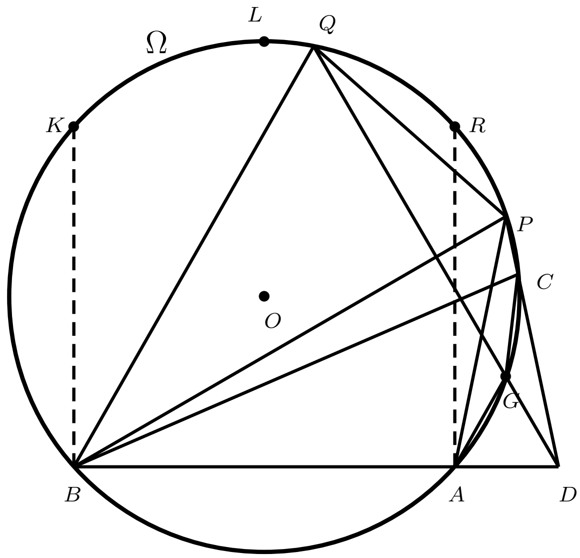

Figure 3(II(i)): Illustrating the proof of II(i)

Since $PD = PA$, we have $\angle PDA = \angle PAD$. But by *Euclid's proposition III-22* we have the sum of each of the two opposite angles PAB and PQB of $PABQ$ and the two opposite angles PCB and PQB of $PCBQ$ is two right angles and hence $\angle PAD = \angle PQB = \angle DCB$. Thus $\angle PDB = \angle PDA = \angle PQB = \angle DCB$. So, from $\angle PDB = \angle PCB$ we get $BD = BC$ and from $PD = PQ$ we get $\angle PDQ = \angle PQD$. Thus from $\angle PDB = \angle PQB$ and $\angle PDQ = \angle PQD$, we get $\angle BQD = \angle BDQ$ and hence $BQ = BD = BC$ as required.

Also, applying *Euclid's proposition III-22* to the cyclic quadrilateral $GQBA$ we get that $\angle GAD = \angle GQB$. But $BD = BQ$. Thus $\angle ADG = \angle BDQ = \angle BQD = \angle GQB = \angle GAD$ and hence $GA = GD$. Since $\angle GCD = \angle GAP$ as exterior angle to the cyclic quadrilateral $GAPC$, $\angle GAP = \angle GQP$ as angles of the same segment with base GP, and $\angle GQP = GDP$ for $PQ = PD$, we have $GC = GD$. Thus we conclude that $GA = GD = GC$ as required.

II(ii). Let P be a point between K and B and $PA = PQ = PD$ and we refer to Fig-

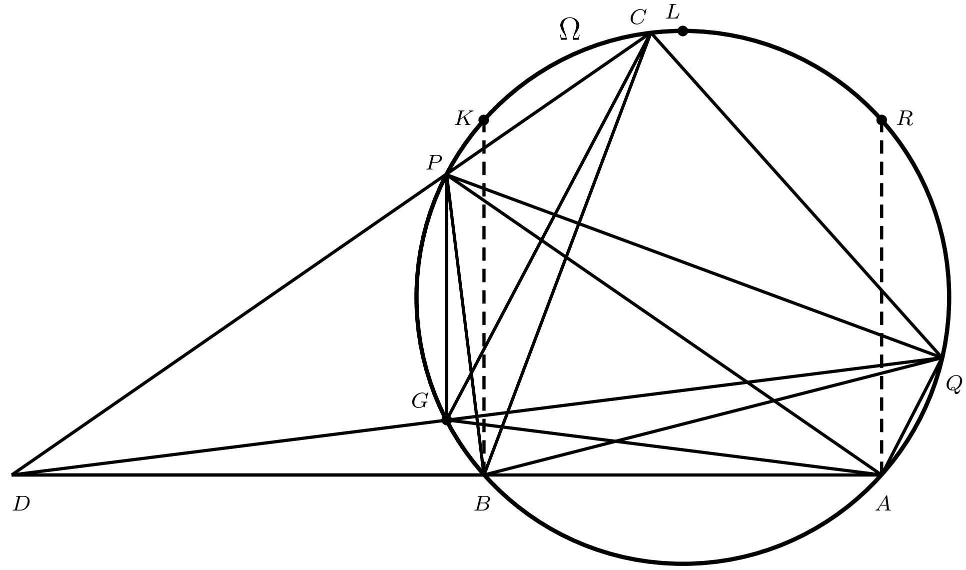

Figure 3(II(ii)): Illustrating the proof of II(ii)

ure 3(II(ii)). Since $PA = PD = PQ$, we have $\angle PDA = \angle PAD$ and $\angle PDG = \angle PDQ = \angle PQG$. But $\angle PQG = \angle PAG$ as angles in the same segment with base PG. Therefore $\angle PDA = \angle PAD$ and $\angle PDG = \angle PAG$ and hence $\angle GDA = \angle GAD$. Thus $GD = GA$. Also, since $\angle PDG = \angle PAG$ and $\angle PAG = \angle PCG$ as angles in the same segment with base PG, we have $\angle PDG = \angle PCG$. So, $\angle CDG = \angle DCG$ and hence $GD = GC$. Thus $GA = GD = GC$ as required.

Next, since $GD = GA$, we have $\angle GDB = \angle GAB$. But $\angle GAB = \angle GQB = \angle DQB$ as angles of the same segment with base BG. Thus $\angle QDB = \angle DQB$ and hence $BD = BQ$. Also, since $PD = PA$, we have $\angle PDB = \angle PAB$. But $\angle PAB = \angle PCB$ as angles of the same segment with base PB. Therefore $\angle PDB = \angle PCB$ and hence $BD = BC$. Thus we conclude that $BD = BQ = BC$ as required. □

Note 1. The three *Euclid's* propositions used in proofs of *Lemmas 4, 7,* and *14.*

- *Euclid's proposition XII-2*, that states that *circles are to one another as the square of their diameters*
- *Euclid's proposition V-16*, that states that *if four magnitudes are proportional, then they are also proportional alternately*
- *Euclid's proposition V-12*, that states that *if any number of magnitudes are proportional, then one of antecedents is to one of the consequents as the sum of the antecedents is to the sum of the consequents*

Lemma 4. *(Archimedean Arbelos 1: Shoemaker's Knife) Let AB be the diameter of a semicircle and N any point on AB, and let semicircles be described within the first semicircle and having AN, BN as diameters respectively, the figure included between the circumferences of the three semicircles is what* Archimedes *called* Arbelos *and its area is equal to the circle on PN as diameter, where PN is perpendicular to AB and meets the original semicircle in P.*

Proof. For convenience, let Ω be the circle with diameter AB, Ω_1 be the circle with diameter AN, Ω_2 be the circle with diameter NB, Γ_1 be the circle with diameter PN, and Γ_2 be the

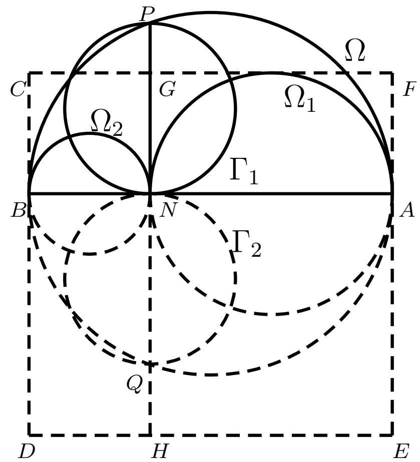

Figure 4: Illustrating the proof of Lemma 4

circle with diameter QN where $PN = NQ$, and let the square $CDEF$ with sides equal to AB and such that CD and FE are tangent to Ω, and $BNGC$, $ANHE$ are squares with sides equal to NB and AN, respectively. Also $BNHD$ and $ANGF$ are equal rectangles with sides equal to AN and BN, respectively, as shown in Figure 4.

It is clear that to prove the Lemma 4 it is sufficient to prove that the sum of the areas of the four circles Ω_1, Ω_2, Γ_1, and Γ_2 is equal to the area of Ω and by *Euclid's proposition XII-2* we show that it is sufficient to prove that the sum of the squares of the diameters of the four circles is equal the square of the diameter of Ω which is clear from the square $CDEF$. So let $[S]$ stands for the area of a polygon or a circle S and to complete the proof we apply *Euclid's proposition XII-2* to get

$$\frac{[\Omega]}{[\Omega_1]} = \frac{(AB)^2}{(AN)^2}, \quad \frac{[\Omega]}{[\Omega_2]} = \frac{(AB)^2}{(BN)^2}, \qquad \frac{[\Omega]}{[\Gamma_1]} = \frac{(AB)^2}{(PN)^2}, \quad \frac{[\Omega]}{[\Gamma_2]} = \frac{(AB)^2}{(QN)^2}. \tag{1}$$

Next, applying *Euclid's proposition V-16* to the ratios in (1) we get

$$\frac{[\Omega]}{(AB)^2} = \frac{[\Omega_1]}{(AN)^2} = \frac{[\Omega_2]}{(BN)^2} = \frac{[\Gamma_1]}{(PN)^2} = \frac{[\Gamma_2]}{(QN)^2} \tag{2}$$

and applying *Euclid's proposition V-12* to (2) we get

$$\frac{[\Omega]}{(AB)^2} = \frac{[\Omega_1] + [\Omega_2] + [\Gamma_1] + [\Gamma_2]}{(AN)^2 + (BN)^2 + (PN)^2 + (QN)^2}. \tag{3}$$

Thus by applying *Euclid's proposition V-16* to (3) we get

$$\frac{(AB)^2}{(AN)^2 + (BN)^2 + (PN)^2 + (QN)^2} = \frac{[\Omega]}{[\Omega_1] + [\Omega_2] + [\Gamma_1] + [\Gamma_2]}. \tag{4}$$

