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Blending of Parametric Curves via Smoothing Functions with Global C^n-Continuity

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Abstract. This paper proposes a geometric modeling approach for blending parametric curves based on a class of smoothing functions ensuring global C^n-continuity. The method constructs a composite curve that smoothly connects two given spatial curves by introducing a transition segment defined as a convex combination of their coordinate functions. The smoothing functions are formulated on the unit interval and extended to arbitrary parameter domains through affine transformations while satisfying prescribed boundary and differentiability conditions of arbitrary order. The resulting framework is independent of the specific parameterizations of the input curves and provides explicit analytical representations of the blended geometry. Several illustrative examples demonstrate the flexibility and robustness of the proposed approach, highlighting its potential for applications in geometric modeling, surface generation, and shape morphing.

Key Words: geometric modeling, curve blending, smoothing function, parametric curve, C^n-continuity

MSC 2020: 51M04

1 Introduction

Geometric modeling is a fundamental component of computer graphics, scientific visualization, and shape representation, providing the mathematical and algorithmic foundations for the construction and manipulation of curves and surfaces [2]. A central task in these areas is the generation of smooth transitions between geometric entities, which is essential for ensuring geometric continuity, visual fairness, and numerical stability of the resulting representations. Accordingly, the problem of constructing blends with prescribed levels of smoothness naturally arises in applications such as animation, surface construction, shape morphing, and geometric design [4].

Curve blending [8] has been extensively studied within computer-aided geometric design, with classical approaches relying on polynomial interpolants, spline constructions [6], or subdivision schemes [1]. Although these methodologies are highly effective and broadly used,

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they are inherently tied to polynomial bases or fixed parameterizations and may encounter difficulties when high-order continuity is required or when curves are defined over nonuniform or incompatible parameter domains. In particular, enforcing C^n-smoothness becomes increasingly challenging as n grows, and the resulting interpolants may exhibit undesirable oscillatory behavior or sensitivity to boundary data [5]. These limitations motivate the development of blending techniques that are both parametrically flexible and capable of achieving arbitrarily high global smoothness in a controlled manner.

From a mathematical perspective, a curve blend may be understood as a composite curve whose endpoint jets match those of the given curves up to a prescribed order. This formulation emphasizes the need for explicit control of boundary behavior, which in turn requires the construction of transition functions satisfying strict differentiability and flatness conditions. Classical Hermite-type blends provide one standard mechanism for achieving such continuity but rely on fixed polynomial basis functions, thereby restricting the available transition profiles and limiting the degree of flexibility in shaping the connecting region.

In this paper, we introduce a general framework for constructing smooth composite curves in three-dimensional space by means of smoothing functions defined on the unit interval. These functions are endowed with prescribed endpoint conditions that guarantee global C^n-continuity for arbitrary $0 \leq n \leq \infty$. A key feature of the proposed approach is that it does not depend on the specific parameterizations of the input curves. Through affine transformations, the smoothing functions are extended to arbitrary parameter domains and used to form convex combinations of coordinate functions. This yields a transition curve that interpolates the endpoint jets of the input curves and achieves the desired smoothness without requiring any reparameterization steps.

Unlike spline- or polynomial-based constructions [3, 10], the proposed method allows the use of nonpolynomial smoothing functions, which significantly enlarges the space of admissible transition profiles. Classical Hermite blending is recovered as a particular case when the smoothing function coincides with the corresponding Hermite polynomial. Thus, the present framework constitutes a strict generalization of traditional blending schemes and enables finer control over shape characteristics such as curvature distribution, smoothness transitions, and the rate at which the blend approaches the boundary curves.

The theoretical properties of the smoothing functions – including their boundary flatness, differentiability, and behavior under affine mappings – are analyzed to establish the global C^n-continuity of the resulting blended curve. The method is applicable to spatial curves of arbitrary parametrization and is illustrated through explicit analytical expressions and representative graphical examples.

The results presented in this paper contribute to the development of robust and flexible techniques for curve blending in geometric modeling. Potential applications include surface construction, shape morphing, and the generation of smooth transitions in multidimensional geometric configurations. From a theoretical standpoint, the proposed framework offers a systematic and extensible basis for constructing smooth composite curves and provides a versatile tool for further research in geometric modeling and computational geometry.

2 Blending One-Dimensional Functions via a Smoothing Function

In this section, we introduce a general smoothing-based mechanism for constructing globally C^n-smooth blends of one-dimensional functions. The construction is independent of any polynomial basis and therefore provides a broad extension of classical Hermite-type blending.

The key idea is to generate a transition function that interpolates between 0 and 1 and whose derivatives up to order n vanish at the endpoints. Such a function enables the formation of smooth convex combinations of two given functions while maintaining precise control over endpoint behavior.

Consider the function $\hat{\sigma}\colon \mathbb{R} \to [0,1]$ defined as

$$\hat{\sigma}(x) = \begin{cases} 0, & x \in (-\infty, 0] \\ \sigma(x), & x \in (0,1) \\ 1, & x \in [1, +\infty) \end{cases} \tag{1}$$

where the function $\sigma\colon [0,1] \to [0,1]$ satisfies the boundary conditions $\sigma(0) = 0$, $\sigma(1) = 1$. A trivial example is the linear function $\sigma(x) = x$, but such a choice does not ensure high-order smoothness. Our goal is to design σ so that it achieves arbitrary prescribed smoothness at the endpoints.

