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Constructing Flexible Polyhedra by Twinning

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Abstract. Polyhedra are generically rigid, but can be made to flex under certain symmetry conditions. We generalise Raoul Bricard's flexible octahedra from 1897 to construct an infinite family of combinatorially distinct flexible polyhedra with self-intersections, forming the only known infinite family of flexible polyhedra. By removing edges from these models, we can make new crinkles of infinitely many different topologies. These crinkles can be used to construct flexible triangulated surfaces. We show this in a particular example to make a flexible embedded polyhedron with a large range of motion.

Key Words: flexible polyhedra, self-intersecting polyhedra, symmetric constructions, Bricard octahedra, crinkles

MSC 2020: 52C25 (primary), 52B10, 51M20, 70B10, 70B15

Introduction

If you make a paper model of any convex polyhedron, it will not flex, despite the fact that each edge functions as a hinge. A count of the degrees of freedom shows that a generic polyhedron is rigid, and it was a longstanding conjecture that any embedded polyhedron would be rigid as well.

The rigidity conjecture for polyhedra was disproved by Robert Connelly in 1977, after he explicitly described an embedded flexible polyhedron in [8]. The result attracted much interest, and the methods used would prove useful in the design of deployable structures. Understanding rigidity holds practical value to engineers and carpenters, who are in need of sturdy structures. Conversely, flexible frameworks can be of use for constructing strong hinges. The book [11] discusses these questions from an engineering viewpoint, and we have used some terminology from there. In the thesis [16], Lijingjiao lists some known results, but explicit examples of flexible polyhedra are rare in the literature. Below, we will generalise a construction of Raoul Bricard from 1897, [6], and introduce an infinite family of combinatorially distinct flexible polyhedra with self-intersections. This is an improvement on the

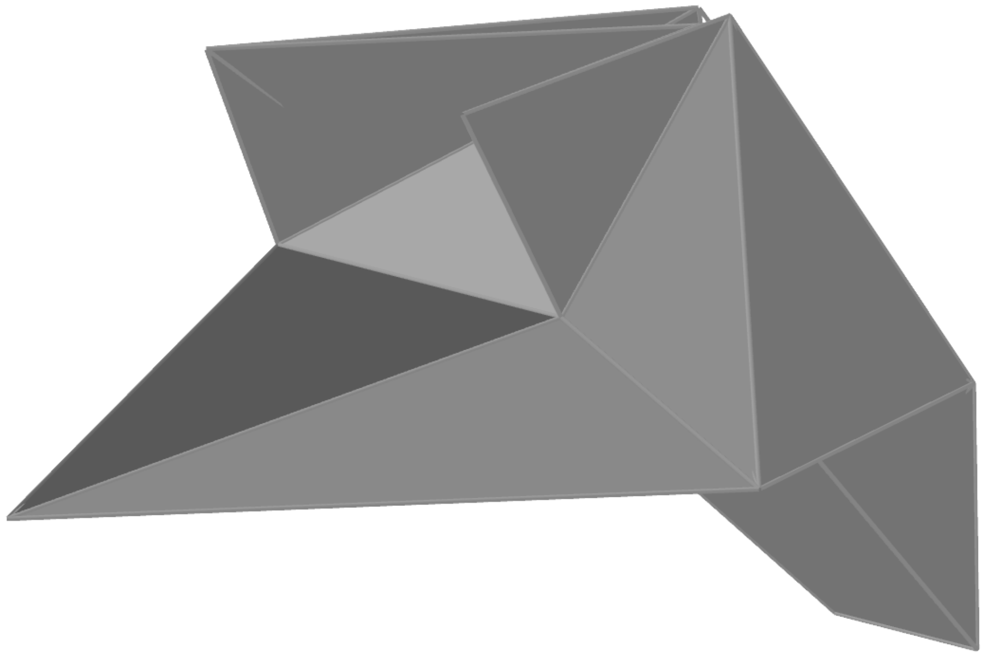

Figure 1: An embedded flexible polyhedron, using a pentagonal crinkle arising from a twinned digonal anticupola. A net is given in figure 8, and a model is available online at `https://www.geogebra.org/m/fq3vdnmy`.

state of the art, and generalises the infinite family from [18]. Further, we can use the flexible polyhedra to produce novel crinkles of arbitrarily complicated topology. Robert Connelly's methods can then be applied to those to construct embedded polyhedra. We will do this in a particular case in section 3.3, leading to the fox-like flexible polyhedron shown in figure 1.

This paper is composed of 3 sections. In section 1 we discuss the rigidity phenomenon for polyhedra. In section 2 we describe the twinning method for constructing infinitely many new flexible polyhedra. In section 3 we work through some simple examples of twinned polyhedra.

1 Rigidity

We say that a polyhedron can *flex* if there is a non-trivial transformation of the vertices keeping all of the edge lengths fixed. Note that simply translating or rotating the polyhedron in $\mathbb{R}^3$ does not count as a flex.