But $PN = QN$ and $(PN)^2 = (QN)^2 = (AN)(BN)$ by *Euclid's proposition III-35.* Therefore

$$(AN)^2 = [ANHE], (BN)^2 = [BNGC], (PN)^2 = [PNAF] = [QNAE] = (QN)^2$$

and hence

$$(AN)^2 + (BN)^2 + (PN)^2 + (QN)^2 = [ANHE] + [BNGC] + [PNAF] + [QNAE]$$

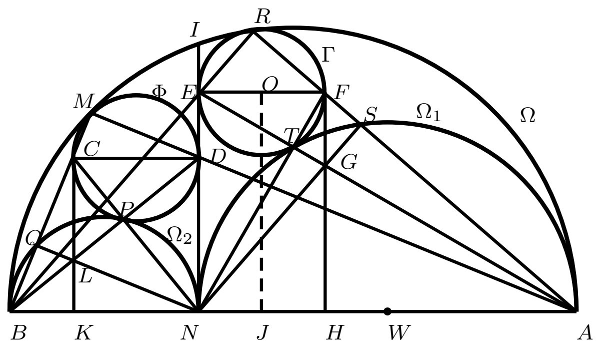

Figure 5: Illustrating the of Lemma 5

But $[ANHE]+[BNGC]+[PNAF]+[QNAE=[CDEF]=(AB)^2$. So, we conclude from (4) that

$$[\Omega_1]+[\Omega_2]+[\Gamma_1]+[\Gamma_2]=[\Omega]$$

as required. □

Note that the proof of Lemma 4 can be seen easily from the square $CDEF$ in Figure 4.

Lemma 5. *(Archimedean Arbelos 2) Let AB be the diameter of a semicircle Ω and N any point on AB, and IN perpendicular to it, and let semicircles Ω_1, Ω_2 be described within the first semicircle and having AN, BN as diameters respectively as shown in Figure 5. Then if two circles Γ, Φ are drawn touching PN on different sides and each touching two of the semicircles, then the circles Γ, Φ so drawn will be equal.*

Proof. Let W be the center of the semicircle Ω_1 and let O be the center of the circle Γ that touches Ω at R, Ω_1 at T, and IN at E. Let EF be the diameter of Γ through O. Then $EF \parallel AB$ and by *Lemma 1* F is a point of RA, E is a point of RB, and T is a point of both FN and EA. Let RA intersect Ω_1 at S, H be the orthogonal projection of F on AB and FH, SN intersect at G, the orthocenter of $\triangle FAN$, and hence EA passes also through G. Since the $\angle ASN, \angle ARB$ are right angles, it follows that $NS \parallel BR$. But also $GH \parallel EN$. Therefore

$$\frac{NH}{AN}=\frac{EG}{EA}=\frac{BN}{AB}.\text{ But } EF=NH \text{ and hence } EF=\frac{(AN)(BN)}{(AB)}. \tag{5}$$

Similarly, for the circle Φ, we have the diameters $CD \parallel AB$ and by *Lemma 1* we have DPB, CPN, MCB and MDA are straight lines. Also L is the orthocenter of $\triangle CBN$ for the angles $\angle CKB, \angle NQB, \angle NPB$, are right angles and hence $LK \parallel DN$ and $NL \parallel AD$ as shown in Figure 5. Thus we have

$$\begin{aligned}&\frac{KN}{BN}=\frac{DL}{DB}=\frac{AN}{AB}.\text{ But } CD=KN \text{ and hence } CD=\frac{(AN)(BN)}{(AB)}.\text{ So,}\\ &CD=EF=\frac{(AN)(BN)}{(AB)} \text{ by (5) and } \Gamma, \Phi \text{ are equal as required.}\end{aligned} \tag{6}$$

Next, we determine the locations of the centers of Γ and Φ. So, let $AN=2a$, $BN=2b$. Then it follows from (6) that Γ and Φ have the same radius $c=\frac{ab}{a+b}$. Hence $AW=a$. Let J be the orthogonal projection of O on AB. Since the centers of two touching circles are collinear with

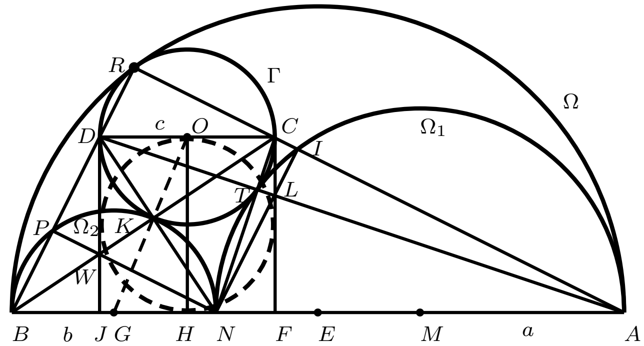

Figure 6: Illustrating the proof of Lemma 6

their point of contact by *Euclid's proposition III-12,* we have O, T, W are collinear. Thus we have

$OW = WT + OT = a + c$ and
$WJ = WN - NJ = a - c$. But by Pythagorean Theorem we have,
$(OJ)^2 = (OW)^2 - (WJ)^2 = (a+c)^2 - (a-c)^2 = 4ac$. So,
$OJ = 2\sqrt{ac}$.

Thus if $B(0,0)$, $A(2a+2b,0)$, then the center of Γ is given by $O(2b+c, 2\sqrt{ac})$ and similarly the center of Φ is given by $(2b-c, 2\sqrt{bc})$ where $c = \frac{ab}{a+b}$. □

Lemma 6. *Let AB, the diameter of a semicircle Ω, be divided at N so that AN is one and half times NB. Semicircles Ω_1 and Ω_2 are constructed within the semicircle Ω with diameters AN and NB, respectively. A circle Γ, with center O and diameter $CD \parallel AB$, is constructed touching the three semicircles Ω, Ω_1, and Ω_2 at R, T and K, as shown in Figure 6. Then it is required to find the ratio of the diameter AB to the diameter CD.*

Proof. Referring to Figure 6, we see by *Lemma 1* that RA passes through C, RB passes through D, CN and DA passes through T, and CB, DN passes through K. Also if F, J are the orthogonal projection of C, D and AR, BR intersect Ω_1, Ω_2 at I, P, respectively, then CF, AT, and NI intersect at the orthocenter L of $\triangle CAN$ and DJ, BK, and NP intersect at the orthocenter W of $\triangle DBN$. Also $LF \parallel DJ$, $LN \parallel DB$, $WJ \parallel CF$, and $WN \parallel CA$. Hence

$\frac{AF}{FJ} = \frac{AL}{LD} = \frac{AN}{NB}$ and $\frac{FJ}{BJ} = \frac{CW}{WB} = \frac{AN}{NB}$. But $CD = FJ$. Thus
$\frac{AB}{CD} = \frac{AF}{FJ} + \frac{FJ}{FJ} + \frac{BJ}{FJ} = \frac{AN}{NB} + 1 + \frac{NB}{AN}$. So, if $AN = rNB$, then
$\frac{AB}{CD} = r + 1 + \frac{1}{r} = \frac{r^2+r+1}{r}$. Therefore, $AB : CD = (r^2+r+1) : r$
and if $r = \frac{3}{2}$, then $AB : CD = 19 : 6$ as wanted.

Note that if $AN = 2a$ and $NB = 2b$, then $r = \frac{a}{b}$ and $\frac{AB}{CD} = \frac{a^2+ab+b^2}{ab}$ and hence $CD = \frac{2ab(a+b)}{a^2+ab+b^2}$.

Next, to prove that the quadrilaterals $CDJF$, $RINP$ are both squares we first prove that

$WJ = CL, DW = LF$, and $NW = NL$. Since $CD \parallel AB, WN \parallel AC$ and $LN \parallel BD$, we have

$\frac{DW}{WJ} = \frac{CW}{BW} = \frac{AN}{BN}$ and $\frac{CL}{LF} = \frac{DL}{AL} = \frac{BN}{AN}$. Therefore
$\frac{DW}{WJ} = \frac{LF}{CL}$ and hence $\frac{DJ}{WJ} = \frac{DW + WJ}{WJ} = \frac{LF + CL}{CL} = \frac{CF}{CL}$. So
$WJ = CL$ and $DW = LF$ as wanted.

Also, since the angles $\angle CIN$ and $\angle CFN$ are equal right angles subtend CN in $CNFI$ and the angles $\angle DPN$ and $\angle DJN$ are equal right angles subtend DN in $DNJP$, it follows from the converse of *Euclid's Proposition IV-21* that $CNFI$ and $DNJP$ are cyclic quadrilaterals. So, by *Euclid's Proposition III-35* of two chords cutting each other and the proved facts that $WJ = CL$ and $DW = LF$, we have

$$(NL)(LI) = (LF)(LC) = (DW)(WJ) = (NW)(PW). \tag{7}$$

Similarly, $CDKT$ is cyclic in Γ and there are two opposite right angles in the quadrilaterals $CILT$ and $DPWK$. Thus, by the converse of *Euclid's Proposition III-22,* the quadrilaterals $CILT$ and $DPWK$ are cyclic. So, by *Euclid's proposition III-36,* we have

$$(NL)(NI) = (NT)(NC) = (NK)(ND) = (NW)(NP). \tag{8}$$

Therefore by (7) and (8) we have

$(NL)(NI) - (NL)(LI) = (NW)(NP) - (NW)(PW)$ and hence
$(NL)^2 = (NW)^2$ and $NL = NW$. Thus $NI = NP$ by (8)

and hence $RINP$ is a square as wanted.

Next, since the angle $\angle INP$ is a right angle, we have in the right triangles $\triangle LFN$ and $\triangle NJW$ that

$NL = WN, \angle LNF = \angle NWJ$ and $\angle NLF = \angle WNJ$. Thus, by *Euclid's Proposition I-26* $\triangle LFN \cong \triangle NJW$ and $LF = NJ, FN = JW$.

