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The Maximum of the Volume of a Part of a Cevian Simplex

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Abstract. The cevians passing through a point in a simplex create a cevian simplex, which is divided by these cevians into smaller simplices. We consider the problem of maximizing the ratio of the sum of the volumes of some of these smaller simplices to the volume of the reference simplex. An analogous question concerning the product of the volumes is also considered. The special case of the tetrahedron is presented as an example.

Key Words: simplex, volume, cevian, barycentric coordinates, optimization

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Given an n-simplex with vertices A_1, $A_2, \ldots, A_{n+1}$ $(n > 2)$ and any point M in its interior, let N_i be the points on the hyperplane opposite A_i such that A_i, M, and N_i are collinear. Let $V = [A_1A_2 \ldots A_{n+1}]$ and

$$V_i = [N_1N_2 \ldots N_{i-1}MN_{i+1} \ldots N_{n+1}] \ (1 \leq i \leq n+1),$$

where the square brackets indicate the volume of the n-simplex. We want to find the point M for which $\frac{V_1+V_2+\cdots+V_k}{V}$ $(1 \leq k \leq n+1)$ is maximal and find this maximum. The extremal cases for the sum of volumes have been established in the literature: the case $k = n+1$ was considered in [9], [4], [7], p. 502, while the boundary cases $k = 1$ and $k = n$ were investigated in [1]. However, the behavior of these volume ratios for the intermediate range $1 < k < n$ has remained an open problem. In the present paper, we address this gap by providing a general solution for all intermediate cases. Furthermore, we extend this investigation to the multiplicative version of the problem. It is shown that, surprisingly, the sum $V_1 + V_2 + \cdots + V_k$ and the product $V_1V_2 \cdots V_k$ are maximized at the same point M for $1 \leq k \leq n$.

Let $(\lambda_1, \lambda_2, \ldots, \lambda_{n+1})$ be the barycentric coordinates of M with respect to the simplex $A_1A_2 \ldots A_{n+1}$. Then $\lambda_1 + \lambda_2 + \cdots + \lambda_{n+1} = 1$. Suppose that $R_i = |MA_i|$, and $s_i = |MN_i|$. Since $\lambda_i = \frac{s_i}{R_i+s_i}$, we have $\frac{s_i}{R_i} = \frac{\lambda_i}{1-\lambda_i}$ (see [8]; [11]; [6]; [2], p. 124–126). Therefore,

$$V_i = [A_1A_2 \ldots A_{i-1}MA_{i+1} \ldots A_{n+1}] \cdot \prod_{j \neq i} \frac{s_j}{R_j}.$$

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Since

$$[A_1A_2 \dots A_{i-1}MA_{i+1} \dots A_{n+1}] = \lambda_i \cdot V$$

and

$$\frac{s_j}{R_j} = \frac{\lambda_j}{1-\lambda_j},$$

we obtain

$$V_i = \lambda_i \cdot V \cdot \prod_{j\neq i} \frac{\lambda_j}{1-\lambda_j} = V \cdot (1-\lambda_i) \cdot \prod_{j=1}^{n+1} \frac{\lambda_j}{1-\lambda_j}.$$

We can also verify the volume ratio V_i/V using the barycentric coordinates of the vertices. The vertices of the sub-simplex V_i are $\{N_1, N_2, \dots, N_{i-1}, M, N_{i+1}, \dots, N_{n+1}\}$, where the point M is positioned in the i-th row to maintain the original orientation of the simplex. Each vertex N_j is obtained by projecting $M(\lambda_1, \dots, \lambda_{n+1})$ from vertex A_j onto the opposite facet. Consequently, the j-th coordinate of N_j is zero, and the remaining coordinates are rescaled by a factor of $1/(1-\lambda_j)$ to satisfy the barycentric normalization condition $\sum \lambda = 1$. The volume ratio is expressed as the following $(n+1) \times (n+1)$ determinant [2], p. 127:

$$\frac{V_i}{V} = \left| \det \begin{pmatrix} 0 & \frac{\lambda_2}{1-\lambda_1} & \dots & \frac{\lambda_{n+1}}{1-\lambda_1} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1 & \lambda_2 & \dots & \lambda_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\lambda_1}{1-\lambda_{n+1}} & \frac{\lambda_2}{1-\lambda_{n+1}} & \dots & 0 \end{pmatrix} \right|.$$