The number of degrees of freedom of a collection of V vertices connected in a generic way by E bars in $\mathbb{R}^3$ is $3V - E$. Each vertex adds three degrees of freedom and each edge removes one. Note that if the edges are not aligned generically, there may be some redundant edges in the arrangement that fail to remove a degree of freedom. In a triangulated polyhedron, we have $V - E + F = 2$ and $3F = 2E$. This implies that the number of degrees of freedom of the polyhedron is

$$3V - E = 6.$$

A rigid body in $\mathbb{R}^3$ has $3 + 3 = 6$ degrees of freedom, corresponding to translation in three independent directions and rotation in three independent angles. This heuristic count of the degrees of freedom suggests that a triangulated polyhedron is rigid. For a more rigorous version of this counting argument, see [14].

The rigidity conjecture for polyhedra is already suggested by definition ten of book XI of Euclid's Elements[1]. No doubt the question of rigidity held practical value in ancient Greece, as it does now to carpenters. Euclid is interested in rigidity, demonstrated by the proof of I.VIII, and VI.def I, but he deals with the topic in more detail in two dimensions. Later scholars have criticised his treatment of the problem of rigidity in three dimension, that of XI.def X, on account of it deserving the role of a proposition to be proved, and also because it turned out to be false. The definition he gives suggests an anticipation of the generic rigidity of polyhedra, and an understanding of how it differs from the two dimensional case. A further discussion of this part of Euclid's Elements can be seen in Thomas L. Heath's notes in his translation, [12].

Augustin-Louis Cauchy's 1813 theorem from [7] states that convex polyhedra are rigid. See also [1] for a simple proof of this fact. However, in the paper [6], Raoul Bricard described families of self-intersecting flexible octahedra that could be constructed out of rods. In 1977, Robert Connelly found a fully embedded flexible polyhedron. He did this by assembling Bricard octahedra to remove the self-intersections, see [10]. We also discuss the relevant methods in section 3.2 of this paper. Further simplification by Klaus Steffen led to a simple working example of nine vertices, discussed in [9]. The simplest possible flexible polyhedron has eight vertices, and was announced in 2024 in the paper [13]. In a later paper, [4], it has been shown how methods based on symmetric quadrilaterals, similar to the present work, can help explain the minimal working example.

Later analysis of flexible polyhedra has shown that during flexion, their volume remains constant. This was called the bellows conjecture, stating that a flexible polyhedron could not be used as bellows, and was first proved in [19]. The proof constructs a polynomial from the combinatorial structure of the polyhedron, such that the volume of the polyhedron is a root. Since the roots of a polynomial form a discrete set, the volume must be constant while the polyhedron goes through a continuous flexing motion. This greatly restricts the type of motion that a flexible polyhedron can undergo.

Bricard's original octahedra come in three types. Type I and II use general symmetry arguments, whereas type III is more involved, relying on a ruling of the hyperboloid. We will focus on the first two types here. A discussion of the type III Bricard octahedron can be found in [17]. Generalising the construction of Bricard's type I and II octahedra, we introduce a method called *twinning* to create an infinite family of self-intersecting flexible polyhedra. Connelly's methods can then be applied to make models without self-intersections. Using the twinning construction, we explicitly construct a new variation of crinkle, as well as a novel flexible polyhedron without self-intersections. The models described have a large range of motion and show some interesting properties, which could be of use in structural engineering.

2 Twinning

We generalise Bricard's flexible octahedra of types I and II. For any triangulated polyhedron P containing a symmetric quadrilateral $ABA'B'$, we remove the edge AA' to create a polyhedral cap on the quadrilateral base $ABA'B'$. If the polyhedron P was previously rigid, this cap will have exactly one degree of freedom. We then make two copies of the cap, apply the symmetry to one of them, and glue the two copies together along their common boundary to create a twinned polyhedron T.

[1] Equal and similar solid figures are those contained by similar planes equal in multitude and magnitude, [12].

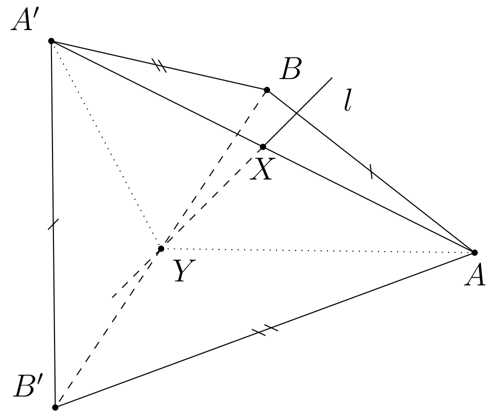

Figure 2: The quadrilateral $ABA'B'$ has rotational symmetry.

2.1 Type I

The type I Bricard octahedron relies on the following theorem.

Theorem 1. *Let A, B, A', B' be four points in $\mathbb{R}^3$, such that $AB = A'B'$ and $AB' = A'B$. Then there exists a line l in $\mathbb{R}^3$ such that a half-rotation in l swaps A with A' and B with B'.*

Proof. Let the four points A, B, A', B' be as described above.