But $JW = CL$. Therefore, $CF = LF + CL = NJ + JW = NJ + FN = FJ$ and $CDJF$ is a square with inscribed circle equals Γ as required.

Finally, to find the location of the center of the circle Γ, let M, G, O be the centers of Ω_1, Ω_2, Γ, respectively and let H be the orthogonal projection of the center O of Γ on AB and $CD = 2c$. Since the centers of two touching circles are collinear with their point of contact, we have $OM = a + c$, $OG = b + c$, $GM = a + b$. Since $CD = 2c = \frac{2ab(a+b)}{a^2+ab+b^2}$ and $CD = CF = OH = 2c$, we have

$$c = \frac{b(a^2 + ba)}{a^2 + b^2 + ab} < b \text{ and } c = \frac{a(ab + b^2)}{a^2 + b^2 + ab} < a. \text{ So if } b \le a, \text{ then } c < b \le a,$$
$$GM = a + b > b + c = OG, GM > a + c = OM \text{ and } c = \frac{2a}{3} \text{ when } a = b.$$

Thus $GM > OM \ge OG$ and H, the orthogonal projection of O on AB, lies between G and M.

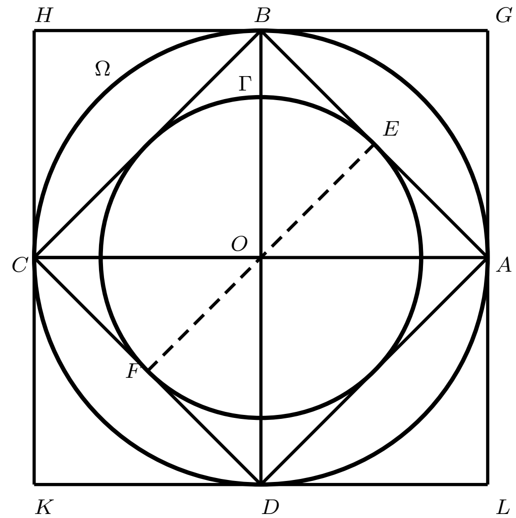

Figure 7: Illustrating the proof of Lemma 7

Applying the Pythagorean theorem on $\triangle OGH$, we get

$$(GH)^2 = (OG)^2 - (OH)^2 = (b+c)^2 - \left(\frac{2ab(a+b)}{a^2+ab+b^2}\right)^2 = \frac{b^4(2a+b)^2}{(a^2+ab+b^2)^2}.$$

Thus,

$$GH = \frac{b^2(2a+b)}{a^2+ab+b^2} \quad \text{and} \quad BH = b + GH = \frac{b(a+b)(a+2b)}{a^2+ab+b^2}.$$

So, the construction of the circle Γ is determined by $BH = \frac{b(a+b)(a+2b)}{a^2+ab+b^2}$ and $OH = 2c = \frac{2ab(a+b)}{a^2+ab+b^2}$, to locate the center O and the radius $c = \frac{ab(a+b)}{a^2+ab+b^2}$ to complete the construction of Γ. □

Lemma 7. *If circles are circumscribed about and inscribed in a square, then the circumscribed circle is double the inscribed circle in the square.*

Proof. Let Γ be the the inscribed circle of the square $ABCD$ and Ω be the circumscribed circle of the square $ABCD$ and also be the inscribed circle of the square $GHKL$, as shown in Figure 7 and let O be their common center. It is clear that the sides of the square $ABCD$ are equal to the diameter EF of the circle Γ and the sides of the square $GHKL$ are equal to the diameter AC of the circle Ω. Applying *Euclid's proposition XII-2*, that states that *circles are to one another as the squares of their diameters,* to the circles Ω and Γ and their corresponding squares $GHKL$ and $ABCD$ we get the ratio of the area of the circle Ω to the area of the circle Γ is equal to the ratio of the area of the square $GHKL$ to the area of the square $ABCD$. But, as seen from Figure 7, the orthogonal diagonals AC and BD of the square $ABCD$ are also orthogonal diameters of the circle Ω and they divide the square $ABCD$ into four equal triangles and divide the square $GHKL$ into four equal squares and each of these four squares is double each of the four triangles. Thus the area of the square $GHKL$ is double the area of the square $ABCD$ and hence the area of the circle Ω is double the area of Γ as required. □

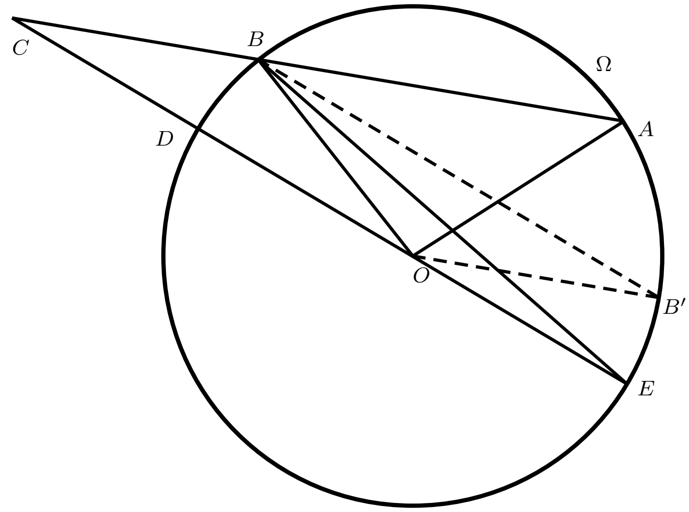

Figure 8: Illustrating the proof of Lemma 8

Lemma 8. *If AB is any chord of a circle Ω whose center is O, and if AB is produced to C so that BC is equal to the radius BO; if CO meets the circle Ω in D and is further produced to meet Ω in E, then the arc AE will be equal to three times the arc BD.*

Proof. By the exterior angle theorem we have

$$\angle ABE = \angle BCE + \angle BEC \text{ but } \angle BCE = \angle BCO = \angle BOC \text{ and}$$
$$\angle BOC = \angle BOD = 2\angle BED \text{ and } \angle BEC = \angle BED. \text{ Therefore}$$
$$\angle ABE = 3\angle BED \text{ but the minor arcs } AE, BD \text{ subtend the}$$
$$\angle ABE, \angle BED, \text{ respectively. Thus arc } AE \text{ is equal three times the arc } BD$$

as required. □

Notice that, by *Euclid's Proposition III-20,* the angle at the center is double the angle at the circumference when they have the same arc. Thus $\angle AOE = 3\angle BOD$. So if we draw $BB' \parallel DE$, then $\angle B'OE = \angle BOD, \angle BED = \angle B'BE$ and hence OB' trisects $\angle AOE$ and BB' trisects $\angle ABE$.

In the next remark we show how by dynamic geometry and *Lemma 8* we can trisect a given angle.

Remark 8(a). Let A, P be two fixed points on the circle Ω with center O. So, to trisect the angle $\angle AOP$ let PH and AK be diameters of Ω, as shown in Figure 8(a), and choose the point B on the minor arc AH of Ω, as in Figure 8, and produce AB to C so that $BC = OB$ and produce CO to meet the minor arc PK of Ω at E and the minor arc BH at D. Draw $BB' \parallel CE$ to meet Ω at B'. Therefore, as in *Lemma 8,* the arc BD is one third the arc EA. So, as B moves on the minor arc BH and A and P are fixed, we see that D moves toward H and it is clear that $\angle AOB$ increases and the $\angle ABO$ decreases. Therefore the $\angle BOD = \frac{1}{2}\angle ABO$ decreases. Thus the arcs BD, AE decreases and when $E = P$, we have $D = H$, $B = F$, $C = G$, and $B' = F'$ and $FF' \parallel GP$ and hence by *Lemma 8* we conclude that OF' trisects $\angle AOP$, as wanted.

Notice that such a process doesn't mean that $\angle AOP$ is trisected by compass and straight edge because the point F is not determined by this rule.

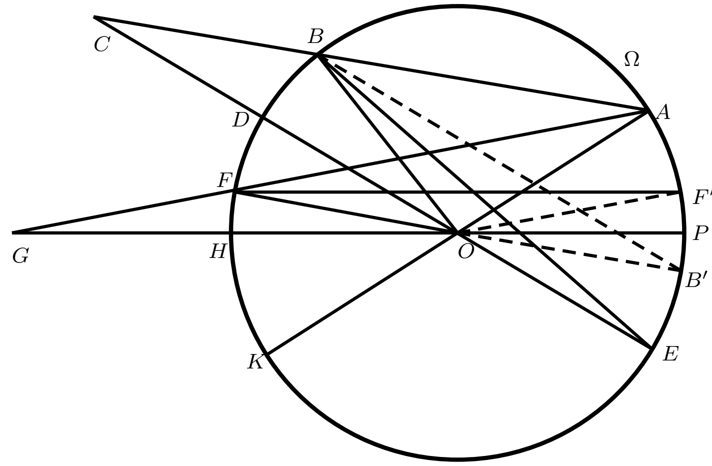

Figure 8(a): Illustrating the *Remark 8(a)*

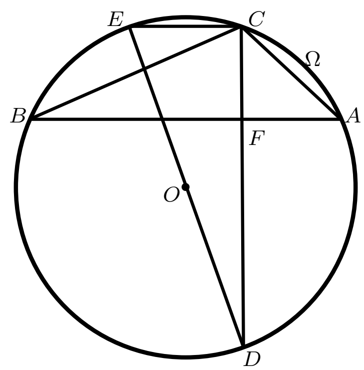

Figure 9: Illustrating the proof of Lemma 9

Lemma 9. *If in a circle two chords AB, CD, which do not pass through the center, intersect at right angles, then the sum of the arcs AD and CB is equal the sum of the arcs AC and DB.*

Proof. Draw the diameter DE and join CE and CA. Since $\angle ECD$ is a right angle in the semicircle DAE by *Euclid's Proposition III-31* and the angle $\angle CFB$ is a right angle, it follows that $CE \parallel FB$ and hence the alternate angles $\angle ECB$ and $\angle CBA$ are equal by *Euclid's Proposition I-29.* Thus the arc AC is equal to the arc EB by *Euclid's Proposition III-26* and hence the sum of the arcs AC and DB is equal the sum of the arcs EB and DB which is the semicircle DBE. Similarly, the sum of the arcs AD and CB is the semicircle DAE. Thus, we have arcs AD and CB are equal the arcs AC and DB as required. □

Lemma 10. *Suppose that TA, TB are two tangents to a circle Ω, while TC cuts it in G and C, as shown in Figures 10(A) and 10(B). Let BD be the chord through B parallel to TC and let AD meet TC in E. Then, if EH is drawn perpendicular to BD, then EH bisects BD in H and AD bisects GC in E.*

Note also that TE bisects the $\angle AEB$.