Given functions $f,\ g \in C^n(\mathbb{R})$, we define their smooth transition using the smoothing function (1) as

$$h(x) = (1 - \hat{\sigma}(x))f(x) + \hat{\sigma}(x)g(x),\ \forall x \in \mathbb{R}, \tag{2}$$

or equivalently,

$$h(x) = f(x) + \hat{\sigma}(x)(g(x) - f(x)). \tag{3}$$

Representation (3) immediately implies that if $\hat{\sigma} \in C^n(\mathbb{R})$, then $h \in C^n(\mathbb{R})$. Explicitly,

$$h(x) = \begin{cases} f(x), & x \in (-\infty, 0] \\ f(x) + \sigma(x)(g(x) - f(x)), & x \in (0,1) \\ g(x), & x \in [1, +\infty) \end{cases} \tag{4}$$

To guarantee C^n-smoothness across the junctions $x = 0$ and $x = 1$, we impose the endpoint flatness conditions:

$$\sigma \in C^n([0,1]),\ \sigma(0) = 0,\ \sigma(1) = 1,\ \sigma^{(k)}(0) = \sigma^{(k)}(1) = 0, \quad k = 1, \dots, n. \tag{5}$$

Using representation (4) and conditions (5), we compute

$$h^{(k)}(x) = f^{(k)}(x) + \sum_{j=0}^{k} \binom{k}{j} \sigma^{(j)}(x)\Big(g^{(k-j)}(x) - f^{(k-j)}(x)\Big),$$

which yields

$$h^{(k)}(0) = f^{(k)}(0),\ h^{(k)}(1) = g^{(k)}(1), \quad k = 1, \dots, n. \tag{6}$$

Thus $h \in C^n(\mathbb{R})$.

Construction of a Smoothing Function. We now build a monotonically increasing function $\sigma\colon [0,1] \to [0,1]$ satisfying (5). Let $\psi \in C^n([0,1])$ be a monotonically increasing function such that $\psi^{(k)}(0) = 0$ for all $k = 0, \dots, n$. Define

$$\sigma(x) = \frac{\psi(x)}{\psi(x) + \psi(1-x)}. \tag{7}$$

Differentiating (7), we obtain

$$\frac{\mathrm{d}\sigma(x)}{\mathrm{d}x} = \frac{\psi'(x)\psi(1-x) + \psi(x)\psi'(1-x)}{(\psi(x)+\psi(1-x))^2} > 0, \tag{8}$$

hence σ is strictly increasing. Define $\eta(x) = 1/(\psi(x)+\psi(1-x))$. Using $\psi^{(k)}(0) = 0$, we have

$$\sigma^{(k)}(0) = \sum_{j=0}^{k} \binom{k}{j} \psi^{(j)}(0)\eta^{(k-j)}(0) = 0.$$

Introduce the auxiliary function

$$\theta(x) = \sigma\Big(x + \tfrac{1}{2}\Big) - \tfrac{1}{2}, \quad x \in [-\tfrac{1}{2}, \tfrac{1}{2}].$$

It is straightforward to check that θ is odd, i.e. $\theta(x) = -\theta(-x)$, implying

$$\theta^{(k)}(x) = (-1)^{k+1}\theta^{(k)}(-x),\ k = 0, \dots, n. \tag{9}$$

From (9) we deduce

$$\sigma(1) = 1,\ \sigma^{(k)}(1) = (-1)^{k+1}\sigma^{(k)}(0) = 0,$$

thus verifying all conditions in (5).

Examples. For a positive integer n, a simple choice is $\psi(x) = x^{n+1}$. For $n = \infty$, one may use the flat function

$$\psi(x) = \begin{cases} e^{-1/x}, & x > 0, \\ 0, & x \le 0. \end{cases}$$

Remark 1. Constructing the blending function $\sigma_{[a;b]}\colon [a,b] \to [0,1]$ reduces to the case $[0,1]$ via the affine rescaling

$$\sigma_{[a;b]}(x) = \sigma\Big(\frac{x-a}{b-a}\Big). \tag{10}$$

If σ satisfies (5), then $\sigma_{[a;b]}$ satisfies the corresponding conditions on $[a,b]$. The resulting blended function

$$h_{[a;b]}(x) = \begin{cases} f(x), & x \in (-\infty, a], \\ (1-\sigma_{[a;b]}(x))f(x) + \sigma_{[a;b]}(x)g(x), & x \in (a,b), \\ g(x), & x \in [b, +\infty), \end{cases}$$

belongs to $C^n(\mathbb{R})$.

Remark 2. The construction extends to functions defined on $[\alpha, \beta]$:

$$h_{[a;b]}(x) = \begin{cases} f(x), & x \in [\alpha, a], \\ (1-\sigma_{[a;b]}(x))f(x) + \sigma_{[a;b]}(x)g(x), & x \in (a,b), \\ g(x), & x \in [b, \beta], \end{cases} \tag{11}$$

yielding a composite function $h_{[a;b]} \in C^n([\alpha, \beta])$.

Remark 3. The general formula (7) yields a broad family of smoothing functions $\sigma \in C^n([0,1])$ satisfying (5). Other representations are also possible. For example:

$$\sigma(x) = \frac{1}{c}\int_0^x \exp\Bigl(-\frac{1}{t(1-t)}\Bigr)\,\mathrm{d}t, \quad c = \int_0^1 \exp\Bigl(-\frac{1}{t(1-t)}\Bigr)\,\mathrm{d}t.$$

If only moderate smoothness is required, polynomial approximations may be used:

$$\sigma(x) = \begin{cases} 3x^2 - 2x^3, & C^1\text{-smooth blend,} \\ 10x^3 - 15x^4 + 6x^5, & C^2\text{-smooth blend.} \end{cases}$$

Remark 4. Although the proposed algorithm employs additive blending, a multiplicative variant can also be used:

$$h(x) = f(x)^{1-\sigma(x)} g(x)^{\sigma(x)}.$$

This approach preserves positivity and proportional scaling, which is useful in applications involving exponential or multiplicative structures. However, it requires $f(x), g(x) > 0$ and may be less intuitive than the additive form.