To evaluate this determinant, we factor out λ_k from each k-th column and $1/(1-\lambda_j)$ from each j-th row (where $j \neq i$). This transformation reduces the expression to:

$$\frac{V_i}{V} = \frac{\prod_{j=1}^{n+1} \lambda_j}{\prod_{j\neq i}(1-\lambda_j)} \cdot |D|$$

where D is the following determinant:

1. **The Structure of D:** D is a matrix where the i-th row (corresponding to M) consists of all ones, while all other rows have 0s on the diagonal and 1s elsewhere:

$$D = \begin{vmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & 0 & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & 0 \end{vmatrix} \begin{matrix} \\ \\ \\ \leftarrow \text{Row } i \\ \\ \\ \end{matrix}$$

2. **Row Operations:** We subtract the i-th row (the row of ones) from all other rows. This operation transforms the matrix such that the diagonal elements in those rows become -1, and all other off-diagonal elements in those rows become 0.
3. **Final Evaluation:** By adding all the resulting rows to the row of ones (the i-th row), the determinant simplifies to a product of its diagonal components. This process yields:

$$D = (-1)^n.$$

By substituting the value of D back into the main expression and applying the identity $\sum_{j=1}^{n+1}\lambda_j = 1$, we obtain:

$$\frac{V_i}{V} = (1-\lambda_i)\prod_{j=1}^{n+1}\frac{\lambda_j}{1-\lambda_j}. \tag{1}$$

Consequently,

$$V_1 + V_2 + \cdots + V_k = V\cdot(k-\lambda_1-\lambda_2-\cdots-\lambda_k)\cdot\prod_{j=1}^{n+1}\frac{\lambda_j}{1-\lambda_j}.$$

In particular, if $k = n+1$, then we obtain

$$V_1 + V_2 + \cdots + V_{n+1} = V\cdot F(\lambda_1,\lambda_2,\ldots,\lambda_{n+1}),$$

where

$$F(\lambda_1,\lambda_2,\ldots,\lambda_{n+1}) = n\cdot\prod_{j=1}^{n+1}\frac{\lambda_j}{1-\lambda_j}.$$

The function $F(\lambda_1,\lambda_2,\ldots,\lambda_{n+1})$ is continuous whenever $\lambda_i \neq 1$ for all $i = 1,\ldots,n+1$. In order to make it continuous at the vertices A_i $(i = 1,\ldots,n+1)$ of the simplex, where $\lambda_i = 1$, we need to define $F(\lambda_1,\lambda_2,\ldots,\lambda_{n+1}) = 0$. After redefining the function $F(\lambda_1,\lambda_2,\ldots,\lambda_{n+1})$, the continuity at the vertex A_i follows from the fact that if $\lambda_i \to 1$, then $\lambda_j \to 0$ for all $j \neq i$, and therefore $\prod_{j\neq i}\lambda_j = o(1-\lambda_i)$. Indeed,

$$\sum_{j\neq i}\lambda_j = 1-\lambda_i,$$

and by the AM-GM inequality

$$\prod_{j\neq i}\lambda_j \le \left(\frac{1-\lambda_i}{n}\right)^n.$$

Since $n > 2$, we have

$$\left(\frac{1-\lambda_i}{n}\right)^n = o(1-\lambda_i).$$

Therefore,

$$\prod_{j\neq i}\lambda_j = o(1-\lambda_i).$$

For $1 < k < n$, we can write $\frac{V_1+V_2+\cdots+V_k}{V} = F_k(\lambda_1,\lambda_2,\ldots,\lambda_{n+1})$, where $F_k = G_kH_k$, and

$$G_k(\lambda_1,\lambda_2,\ldots,\lambda_k) = (k-\lambda_1-\lambda_2-\cdots-\lambda_k)\cdot\prod_{j=1}^{k}\frac{\lambda_j}{1-\lambda_j},$$

$$H_k(\lambda_{k+1},\lambda_{k+2},\ldots,\lambda_{n+1}) = \prod_{j=k+1}^{n+1}\frac{\lambda_j}{1-\lambda_j}.$$

