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Application of Heron's Formula

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Abstract. We apply Heron's formula to obtain a generalized Ptolemy's theorem, to prove Euler's formula for the volume of a tetrahedron, and to prove Klain's theorem on reversible tetrahedra. After that, we prove triangle inequalities derived from a tetrahedron, find an inequality on the circumradius of a tetrahedron, find a volume expression of an isosceles tetrahedron, and give a condition for four points in the space to be co-planar.

Key Words: Heron's formula, Euler's formula, reversible tetrahedron, isosceles tetrahedron, Bang's theorem, Klain's theorem, circumradius of a tetrahedron, parallelogram law

MSC 2020: 51M04

1 Introduction

Heron of Alexandria, lived during the 1st century AD, is known for the following formula to evaluate the area of a triangle.

Heron's formula. The area $\mathcal{A}$ of a triangle with edge lengths of a, b, c is given by

$$\begin{aligned}16\mathcal{A}^2 &= (a+b+c)(-a+b+c)(a-b+c)(a+b-c)\\ &= 2a^2b^2+2b^2c^2+2c^2a^2-a^4-b^4-c^4\\ &= 4a^2b^2-[c^2-(a^2+b^2)]^2.\end{aligned}$$

In [12], Pratt gives a nice proof showing the equivalence of Heron's formula and Pythagorean theorem. We will apply Heron's formula to various situations in three-dimensional geometry.

Let A, B, C, D be four distinct points in $\mathbb{R}^3$. The line segment AB with its endpoints is denoted by $[AB]$, and its length is denoted by $|AB|$. By joining these points by edges $[AB]$, $[BC]$, $[CD]$, $[DC]$, and if $[AB]\cap[CD]=\varnothing=[AD]\cap[BC]$, then the resulting figure is called a *skew quadrilateral*, and it is denoted by $SQ(ABCD)$. If $SQ(ABCD)$ is *planar*, that is, if $[AC]\cap[BD]\neq\varnothing$, then it is called a *quadrilateral*, and is denoted by $Q(ABCD)$. A quadrilateral is *cyclic* if it can be inscribed in a circle. Here, we do *not* consider a line to be a circle of infinite radius since they do not form a quadrilateral by our definition if A, B, C, D are on a line. Ptolemy of Alexandrea, who lived in the 2^{nd} century AD, wrote the next theorem in his classical book known by the name *The Almagest.*

Ptolemy's Theorem. *Suppose $Q(ABCD)$ is a cyclic quadrilateral. Then*

$$|AC| \cdot |BD| = |AB| \cdot |CD| + |AD| \cdot |BC|.$$

Our first application of Heron's formula is Theorem 1 in Section 2 that makes Ptolemy's theorem more detailed.

If $SQ(ABCD)$ is a non-planar skew quadrilateral, we obtain a solid by adding two edges $[AC]$ and $[BD]$, and this solid is called a *tetrahedron,* and is denoted by $\nabla ABCD$. By a tetrahedron, we assume it to have a positive volume. A tetrahedron with zero volume is said to be *degenerate* and it is planar. A triangle ABC is denoted by $\triangle ABC$. We write $\triangle ABC \equiv \triangle A'B'C'$ to mean $\triangle ABC$ and $\triangle A'B'C'$ are congruent such that $|AB| = |A'B'|$, $|BC| = |B'C'|$ and $|CA| = |C'A'|$.

A formula for the volume of a tetrahedron, analogous to Heron's formula, does not seem well known. Dörrie [3] calls it Euler's formula, but he does not provide any reference to Euler's published work. He derives Euler's formula using matrices and expresses it in a 5 × 5 matrix determinant. Tran [13] presents a volume formula for a special type of tetrahedron by "a purely geometric proof". In Section 3, we will derive Euler's formula from Heron's formula that may qualify as "a purely geometric proof". The following Euler' formula is the same format written in [10] by Lukarevski without determinants.

Euler's formula. Let $a = |BC|$, $b = |CA|$, $c = |AB|$, $\alpha = |DA|$, $\beta = |DB|$, $\gamma = |DC|$ for a tetrahedron $\nabla ABCD$. The volume $\mathcal{V}$ of $\nabla ABCD$ is given by

$$\begin{aligned}144\mathcal{V}^2 = a^2\alpha^2(-a^2+b^2+c^2-\alpha^2+\beta^2+\gamma^2) + b^2\beta^2(a^2-b^2+c^2+\alpha^2-\beta^2+\gamma^2)\\ + c^2\gamma^2(a^2+b^2-c^2+\alpha^2+\beta^2-\gamma^2) - a^2\beta^2\gamma^2 - b^2\alpha^2\gamma^2 - c^2\alpha^2\beta^2 - a^2b^2c^2.\end{aligned}$$

A skew quadrilateral $SQ(ABCD)$ is said to be *reversible* if $|AB| = |CD|$ and $|AD| = |BC|$. A tetrahedron is *reversible* if it can be labelled $\nabla ABCD$ so that $\triangle ABC \equiv \triangle DCB$ and $\triangle BAD \equiv \triangle CDA$. (Equivalently, a tetrahedron is *reversible* if it can be labelled $\nabla ABCD$ so that $|AB| = |CD|$ and $|AC| = |DB|$.) Note that a tetrahedron $\nabla ABCD$ has three skew quadrilaterals $SQ(ABCD)$, $SQ(ABDC)$, and $SQ(ACBD)$. Hence, $\nabla ABCD$ is reversible if and only if one of $SQ(ABCD)$, $SQ(ABDC)$, or $SQ(ACBD)$ is reversible. A tetrahedron $\nabla ABCD$ is said to be *isosceles* if $|AB| = |CD|$, $|AD| = |BC|$, and $|AC| = |BD|$. Equivalently, $\nabla ABCD$ is isosceles if two of $SQ(ABCD)$, $SQ(ABDC)$, or $SQ(ACBD)$ are reversible. Also, $\nabla ABCD$ is isosceles if all triangular faces are of the same area (Bang's theorem).