In the case when the diagonals AA' and BB' intersect, the quadrilateral $ABA'B'$ is planar, and thus a parallelogram. It has a rotational symmetry in a line perpendicular to the plane of the four points.

Assuming the diagonals do not intersect, let X be the midpoint of AA' and Y be the midpoint of BB', as shown in figure 2. Let l be the line through X and Y.

Since $\triangle ABB' \cong \triangle BA'B'$, and Y is the midpoint of BB', we have

$$AY = A'Y,$$

since they are corresponding medians in congruent triangles. Therefore, $\triangle AXY \cong \triangle A'XY$, since the corresponding sides in these triangles are equal. In particular,

$$\angle AXY = \angle A'XY,$$

and since A', X, A lie on a line, we have

$$XY \perp AA'.$$

Similarly, we obtain $XY \perp BB'$, and so it follows that l is the line of symmetry of the quadrilateral $ABA'B'$. □

Let $ABA'B'$ be a quadrilateral in $\mathbb{R}^3$ such that the edges satisfy $AB = A'B'$ and $AB' = A'B$, as in the theorem above. To construct the Bricard octahedron of type I, call the line of symmetry l, and pick some point C not on l. We then draw the edges CA, CB, CA', CB', to create a pyramid with a missing bottom. This is a chain of four triangles, and it has a single degree of freedom. Throughout the flexing motion, the base $ABA'B'$ of the pyramid has a line of symmetry, l. If we rotate the pyramid in that line, we send C to C', A to A' and B to B'. Note that by construction, the base is fixed by this rotation. So we can glue the rotated pyramid onto the original pyramid to obtain a closed octahedron. This is a flexible polyhedron with self-intersection, the first of three types of flexible octahedra described by Raoul Bricard in 1897, [6]. See a depiction in figure 3. A model is also available at `https://www.geogebra.org/m/pshaep9q`.

More generally, let P be a polyhedron with two faces $\triangle ABA'$ and $\triangle A'B'A$ along the edge AA', such that $AB = A'B'$ and $BA' = B'A$. We restrict to examples where every edge removes exactly one degree of freedom, in line with our parameter count from section 1 for the degrees of freedom of a polyhedron. This is the case when a framework is infinitesimally rigid with no self-stress, which we refer to as being isostatic. For a more detailed discussion of this terminology, see section 8 of [11]. So if we remove the edge AA', the resulting polyhedral cap on a quadrilateral hole has one degree of freedom. Throughout the flexing motion of this cap, the boundary $ABA'B'$ retains a rotational symmetry in some line l by theorem 1. Let P_{cap} be the polyhedral cap obtained from P after removing the edge AA'. Let P'_{cap} be a copy of the polyhedral cap P_{cap}, obtained by rotating it by 180° around the line l. By construction, the boundary of P_{cap} aligns with the boundary of P'_{cap}, so we can glue P_{cap} to P'_{cap} along their common boundary, obtaining a flexible polyhedron, T. See the top of figure 4 for a specific example of this type I twinning. The construction can also be realised in the following GeoGebra file, `https://www.geogebra.org/m/vabhua2b`. Here, by letting the slider *distance* go to zero, the two caps are glued together along their common quadrilateral bases.

A skew quadrilateral in $\mathbb{R}^3$ has $3 \cdot 4 - 4 = 8$ degrees of freedom, so disregarding the Euclidean symmetry group, there are two flexing degrees of freedom. The hole we create in P by removing a single edge has one degree of freedom, so it may be regarded as a path in the two-dimensional space of all realisations of the quadrilateral. In other words, the polyhedral cap on top of the quadrilateral restricts it from having two degrees of freedom to just one. By copying the cap and rotating it, we obtain a different cap on the quadrilateral base which has exactly the same movement, allowing us to glue the two caps together to obtain a closed polyhedron.

Borrowing some nomenclature from crystallography, we call this action *twinning*[2]. We call T the *twinned* polyhedron of the original polyhedron P around the *equator* $ABA'B'$.

Let us summarise our results in a theorem.

Theorem 2. *Let P be a polyhedron such that the edges form an isostatic framework. Assume that there are some faces $\triangle ABA'$ and $\triangle A'B'A$ in P, such that $AB = A'B'$ and $BA' = B'A$. Let T be a twinned polyhedron of P around the equator $ABA'B'$. Then T has a single flex.*

Proof. Since P is a polyhedron such that the edges form an isostatic framework, every edge accounts for exactly one degree of freedom. Removing the edge AA', we get a framework with exactly one degree of freedom, and so the twinned polyhedron T also has one degree of freedom. □

By an application of Cauchy's rigidity theorem for convex polyhedra, we get a corollary.

Corollary 1. *If P is convex, the twinned polyhedron T has a single flex.*

The polyhedron P is not a generic polyhedron as we have restricted some edge lengths to be equal, however, we do not expect that to affect the flexibility of the generic example. We therefore have a conjecture.

[2]A twinned crystal is a pair of crystals growing in different directions from the same base.