First proof. Referring to Figure 10(A) we have by *alternate segment theorem* (i.e.*Euclid's Proposition III-32*) that $\angle TBA = \angle ADB$. But $TE \parallel BD$. Hence $\angle ADB = \angle AET$ as exterior and interior opposite angles by *Euclid's Proposition I-29.* Thus $\angle TBA = \angle TEA$ and hence the quadrilateral $AEBT$ is cyclic by the converse of *the angles in the same segment*

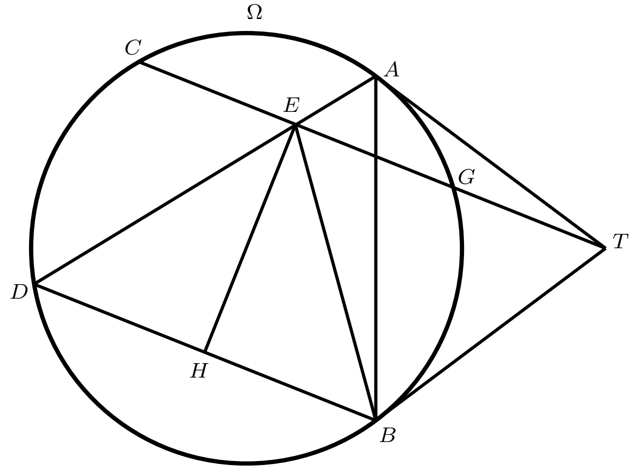

Figure 10(A): Illustrating the First proof of Lemma 10

theorem (i.e. *Euclid's Proposition III-21.*) Thus by the same proposition we have $\angle TAB = \angle TEB$ But $TE \parallel BD$. Hence $\angle TEB = \angle EBD$ as alternate angles by *Euclid's Proposition I-29.* Therefore $\angle TAB = \angle EBD$. But also $\angle TAB = \angle ADB = \angle EDB$ by *alternate segment theorem.* Thus $\angle EBD = \angle EDB$ and hence $EB = ED$ and $\angle BEH = \angle DEH$. So, by *Euclid's Proposition I-4,* $\triangle BEH \cong \triangle DEH$ and the line EH bisects BD and passes through the center O of Ω. Also the line EH cuts GC at a right angle and hence it bisects GC by *Euclid's Proposition III-3* as required. □

It is clear also that TE bisects the $\angle AEB$ and AD bisects CG.

Second proof. Let Γ be the circle with diameter OT and Φ be the circle with diameter OB as shown in Figure 10(B). Since TA and TB are tangents to Ω at A and B and OT is perpendicular to AB at F, it follows that the angles $\angle OAT$, $\angle OBT$, and $\angle OFB$ are right angle and hence Γ passes through A and B and Φ passes through F. So, suppose that TC meets Γ at E_1 and BD which is parallel to TC meets Φ at H_1. Join DE_1 and AE_1. We want to prove that $E_1 = E$. Since the quadrilateral E_1ATB is cyclic, it follows that $\angle TAE_1 = \angle TBA$ by *the angles in the same segment theorem.* But $\angle TBA = \angle BDA$ by *the alternate segment theorem* and $\angle BDA = \angle TEA$ by *Euclid's Proposition I-29* as equal opposite angles on the same side of the line AD falling on the parallel lines TC and BD and hence $\angle TE_1A = \angle TEA$. Therefore $E_1 = E$ and AD meets Γ at E. Next, we show that EOH_1 is a straight line and $EOH_1 = EH$. Let EO produced meet BD at H_2 and since $\angle H_2ET$ is the same right angle $\angle OET$ in the semicircle OET of Γ and $BD \parallel TC$, it follows by *Euclid's Proposition I-29* that the sum of the interior angles $\angle H_2ET$ and $\angle EH_2B$ is two right angles and hence $\angle OH_2B$ is a right angle. But $\angle OH_1B$ is also a right angle in the semicircle OH_1B of Φ and hence $\angle OH_1B = \angle OH_2B$. Therefore $H_1 = H_2 = H$ and EH, the perpendicular to BD, passes through the center O of Ω and bisects the chord BD by *Euclid's Proposition III-3* as required. □

Note that *Lemma 10* can be also proved by letting E be the point of intersection of TC with Γ and producing AE to meet Ω at D. Then join BD to meet Φ at H and prove that BD is parallel to TC and EOH is a straight line that is perpendicular to both TC and BD and bisects GC and BD. This approach will be applied in the next remark for any point E of the circle Γ.

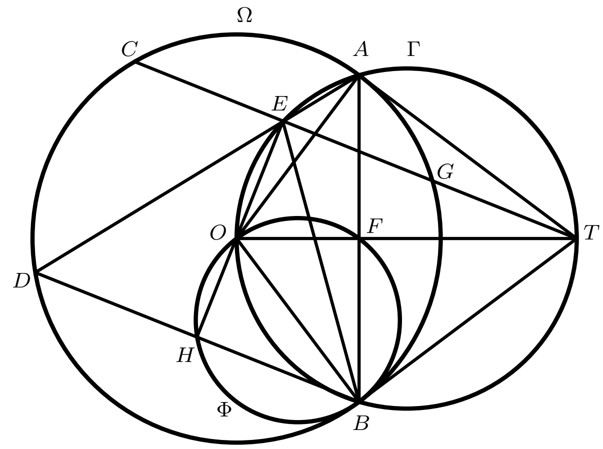

Figure 10(B): Illustrating the Second proof of Lemma 10

Remark 10(a). Let Ω, as in Lemma 10, be the circle with center O and TA and TB are tangents to Ω, Γ the circle with diameter OT, and Φ the circle with diameter OB as shown in Figures 10(I), 10(II), and 10(III). Then OA is perpendicular to TA, OB is perpendicular to TB, OT is perpendicular to AB at F, and Γ passes through A and B and Φ passes through F and B. Also, if E is any point of Γ and the line EA produced meets Ω at D, then the line BD that meets Φ at H is parallel to the line TE and the line EH passes through O and is perpendicular to both TE and BD and EH bisects BD.

Proof. It is clear that Γ passes through A and B and Φ passes through F and B. So to prove this remark we distinguish three cases depending on the angle $\angle AOB$ is either greater than a right angle, equal to a right angle, or less than a right angle. We refer, to Case (I) and Figure 10(I) when $\angle AOB$ is greater than a right angle, to Case (II) and Figure 10(II) when $\angle AOB$ is a right angle, and to to Case (III) and Figure 10(III) when $\angle AOB$ is less than a right angle. Let KL be the line tangent to the circle Φ at O and meet the circle Γ at K and the circle Ω at L as shown below in the Figures 10(I), 10(II), and 10(III).

Case (I). Let $\angle AOB$ be greater than a right angle. Then the line KL that is tangent to the circle Φ at O meets Ω at L and Γ at K of the minor arc TA between T and A as shown in Figure 10(I).

First, we show that as E moves on Γ from T to A, the point H of Φ moves from F to J. To see this we consider the following possible positions of the point E and we refer to Figure 10(I):

I(i). $E = T$. Then EA intersects Ω at $D = A$ and hence $DB = AB$. Thus it is clear that AB is parallel to the tangent TS of Γ and AB meets Φ at F and TF is perpendicular to AB and bisects AB as required.

I(ii). E is any point of the minor arc TA of Γ. First, we consider the positions E is a point of the arc TK, $E = K$, and $E = E'$ is a point of the arc KA. So, let the line EA produced meet Ω at D and BD meet Φ at H and join EO and similarly for KA and $E'A$ as shown in Figure 10(I). Then we have in the quadrilaterals $TEAB$, $TKAB$, and $TE'AB$ that each of the sums $\angle TEA + \angle TBA$, $\angle TKA + \angle TBA$, and $\angle TE'A + \angle TBA$ is equal to two right angles by *Euclid's Proposition III-22.* But $\angle TBA = \angle ADB$ by *the alternate segment theorem,* $\angle ADB = \angle AQB = \angle AD'B$ by angles in the same segment with base AB of Ω and it is clear that $\angle ADB = \angle EDB$ and $\angle TEA = \angle TED$. Therefore the sum $\angle TED + \angle EDB =$

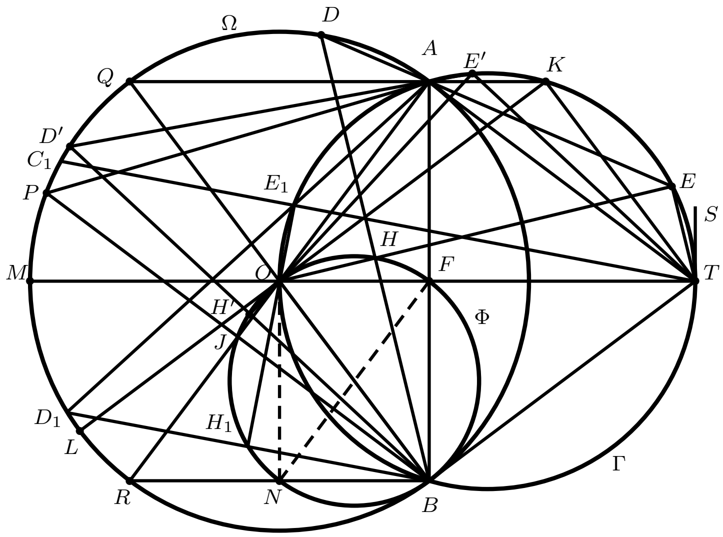

Figure 10(I): Illustrating the proof of Case (I) of *Remark 10(a)*

$\angle TEA + \angle ADB = \angle TEA + \angle TBA$ is equal to two right angles and hence $TE \parallel BD$ by *Euclid's Proposition I-28* as required. Similarly, the sum $\angle TKQ + \angle KQB = \angle TKA + \angle TBA$ is equal to two right angles and also the sum $\angle TE'D' + \angle E'D'B = \angle TE'A + \angle TBA$ is equal to two right angles and hence $TK \parallel BQ$ and $TE' \parallel BD'$ by *Euclid's Proposition I-28* as required.