Klain's theorem (See [8]). *Let $\mathcal{A}_A$, $\mathcal{A}_B$, $\mathcal{A}_C$, $\mathcal{A}_D$ be the areas of triangular faces $\triangle BCD$, $\triangle ACD$, $\triangle ABD$ $\triangle ABC$, respectively, of a tetrahedron $\nabla ABCD$. Then $\nabla ABCD$ is reversible such that $\triangle ABC \equiv \triangle DCB$ and $\triangle BAD \equiv \triangle CDA$ if and only if $\mathcal{A}_A = \mathcal{A}_D$ and $\mathcal{A}_B = \mathcal{A}_C$.*

Klain proves his theorem using rhombuses generated by normal vectors to the faces of a tetrahedron with its areas as their magnitudes. In Section 4, we will prove Klain's theorem using Heron's and Euler's formulas. Our proofs of both Euler's formula and Klain's theorem are lengthy. However, since Euler's formula is a three-dimensional analogue of Heron's formula, and since Klain's theorem is about the triangular face areas of a tetrahedron, we thought it is desirable to have proofs of these based on Heron's Formula.

In Section 5, we will prove inequalities in Theorem 2 motivated by Crelle's theorem and Euler's formula using the parallelogram law. Then we will obtain an inequality on the circumradius of a tetrahedron in Theorem 3 using Mazur's theorem, Crelle's theorem and Heron's formula in Section 6. We will apply this theorem and Heron's formula to obtain Theorem 4 for an expression of the volume for an isosceles tetrahedron that we think is new. In Section 7, as an application of Euler's formula and the parallelogram law, we will give a condition for four points in $\mathbb{R}^3$ to be co-planar in Theorem 5.

2 A Detailed Ptolemy's Theorem from Heron's Formula

Suppose $a, b, c, d > 0$ are numbers such that there is a quadrilateral $Q(ABCD)$ with $|AB| = a$, $|BC| = b$, $|CD| = c$, $|DA| = d$. In this case, $Q(ABCD)$ is said to be a (a, b, c, d)*-quadrilateral*. We will prove the following theorem.

Theorem 1. *Suppose $SQ(ABCD)$ is a skew quadrilateral. Let $a = |AB|$, $b = |BC|$, $c = |CD|$, $d = |DA|$.*

(i) There is a new skew quadrilateral $SQ(ABCD)$ such that $a = |AB|$, $b = |BC|$, $c = |CD|$, $d = |DA|$,

$$|AC| = \sqrt{\frac{(ac+bd)(ad+bc)}{ab+cd}} \quad \text{and} \quad |BD| = \sqrt{\frac{(ac+bd)(ab+cd)}{ad+bc}}.$$

(ii) $SQ(ABCD)$ is a cyclic (planar) quadrilateral *if, and only if*

$$|AC| = \sqrt{\frac{(ac+bd)(ad+bc)}{ab+cd}} \quad \text{and} \quad |BD| = \sqrt{\frac{(ac+bd)(ab+cd)}{ad+bc}}.$$

(iii) A cyclic *(a, b, c, d)-quadrilateral has the largest area among all (a, b, c, d)-quadrilaterals.*

(iv) A cyclic *(a, b, c, d)-quadrilateral has the circumradius*

$$\hat{r} = \sqrt{\frac{(ac+bd)(ab+cd)(ad+bc)}{4(ab+cd)^2 - \{a^2+b^2-c^2-d^2\}^2}}.$$

Remark 1. In Theorem 1, since A, B, C, D are distinct points, we have $ab + cd \neq 0$ and $ad + bc \neq 0$. Also, note that $|AC| = \sqrt{\frac{(ac+bd)(ad+bc)}{ab+cd}}$ and $|BD| = \sqrt{\frac{(ac+bd)(ab+cd)}{ad+bc}}$ implies that $|AC| \cdot |BD| = |AB| \cdot |CD| + |AD| \cdot |BC|$. Thus, Theorem 1 is a generalization of Ptolemy's theorem. We will need three lemmas to prove this.

Lemma 1 (See [2, Page 3] and Remark 3 in Section 3)**.** *Let $\hat{r}$ be the circumradius of $\triangle ABC$, and let $|AB| = c$, $|BC| = a$, $|CA| = b$. If $\mathcal{A}$ is the area of $\triangle ABC$, then $4\mathcal{A}\hat{r} = abc$.*

Lemma 2. *Let $Q(ABCD)$ be a quadrilateral. Let $a = |AB|$, $b = |BC|$, $c = |CD|$, $d = |DA|$. And, let $x = |AD|$ and $y = |BD|$. Let*

$$g(x) = \sqrt{4a^2b^2 - [x^2 - (a^2+b^2)]^2}, \text{ and } h(x) = \sqrt{4c^2d^2 - [x^2 - (c^2+d^2)]^2}, \tag{1}$$

$$G(y) = \sqrt{4a^2d^2 - [y^2 - (a^2+d^2)]^2}, \text{ and } H(y) = \sqrt{4c^2b^2 - [y^2 - (c^2+b^2)]^2}. \tag{2}$$

Then the areas of $\triangle ABC$, $\triangle ACD$, $\triangle ABD$, $\triangle BCD$ are given by $\frac{1}{4}g(x)$, $\frac{1}{4}h(x)$, $\frac{1}{4}G(y)$, and $\frac{1}{4}H(y)$, respectively.

Proof. These equations are restatements of Heron's formula. □

Lemma 3 (See Theorem 4 of [6])**.** *If a skew quadrilateral $SQ(ABCD)$ has a property $|AC| \cdot |BD| = |AB| \cdot |CD| + |AD| \cdot |BC|$, then it is a (planar) cyclic quadrilateral.*

Now, we are ready to prove Theorem 1.