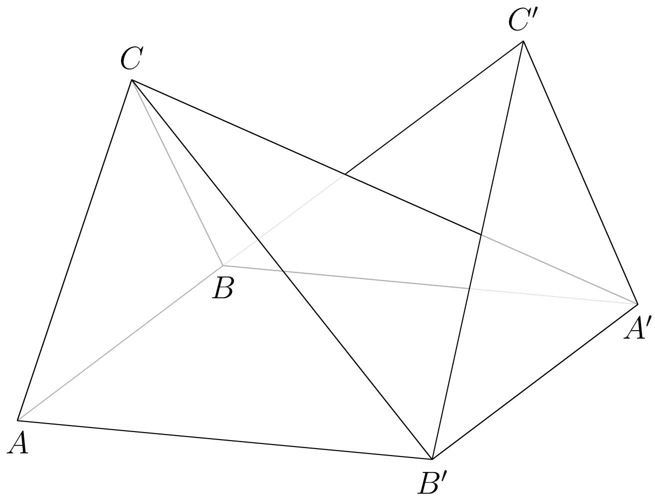

Figure 3: The Bricard octahedron, or twinned pyramid. Two faces have been omitted to remove the self-inersections, creating a crinkle.

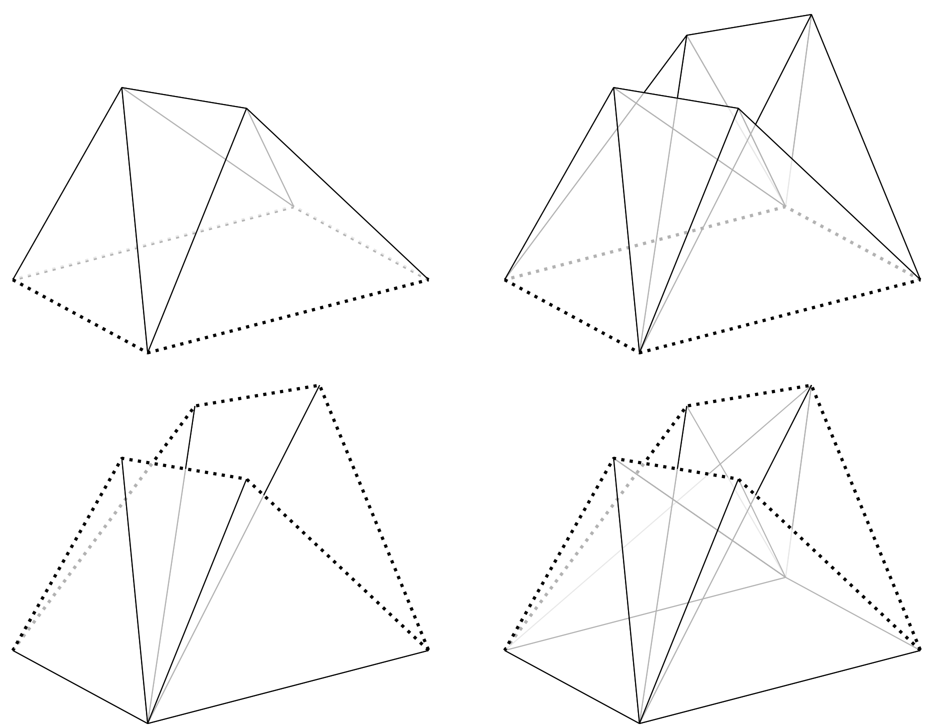

Figure 4: A digonal anticupola and its twinned version, obtained by removing a diagonal of the square base with dotted boundary and rotating a copy around its axis of symmetry. The equator is depicted as a dotted line. Below, see how the same flexible polyhedron can be realised as a twinned hexagonal cone, with a hexagonal equator instead.

Conjecture 1 (Generic case)**.** *For almost all triangulated polyhedra P with faces $\triangle ABA'$ and $\triangle A'B'A$ in P, such that $AB = A'B'$ and $AB' = BA'$, the twinned polyhedron T of P has a single flex.*

Our method works on any shape with a symmetric boundary such that the symmetry is preserved in the flexing motion. The cases here described have quadrilateral boundaries.

Larger boundaries may be achieved as the following simple example shows. Take a digonal anticupola on a square, remove the square bottom, and twin it on its square boundary, as shown in figure 4. This has a flexing motion, and the equator is a dotted quadrilateral. Note that in the final model, the equator may not be uniquely defined, and various different boundaries of symmetry may be used. To illustrate this, we have chosen a different possible equator in the figure below. Working backwards, we see that this shape may also be realised as a twinned hexagonal cone. In twinning the cone, the symmetry on the base is quite difficult to realise, and the first explanation gives a clearer view of the flexibility.

Above, we have generalised the method for constructing the Bricard type I octahedron. We get infinitely many examples, where every polyhedron P with the required symmetry property of two adjacent triangular faces gives rise to a flexible polyhedron. The Bricard type I octahedron is part of this family, being the twinned pyramid.