3 Blending of Functions Containing Singularities

If the functions $f, g\colon \mathbb{R} \to \mathbb{R}$ exhibit irregularities within the blending interval $[0,1]$ – for example, loss of smoothness, insufficient regularity, or the presence of isolated singularities – then formula (3) will naturally transfer these undesirable features to the blended function $h\colon \mathbb{R} \to \mathbb{R}$. In many practical settings, however, only the smoothness of f and g at the endpoints of the interval is required in order to construct a transition with a prescribed degree of differentiability. To obtain a more robust blending algorithm, we therefore deliberately disregard the values of f and g inside $[0,1]$, which in particular allows the method to accommodate functions with various types of interior irregularities.

3.1 Cut-off Functions and Their Application

Consider the following compactly supported C^∞ "hat" function defined on the real axis:

$$\omega(x) = \begin{cases} \frac{1}{c}\exp\bigl(\frac{1}{x^2-1}\bigr), & \text{for } |x| < 1, \\ 0, & \text{for } |x| \geq 1, \end{cases} \tag{12}$$

where

$$c = \int_{-1}^1 \exp\Bigl(\frac{1}{x^2-1}\Bigr)\,\mathrm{d}x.$$

We also introduce the scaled averaging kernel

$$\omega_h(x) = \frac{\omega(x/h)}{h}, \ \forall h > 0, \tag{13}$$

which satisfies

$$\omega_h \in C^\infty(\mathbb{R}), \quad \int_{-\infty}^{\infty} \omega_h(x)\,\mathrm{d}x = 1, \quad \forall h > 0.$$

Using this kernel, we define on $[a,b]$ the cut-off function [7]

$$\xi_{\delta,a,b}(x)=\int_{a+\frac{3\delta}{4}}^{b-\frac{3\delta}{4}}\omega_{\frac{\delta}{4}}(x-y)\,\mathrm{d}y,\quad x\in\mathbb{R}. \tag{14}$$

The cut-off function $\xi_{\delta,a,b}(x)$ has the following properties: (i) $\xi_{\delta,a,b}(x)=1$ for $a+\delta\le x\le b-\delta$; (ii) $\xi_{\delta,a,b}(x)=0$ for $x\in(-\infty,a]\cup[b,\infty)$; (iii) $\xi_{\delta,a,b}\in C^{\infty}(\mathbb{R})$ for all $\delta>0$.

The function $\xi_{\delta,a,b}(x)$ can also be constructed without integrals, using the previously defined smoothing function $\sigma\in C^n([0,1])$ satisfying (5), (7), and (10). In this case,

$$\xi_{\delta,a,b}(x)=\begin{cases}\sigma\left(\frac{x-a}{\delta}\right), & a\le x\le a+\delta,\\ 1, & a+\delta<x<b-\delta,\\ \sigma\left(\frac{b-x}{\delta}\right), & b-\delta\le x\le b.\end{cases} \tag{15}$$

Let $\hat{\sigma}\colon\mathbb{R}\to\mathbb{R}$ be defined by (1). Since $\hat{\sigma}\in C^n(\mathbb{R})$, we may rewrite (15) in the equivalent form

$$\xi_{\delta,a,b}(x)=\begin{cases}\hat{\sigma}\left(\frac{x-a}{\delta}\right), & x\in(-\infty,a+\delta),\\ 1, & x\in[a+\delta,b-\delta],\\ \hat{\sigma}\left(\frac{b-x}{\delta}\right), & x\in(b-\delta,\infty).\end{cases} \tag{16}$$

To exclude the interior portions of the functions f and g when constructing a smooth blending function, we use the cut-off functions $\xi_{\delta,0,\infty}$ and $\xi_{\delta,-\infty,1}$:

$$f_\delta(x)=\Big(1-\xi_{\delta,0,\infty}(x)\Big)f(x),\quad g_\delta(x)=\Big(1-\xi_{\delta,-\infty,1}(x)\Big)g(x). \tag{17}$$

Finally, define the δ-regularized blended function

$$h_\delta(x)=f_\delta(x)+\hat{\sigma}(x)\Big(g_\delta(x)-f_\delta(x)\Big). \tag{18}$$

Thus, representation (18) completely excludes the values of f and g inside the interval $[0,1]$. Using (14)–(16), we obtain the explicit forms:

$$\xi_{\delta,0,\infty}(x)=\int_{\frac{3\delta}{4}}^{\infty}\omega_{\frac{\delta}{4}}(x-y)\,\mathrm{d}y,\quad \xi_{\delta,0,\infty}(x)=\begin{cases}\hat{\sigma}\left(\frac{x}{\delta}\right), & x\in(-\infty,\delta),\\ 1, & x\in[\delta,\infty).\end{cases}$$

$$\xi_{\delta,-\infty,1}(x)=\int_{-\infty}^{1-\frac{3\delta}{4}}\omega_{\frac{\delta}{4}}(x-y)\,\mathrm{d}y,\quad \xi_{\delta,-\infty,1}(x)=\begin{cases}1, & x\in(-\infty,1-\delta],\\ \hat{\sigma}\left(\frac{1-x}{\delta}\right), & x\in(1-\delta,\infty).\end{cases}$$

4 Smooth Blending of Two Three-Dimensional Curves

We consider the task of constructing a composite parametric curve in three-dimensional space that joins selected points on two given smooth curves while preserving a prescribed order of smoothness. The construction relies on the one-dimensional blending mechanism developed earlier and inherits its continuity guarantees.