Proof of Theorem 1 (i). Let $x = |AC|$ of the given $SQ(ABCD)$ that we know its existence. Then $\min\{a+b, c+d\} \geq x$. We will show that $\sqrt{\frac{(ac+bd)(ad+bc)}{ab+cd}} \leq a+b$, or equivalently, $(a+b)^2(ab+cd) - (ac+bd)(ad+bc) \geq 0$.

$$\begin{aligned}
&(a+b)^2(ab+cd) - (ac+bd)(ad+bc) \\
&\qquad = (a^2+b^2+2ab)(ab+cd) - (ac+bd)(ad+bc) \\
&\qquad = (a^3b + a^2cd + ab^3 + b^2cd + 2a^2b^2 + 2abcd) - (a^2cd + abc^2 + abd^2 + b^2cd) \\
&\qquad = ab(a^2 + b^2 + 2ab + 2cd - c^2 - d^2) \\
&\qquad = ab\{(a+b)^2 - (c-d)^2\} \\
&\qquad \geq ab\{x^2 - (c-d)^2\} = ab(x-c+d)(x+c-d).
\end{aligned}$$

Since $SQ(ABCD)$ is a skew quadrilateral, vertices A, C, D from a triangle $\triangle ACD$ so that $x - c + d \geq 0$ and $x + c - d \geq 0$. Hence, $(a+b)^2(ab+cd) - (ac+bd)(ad+bc) \geq 0$.

Next, we will show that $a \leq \sqrt{\frac{(ac+bd)(ad+bc)}{ab+cd}} + b$. This is equivalent to showing that $(ac+bd)(ad-bc) - (a-b)^2(ab+cd) \geq 0$. And this is like the above inequality.

$$\begin{aligned}
&(ac+bd)(ad-bc) - (a-b)^2(ab+cd) \\
&\quad = (a^2cd + abc^2 + abd^2 + b^2cd) - (a^3b + a^2cd + ab^3 + b^2cd - 2a^2b^2 - 2abcd) \\
&\quad = ab(c^2 + d^2 - a^2 - b^2 + 2ab + 2cd) \\
&\quad = ab\{(c+d)^2 - (a-b)^2\} \\
&\quad \geq ab\{x^2 - (a-b)^2\} = ab(x-a+b)(x+a-b).
\end{aligned}$$

Again, since $SQ(ABCD)$ is a skew quadrilateral, vertices A, B, C form a triangle $\triangle ABC$ so that $x - a + b \geq 0$ and $x + a - b \geq 0$. Hence, $(ac+bd)(ad+bc) - (a-b)^2(ab+cd) \geq 0$. Thus, $a \leq \sqrt{\frac{(ac+bd)(ad+bc)}{ab+cd}} + b$. Similarly, we can show that $b \leq \sqrt{\frac{(ac+bd)(ad+bc)}{ab+cd}} + a$. Therefore, $\left\{a, b, \sqrt{\frac{(ac+bd)(ad+bc)}{ab+cd}}\right\}$ form edges of a triangle. Similarly, for $\left\{c, d, \sqrt{\frac{(ac+bd)(ad+bc)}{ab+cd}}\right\}$, $\left\{a, d, \sqrt{\frac{(ac+bd)(ab+cd)}{ad+bc}}\right\}$, $\left\{b, c, \sqrt{\frac{(ac+bd)(ab+cd)}{ad+bc}}\right\}$. Therefore, we can form new triangles $\triangle ABC$, $\triangle ACD$, $\triangle ABD$, $\triangle BCD$ such that $|AC| = \sqrt{\frac{(ac+bd)(ad+bc)}{ab+cd}}$ and $|BD| = \sqrt{\frac{(ac+bd)(ab+cd)}{ad+bc}}$. That is, this guarantees the existence of a new skew quadrilateral $SQ(ABCD)$ such that

$$|AC| = \sqrt{\frac{(ac+bd)(ad+bc)}{ab+cd}} \qquad \text{and} \qquad |BD| = \sqrt{\frac{(ac+bd)(ab+cd)}{ad+bc}}. \qquad \square$$

Proof of Theorem 1 (ii): Suppose $SQ(ABCD)$ is a skew quadrilateral such that

$$|AC| = \sqrt{\frac{(ac+bd)(ad+bc)}{ab+cd}} \quad \text{and} \quad |BD| = \sqrt{\frac{(ac+bd)(ab+cd)}{ad+bc}}. \qquad \text{Then}$$

$$\begin{aligned}
|AC| \cdot |BD| = \sqrt{\frac{(ac+bd)(ad+bc)}{ab+cd}} \cdot \sqrt{\frac{(ac+bd)(ab+cd)}{ad+bc}} &= ac + bd \\
&= |AB| \cdot |CD| + |AD| \cdot |BC|.
\end{aligned}$$

By Lemma 3, $SQ(ABCD)$ is a cyclic quadrilateral.

Conversely, suppose $SQ(ABCD)$ is a cyclic quadrilateral. So, $SQ(ABCD) = Q(ABCD)$. Let $\hat{r}$ be the circumradius of $Q(ABCD)$ and $x = |AC|$. Then $\hat{r}$ is the circumradius of $\triangle ABC$ and $\triangle ACD$. So, by Lemma 1, we have $\hat{r} = \frac{abx}{g(x)} = \frac{cdx}{h(x)}$, or $ab \cdot h(x) = cd \cdot g(x)$, where $h(x)$ and $g(x)$ are functions in Lemma 2. And $ab \cdot h(x) = cd \cdot g(x)$ becomes

$$(ab)^2\{4c^2d^2 - [x^2 - (c^2 + d^2)]^2\} = (cd)^2\{4a^2b^2 - [x^2 - (a^2 + b^2)]^2\}.$$

This simplifies to

$$(ab)^2[x^2 - (c^2 + d^2)]^2 = (cd)^2[x^2 - (a^2 + b^2)]^2. \tag{3}$$

Since both $\triangle ABC$ and $\triangle ACD$ are inscribed in the same circle, and since B and D are on opposite sides of the line $\overline{AC}$, we have $\sphericalangle ABC + \sphericalangle ADC = \pi$. Let $\sphericalangle ABC = \theta$, where $0 < \theta < \pi$. Then $\sphericalangle ADC = \pi - \theta$. Since $\cos(\pi - \theta) = -\cos\theta$, the law of cosines applied to $\triangle ABC$ and $\triangle ADC$ shows that

$$x^2 = a^2 + b^2 - 2ab \cdot \cos\theta \quad \text{and} \quad x^2 = c^2 + d^2 + 2cd \cdot \cos\theta. \tag{4}$$