All examples described by this method have zero volume, as the rotated cap has a negative volume equal in size to the original cap. Also note that if P was symmetric in the line l, the construction simply gives a double cover of P, but in the generic case, this will not happen. Since the polyhedron has zero volume, it must self-intersect. This can also be seen in the following way. The line l goes through the polyhedron P at the edge AA', so it must intersect the polyhedron again at some point. This point will lie on both P and P', and so will be a point of self-intersection. In section 3.2, we will discuss how to eliminate these self-intersections in many cases. In section 3.3, we show this by example by constructing a novel flexible polyhedron without self-intersections.

The novelty of this method is surprising, considering how simple the construction is. This can be explained by the fact that most accounts describing the flexibility of the Bricard I octahedron do not use theorem 1 and the line of rotational symmetry, but rather the more algebraic methods of R. Bricard from the paper [6]. Those rely heavily on the specifics of the movement of the four triangle linkage, and obscure this possible generalisation. An early discussion of Bricard's work is given by G.T. Bennett in [5], which mentions without proofs the importance of symmetry. The author uses the term twin for half of the octahedron when describing the flexibility, but his theorem is not general enough to allow the introduction of a general twin. Bennett mentions the possibility of adding rigid tetrahedra on a flexible polyhedron to make more complicated topologies, but beyond that, no attempt is made to generalise in the way we have done. The most relevant place where we could see this rotational symmetry directly used to show the flexibility of the Bricard I octahedron can be seen in Robert Connelly's paper [10], but he also does not stop to make this generalisation, making do with the flexible octahedron. The only other mention of an infinite family of combinatorially distinct flexible polyhedra is made in the paper [18]. The polyhedra there form a subset of our twinned polyhedra, each of them being a twinned pyramid on a $2n$-gonal base.

2.2 Type II

In an analogous way, we can adapt the method of the Bricard type II octahedron. The construction of the type II octahedron is based on the following theorem, which has a similar proof to theorem 1 above.

Theorem 3. *Let A, B, A', B' be four points in $\mathbb{R}^3$ such that $AB = AB'$ and $A'B = A'B'$. Then there exists a plane π in $\mathbb{R}^3$ through A and A', such that reflecting in π swaps B and B'.*

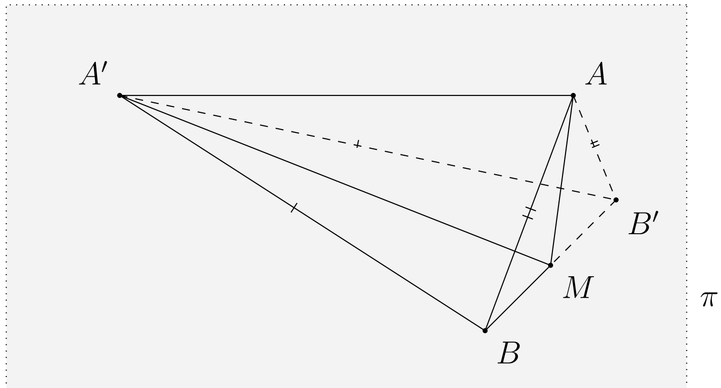

Figure 5: The quadrilateral $ABA'B'$ has reflective symmetry. The point B' lies behind the grey plane labelled π.

Proof. If the four points are coplanar, the quadrilateral $ABA'B'$ is a kite, and so a plane perpendicular to the plane of the quadrilateral will work.

In the other case, refer to figure 5. Let M be the midpoint of BB', and π be the plane through A, A', and M. Since $\triangle BA'B'$ is isosceles,

$$\angle A'MB = \angle B'MA' = 90^\circ.$$

Similarly, since $\triangle BAB'$ is isosceles, we also have

$$\angle AMB = \angle B'MA = 90^\circ.$$

This shows $BB' \perp \pi$, and the mirror symmetry in the plane π follows as M is the midpoint of BB'. □

To get a Bricard type II octahedron, we start with a pyramid with base $ABA'B'$ and vertex E such that $AB = AB'$ and $A'B = A'B'$. Reflecting the vertex E in the symmetric plane of the boundary and gluing along the common boundary creates the flexible octahedron. Again, we see that this process works for any triangulated polyhedron, giving us a type II twinning. As before, this yields infinitely many combinatorially distinct self-intersecting, flexible polyhedra.

We get an analogous theorem to theorem 2. In analogy with plane geometry, we call the reflectionally symmetric quadrilateral a *kite*.

Theorem 4. *Let P be a polyhedron such that the edges form an isostatic framework. Assume that there are some faces $\triangle ABA'$ and $\triangle A'B'A$ in P, such that $ABA'B'$ is a kite. Let T be a twinned polyhedron of P around the equator $ABA'B'$. Then T has a single flex.*

3 Examples of Twinned Polyhedra

3.1 A New Crinkle

We next focus on a particular case. Let the starting shape be a digonal anticupola, as depicted in the top left corner of figure 4. We realise the square face by adding a diagonal in that square as an edge, and when twinning, we remove that edge. The twinning action results in