Consider two arbitrary smooth curves in parametric form:

$$\vec{\mathbf{r}}_f(t)=(f_i(t))_{i=\overline{1;3}}\in\mathbb{R}^3,\ \vec{\mathbf{r}}_g(t)=(g_i(t))_{i=\overline{1;3}}\in\mathbb{R}^3,\quad \forall t\in[0,1]. \tag{19}$$

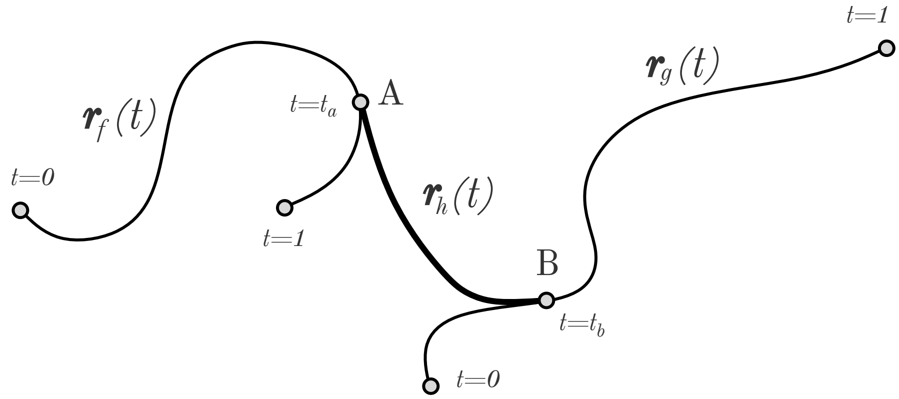


Figure 1: Blending of two three-dimensional curves.

Assume that the component functions satisfy

$$f_i, g_i \in C^n\Big([0,1]\Big),\ \forall i = \overline{1;3}.$$

Let $\mathbf{A} = \vec{\mathbf{r}}_f(t_a)$ and $\mathbf{B} = \vec{\mathbf{r}}_g(t_b)$ be arbitrarily chosen points on $\vec{\mathbf{r}}_f$ and $\vec{\mathbf{r}}_g$, respectively, with $t_a, t_b \in [0,1]$. Our goal is to construct a curve

$$\vec{\mathbf{r}}_h(t) = \Big(h_i(t)\Big)_{i=\overline{1;3}}$$

(see Fig. 1) that consists of three parts: (i) the initial segment of $\vec{\mathbf{r}}_f$ from its start to $\mathbf{A}$; (ii) a transition segment joining $\mathbf{A}$ to $\mathbf{B}$; (iii) the terminal segment of $\vec{\mathbf{r}}_g$ from $\mathbf{B}$ to the end of $\vec{\mathbf{r}}_g$. By construction, each h_i must belong to C^n, ensuring the overall C^n-smoothness of the composite path.

To define the transition interval, introduce nested intervals $[t_a, t_b + \tau] \subset (\tau, 1)$, where (see Fig. 2)

$$\tau = \Delta_1 + \varepsilon\Delta_2,\ \Delta_1 = \max\{0, t_a - t_b\},\ \Delta_2 = \min\{t_b, 1 - t_a\},\ 0 < \varepsilon < 1.$$

The parameter τ guarantees sufficient separation of the transition zone, while ε modulates its extent. On $[\tau, 1]$ define, for all $i = \overline{1;3}$,

$$f_i \colon [\tau, 1] \to \mathbb{R}, \tag{20}$$

and

$$g_i^\tau \colon [\tau, 1] \to \mathbb{R},\ g_i^\tau(t) = g_i(t - \tau),\ \forall t \in [\tau, 1]. \tag{21}$$

Here (20) is the restriction of f_i to $[\tau, 1]$, while (21) is the restriction of g_i shifted to the right by τ (see Fig. 2). Applying the blending construction (11) on $[t_a, t_b + \tau]$, we set, for each $i = \overline{1;3}$,

$$h_i(t) = \begin{cases} f_i(t), & t \in [0, t_a], \\ (1 - \sigma_{[t_a, t_b+\tau]}(t)) f_i(t) + \sigma_{[t_a, t_b+\tau]}(t) g_i^\tau(t), & t \in (t_a, t_b + \tau), \\ g_i^\tau(t), & t \in [t_b + \tau, 1 + \tau]. \end{cases} \tag{22}$$

By construction, the functions in (22) are in C^n and yield a parametric representation $\vec{\mathbf{r}}_h(t) = \Big(h_i(t)\Big)_{i=\overline{1;3}}$ on $[0, 1+\tau]$ (see Fig. 2). The resulting curve coincides with $\vec{\mathbf{r}}_f$ on $[0, t_a]$, follows a C^n-smooth transition, and continues as the shifted $\vec{\mathbf{r}}_g$ on $[t_b + \tau, 1 + \tau]$.