Hence, $2ab \cdot \cos\theta = a^2 + b^2 - x^2$ and $2cd \cdot \cos\theta = x^2 - (c^2 + d^2)$. Thus

$$\begin{aligned} &(x^2 \geq a^2 + b^2 \text{ and } x^2 \leq c^2 + d^2) \text{ if } \cos\theta \leq 0, \text{ or} \\ &(x^2 \leq a^2 + b^2 \text{ and } x^2 \geq c^2 + d^2) \text{ if } \cos\theta \geq 0. \end{aligned} \tag{5}$$

Because of (5), equation (3) simplifies to

$$-cd[x^2 - (a^2 + b^2)] = ab[x^2 - (c^2 + d^2)]. \tag{6}$$

Solving (6) for x^2, we have $x^2 = \frac{(ac+bd)(ad+bc)}{ab+cd}$, or $|AC| = \sqrt{\frac{(ac+bd)(ad+bc)}{ab+cd}}$.

Similarly, by letting $y = |BD|$, the functions $\frac{1}{4}G(y)$ and $\frac{1}{4}H(y)$ in Lemma 2 give the areas of $\triangle ABD$ and $\triangle BCD$, respectively. Then as in (3)–(6), we have $|BD| = \sqrt{\frac{(ac+bd)(ad+bc)}{ab+cd}}$. $\square$

Proof of Theorem 1 (iii): Let $x = |AC|$ and $f(x) = g(x) + h(x)$. Then $\frac{1}{4}f(x)$ is the area of an (a, b, c, d)-quadrilateral $Q(ABCD)$ by Lemma 2. By taking the derivative of $f(x)$, we have $f'(x) = \frac{-2x}{g(x)h(x)} \cdot \{h(x) \cdot [x^2 - (a^2 + b^2)] + g(x) \cdot [x^2 - (c^2 + d^2)]\}$.

Hence, the positive critical point of $f(x)$ is given by the solution to the equation

$$h(x) \cdot [(a^2 + b^2) - x^2] = g(x) \cdot [x^2 - (c^2 + d^2)]. \tag{7}$$

By squaring both sides, (7) simplifies to

$$c^2d^2 \cdot [(a^2 + b^2) - x^2]^2 = a^2b^2 \cdot [x^2 - (c^2 + d^2)]^2. \tag{8}$$

Since $g(x) \geq 0$ and $h(x) \geq 0$, the equation (7) implies

$$(x^2 \geq a^2 + b^2 \text{ and } x^2 \leq c^2 + d^2), \quad \text{or} \quad (x^2 \leq a^2 + b^2 \text{ and } x^2 \geq c^2 + d^2). \tag{9}$$

By (9), equation (8) simplifies to

$$-cd[x^2 - (a^2 + b^2)] = ab[x^2 - (c^2 + d^2)]. \qquad \text{(This is the above equation (6).)} \tag{10}$$

Hence, the unique critical point of f is $x = \sqrt{\frac{(ac+bd)(ad+bc)}{ab+cd}}$, and the area of $Q(ABCD)$ is maximized when $|AC| = \sqrt{\frac{(ac+bd)(ad+bc)}{ab+cd}}$.

Similarly, let $y = |BD|$ and $F(y) = G(y) + H(y)$. Then $\frac{1}{4}F(y)$ is also the area of an (a,b,c,d)-quadrilateral $Q(ABCD)$ by Lemma 2. As in (7)–(10), we can see that the unique critical point of F is $y = \sqrt{\frac{(ac+bd)(ad+bc)}{ab+cd}}$. Hence, the area of $Q(ABCD)$ is maximized when $|BD| = \sqrt{\frac{(ac+bd)(ad+bc)}{ab+cd}}$. And the maximum values of f and F must be the same. By part (ii), we know the maximum area of an (a,b,c,d)-quadrilateral is obtained when it is cyclic. ☐

Proof. Proof of Theorem 1 (iv): A *cyclic* (a,b,c,d)-quadrilateral $Q(ABCD)$ has the same circumcircle as $\triangle ABC$. Since $|AB| = a$, $|BC| = b$, and $|AC| = \sqrt{\frac{(ac+bd)(ad+bc)}{ab+cd}}$. Hence, Lemma 1 and Heron's formula give

$$16\mathcal{A}^2\hat{r}^2 = (a+b+|AC|)(-a+b+|AC|)(a-b+|AC|)(a+b-|AC|)\hat{r}^2 = a^2b^2|AC|^2.$$

$$\begin{aligned}\hat{r}^2 &= \frac{a^2b^2|AC|^2\cdot(ab+cd)^2}{[(a+b)^2-|AC|^2][|AC|^2-(a-b)^2]\cdot(ab+cd)^2}\\ &= \frac{a^2b^2(ab+cd)(ac+bd)(ad+bc)}{[(a+b)^2(ab+cd)-(ac+bd)(ad+bc)][(ac+bd)(ad+bc)-(a-b)^2(ab+cd)]}\\ &= \frac{(ab+cd)(ac+bd)(ad+bc)}{[(a+b)^2-(c-d)^2][(c+d)^2-(a-b)^2]}\\ &= \frac{(ab+cd)(ac+bd)(ad+bc)}{[(a^2+b^2-c^2-d^2)+2(ab+cd)][2(ab+cd)-(a^2+b^2-c^2-d^2)]}\\ &= \frac{(ac+bd)(ab+cd)(ad+bc)}{4(ab+cd)^2-\{a^2+b^2-c^2-d^2\}}.\end{aligned}$$

$$\text{Therefore, } \hat{r} = \sqrt{\frac{(ac+bd)(ab+cd)(ad+bc)}{4(ab+cd)^2-\{a^2+b^2-c^2-d^2\}}}.$$ ☐

Remark 2. As we mentioned above, Pythagorean theorem and Heron's formula are equivalent. We just showed that Heron's formula implies Ptolemy's theorem. It is not difficult to see that Ptolemy's theorem implies Pythagorean theorem. Therefore, Pythagorean theorem, Heron's formula, and Ptolemy's theorem are equivalent.