Next, to prove that EO meets BD at H, suppose that EO meets BD at H_0. Since $\angle OET$ is a right angle by *Euclid's Proposition III-31* as an angle in the semicircle OET of Γ and $TE \parallel BD$, it follows that the angles $\angle OET$ and $\angle OH_0B$ are equal opposite angles on the same side of EO falling on the parallel lines TC and BD by *Euclid's Proposition I-29* and hence $\angle OH_0B$ is a right angle and is equal to the $\angle OHB$. Thus $H_0 = H$ and EH is perpendicular to BD and bisects BD as also required.

Also, to prove that BQ meets KL at O we assume that BQ meets KL at O', since O and O' are points of KL and $\angle TKO$ is a right angle and $TK \parallel BO'$, we have $\angle TKO' = \angle TKO$ is a right angle and the sum of $\angle TKO' + \angle KO'B$ is two right angles and hence $\angle KO'B$ is a right angle. But also $\angle KOB$ is a right angle. Thus $O' = O$, BQ meets Φ at O, $BQ \parallel TK$, and KO is perpendicular to BQ and bisects BQ as wanted.

Next, to prove that $E'O$ produced meets BD' at H', suppose that $E'O$ produced meets BD' at H_0. Since $\angle OE'T$ is a right angle and $TE' \parallel BD'$, it follows that $\angle OH_0B$ is a right angle and is equal to the $\angle OH'B$. Thus $H' = H_0$ and $E'H'$ that passes through the center O of Ω is perpendicular to BD' and bisects BD' as also required.

Finally, we consider the position $E = A$. So, let AO produced meet Φ at J and BJ produced meet Ω at P. Then, to prove that $BP \parallel TA$ and AJ is perpendicular and bisects BP. Note that $\angle TAJ$ is equal to the right angle $\angle TAO$ inscribed in the semicircle of TAO of Γ. Similarly, $\angle AJB$ is equal to the right $\angle OJB$ in Φ. Therefore the sum of the angles $\angle TAJ$ and $\angle AJB$ is two right angles and hence $BP \parallel TA$ as required. Since the line AJ cuts the chord BP at a right angle, it follows that AJ bisects BP by *Euclid's Proposition III-3* as wanted.

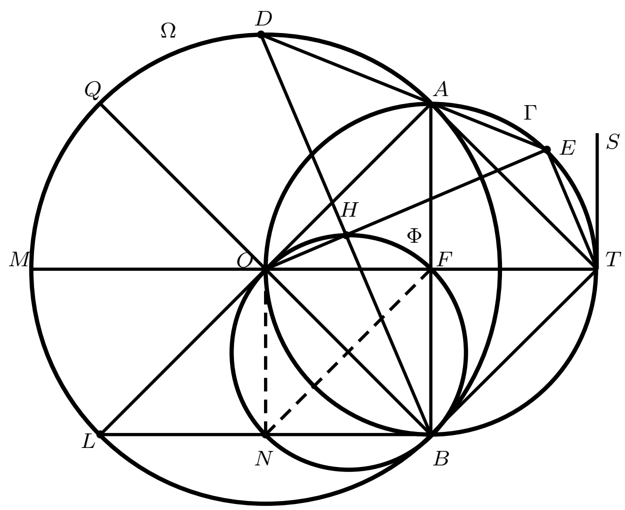

Figure 10(II): Illustrating the proof of Case (II) of *Remark 10(a)*

I(iii). $E = E_1$ is a point of the minor arc AO of Γ. So, if TE_1 produced meets Ω at C_1, E_1A produced meets Ω at D_1 and BD_1 meets Φ at H_1 between J and N of the minor arc JN, then, to prove that $BD_1 \parallel TC_1$, note that $\angle AE_1T = \angle ABT$ in Γ by *the angles in the same segment theorem with base AT* and $\angle ABT = \angle AD_1B$ by *the alternate segment theorem.* Therefore $\angle AE_1T = \angle AD_1B$ and since they are equal opposite angles on the same side of the line AD_1 falling on the lines TC_1 and BD_1, we conclude by *Euclid's Proposition I-28* that $TC_1 \parallel BD_1$ as required. Next, to prove that E_1O produced meets BD_1 at H_1, suppose that E_1O produced meets BD_1 at H_0. Since $\angle OE_1T$ is a right angle in the semicircle TAO of Γ, it follows that $\angle H_0ET = \angle OE_1T$ is a right angle. But $TE_1 \parallel BD_1$. Thus the sum $\angle H_0E_1T + \angle E_1H_0B$ is two right angles and hence $\angle OH_0B = \angle E_1H_0B$ is a right angle and is equal to the $\angle OH_1B$. Thus $H_1 = H_0$ and E_1H_1 passes through O and is perpendicular to BD_1 and hence bisects BD_1 as also required.

Note also that if $E = O$ and AO produced meets Ω at R and BR meets Φ at N, then for $E = O$, we have $\angle AOT = \angle ABT$ by *angles in the same segment theorem with base AT* of Γ and $\angle ABT = \angle ARB$ by *alternate segment theorem with base AB.* Therefore $\angle AOT$ and $\angle ARB$ are equal opposite angles on the same side of the line AR falling on the lines TO and BR and hence $BR \parallel TM$ where the line TO meets Ω at M. Also since $\angle ONT$ is a right angle in the semicircle ONB, it follows that $EN = ON$ passes through the center O of Ω and cuts BR at a right angle and hence it bisects BR as required.

Thus we conclude from **I(i)**, **I(ii)**, and **I(iii)** that as E moves from T to O on the semicircle TAO of Γ, H moves from F to N on the semicircle FON of Φ.

Case (II). Let $\angle AOB$ be a right angle. Then AL is tangent to Φ at O and meets Ω at L as seen in Figure 10(II) and as E moves from T to A on Γ, H moves from F to O on Φ and as E moves from A to O on Γ, H moves from O to N on Φ. The proofs are the same as the proofs in **Case (I)** from T to A and from A to O. Also, as seen from Figure 10(II), $TE \parallel BD$ for E between T and A, $TA \parallel BQ$ for $E = A$, and $TM \parallel BL$ for $E = O$.

Case (III). Let $\angle AOB$ be less than a right angle. Then the diameter AR of Ω meets Φ at the point J of the minor arc FO and $BR \parallel TM$ and meets Φ at N as seen from

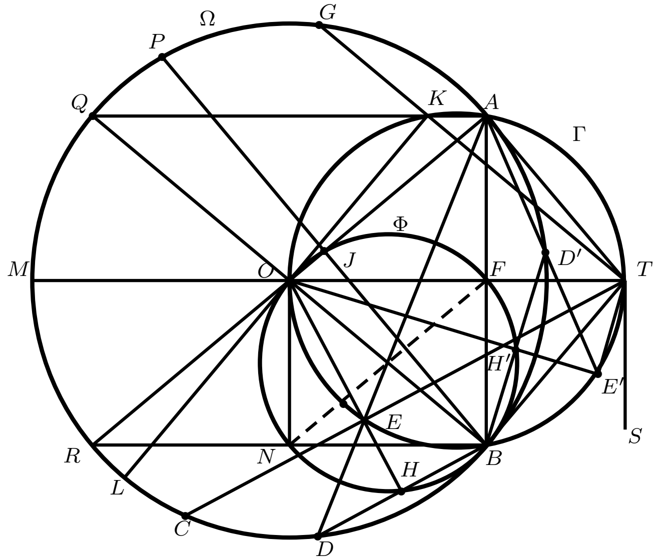

Figure 10(III): Illustrating the proof of Case (III) of *Remark 10(a)*

Figure 10(III). Also the tangent KL of Φ at O is perpendicular to OB and meets Γ at the point K of the minor arc AO and meets Ω at L.

Note that $TA \parallel BP$, BP passes through J, and as the point E of the minor arc TA of Γ moves from T to A, the point H of Φ moves on the minor arc FJ from F to J. Also as the point E of the minor arc AO of Γ moves from A to O, the point H of Φ moves on the minor arc JN from J to N and the proofs are the same as in **Case (I)** and **Case (II)**.

We prove in what follows that in all cases as E moves on the semicircle OBT from O to B, the point H of Φ moves on the semicircle NBF from N to B and also as E moves on the semicircle OBT from B to T, the point H of Φ moves on the semicircle NBF from B to F.