Since $F_k(\lambda_1,\lambda_2,\ldots,\lambda_{n+1}) \le F(\lambda_1,\lambda_2,\ldots,\lambda_{n+1})$, we can redefine $F_k(\lambda_1,\lambda_2,\ldots,\lambda_{n+1}) = 0$ at the vertices A_i $(i = 1,\ldots,n+1)$ of the simplex to make it continuous at these vertices, too. Now the function $F_k(\lambda_1,\lambda_2,\ldots,\lambda_{n+1})$ is continuous in the simplex $A_1A_2\ldots A_{n+1}$, including its boundary, and therefore by the classical result in analysis (see e.g. [10], Theorem 4.16). $F_k(\lambda_1,\lambda_2,\ldots,\lambda_{n+1})$ attains both maximum and minimum values in this simplex, which is a compact set. At the boundary (vertices, edges, faces, etc.) $F_k(\lambda_1, \lambda_2, \ldots, \lambda_{n+1}) = 0$, which is the minimum. So, the maximum occurs in the interior of the simplex.

Suppose the function $F_k(\lambda_1, \lambda_2, \ldots, \lambda_{n+1})$ reaches its absolute maximum at $(\lambda_1, \lambda_2, \ldots, \lambda_{n+1})$. We now investigate if a greater value is achieved at any other point. For fixed $\lambda_i + \lambda_j$ the product $\frac{\lambda_i}{1-\lambda_i} \cdot \frac{\lambda_j}{1-\lambda_j}$ reaches the maximum when $\lambda_i = \lambda_j$. Indeed, let $\lambda_i + \lambda_j$ be constant. Observe that the product can be rewritten as:

$$\frac{\lambda_i}{1-\lambda_i} \cdot \frac{\lambda_j}{1-\lambda_j} = \frac{\lambda_i \cdot \lambda_j}{1-\lambda_i-\lambda_j+\lambda_i\lambda_j} = \frac{1}{\frac{1-\lambda_i-\lambda_j}{\lambda_i \cdot \lambda_j}+1}.$$

Since $1-\lambda_i-\lambda_j$ is also constant and $1-\lambda_i-\lambda_j > 0$, to maximize the fraction, it suffices to maximize the product $\lambda_i \cdot \lambda_j$. Therefore, this product attains its maximum value when $\lambda_i = \lambda_j$. This optimization argument goes back to Cauchy [3] and Maclaurin [5].

Therefore, for fixed $\sum_{i=1}^{k} \lambda_i = kx$ the function G_k reaches the maximum when $\lambda_1 = \lambda_2 = \cdots = \lambda_k = x$. Similarly, for fixed $\sum_{i=k+1}^{n+1} \lambda_i = my$, where $m = n+1-k > 0$, the function H_k reaches the maximum when $\lambda_{k+1} = \lambda_{k+2} = \cdots = \lambda_{n+1} = y$. Then $y = \frac{1-kx}{m}$, $1-y = \frac{m-1+kx}{m}$, and

$$F_k(\lambda_1, \lambda_2, \ldots, \lambda_{n+1}) = k \cdot (1-x) \cdot \Big(\frac{x}{1-x}\Big)^k \cdot \Big(\frac{1-kx}{m-1+kx}\Big)^m.$$

Note that the variable x is constrained by the geometry of the simplex. Since $\lambda_i \geq 0$ for all i and their sum is unity, i.e., $\sum_{i=1}^{n+1} \lambda_i = 1$, it follows that for the first k variables we have $\sum_{i=1}^{k} \lambda_i \leq 1$. Under the assumption $\lambda_1 = \cdots = \lambda_k = x$, this sum simplifies to $kx \leq 1$, which implies $0 \leq x \leq \frac{1}{k}$. By studying the function $f(x) = k \cdot \frac{x^k}{(1-x)^{k-1}} \cdot \big(\frac{1-kx}{m-1+kx}\big)^m$ for the maximum in interval $[0, \frac{1}{k}]$ we find that

$$f(x) = k \cdot \frac{x^k}{(1-x)^{k-1}} \cdot \Big(\frac{1-kx}{m-1+kx}\Big)^m,$$