Bretschneider's formula states that the area K of a convex quadrilateral, whose consecutive sides are given by a, b, c, d, is given by $K = \sqrt{(s-a)(s-b)(s-c)(s-d) - abcd\cos^2\theta}$, where $s = \frac{1}{2}(a+b+c+d)$ and θ is the half of the sum of any two opposite angles (see [9]). From this, we can observe that K is maximized when $\theta = \frac{\pi}{2}$, that is, when the quadrilateral is cyclic. However, the existence of a convex quadrilateral with consecutive sides a, b, c, d, does not guarantee the existence of a cyclic quadrilateral with sides a, b, c, d without Theorem 1 (i) and (ii).

3 Euler's Formula from Heron's Formula

An *angle* of a tetrahedron is an angle of its triangular faces. Let $\nabla ABCD$ be a tetrahedron. Then the inner angle between the faces $\triangle ADB$ and $\triangle CDB$ is said to be the *dihedral angle* of $\nabla ABCD$ at the edge $[DB]$, and it is denoted by $\sphericalangle DB$. (See Figure 1.)

We need two lemmas to prove Euler's formula.

Lemma 4 (See [7]). *A tetrahedron must have a vertex at which all three angles are acute.*

We prove the next known theorem since its proof is difficult to find.

Lemma 5 (Spherical law of cosines)**.** *For a tetrahedron $\nabla ABCD$, we have*

$$\cos\sphericalangle DB = \frac{\cos\sphericalangle ADC - \cos\sphericalangle ADB\cdot\cos\sphericalangle BDC}{\sin\sphericalangle ADB\cdot\sin\sphericalangle BDC}.$$

Proof. Let $\sphericalangle BDC = \alpha$, $\sphericalangle ADC = \beta$, $\sphericalangle ADB = \gamma$. Without loss of generality, we assume that $|AD| = |BD| = |CD| = 1$. Let $A' \in \overline{DA}$ such that $\sphericalangle A'BD = \frac{\pi}{2}$. Let $C' \in \overline{DC}$ such that $\sphericalangle C'BD = \frac{\pi}{2}$. Then $\sphericalangle A'BC' = \sphericalangle DB$. Since $1 = |DB| = |DA'|\cos\gamma$ and $1 = |DB| = |DC'|\cos\alpha$, we have $|DA'| = \sec\gamma$ and $|DC'| = \sec\alpha$. Also $\tan\gamma = |A'B|$ and $\tan\alpha = |C'B|$. Applying the law of cosines to $\triangle A'DC'$ and $\triangle A'BC'$, we have

$$\begin{aligned}|A'C'|^2 &= \sec^2\gamma + \sec^2\alpha - 2\sec\gamma\sec\alpha\cos\beta, \quad \text{and}\\ |A'C'|^2 &= \tan^2\gamma + \tan^2\alpha - 2\tan\gamma\tan\alpha\cos\sphericalangle A'BC'\\ &= \tan^2\gamma + \tan^2\alpha - 2\tan\gamma\tan\alpha\cos\sphericalangle DB.\end{aligned}$$

Hence, $\sec^2\gamma + \sec^2\alpha - 2\sec\gamma\sec\alpha\cos\beta = \tan^2\gamma + \tan^2\alpha - 2\tan\gamma\tan\alpha\cos\sphericalangle DB$. Solving this equation for $\cos\sphericalangle DB$, we have

$$\begin{aligned}\cos\sphericalangle DB &= \frac{\tan^2\gamma + \tan^2\alpha - \sec^2\gamma - \sec^2\alpha + 2\sec\gamma\sec\alpha\cos\beta}{2\tan\gamma\tan\alpha}\\ = \frac{-2 + 2\sec\gamma\sec\alpha\cos\beta}{2\tan\gamma\tan\alpha} &= \frac{\cos\beta - \cos\alpha\cos\gamma}{\sin\alpha\sin\gamma} = \frac{\cos\sphericalangle ADC - \cos\sphericalangle ADB\cdot\cos\sphericalangle BDC}{\sin\sphericalangle ADC\cdot\sin\sphericalangle BDC}. \quad \square\end{aligned}$$

Proof of Euler's Formula. By Lemma 4, we assume that $\sphericalangle ADB$, $\sphericalangle ADC$, and $\sphericalangle BDC$ are acute angles at the vertex D of $\nabla ABCD$. Recall that $\mathcal{A}_A$ is the area of $\triangle BCD$. Let h be the height of $\nabla ABCD$ from the vertex A. Then $\mathcal{V} = \frac{1}{3}h\mathcal{A}_A$, or $\mathcal{V}^2 = \frac{h^2\mathcal{A}_A^2}{9}$. The height $\hat{h}$ of $\triangle ABD$ from A is given by $\hat{h} = \frac{2\mathcal{A}_C}{\beta}$. Thus, $h = \hat{h}\sin\sphericalangle DB = \frac{2\mathcal{A}_C}{\beta}\sin\sphericalangle DB$ so that

$$\mathcal{V}^2 = \frac{h^2\mathcal{A}_C^2}{9} = \frac{4\mathcal{A}_A^2\mathcal{A}_C^2}{9\beta^2}\cdot\sin^2\sphericalangle DB. \quad \text{(See Figure 1.)} \tag{a}$$

We will calculate $\cos\sphericalangle DB$ by Lemma 5. $\mathcal{A}_B = \frac{1}{2}\alpha\gamma\sin\sphericalangle ADC$ so that $\sin\sphericalangle ADC = \frac{2\mathcal{A}_B}{\alpha\gamma}$. Similarly, $\sin\sphericalangle ADB = \frac{2\mathcal{A}_C}{\alpha\beta}$ and $\sin\sphericalangle BDC = \frac{2\mathcal{A}_A}{\beta\gamma}$.