First, let E be any point of the minor arc OB of Γ and we refer to Figure 10(III). So, let TE produced meet Ω at C, EA produced meet Ω at D and BD meet Φ at H between N and B of the minor arc NB. Then, to prove that $BD \parallel TC$, note that $\angle AET = \angle ABT$ in Γ by *the angles in the same segment theorem with base* AT and $\angle ABT = \angle ADB$ by *the alternate segment theorem.* Therefore $\angle AET = \angle ADB$ and since they are equal opposite angles on the same side of the line AD falling on the lines TC and BD, we conclude that $TC \parallel BD$ as required. Next, to prove that OE produced meets BD at H, suppose that OE produced meets BD at H_0. Since $\angle OET$ is a right angle in the semicircle TBO of Γ and $TE \parallel BD$, it follows that $\angle OET$ and $\angle OH_0B$ are equal opposite right angles on the same side of the line OH_0 falling on the parallel lines TC and BD. Thus $\angle OH_0B$ is a right angle. But also $\angle OHB$ is a right angle in the semicircle OHB of Φ. Therefore $\angle OHB = \angle OH_0B$ and hence $H = H_0$. So, we conclude that OH passes through E and is perpendicular to BD and hence EH bisects BD as required.

Note that if $E = B$, then $E = B = C = D = H$ and E is the only common point of the three circles Ω, Γ, and Φ.

Finally, we consider the position E' is a point of the minor arc BT of Γ and prove that as E' moves from B to T on the arc BT of Γ, the point H' moves from B to F on the arc BF of Φ.

So, let $E = E'$ is a point of the minor arc BT of Γ, $E'A$ meets Ω at the point D' of the minor arc AB, and BD' meet Φ at H' . So, we want to prove that $BD' \parallel TE'$. Since the

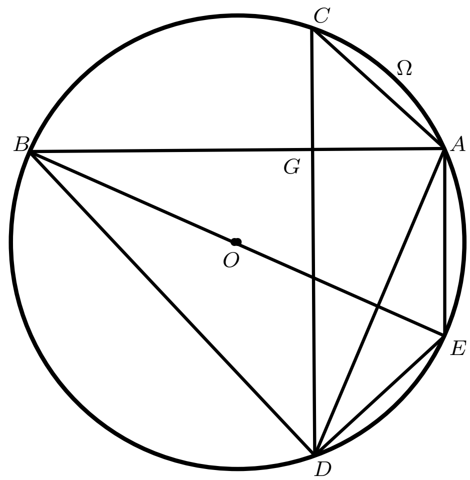

Figure 11: Illustrating the proof of Lemma 11

angles $\angle E'TB$ and $\angle E'AB$ in Γ are equal angles by *the angles in the same segment theorem with base* $E'B$ and $\angle E'AB = \angle D'AB = \angle TBD'$ by *the alternate segment theorem,* it follows that two alternate angles $\angle E'TB$ and $\angle TBD'$ are equal. Therefore $BD' \parallel TE'$ by *Euclid's Proposition I-27* as required. Next, we prove that OE' meets BD' at H'. Suppose that OE' meets BD' at H_0. Since $\angle OE'T$ is a right angle in the semicircle TOB of Γ and OE' falling on the parallel lines TE' and BD' and the angles $\angle OE'T$ and $\angle E'H_0B$ are alternate angles, it follows that the angles $\angle E'H_0B$ and $\angle H_0E'T = \angle OE'T$ are both right angles. Therefore also $\angle OH_0B$ is a right angle. But $\angle OH'B$ is a right angle in the semicircle OBF of Φ. Thus the angles $\angle OH_0B$ and $\angle OH'B$ are both right angles and hence $H_0 = H'$, OE' is perpendicular to BD', meets BD' at H', and bisects BD' as required.

Thus, by proving that as E moves on the semicircle OBT of Γ from O to T, the point H moves on the semicircle NBF of Φ from N to F, we completed the proof of *Remark 10(a)* as required. □

Lemma 11. *If two chords AB, CD in a circle Ω intersect at right angles at a point G, not being the center, then*

$$(AG)^2 + (BG)^2 + (CG)^2 + (DG)^2 = (\mathit{diameter})^2.$$

Proof. Draw the diameter BE and join AC, AD, AE, and BD, as shown in Figure 11. Since $AB \perp CD$ and $\angle BAE$ is a right angle in the semicircle BAE by *Euclid's proposition III-31,* it follows that $AE \parallel CD$, and hence the alternate angles $\angle CDA$ and $\angle EAD$ are equal by *Euclid's Proposition I-29* and hence $AC = ED$ by *Euclid's Proposition III-26.* But the angle $\angle EDB$ is a right angle in the semicircle EDB. Thus by *Pythagorean Theorem (i.e. Euclid's Proposition I-47)* we have

$$(AG)^2 + (CG)^2 = (AC)^2 = (ED)^2 \text{ and } (BG)^2 + (DG)2 = (BD)^2. \text{ Therefore}$$
$$(AG)^2 + (BG)^2 + (CG)^2 + (DG)^2 = (AC)^2 + (BD)^2 = (BE)^2,$$

the square of the diameter BE of the circle Ω as required. □

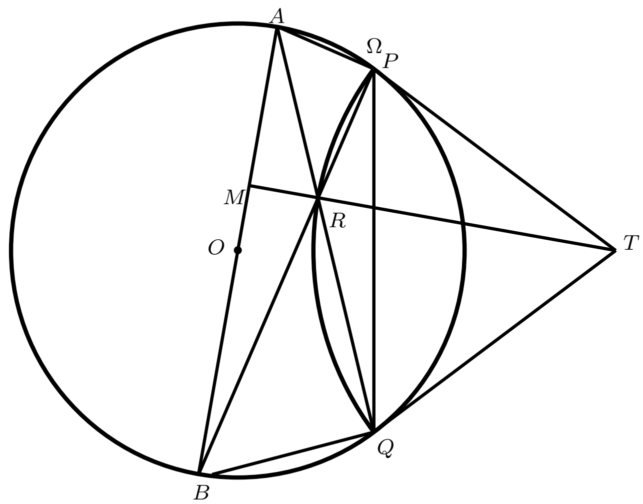

Figure 12: Illustrating the proof of Lemma 12

Lemma 12. *If AB be the diameter of a circle Ω, and TP, TQ the tangents to it from any point T, and if AQ, BP be joined meeting in R, then TR produced is perpendicular to AB. In fact, also $TR = TP = TQ$ (i.e. R lies on the circular arc PRQ with center T, as A moves away from P and B approaching Q and M shall be the center of Ω when $AB \parallel PQ$.*

Proof. Draw $TM \perp AB$ and join AQ and suppose that AQ meets TM in R, as shown in Figure 12. So, since the sum of the opposite angles $\angle AQB$ and $\angle RMB$ in the quadrilateral $RMBQ$ equals two right angles, it follows by the converse of *Euclid's Proposition III-22* that $RMBQ$ is cyclic and since the sum of the angles $\angle MBQ$ and $\angle MRQ$ equals two right angles, we have $\angle TRQ = \angle MBQ$. But by *alternate segment theorem* we have $\angle TQA = \angle ABQ$ and since $\angle TQA = \angle TQR$, $\angle ABQ = \angle MBQ$, we have $\angle TQR = \angle TRQ$. Thus $TR = TQ$ and similarly, PB will meet TM at a point, say, R' such that $TR' = TP$. But $TP = TQ$. Therefore $TR = TR'$. Thus $R = R'$ and if AQ, BP intersect at R, then TR produced must be perpendicular to the diameter AB and $TP = TQ = TR$ as required. □

Note that the proof of *Lemma 12* is different than that given in [1] and in the next remark we show that the result is also true when the diameter AB intersects PQ.

Remark 12(a). Note that, as seen from *Lemma 12* and as A moves counter-clock wise on the semicircle CGD of Γ from C to D, we see that B moves counter-clockwise on the semicircle DHC from D to C, R moves counter-clockwise on the semicircle ENF, with radius TP, from E to F, and M moves counter-clockwise on the circle Φ, with diameter TO, from T through P, O, and Q to T.

Note that if the point A of the diameter AB is a point of the minor arc CP and the perpendicular TM meets BA produced at M and QA produced meets TM produced at R as shown in Figure 12(a), then the two equal right angles $\angle RMB$ and $\angle RQB = \angle AQB$ subtend RB. Thus the quadrilateral $RMQB$ is cyclic and hence $\angle MRQ = \angle MBQ$ for they both subtend MQ and also $\angle MBQ = \angle ABQ = \angle TQR$ by *alternate segment theorem.* Therefore $\angle TRQ = \angle MRQ = \angle MBQ = \angle ABQ = \angle TQA = \angle TQR$ and hence $TQ = TR$. Similarly if BP produced meets TM produced at R', then the quadrilateral $AMR'P$ is cyclic for the angles $\angle AMR'$ and $\angle APR' = \angle APB$ are right angles. Thus $\angle TR'P = \angle MR'P = \angle PAB = \angle SPB = \angle TPR'$ and hence $TR' = TP = TQ = TR$ and $R' = R$ as in the proof of *Lemma 12.*

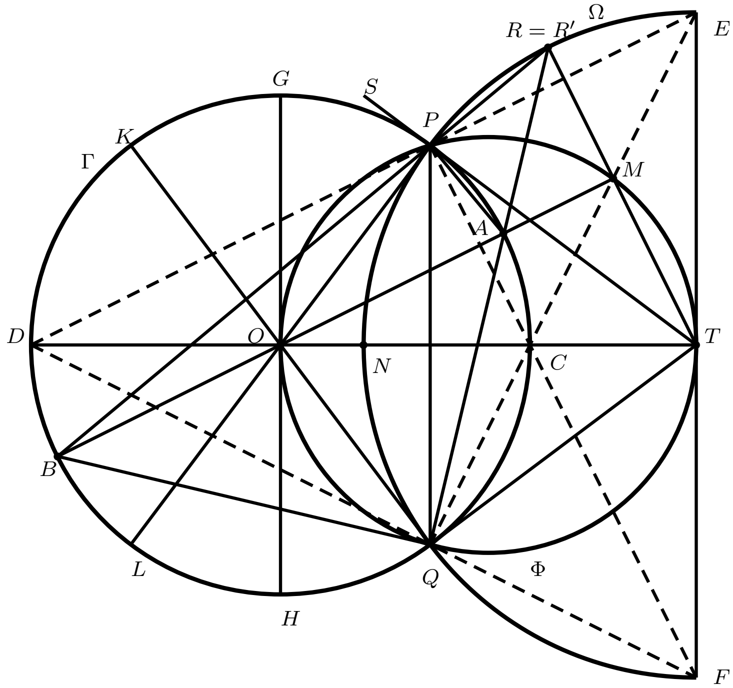

Figure 12(a): Illustrating the proof of *Remark 12(a)*

Lemma 13. *If a diameter AB of a circle Ω with center O meets any chord CD, not a diameter, in E, and if AM, BN be drawn perpendicular to CD, as shown in Figure 13, then $DM = CN$ (i.e. $DN = CM$).*