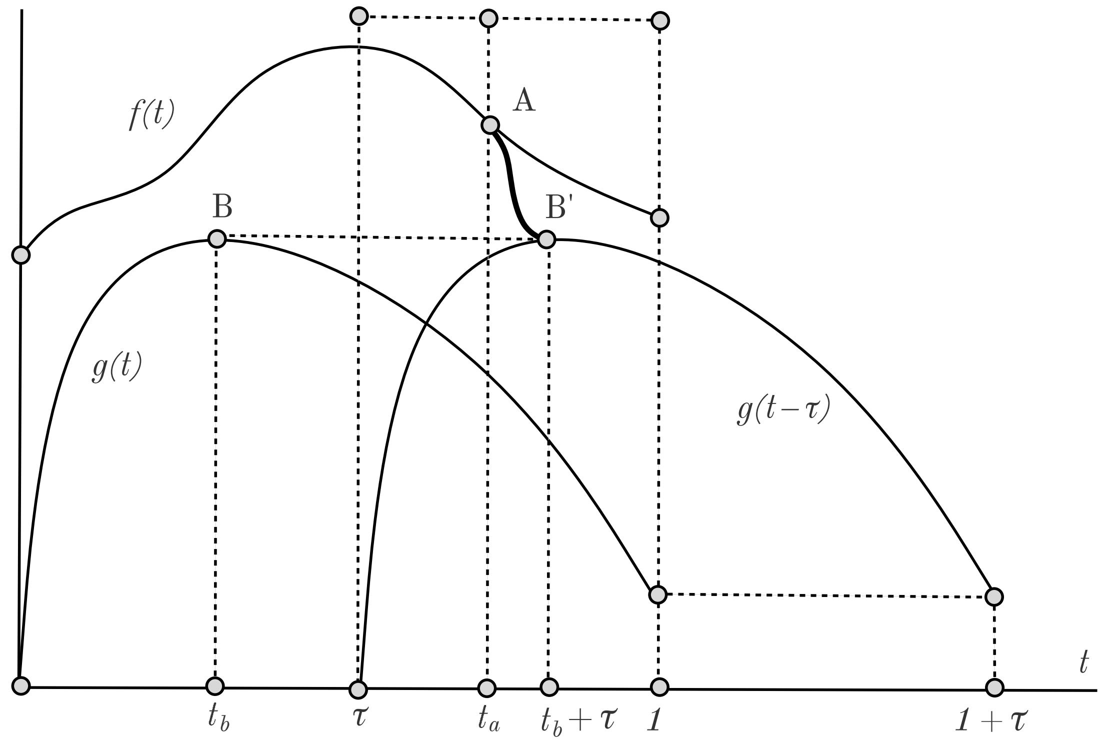


Figure 2: Graphical depiction of the algorithm for constructing a blending curve.

Remark 5. Formulas (22) define a family of curves $\overrightarrow{\mathbf{r}_{h,\varepsilon}}(t)$ parameterized by $0 < \varepsilon < 1$. For each $\varepsilon \in (0, 1)$, the curve $\overrightarrow{\mathbf{r}_{h,\varepsilon}}$ is C^n-smooth; the parameter ε changes the geometry of the transition between **A** and **B′** (see Fig. 2), effectively tuning the width and location of the blending zone.

5 Applications of the Proposed Algorithm

We illustrate several representative use cases for the smoothing-based blending of parametric curves. When a C^∞ smoothing function is employed, the construction yields a C^∞ transition; in general, the achieved smoothness matches the class of the chosen smoothing function.

5.1 Blending Two Parametric Curves

Figure 3 shows a smooth connection between $\Gamma_f = \{(t, e^{-t}) : t \in (-\infty, 0)\}$ and $\Gamma_g = \{(t, \sin(10t)) : t \in (1, \infty)\}$. The transition is implemented using the representation (4), producing a C^∞ blend across the junction. Such connections suppress abrupt curvature variations and are suitable for CAD/CAM tasks that require high-order continuity (e.g., C^2/G^2 or higher) in downstream surface generation and toolpath planning.

5.2 Effect of the δ Parameter

Figure 4 illustrates the influence of the parameter δ on the blended curve (18). We compare two settings: (a) $\delta = 0.5$, which yields a relatively narrow transition, and (b) $\delta = 0.95$,

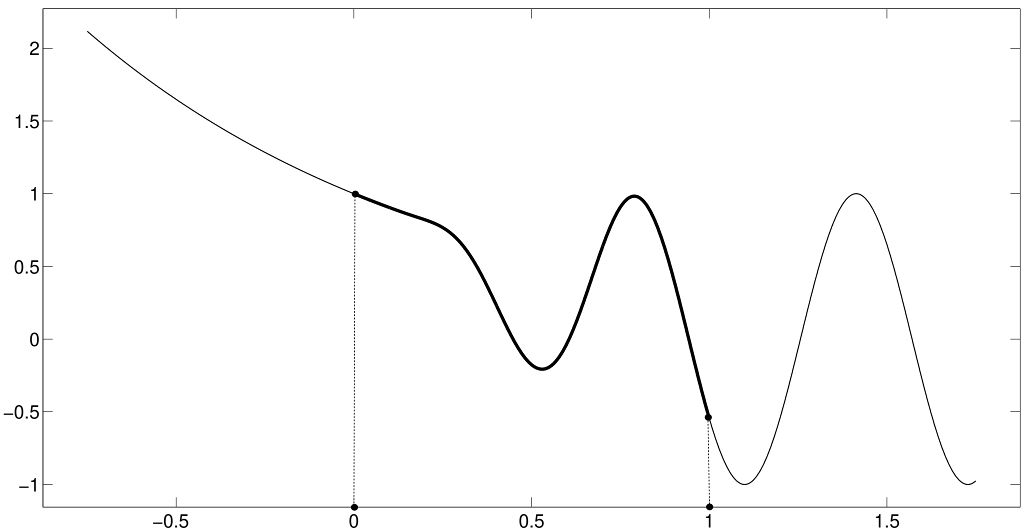


Figure 3: Smooth connection (C^∞) of two curves: $\Gamma_f = \{(t, e^{-t}) : t \in (-\infty, 0)\}$ and $\Gamma_g = \{(t, \sin(10t)) : t \in (1, \infty)\}$.

producing a wider blending region. This degree of control is valuable in applications such as trajectory planning, where curvature and acceleration constraints must be accommodated. In the example,

$$\Gamma_f = \Big\{\Big(t,\ 2 + \tfrac{1}{t-1}\Big) : t \in (-\infty, 1)\Big\}, \quad \Gamma_g = \Big\{\Big(t,\ \sin(\tfrac{1}{t})\Big) : t \in (0, \infty)\Big\}.$$

5.3 Squircle Deformation

Figure 5 demonstrates a deformation of a squircle [9] induced by varying the length of the tangent segment to a circle. The smoothing-based algorithm adapts the transition profile while preserving the desired continuity, which is useful in shape design and optimization tasks that require gradual curvature changes for both structural and aesthetic reasons. The same strategy extends to spatial profiles for surface modeling in industrial design.