$$\begin{aligned}
f'(x) &= \Big(\frac{1-kx}{m-1+kx}\Big)^m \Big[k \cdot kx^{k-1}(1-x)^{1-k} - kx^k(1-k)(1-x)^{-k}\Big] \\
&\quad + \frac{kx^k}{(1-x)^{k-1}} \cdot m\Big(\frac{1-kx}{m-1+kx}\Big)^{m-1} \cdot \frac{-k(m-1+kx)-k(1-kx)}{(m-1+kx)^2} \\
&= \Big(\frac{1-kx}{m-1+kx}\Big)^m \Big[k^2x^{k-1}(1-x)^{1-k} - kx^k(1-x)^{-k} + k^2x^k(1-x)^{-k}\Big] \\
&\quad + \frac{mkx^k}{(1-x)^{k-1}}\Big(\frac{1-kx}{m-1+kx}\Big)^{m-1} \cdot \frac{-km+k-k^2x-k+k^2x}{(m-1+kx)^2} \\
&= k \cdot \frac{x^{k-1}}{(1-x)^k} \cdot \frac{(1-kx)^{m-1}}{(m-1+kx)^{m+1}} \cdot \Big[(1-kx)(m-1+kx)(k-x) - km^2x(1-x)\Big],
\end{aligned}$$

$$f'(x) = k \cdot \frac{x^{k-1}}{(1-x)^k} \cdot \frac{(1-kx)^{m-1}}{(m-1+kx)^{m+1}} \cdot q(x)$$

where

$$q(x) = (k-x)(1-kx)(m-1+kx) - km^2x(1-x).$$

The factor $k \cdot \frac{x^{k-1}}{(1-x)^k} \cdot \frac{(1-kx)^{m-1}}{(m-1+kx)^{m+1}}$ is strictly positive in the interior of the interval $(0, \frac{1}{k})$. Consequently, the critical points of $f(x)$ are determined solely by the zeros of $q(x)$. Note that

$q(-\infty) = -\infty$, $q(0) = k(m-1) > 0$, $q(\frac{1}{k}) = -\frac{m^2(k-1)}{k} < 0$, and $q(+\infty) = +\infty$. Therefore, the cubic $q(x)$ has exactly one zero in each of the intervals $(-\infty, 0)$, $(0, \frac{1}{k})$, and $(\frac{1}{k}, +\infty)$. Let us denote by θ the zero of $q(x)$ in $(0, \frac{1}{k})$. Since $f(x)$ is continuous on the closed interval $[0, \frac{1}{k}]$ and vanishes at its endpoints, the fact that θ is the unique critical point in the interior $(0, \frac{1}{k})$ ensures that it must be the global maximum point. Consequently,

$$f(x) \le f(\theta) = k \cdot \frac{\theta^k}{(1-\theta)^{k-1}} \cdot \left(\frac{1-k\theta}{m-1+k\theta}\right)^{n+1-k}.$$

Thus we proved the following result.

Theorem 1. *Let $A_1A_2\ldots A_{n+1}$ $(n > 2)$ be an n-simplex and M be any point in its interior. Let N_i be the points on the hyperplane opposite A_i such that A_i, M, and N_i are collinear. Let V be the volume of $A_1A_2\ldots A_{n+1}$ and let V_i be the volume of $MN_1N_2\ldots N_{i-1}N_{i+1}\ldots N_{n+1}$ $(1 \le i \le n+1)$. Then for $1 < k < n$*

$$\frac{V_1 + V_2 + \cdots + V_k}{V} \le \frac{k\theta^k}{(1-\theta)^{k-1}} \cdot \left(\frac{1-k\theta}{m-1+k\theta}\right)^m,$$

where $m = n+1-k$, and θ is the zero of the cubic

$$q(x) = k^2x^3 - k(k^2 - m^2 - m + 2)x^2 + (2k^2 - k^2m - km^2 - m + 1)x + k(m-1)$$

in $(0, \frac{1}{k})$. Equality is attained at the unique point P with barycentric coordinates $\lambda_1 = \lambda_2 = \cdots = \lambda_k = \theta$, $\lambda_{k+1} = \lambda_{k+2} = \cdots = \lambda_{n+1} = \frac{1-k\theta}{m}$.