First proof. Produce AM, BN to cut the circle at G, F, respectively. Join BG, AF, AD, AC, FD, and FC. Since $\angle AGB$ and $\angle AFB$ are right angles, it follows that $AFNM$ and $MNBG$ are rectangles. Hence

$AM = FN, MG = NB,$ but by intersecting chords Theorem
(i.e. Euclid's proposition III-35), we have
$(AM)(MG) = (CM)(MN + DN) = (CM)(MN) + (CM)(DN)$ and
$(FN)(NB) = (DN)(MN + CM) = (DN)(MN) + (CM)(DN)$. Hence
$(CM)(MN) = (DN)(MN)$. Thus $CM = DN$ and hence
$CN = CM + MN = DN + MN = DM$ as required. □

Second proof. Since $AMNF$ is a rectangle, $\angle FCD = \angle FAD$ by angles in the same segment theorem (i.e. *Euclid's proposition III-21*), and $\angle FAD = \angle ADC$, for $AF \parallel CD$, we have $AM = FN$, $\angle ADM = \angle ADC = \angle FCD = \angle FCN$. Thus $\angle DAM = \angle CFN$ and hence $\triangle ADM \cong \triangle FCN$ and $DM = CN$ as required. □

Third proof. Since $AFNM$ is a rectangle and $AF \parallel CD$, it follows that

$AM = FN$ and $\angle ADC = \angle FAD$. Hence
$AC = FD$ by the converse of angles in the same segment theorem. Thus
$CM = DN,$ by Pythagorean Theorem. So, we have
$CN = CM + MN = DN + MN = DM$ as required. □

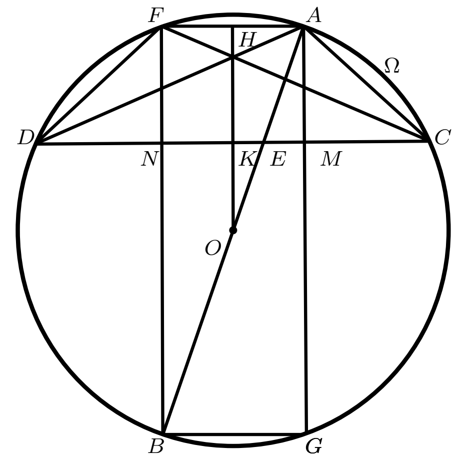

Figure 13: Illustrating the proof of Lemma 13

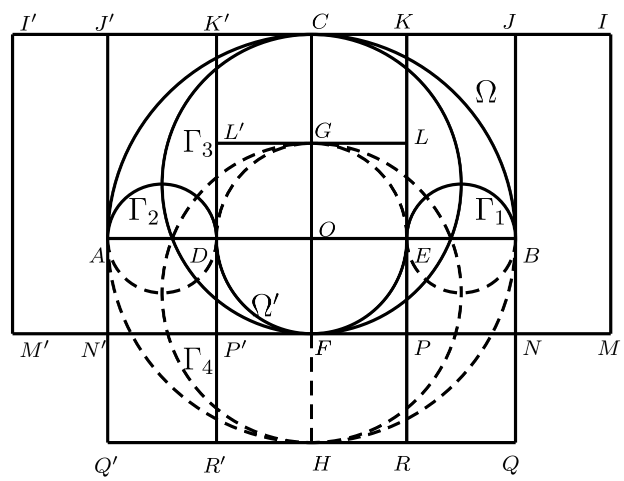

Figure 14: Illustrating the proof of Lemma 14

Fourth proof. Draw $OH \perp AF$ and intersects CD at K. Since $AFNM$ is a rectangle, $AF \parallel CD$, and $OH \perp AF$, we get by *Euclid's proposition III-3* that OH bisects AF and CD. But $AF = MN$. Therefore K is the midpoint of both CD and MN and hence $CM = CK - MK = DK - NK = DN$. So, $CN = CM + MN = DN + NM = DM$ as required. □

Lemma 14. *(Salinon) Let ACB be a semicircle Ω on AB as diameter, and let AD, BE be equal lengths measured along AB from A, B, respectively. On BE, AD as diameters describe semicircles, Γ_1 and Γ_2 on the side towards C, respectively, and on DE, as diameter, a semicircle Ω' on the opposite side. Let the perpendicular to AB through O, the center of Ω and Ω', meet the opposite semicircles in C, F, respectively. Then shall the area of the figure bounded by the circumferences of all the semicircles be equal to the area of the circle Γ_3 on CF as diameter (i. e. the sum of the areas of the semicircles Ω and Ω' is the same as the sum of the areas of the two equal semicircles Γ_1, Γ_2 and the area of the circle Γ_3.*

Proof. For convenience denote also by Ω, Γ_1, Ω' and Γ_2 the complete circles with diameters AB, BE, DE and AD, respectively, as shown in Figure 14. Denote also the area of a circle or a polygon S by $[S]$. Let G be the point of intersection of CF and the circle Ω' and produce CF to H so that $GH = CF$ and denote the circle with diameter GH by Γ_4. So,

$CG = FH = BE = AD$ and the areas of the circles Γ_3 and Γ_4 are equal. Thus to prove *Lemma 14* it is sufficient to prove that

$$[\Omega] + [\Omega'] = [\Gamma_1] + [\Gamma_2] + [\Gamma_3] + [\Gamma_4] \tag{9}$$

So, by *Euclid's propositions XII-2, V-16, V-12,* stated in *Note 1,* and as proved in *Lemma 4,* to prove (9) it is sufficient to prove that the sum of the squares of diameters of Ω and Ω' is equal to the sum of the squares of the diameters of Γ_1, Γ_2, Γ_3, and Γ_4 which can be seen from Figure 14.

So, to prove that $(AB)^2+(DE)^2 = (EB)^2+(AD)^2+(CF)^2+(GH)^2$, we draw in Figure 14 the squares $JQQ'J'$, $LPP'L'$, $PRQN$, $P'R'Q'N'$, $CFM'I'$, and $CFMI$ with sides equal to the diameters AB, DE, BE, $AD = BE$, CF, and $GH = CF$, respectively. Therefore

$$(EB)^2 = [PRQN], \quad (AD)^2 = [P'R'Q'N'], \quad (CF)^2 = [CFM'I'], \quad (GH)^2 = [CFMI],$$
$$(AB)^2 = [JQQ'J'], \quad (DE)^2 = [LPP'L']$$

and hence

$$(AB)^2 + (DE)^2 = [JQQ'J'] + [LPP'L'] = [JNN'J'] + [LRR'L'] + [P'R'Q'N'] + [PRQN].$$

But $[LRR'L'] = [J'N'M'I'] + [JNMI]$. So,

$$[JNN'J'] + [LRR'L'] = [JNN'J'] + [J'N'M'I'] + [JNMI] = [IMM'I'] = (CF)^2 + (GH)^2.$$

Thus $[PRQN] + [P'R'Q'N'] + [IMM'I'] = (EB)^2 + (AD)^2 + (CF)^2 + (GH)^2$ and hence

$$(AB)^2 + (DE)^2 = (EB)^2 + (AD)^2 + (CF)^2 + (GH)^2 \tag{10}$$

Therefore we conclude from (10) that

$$[\Omega] + [\Omega'] = [\Gamma_1] + [\Gamma_2] + [\Gamma_3] + [\Gamma_4] \text{ as required in (9).}$$ □

Lemma 15. *Let AB be the diameter of a circle Ω, AC a side of an inscribed regular pentagon, D the midpoint of the arc AC. Join CD and produce it to meet BA produced in E; join AC, DB meeting in F, and draw FM perpendicular to AB. Then EM is equal to the radius of the circle. In fact, the two triangles $\triangle EMD$ and $\triangle OAD$ are isosceles triangles and congruent, the triangles $\triangle EAD$ and $\triangle OMD$ are isosceles triangles and congruent (i.e. $EM = ED = OA = OD$ and $EA = AD = DM = MO$). Also, if $\angle DBA = \alpha$, then $\angle EDO = 6\alpha$ and is trisected by DA, $\angle MDB = 3\alpha$ is trisected by DO and DM, and $\angle ECB = \angle EDO = \angle EAD = \angle OMD = 6\alpha$.*

Note also that the minor arc AD is one third of the minor arc CB.