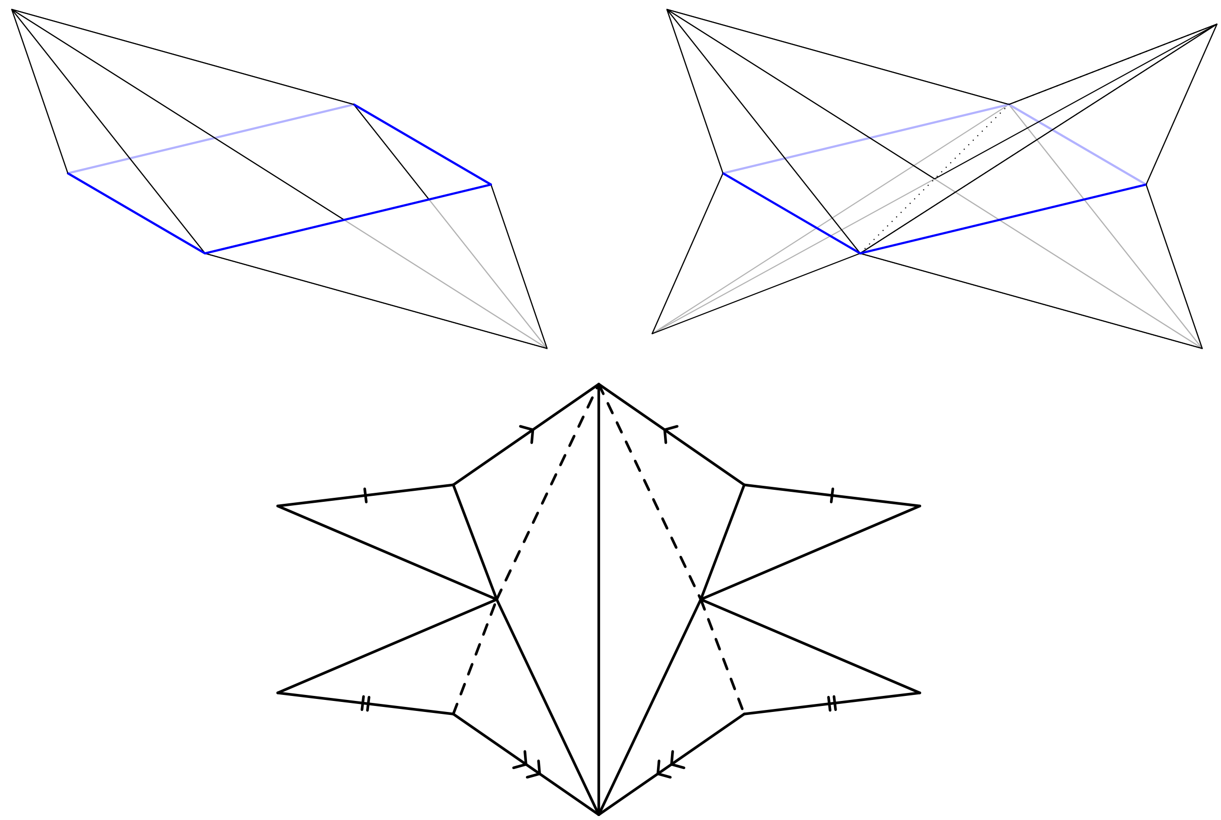

Figure 6: Above, a flexible dodecahedron constructed by twinning a digonal anticupola along the blue equator. Below, a net for assembling a crinkle obtained from this polyhedron. Labels show gluing instructions.

a flexible, triangulated dodecahedron. This flexible dodecahedron has been described before in [2] and [18], but not as a part of this larger family of twinned polyhedra.

See figure 6 for an example of the above, where we start out with the two vertices of the digonal anticupola on different sides of the base. The original polyhedron is self-intersecting, as the two peaks of the digonal anticupola lie on different sides of the quadrilateral base. The equator is drawn in blue, and the star shaped flexible dodecahedron self-intersects along the dotted line. By removing an edge, we get a new crinkle, with a net given on the bottom in figure 6. This crinkle looks similar to the Bricard crinkle, but has an indentation on one of the sides, meaning it can be constructed to take less space. The parameters are chosen so that the crinkle can lie flat.

This design takes up less space than the usual Bricard crinkle, with the cost of adding one vertex. It could be of use in mechanisms where the Bricard crinkle cannot fit.

3.2 Methods of Removing Self-Intersections

One type of self-intersection is a vertex passing through a face. We can fix this by removing the face, adding a vertex above the face, and adding a cone on the face with the new vertex as apex. This is called *erecting a tent*.