Remark 1. For the tetrahedral case where $n = 3$, $k = 2$, and $m = 2$, the general cubic equation $q(x)$ from Theorem 1 simplifies to:

$$q(x) = 4x^3 - 9x + 2 = 0.$$

To solve this equation analytically, we apply the trigonometric substitution $x = \sqrt{3}\cos\phi$. Substituting this into the equation and utilizing the triple-angle identity $4\cos^3\phi - 3\cos\phi = \cos(3\phi)$, the equation transforms into:

$$\cos(3\phi) = -\frac{2}{3\sqrt{3}} \implies \cos(3\phi - \pi) = \frac{2}{3\sqrt{3}} \implies \phi = \frac{\arccos\left(\frac{2}{3\sqrt{3}}\right)}{3} + \frac{\pi}{3}.$$

This leads to the explicit root θ_0 in the required interval $(0, 1/2)$, as stated in the following corollary.

Corollary 2. *Let $A_1A_2A_3A_4$ be a tetrahedron and M be any point in its interior. Let N_i be the points on the plane opposite A_i such that A_i, M, and N_i are collinear. Let V, V_1, and V_2 be the volumes of tetrahedra $A_1A_2A_3A_4$, $MN_2N_3N_4$, and $MN_1N_3N_4$, respectively (see Fig. 1 or* `https://www.geogebra.org/3d/htesqury`*). Then:*

$$\frac{V_1 + V_2}{V} \le \frac{2\theta_0^2(1-2\theta_0)}{3} \approx 0.01880,$$

where

$$\theta_0 = \sqrt{3}\cos\left(\frac{\arccos\left(\frac{2}{3\sqrt{3}}\right)}{3} + \frac{\pi}{3}\right) \approx 0.22745.$$

Proof. To derive the simplified form for the tetrahedral case ($n = 3$, $k = 2$, $m = 2$), we substitute these parameters into the general expression from Theorem 1:

$$f(\theta_0) = \frac{2\theta_0^2}{1-\theta_0} \cdot \left(\frac{1-2\theta_0}{1+2\theta_0}\right)^2.$$

Recall that θ_0 is the root of the cubic equation $q(x) = 4x^3 - 9x + 2 = 0$. This cubic relation allows for a significant algebraic reduction. Specifically, we observe the following identity:

$$(1-\theta_0)(1+2\theta_0)^2 = 1 + 3\theta_0 - 4\theta_0^3.$$

Substituting $4\theta_0^3 = 9\theta_0 - 2$ from the cubic equation into the identity above, we obtain:

$$(1-\theta_0)(1+2\theta_0)^2 = 1 + 3\theta_0 - (9\theta_0 - 2) = 3 - 6\theta_0 = 3(1-2\theta_0).$$

Consequently, the rational part of the expression simplifies as:

$$\frac{1-2\theta_0}{(1-\theta_0)(1+2\theta_0)^2} = \frac{1-2\theta_0}{3(1-2\theta_0)} = \frac{1}{3}.$$

Substituting this back into the original function, we arrive at the concise form:

$$f(\theta_0) = \frac{2\theta_0^2(1-2\theta_0)}{3}.$$

□

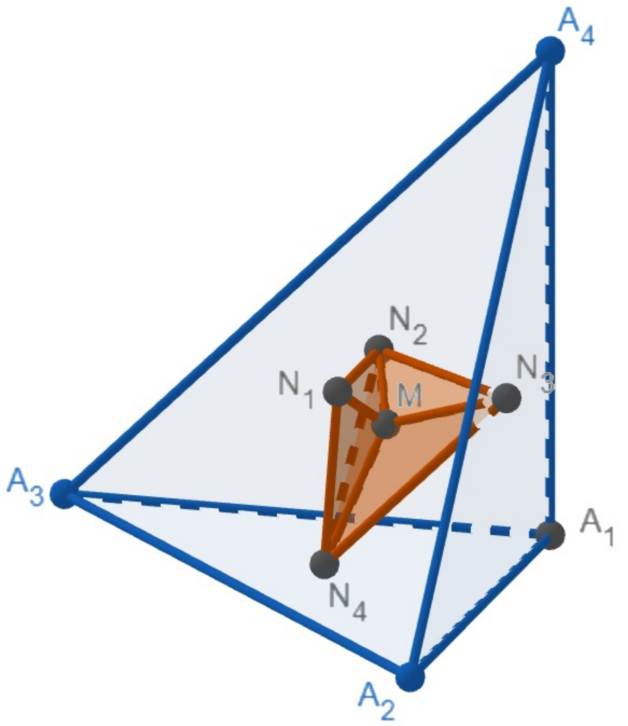

Figure 1: The tetrahedron and its optimal cevian parts

Remark 2. If $k = 0$, $k = 1$, $m = 0$, or $m = 1$, then the cubic $q(x)$ has a rational zero. It would be interesting to find $k > 1$ and $m > 1$ such that the cubic $q(x)$ has a rational zero or prove that such k and m do not exist.

Next, we examine the multiplicative variants of the considered problems.

Theorem 3. *In the notations of Theorem 1, for* $1 < k < n$

$$\frac{V_1 V_2 \cdots V_k}{V^k} \leq \left(\frac{f(\theta)}{k}\right)^k .$$

Equality occurs at the same unique point P.

Proof. Recall that by (1) the volume of each sub-simplex $MN_1 \dots N_i \dots N_{n+1}$ can be expressed in terms of barycentric coordinates as:

$$V_i = V \cdot (1 - \lambda_i) \cdot \prod_{j=1}^{n+1} \frac{\lambda_j}{1 - \lambda_j} .$$

Taking the product of the first k volumes, we obtain:

$$\frac{V_1 V_2 \cdots V_k}{V^k} = \prod_{j=1}^{k} \frac{\lambda_j}{1 - \lambda_j} \cdot \left(\prod_{j=k+1}^{n+1} \frac{\lambda_j}{1 - \lambda_j} \right)^k .$$

Rearranging the terms, we can separate the expression into a part involving the first k coordinates and a part involving the remaining $m = n + 1 - k$ coordinates:

$$\frac{V_1 V_2 \cdots V_k}{V^k} = \prod_{j=1}^{k} \frac{\lambda_j^k}{(1 - \lambda_j)^{k-1}} \cdot \left(\prod_{j=k+1}^{n+1} \frac{\lambda_j}{1 - \lambda_j} \right)^k .$$

To maximize the first part, let us consider the product $\prod_{j=1}^{k} \phi(\lambda_j)$ where $\phi(\lambda) = \frac{\lambda^k}{(1-\lambda)^{k-1}}$. For any pair λ_i, λ_j with a constant sum, the product is maximized when $\lambda_i = \lambda_j$. Indeed, analyzing the expression

$$\frac{\lambda_i \lambda_j}{\left(\frac{1-\lambda_i-\lambda_j}{\lambda_i \lambda_j} + 1\right)^{k-1}}$$

shows that maximizing the product $\lambda_i \lambda_j$ leads to the maximum of the overall term. Thus, the global maximum occurs at the symmetric point $\lambda_1 = \lambda_2 = \cdots = \lambda_k = x$. From the symmetry and the results established in Theorem 1, the maximum of the second part, $\prod_{j=k+1}^{n+1} \frac{\lambda_j}{1-\lambda_j}$, occurs when $\lambda_{k+1} = \cdots = \lambda_{n+1} = \frac{1-kx}{m}$. Substituting these optimal coordinates into the product formula:

$$\frac{V_1 V_2 \cdots V_k}{V^k} = \left(\frac{x^k}{(1-x)^{k-1}} \cdot \left(\frac{1 - kx}{m - 1 + kx} \right)^m \right)^k = \left(\frac{f(x)}{k}\right)^k .$$

Taking $x = \theta$ as the root that maximizes the original function $f(x)$, we conclude the proof of the inequality. Equality holds when P is the same point defined in Theorem 1. ☐

Remark 3. If $n = 3$, $k = 2$ and $m = 2$, then $\frac{V_1 V_2}{V^2}$ attains its maximum at the same point with barycentric coordinates $(\theta_0, \theta_0, \frac{1-\theta_0}{2}, \frac{1-\theta_0}{2})$, where $\frac{V_1+V_2}{V}$ also reaches its maximum. The cases $k = n$ and $k = n + 1$ for the product $V_1 V_2 \cdots V_k$ can be considered similarly using the results from [1].

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