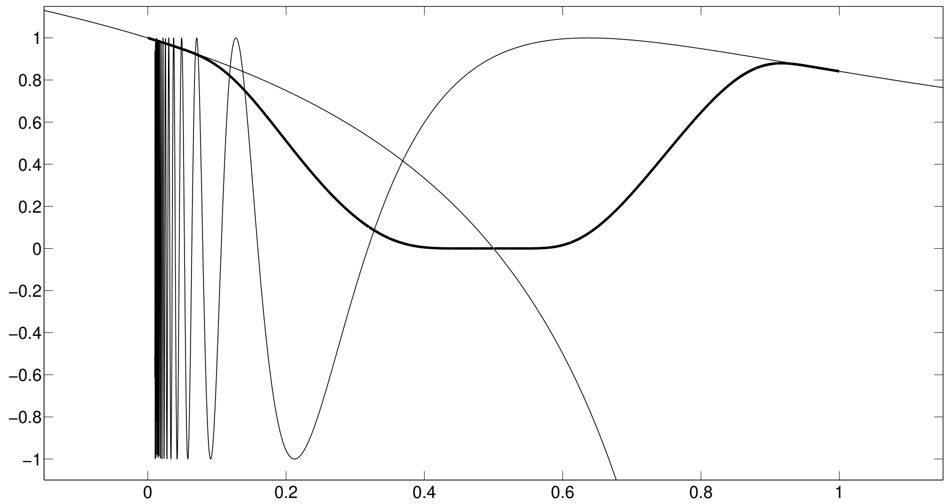
5.4 Blending Two Spatial Helical Curves

To demonstrate the capabilities of the proposed smoothing-based blending framework in three-dimensional space, we consider an example involving two spatial helical curves with different screw axes and incompatible parameterizations (see Fig. 6). This example illustrates how the method naturally extends to genuinely spatial geometries and does not require any reparameterization of the original curves.

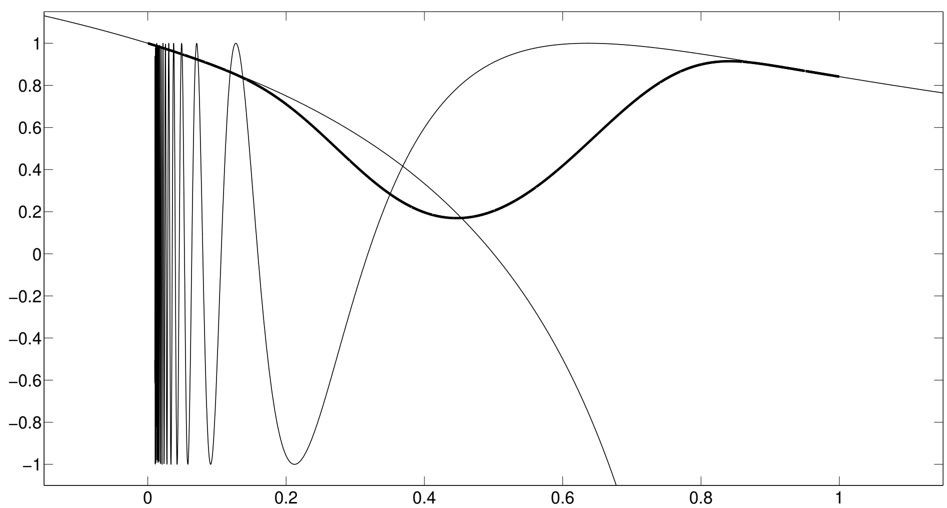
Consider the following helical curves in $\mathbb{R}^3$:

$$\vec{\mathbf{r}}_f(s) = \Big(\cos(2\pi s), \sin(2\pi s), s\Big), \qquad s \in [-2, 0],$$
$$\vec{\mathbf{r}}_g(u) = \Big(1 + 0.5\cos(2\pi u), 1 + 0.5\sin(2\pi u), 3 - 0.8u\Big), \qquad u \in [1, 3].$$

The first helix $\vec{\mathbf{r}}_f$ follows a vertical axis aligned with the z-direction and has unit radius, ascending as the parameter s increases. The second helix $\vec{\mathbf{r}}_g$ is centered at $(1, 1, 3)$, has radius



(a) $\delta = 0.5$



(b) $\delta = 0.95$

Figure 4: Effect of the parameter δ on the blending curve.

0.5, and descends along a displaced and skew axis, resulting in a pronounced spatial separation and orientation mismatch between the two curves.

To construct a smooth composite curve, we introduce a global parameter $t \in [-2, 3]$ and define a transition interval $t \in [0, 1]$. Outside this interval, the composite curve coincides exactly with the original helices, while within $[0, 1]$ a smooth transition is performed. The transition curve $\vec{\mathbf{r}}_h$ on $[0, 1]$ is defined using the parameter shifts

$$s(t) = t - 1 \in [-1, 0], \qquad u(t) = t + 1 \in [1, 2], \qquad \forall\, t \in [0, 1],$$

so that both original curves actively participate in the blending process. The composite curve