Another type of self-intersection is an edge lying below another edge. This can often be remedied by swapping out edges for crinkles. The Bricard crinkle is the basis of most known methods of constructing flexible polyhedra, as discussed in [16]. It is a surface with

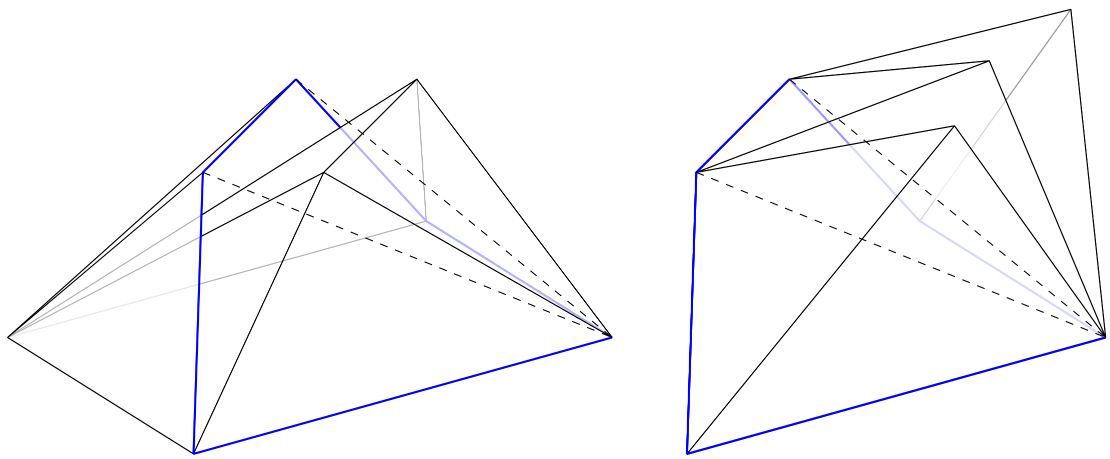

Figure 7: Left, a twinned digonal anticupola with two edges removed gives a pentagonal crinkle without self-intersections. The boundary is coloured blue, and the removed edges are dashed. Right, a tetrahedral chain as base to glue the pentagonal crinkle onto, gluing blue on blue, and so that the dashed lines meet.

the topology of a disk that has a range of motion such that two of the vertices keep a fixed distance throughout the motion, as if they were connected by a phantom edge. This gadget may be used to remove double covers of an edge, or a hinge participating in a flexing motion. This is done by swapping the edge for two opposing crinkles, arranged such that they do not intersect. We see this implemented when Steffen's polyhedron is constructed by adding crinkles to the system of a tetrahedron and a triangle glued together along an edge. The edge corresponds to a hinge, which is then replaced by crinkles. This can be considered as an example of the base+crinkle method, as discussed in [15]. Here, you start out with a flexible base that is not a polyhedron, and turn it into a polyhedron by adding crinkles. All of the crinkles have zero volume, so the base must have positive volume in order for the result to stand a chance of being embedded.

To summarise, the two potential issues are that of vertices sticking through faces, and edges meeting edges. The first problem can be resolved by erecting a tent on the face, and the second by swapping the edges for crinkles. This is the idea of Connelly's methods as described in [10]. We may apply this method using crinkles arising from our twinned polyhedra, giving fundamentally different constructions of flexible polyhedra. We can also use our twinned polyhedra to obtain new crinkles by removing edges in the right way.

Some trial and error is necessary to find a good choice of parameters. This is best done by the aid of computer models, constructing examples for a range of parameters and judging which yield a polyhedron without self-intersection and a large range of motion. Verifying that models have no self-intersection is a finite check, best executed in particular cases by looking at an accurately drawn image. For this paper, the authors used models in MATLAB and GeoGebra. A symbolic and computational proof, following the algorithm described in [3] would address this formally. This is done by first noting that it is enough to check that no edge passes through a face, and then going through all pairs of faces and edges and checking on which sides of a face the endpoints of an edge land.

3.3 A Novel Flexible Polyhedron

Starting with a twinned digonal anticupola, we may remove two edges to create a crinkle, as seen in the left of figure 7. This crinkle of pentagonal boundary may be regarded as three triangles glued together, two at an edge, all flexing in a joined motion. Note that in creating it we have removed two edges, so the system itself has two degrees of freedom, but we will only use one of them.

To make a flexible polyhedron, we glue the pentagonal crinkle depicted in figure 7 to three linked tetrahedral caps as the figure shows. The tetrahedra are not closed, their bottoms are missing, and these holes align with the holes in the crinkle. We get rid of further self-intersections by adding a Bricard crinkle between each of the two adjacent tetrahedra. Finally, we dispose of any additional vertices poking through faces by adding tents on those faces. By trial and error, we found a selection of edge lengths without self-intersection that showed a reasonable range of motion. A picture of the assembled model can be see in figure 1, and we include a net for assembling the polyhedron in figure 8. The model resembles a fox, and the working name for the polyhedron was the foxtrot, after a popular American dance. A three dimensional version, showing the flexing motion, is available online at `https://www.geogebra.org/m/fq3vdnmy`. By rotating this model around, it is easy to convince yourself that it is free of self-intersection. For the sake of brevity, we do not include a proof that the foxtrot is embedded. A formal proof of this fact could follow the algorithm of [3].

The construction we have described bears a likeness to that of Steffen's polyhedron, and is an example of the base+crinkle method outlined in [15]. Here, the base is three linked tetrahedra, and the crinkles are two Bricard crinkles and one pentagonal crinkle. Note that the volume of the polyhedron comes from the three tetrahedra, and remains constant during the flexing motion as predicted by the Bellows theorem.