Hence, we have

$$\cos^2\sphericalangle ADC = 1 - \sin^2\sphericalangle ADC = \frac{(\alpha\gamma)^2 - 4\mathcal{A}_B^2}{(\alpha\gamma)^2}. \tag{b}$$

By Heron's formula, $16\mathcal{A}_B^2 = 4\alpha^2\gamma^2 - (\alpha^2 + \gamma^2 - b^2)^2$. This can be re-written as

$$(\alpha\gamma)^2 - 4\mathcal{A}_B^2 = \frac{1}{4}(\alpha^2 + \gamma^2 - b^2)^2. \tag{c}$$

From (b) and (c), $\cos^2\sphericalangle ADC = \frac{(\alpha^2+\gamma^2-b^2)^2}{4(\alpha\gamma)^2}$. Since $\sphericalangle ADC$ is an acute angle, we have $\alpha^2 + \gamma^2 - b^2 > 0$ so that $\cos\sphericalangle ADC = \frac{\alpha^2+\gamma^2-b^2}{2\alpha\gamma}$.

Similarly, $\cos\sphericalangle ADB = \frac{\alpha^2+\beta^2-c^2}{2\alpha\beta}$ and $\cos\sphericalangle BDC = \frac{\beta^2+\gamma^2-a^2}{2\alpha\gamma}$ because $\sphericalangle ADB$ and $\sphericalangle BDC$ are acute angles. Since $\sin\sphericalangle ADB = \frac{2\mathcal{A}_C}{\alpha\beta}$ and $\sin\sphericalangle BDC = \frac{2\alpha A_A}{\beta\gamma}$, we have

$$\begin{aligned}\cos\sphericalangle DB &= \frac{\cos\sphericalangle ADC - \cos\sphericalangle ADB\cdot\cos\sphericalangle BDC}{\sin\sphericalangle ADC\cdot\sin\sphericalangle BDC} = \frac{\frac{\alpha^2+\gamma^2-b^2}{2\alpha\gamma} - \frac{\alpha^2+\beta 2-c^2}{2\alpha\beta}\cdot\frac{\beta^2+\gamma^2-a^2}{2\beta\gamma}}{\frac{2\mathcal{A}_C}{\alpha\beta}\cdot\frac{2\mathcal{A}_A}{\beta\gamma}}\\ &= \frac{2\beta^2(\alpha^2+\gamma^2-b^2) - (\alpha^2+\beta 2-c^2)(\beta^2+\gamma^2-a^2)}{16\mathcal{A}_A\mathcal{A}_C} \qquad \text{by Lemma 5.}\end{aligned} \tag{d}$$

By (d), we have

$$\begin{aligned}
&16^2\mathcal{A}_A^2\mathcal{A}_C^2 \cdot \sin \sphericalangle DB = 16^2\mathcal{A}_A^2\mathcal{A}_C^2(1-\cos^2 \sphericalangle DB)\\
&= 16^2\mathcal{A}_A^2\mathcal{A}_C^2 - 16^2\mathcal{A}_A^2\mathcal{A}_C^2 \cos^2 \sphericalangle DB\\
&= 16^2\mathcal{A}_A^2\mathcal{A}_C^2 - \{2\beta^2(\alpha^2+\gamma^2-b^2)-(\alpha^2+\beta^2-c^2)(\beta^2+\gamma^2-a^2)\}^2 \qquad \text{(e)}\\
&= 16^2\mathcal{A}_A^2\mathcal{A}_C^2 - 4\beta^4(\alpha^2+\gamma^2-b^2)^2 + 4\beta^2(\alpha^2+\gamma^2-b^2)(\alpha^2+\beta^2-c^2)(\beta^2+\gamma^2-a^2)\\
&\quad -(\alpha^2+\beta^2-c^2)^2(\beta^2+\gamma^2-a^2)^2.
\end{aligned}$$

From Heron's formula, we have

$$\begin{aligned}
16\mathcal{A}_A^2 &= 4\beta^2\gamma^2-(\beta^2+\gamma^2-a^2)^2 \qquad \text{and} && \text{(f)}\\
16\mathcal{A}_C^2 &= 4\alpha^2\beta^2-(\alpha^2+\beta^2-c^2)^2. && \text{(g)}
\end{aligned}$$

Substituting (f) and (g) into $16^2\mathcal{A}_A^2\mathcal{A}_C^2$ in the right-hand side of equation (e), we have

$$\begin{aligned}
&16^2\mathcal{A}_A^2\mathcal{A}_C^2 \cdot \sin^2 \sphericalangle DB\\
&= \{4\beta^2\gamma^2-(\beta^2+\gamma^2-a^2)^2\}\{4\alpha^2\beta^2-(\alpha^2+\beta^2-c^2)^2\} - 4\beta^4(\alpha^2+\gamma^2-b^2)^2\\
&\quad + 4\beta^2(\alpha^2+\gamma^2-b^2)(\alpha^2+\beta^2-c^2)(\beta^2+\gamma^2-a^2) - (\alpha^2+\beta^2-c^2)^2(\beta^2+\gamma^2-a^2)^2\\
&= 16\alpha^2\beta^4\gamma^2 - 4\beta^2\gamma^2(\alpha^2+\beta^2-c^2)^2 - 4\alpha^2\beta^2(\beta^2+\gamma^2-a^2)^2\\
&\quad - \beta^4(\alpha^2+\gamma^2-b^2)^2 + 4\beta^2(\alpha^2+\gamma^2-b^2)(\alpha^2+\beta^2-c^2)(\beta^2+\gamma^2-a^2).
\end{aligned}$$