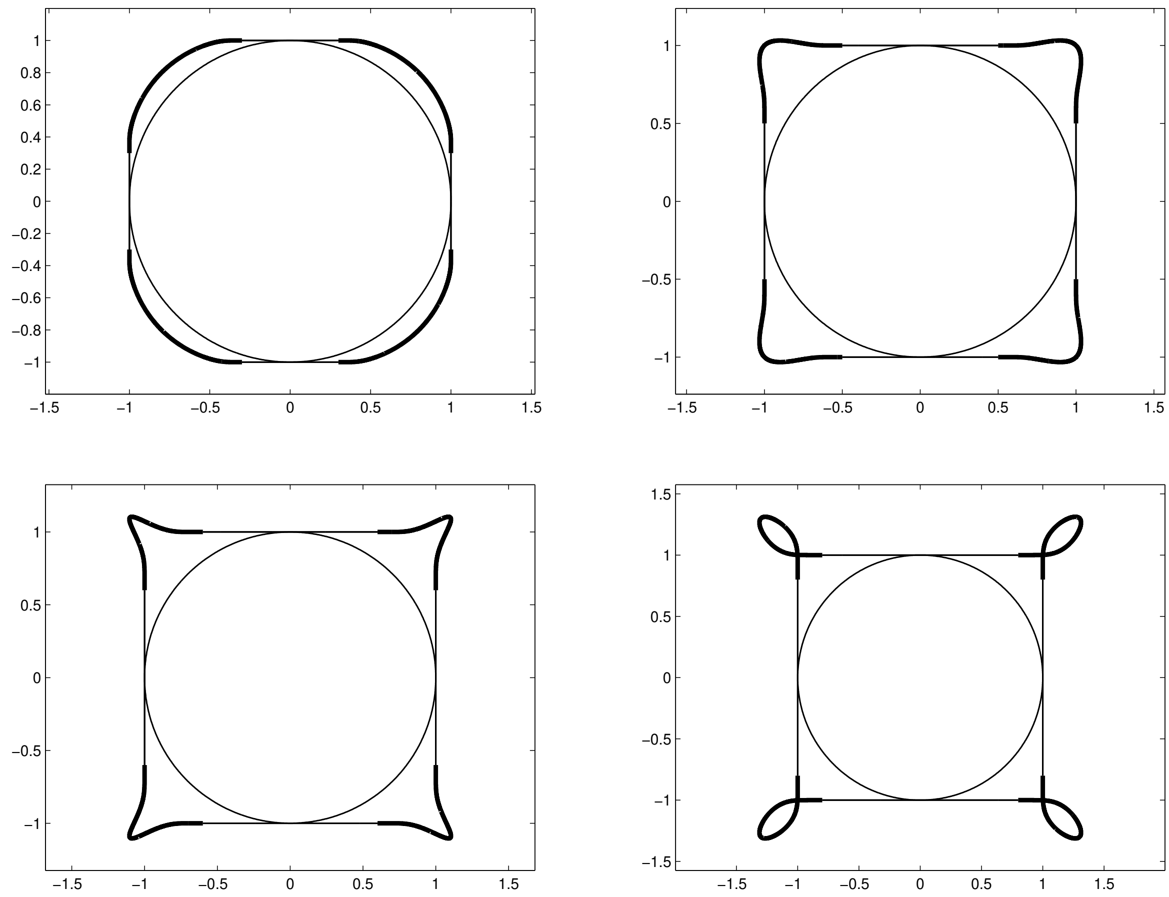


Figure 5: Squircle deformation as the tangent segment length to the circle varies.

$\vec{\mathbf{r}}_h(t)$ is then given by

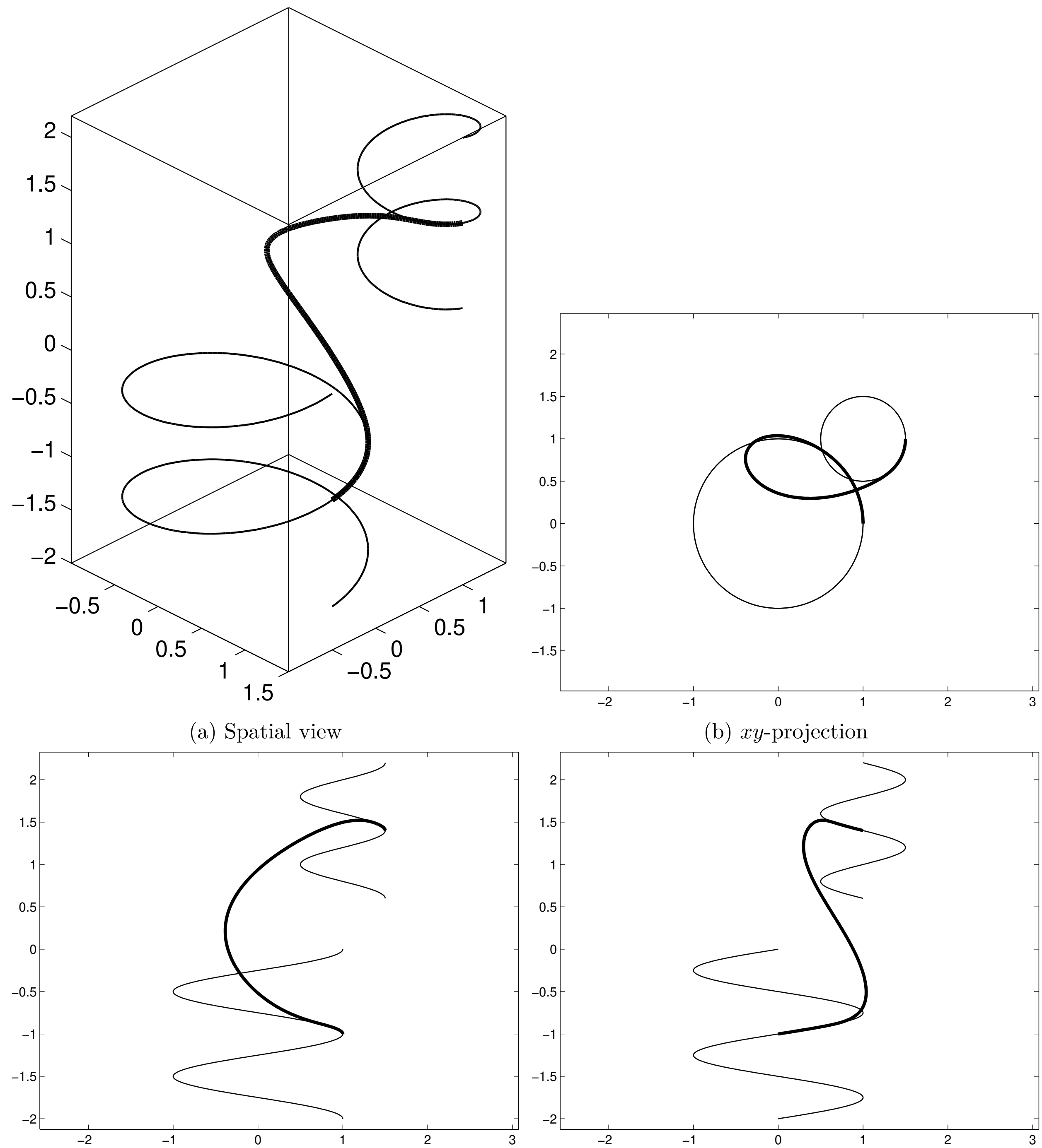
$$\vec{\mathbf{r}}_h(t) = \begin{cases} \vec{\mathbf{r}}_f(t), & t \in [-2, 0], \\ (1 - \sigma(t))\,\vec{\mathbf{r}}_f(t-1) + \sigma(t)\,\vec{\mathbf{r}}_g(t+1), & t \in [0, 1], \\ \vec{\mathbf{r}}_g(t), & t \in [1, 3]. \end{cases}$$