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Conclusion

We have generalised Raoul Bricard's 1897 construction of flexible octahedra, creating infinitely many combinatorially distinct flexible polyhedra with self-intersection. Using these, Connelly's methods from [10] may be used to remove the self-intersections in many cases. We described two explicit examples of notable flexible surfaces without self-intersections. Many further examples of twinned polyhedra could be considered. Each such example leads to a crinkle by the removal of edges. These crinkles, despite their simplicity, can allow for a variety of interesting motions. Further application of this work could use them in the design of flexible triangulated surfaces.

Above, we have generalised the Bricard octahedra of types I and II, but further research could consider possible generalisations of the third type. The equator of the Bricard III octahedron can be realised as a subset of rods lying on a ruled surface undergoing a motion,

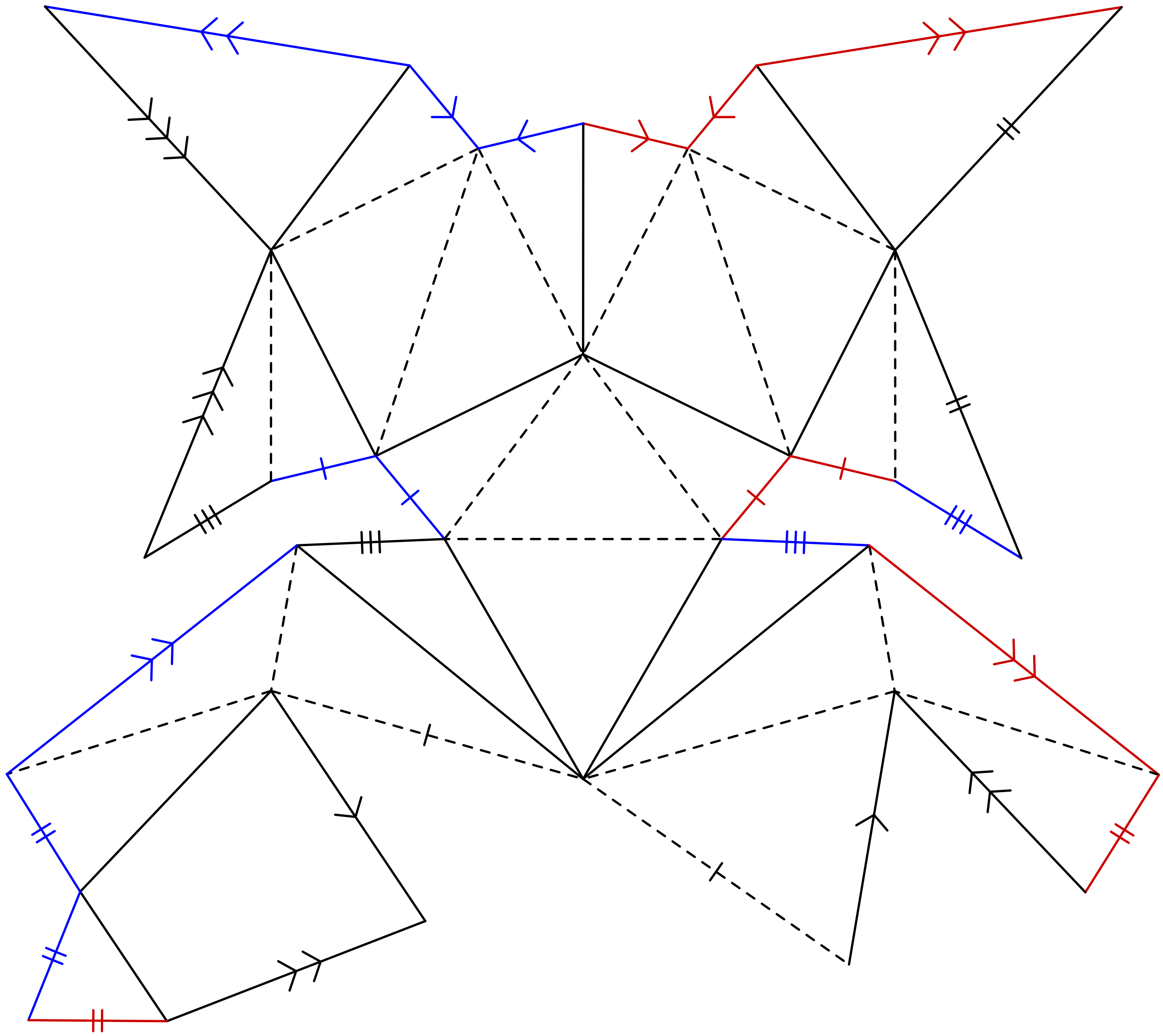

Figure 8: A net for the foxtrot, a flexible polyhedron constructed with a crinkle arising from the twinned digonal anticupola. Dashed lines correspond to valley folds and solid lines to mountain folds. Gluing instructions are indicated by a symbol and colour.

and a generalisation might look for other ways of erecting polyhedra onto those equators. A different direction could be to use equators of more than four sides.

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