Therefore, from (a), we have

$$\begin{aligned}
144\mathcal{V}^2 &= \frac{1}{4\beta^2}16^2\mathcal{A}_A^2\mathcal{A}_C^2 \cdot \sin^2 \sphericalangle DC\\
&= 4\alpha^2\beta^2\gamma^2 - \gamma^2(\alpha^2+\beta^2-c^2)^2 - \alpha^2(\beta^2+\gamma^2-a^2)^2 - \beta^2(\alpha^2\gamma^2-b^2)^2\\
&\qquad + (\alpha^2+\gamma^2-b^2)(\alpha^2+\beta^2-c^2)(\beta^2+\gamma^2-a^2)\\
&= -c^4\gamma^2 + \alpha^2\gamma^2c^2 + \beta^2\gamma^2c^2 - \alpha^2a^4 + \alpha^2\beta^2a^2 + \alpha^2\gamma^2a^2 - \beta^2b^4 + \alpha^2\beta^2b^2 + \beta^2\gamma^2b^2\\
&\quad - \alpha^4a^2 - \alpha^2\beta^2c^2 + \alpha^2a^2c^2 - \beta^2\gamma^2a^2 - \gamma^4c^2 + \gamma^2a^2c^2 - \alpha^2\gamma^2b^2\\
&\qquad + \alpha^2a^2b^2 - \beta^4b^2 + \beta^2a^2b^2 + \beta^2b^2c^2 + \gamma^2b^2c^2 - a^2b^2c^2\\
&= a^2\alpha^2(-a^2+b^2+c^2-\alpha^2+\beta^2+\gamma^2) + b^2\beta^2(a^2-b^2+c^2+\alpha^2-\beta^2+\gamma^2)\\
&\quad + c^2\gamma^2(a^2+b^2-c^2+\alpha^2+\beta^2-\gamma^2) - a^2\beta^2\gamma^2 - b^2\alpha^2\gamma^2 - c^2\alpha^2\beta^2 - a^2b^2c^2.
\end{aligned}$$

This proves Euler's formula. □

Remark 3. Let $\mathcal{V}'$ be the volume of $\nabla A'B'C'D'$ such that $\alpha = |A'D'| = |B'D'| = |C'D'|$, $a = |B'C'|$, $b = |C'A'|$, $c = |A'B'|$. Let $\mathcal{A}$ be the area of $\nabla A'B'C'$. By Euler's and Heron's formulas, we have

$$144\mathcal{V}'^2 = \alpha^2[2a^2b^2+2b^2c^2+2c^2a^2-a^4-b^4-c^4] - a^2b^2c^2 = 16\alpha^2\mathcal{A}^2 - a^2b^2c^2.$$

Hence, $144\mathcal{V}'^2 = 16\alpha^2\mathcal{A}^2 - a^2b^2c^2$.

This gives us an alternate *proof* of Lemma 1. Let D be the circumcenter of $\triangle ABC$. Then $\nabla ABCD$ is a degenerate tetrahedron and its volume is 0. Then this formula gives us $16\hat{r}^2\mathcal{A}^2 - a^2b^2c^2 = 0$ or $4\mathcal{A}\hat{r} = abc$.

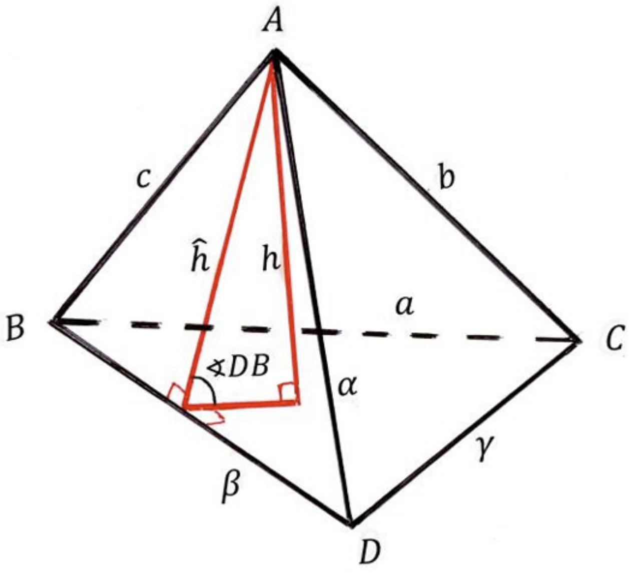

Figure 1: The dihedral angle $\sphericalangle DB$ and the height h of $\nabla ABCD$ from A and the height $\hat{h}$ of $\triangle ABD$ from A are shown.

4 Klain's Theorem from Heron and Euler's Formulas

Bang's Theorem is a theorem on an isosceles tetrahedron older than and implied by Klain's Theorem.

Bang's Theorem (See [1], [4] or [5, Chapter 9])**.** *A tetrahedron $\nabla ABCD$ is isosceles if and only if four faces of the tetrahedron have the same area.*

According to Honsberger [5], Bang conjectured the above theorem in 1897, and Gehrke proved it in the same year. In 2023, Klain improved Bang's theorem. Let $|AB| = c$, $|BC| = a$, $|CA| = b$, $|DA| = \alpha$, $|DB| = \beta$, $|DC| = \gamma$ for a tetrahedron $\nabla ABCD$.

Proof of Klain's Theorem. This proof is lengthy, but it is elementary. Suppose a tetrahedron $\nabla ABCD$ is reversible such that $\triangle BCD$ is congruent to $\triangle ABC$, and $\triangle ACD$ is congruent to $\triangle ABD$. Then $\mathcal{A}_A = \mathcal{A}_D$ and $\mathcal{A}_B = \mathcal{A}_C$.

Conversely, suppose $\mathcal{A}_A = \mathcal{A}_D$ and $\mathcal{A}_B = \mathcal{A}_C$. To prove that $\triangle BCD \equiv \triangle ABC$, and $\triangle ACD \equiv \triangle ABD$, it is sufficient to show that $|AB| = |CD|$ and $|AC| = |BD|$. This is equivalent to showing $c = \gamma$ and $b = \beta$. By Heron's formula, we have

$$\begin{aligned}
16\mathcal{A}_A^2 &= 2a^2\beta^2 + 2\beta^2\gamma^2 + 2\gamma^2a^2 - a^4 - \beta^4 - \gamma^4,\\
16\mathcal{A}_D^2 &= 2a^2b^2 + 2b^2c^2 + 2c^2a^2 - a^4 - b^4 - c^4,\\
16\mathcal{A}_B^2 &= 2\alpha^2b^2 + 2\beta^2b^2 + 2\gamma^2\alpha^2 - \alpha^4 - b^4 - \gamma^4,\\
16\mathcal{A}_C^2 &= 2\alpha^2\beta^2 + 2\beta^2c^2 + 2c^2\alpha^2 - \alpha^4 - \beta^4 - c^4.
\end{aligned}$$