Since the smoothing function σ is flat at the endpoints, all derivatives of arbitrary order match at $t = 0$ and $t = 1$. Consequently, the resulting composite curve $\vec{\mathbf{r}}_h$ belongs to $C^\infty([-2, 3])$ and exhibits no discontinuities in position, tangent, curvature, or higher-order geometric quantities.

This example demonstrates that the proposed blending framework operates naturally in three-dimensional space and accommodates curves with different parameter domains, non-coplanar axes, and distinct geometric characteristics. Such constructions are particularly relevant for applications in robotics path planning, spatial mechanism design, and the generation of smooth trajectories for computer-controlled manufacturing processes.

Conclusions

This work has introduced a general geometric modeling framework for the smooth blending of three-dimensional parametric curves based on a class of smoothing functions ensuring



(a) Spatial view

(b) xy-projection

(c) xz-projection

(d) yz-projection

Figure 6: Smooth C^∞ blending of two spatial helical curves with skew screw axes.

global C^n-continuity, including the case $n = \infty$. The construction produces a composite curve containing a transition segment inserted between arbitrary points on two given spatial curves. This transition is defined through convex combinations of coordinate functions, controlled by smoothing functions that satisfy prescribed boundary and differentiability conditions of arbitrary order. Prototype smoothing functions defined on the unit interval are extended to any parameter domain through affine transformations, preserving the imposed smoothness requirements.

A salient feature of the proposed framework is its independence from the parameterization of the input curves. The method requires neither reparameterization nor knot insertion, nor any modification of the original curves. The desired order of smoothness is achieved solely by the appropriate choice of the smoothing function. This distinguishes the approach from classical spline-based or polynomial interpolation schemes, where continuity is typically enforced by raising polynomial degree, adjusting control points, or introducing additional degrees of freedom. Consequently, the framework affords direct and explicit control over smoothness while preserving the geometric fidelity of the original curves. Beyond classical approaches, the proposed method provides a strict generalization of Hermite blending. Classical Hermite interpolation emerges as a special case corresponding to a particular choice of the smoothing function, whereas the present construction allows the use of arbitrary C^n-smooth functions satisfying endpoint flatness conditions. This decouples the smoothness order from the algebraic structure of polynomial bases and enables blending profiles that cannot be reproduced by Hermite polynomials alone. In particular, non-polynomial transition functions may be employed to control curvature evolution, locality, and fairness properties in ways unavailable to purely polynomial schemes. Thus, while the derivation of endpoint matching conditions is analytically straightforward, the flexibility and generality of the smoothing-function framework constitute a substantive conceptual extension beyond traditional methods. From the standpoint of geometric modeling, the method provides a simple, robust, and flexible tool for constructing smooth transitions between spatial parametric curves. The geometry of the transition region is easily adjustable through the selection of smoothing functions, allowing precise control over locality, regularity, and curvature behavior. Given its explicit analytical formulation and conceptual clarity, the technique is well suited for incorporation into modeling pipelines, CAD/CAM systems, and shape processing workflows. Beyond its practical significance, the proposed construction also contributes to the theoretical study of smooth transitions between parametric objects. The use of smoothing functions satisfying high-order boundary conditions naturally suggests extensions to surface and volume blending, as well as to higher-dimensional geometric structures. Prospective research directions include the development of criteria for selecting optimal smoothing functions with respect to curvature or fairness measures, the incorporation of additional geometric constraints such as prescribed tangents or curvature continuity, and the extension of the present framework to surface blending, geometric deformation, and multidimensional shape manipulation.

References

[1] E. Cohen, R. F. Riesenfeld, and G. Elber: *Geometric Modeling with Splines: An Introduction*. A K Peters, 2001.

[2] G. Farin: *Curves and Surfaces for CAGD: A Practical Guide*. Morgan Kaufmann, 5 ed., 2002.

[3] E. Hartmann: *Parametric G^n blending of curves and surfaces*. The Visual Computer **17**(1), 1–13, 2001. doi: 10.1007/PL00013398.

[4] J. Hoschek and D. Lasser: *Fundamentals of Computer Aided Geometric Design*. A K Peters, 1993.

[5] K. C. Hui: *Shape blending of curves and surfaces with geometric continuity*. Computer-Aided Design **31**(13), 819–828, 1999. doi: 10.1016/S0010-4485(99)00059-7.

[6] L. Piegl and W. Tiller: *The NURBS Book*. Springer, 1997.

[7] V. A. Trenogin: *Functional Analysis*. Fizmatlit, Moscow, 2002. 488 pp.

[8] G. Wang and W. Wang: *Blending two parametric curves*. Computer-Aided Design **41**(6), 423–431, 2009. doi: 10.1016/j.cad.2009.02.006.

[9] E. W. Weisstein: *Squircle*. MathWorld–A Wolfram Web Resource. http://mathworld.wolfram.com/Squircle.html. N.d.

[10] A. Wiltsche: *Blending curves*. Journal for Geometry and Graphics **9**(1), 67–75, 2005.

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