Proof. Join DA, DM, DC, DO, and CB, as shown in Figure 15. For convenience, let $\angle DBA = \alpha$. Then $\angle DBC = \angle DBA = \alpha$ as angles subtending the equal line segments $DA = DC$ in Ω. Thus $\angle CBA = 2\alpha$ and hence the central $\angle AOC = 4\alpha$ subtending the side AC of an inscribed regular pentagon in Ω. So α is one fifth of a right angle, a right angle is equal to 5α, and the sum of the angles of a triangle is 10α. Since the sum of the opposite angles $\angle ABC$ and $\angle ADC$ of the cyclic quadrilateral $ABCD$ is equal to two right angles, by *Euclid's Proposition III-22*, it follows that the exterior angle $\angle EDA = \angle ABC = 2\alpha$ and

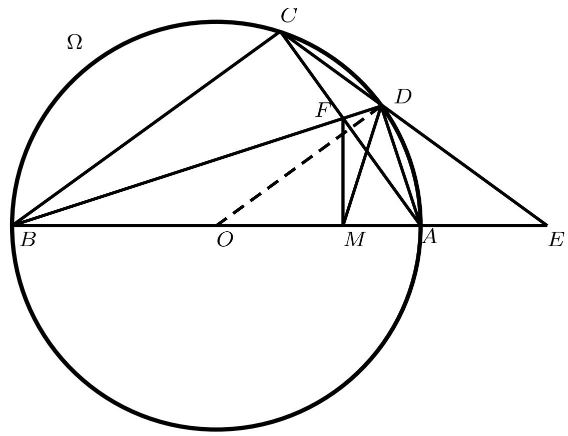

Figure 15: Illustrating the proof of Lemma 15

by *Euclid's Proposition III-21*, the two angles $\angle DAC$ and $\angle DBC$ are each equal to α, for they are in the same segment DC. Also, since in the quadrilateral $FMBC$ we have the sum of $\angle BMF$ and $\angle BCF$ is equal to two right angles and in $AMFD$ we have the sum of the angles $\angle AMF$ and $\angle ADF$ is equal to two right angles, it follows that $AMFD$ and $FMBC$ are cyclic, by the converse of *Euclid's proposition III-22.* So, by using the fact that $\angle DAC = \angle DBC = \alpha$ and by applying *Euclid's proposition III-21 (i.e. angles in the same segment theorem)* we get

$$\angle ADM = \angle AFM = \angle CBM = 2\alpha = \angle EDA. \text{ So, } \angle EDM = 4\alpha \text{ and}$$
$$\angle DMF = \angle DAF = \angle DAC = \alpha \text{ and } \angle AMD = \angle EMD \text{ but}$$
$$\angle EMD = \angle EMF - \angle DMF = 5\alpha - \alpha = 4\alpha. \text{ So in,}$$
$$\triangle EMD \text{ we have } \angle EDM = \angle EMD = 4\alpha \text{ and } \angle DEA = 10\alpha - 8\alpha = 2\alpha.$$
$$\text{But } \angle DOE = \angle DOM = 2\alpha. \text{ Therefore } EM = ED = DO = OA \text{ and}$$
$$EM \text{ is equal to the radius of } \Omega \text{ and the triangles } \triangle EMD \text{ and } \triangle OAD$$

are isosceles and congruent as required.

Next, since $\angle EDA = 2\alpha = \angle DEM = \angle DEA$ and $EM = EA + AM$, $OA = OM + AM$, and $EM = OA$, we have in triangles $\triangle DEA$ and $\triangle DOM$ that $EA = OM$, $ED = OD$, and $\angle DEA = \angle DOM$. Thus the triangles $\triangle EAD$ and $\triangle OMD$ are isosceles and congruent and $EA = AD = DC = DM = OM$ as required. □

Note also that $\angle EDA = \angle ADM = \angle MDO = 2\alpha$ and $DO \parallel CB$. So, $\angle EDO = \angle DCB = 6\alpha$, $\angle EDO$ is trisected by DA and DM, DA bisects $\angle EDM$, $\angle MDB = 3\alpha$ is also trisected by DO, and $\angle ECB = \angle EDO = \angle EAD = \angle OMD = 6\alpha$.

Since ED is equal to the radius of Ω, it follows by *Lemma 8* that the minor arc AD is equal to one third the minor arc CB as required. This is also seen from the fact that $\angle DCA = \alpha$ and $\angle CAB = 3\alpha$.

We prove in the next remark the converse of *Lemma 15* so that if EM is equal to the radius of Ω, then AC is a side of a regular pentagon. Also we prove that $DA = \frac{1}{2}(\sqrt{5}-1)(OA)$ and DA can be constructed by compass and a straight edge.

Remark 15(a). Let AB be a diameter of a circle Ω withe center O, AC be a chord of Ω, D the midpoint of the arc AC such that CD produced meet OA produced at E, and M a point of OA such that $DM = DA$ and EM is equal to the radius OA of Ω. Then AC is a side of a regular pentagon and $DA = \frac{1}{2}(OA)(\sqrt{5} - 1)$ which can be constructed by compass and a straight edge.

Proof. Referring to Figure 15, let AC meet DB at F. Since $DA = DC = DM$, it follows by Lemma 3 that $BM = BC$ and DB bisects the $\angle CBA$. Thus $\triangle FBM \cong \triangle FBC$ by SAS rule and hence $\angle FCB = \angle FMB$. But $\angle FCB = \angle ACB$ and $\angle ACB$ is a right angle in the semicircle ACB of Ω. Therefore $\angle FMB$ is a right angle and FM is perpendicular to AB if and only if $DM = DA$ and hence the quadrilaterals $CBMF$ and $DFMA$ are cyclic. Also since in triangles $\triangle DME$ and $\triangle DAO$ we have $EM = OA$, $DM = DA$, and $\angle DME = \angle DAO$, it follows by SAS rule that $\triangle DME \cong \triangle DAO$ and hence $DE = DO = OA = EM$. So, if $\angle DBA = \alpha$, then $\angle CBA = 2\alpha = \angle DOM = \angle DEA$. Also since $\angle EDA = \angle CBA = 2\alpha$ as exterior angle equal to the opposite angle in $CBAD$ to its adjacent angle. Therefore $DA = EA = OM = DM$, $\angle DMA = \angle DAM = 4\alpha$. Also the $\angle ADM = \angle AFM$ as equal angles having the same base AM in the cyclic quadrilateral $FMAD$ and $\angle AFM = \angle CBM = 2\alpha$ as exterior angle of the cyclic quadrilateral $FCBM$ equal the opposite angle to its adjacent angle $\angle CFM$. Therefore $\angle ADM = 2\alpha$, $\angle DAM = 4\alpha$, and $\angle DMA = 4\alpha$ and hence the sum of the angles of $\triangle DAM = 10\alpha = 180°$. So, the central angle that subtend AC is $\angle COA = 4\alpha = 72°$ and hence AC is a side of a regular pentagon as required.

Since $\angle EDA = \angle ADM = 2\alpha$, it follows that DA is the angle bisector of $\angle EDM$ of $\triangle EDM$. So, by the angle bisector theorem (i.e.*Euclid's Proposition VI-3*) we have $\frac{DM}{DE} = \frac{MA}{EA}$. But $DM = DA = AE = OM$ and the radius OA of Ω is equal to ME and DE. So, $MA = OA - OM = OA - DA$ and hence $\frac{DA}{OA} = \frac{OA-DA}{DA}$. Therefore $(DA)^2 + (OA)(DA) - (OA)^2 = 0$ which can also be derived from a corollary of *Euclid's Proposition III-36* applied to the lines EC and EB falling on Ω to get $(ED)(EC) = (EA)(EB)$ and hence $(OA)(OA + DA) = (DA)(DA + 2OA)$. Thus $DA = \frac{1}{2}(\sqrt{5} - 1)(OA)$ as required. So, it is clear to draw DA by compass and straight edge draw OT perpendicular to OA with length equal to $2OA$, join $TA = (OA)\sqrt{5}$ and draw the point Q on TA such that $TQ = OA$ and $AQ = (OA)(\sqrt{5} - 1)$. Then draw the midpoint P of AQ, the point D of Ω such that $AD = AP$, and the point C of Ω such that $AD = DC$. Thus AC is a side of a regular pentagon as wanted. □

3 Conclusions

As a summary of the results we mention the following:

(1) *Lemma 1* is essential in proving *Lemmas 5* and *6* and it states that if two diameters of two touching circles are parallel, then each of the corresponding endpoints of the two parallel diameters and the point of contact of circles lie on a line.

(2) *Lemma 3* is used to proof a part of *Remark 15(a)* that if $DA = DM = DC$, then $BM = BC$ and trisecting angles appeared in both *Lemma 8* and in *Lemma 15.*

(3) Visual diagrams that are drawn in *Lemmas 4, 7,* and *14* as squares to make the proofs of these Lemmas more feasible without computations.

(4) In *Lemma 6* it is proved that the quadrilaterals $CDJF$ and $RINP$ are squares and the inscribed circle of the square $CDJF$ and the circle Γ are equal.

(5) College and high school students and teachers can benefit from the idea of using Dynamic geometry, as in *Remark 8(a)* to trisect an angle, to study problems looking for new proofs and new generalizations similar to *Remarks 2(a), 3(a), 10(a),* and *12(a),* where moving points or changing positions of points on circular arcs or other curves are

considered. Also it is good for students to know the *Euclid's Propositions* that are applied in the proofs of lemmas and remarks. High school teachers can play an important role in helping students in this direction.

(6) In *Lemma 2* it is proved that if D is a point of a semicircle with diameter AB and the tangents at B and D meet at T, then AT bisects the perpendicular DE to AB and in Case (II) of *Remark 2(a)* it is proved that the conclusion of *Lemma 2* is also true for semi-ellipses and also as the point D moves from A to B, the locus of the midpoint F of DE is also a semi-ellipse in both cases. This opens the way for other generalizations to other lemmas.

(7) In *Lemma 15* it is proved that if AB is the diameter of a circle Ω with center O, AC a side of an inscribed regular pentagon, D the midpoint of the arc AC, CD produced to meet BA produced in E, AC and DB meet at F, and FM perpendicular to AB, then $EM = AO$ and in *Remark 15(a)*, the converse of *Lemma 15* is proved and it states if $EM = AO$, then AC is a side of a regular pentagon. Also a new method to inscribe a regular pentagon in a circle by compass and straight edge is proved.

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