Factoring the equation $16\mathcal{A}_A^2 = 16\mathcal{A}_D^2$ gives us

$$2a^2[(\beta^2 - b^2) + (\gamma^2 - c^2)] = [(\beta^2 - b^2) - (\gamma^2 - c^2)][(\beta^2 + b^2) - (\gamma^2 + c^2)]. \tag{11}$$

Similarly, the equation $16\mathcal{A}_B^2 = 16\mathcal{A}_C^2$ gives us

$$2\alpha^2[(\beta^2 - b^2) - (\gamma^2 - c^2)] = [(\beta^2 - b^2) + (\gamma^2 - c^2)][(\beta^2 + b^2) - (\gamma^2 + c^2)]. \tag{12}$$

Let $x = \beta^2 + b^2$, $y = \gamma^2 + c^2$, *and* $z = \beta^2 - b^2$, $w = \gamma^2 - c^2$.

Then (11) and (12) become

$$\begin{aligned} 2a^2(z+w) &= (z-w)(x-y), \quad \text{and} && (13)\\ 2\alpha^2(z-w) &= (z+w)(x-y). && (14) \end{aligned}$$

We will prove that $z + w = 0$ *or* $z - w = 0$.

On the contrary, suppose $z + w \neq 0$ and $z - w \neq 0$. Then $x - y \neq 0$ by (13). From (13) and (14), we have

$$a^2\alpha^2 = \frac{1}{4}(x-y)^2 \quad \text{and} \quad a^2 + \alpha^2 = \frac{(x-y)\{(z+w)^2+(z-w)^2\}}{2(z+w)(z-w)}. \tag{15}$$

Hence,

$$\begin{aligned} a^2\alpha^2[-(a^2+\alpha^2) + (\beta^2+b^2) + (\gamma^2+c^2)] &= \tfrac{1}{4}(x-y)^2\Big[-\tfrac{(x-y)\{(z+w)^2+(z-w)^2\}}{2(z+w)(z-w)} + x + y\Big]\\ &= \tfrac{1}{4}(x-y)^2 \cdot \tfrac{-x\{(z+w)^2+(z-w)^2-2(z+w)(z-w)\}+y\{(z+w)^2+(z-w)^2+2(z+w)(z-w)\}}{2(z+w)(z-w)}\\ &= \tfrac{1}{4}(x-y)^2 \cdot \tfrac{-4xw^2+4yz^2}{2(z+w)(z-w)}\\ &= \tfrac{(x-y)^2(-xw^2+yz^2)}{2(z+w)(z-w)}. \end{aligned} \tag{16}$$

Next, $x = (\beta^2 + b^2)$ and $z = (\beta^2 - b^2)$ imply $2\beta^2 = x + z$ and $2b^2 = x - z$. Hence, $b^2\beta^2 = \frac{1}{4}(x+z)(x-z)$. From this, we have

$$\begin{aligned} b^2\beta^2[(a^2+\alpha^2) - (\beta^2+b^2) + (\gamma^2+c^2)] &= \tfrac{1}{4}(x+z)(x-z)\Big[\tfrac{(x-y)\{(z+w)^2+(z-w)^2\}}{2(z+w)(z-w)} - x + y\Big]\\ &= \tfrac{1}{4}(x+z)(x-z) \cdot \tfrac{x\{(z+w)^2+(z-w)^2-2(z+w)(z-w)\}-y\{(z+w)^2+(z-w)^2-2(z+w)(z-w)\}}{2(z+w)(z-w)}\\ &= \tfrac{(x+z)(x-z)(x-y)w^2}{2(z+w)(z-w)}. \end{aligned} \tag{17}$$

Similarly, $y = (\gamma^2 + c^2)$ and $w = (\gamma^2 - c^2)$ imply $c^2\gamma^2 = \frac{1}{4}(y+w)(y-w)$. So

$$\begin{aligned} c^2\gamma^2[(a^2+\alpha^2) + (\beta^2+b^2) - (\gamma^2+c^2)] &= \tfrac{1}{4}(y+w)(y-w)\Big[\tfrac{(x-y)\{(z+w)^2+(z-w)^2\}}{2(z+w)(z-w)} + x - y\Big]\\ &= \tfrac{1}{4}(y+w)(y-w) \cdot \tfrac{x\{(z+w)^2+(z-w)^2+2(z+w)(z-w)\}-y\{(z+w)^2+(z-w)^2+2(z+w)(z-w)\}}{2(z+w)(z-w)}\\ &= \tfrac{(y+w)(y-w)(x-y)z^2}{2(z+w)(z-w)} \end{aligned} \tag{18}$$

Next,

$$\begin{aligned} a^2b^2c^2 + a^2\beta^2\gamma^2 + \alpha^2b^2\gamma^2 + \alpha 2c^2\beta^2 &= a^2(b^2c^2 + \beta^2\gamma^2) + \alpha^2(b^2\gamma^2 + c^2\beta^2)\\ &= \tfrac{(z-w)(x-y)}{2(z+w)}(b^2c^2 + \beta^2\gamma^2) + \tfrac{(z+w)(x-y)}{2(z-w)}(b^2\gamma^2 + c^2\beta^2) \qquad \text{by (13) and (14)}\\ &= \tfrac{(x-y)}{2(z+w)(z-w)}\{(z-w)^2(b^2c^2+\beta^2\gamma^2) + (z+w)^2(b^2\gamma^2 + c^2\beta^2)\}\\ &= \tfrac{(x-y)}{2(z+w)(z-w)}\{(z^2+w^2)(b^2c^2+\beta^2\gamma^2+b^2\gamma^2+c^2\beta^2) - 2zw(b^2c^2+\beta^2\gamma^2-b^2\gamma^2-c^2\beta^2)\}. \end{aligned}$$

Since $(b^2c^2 + \beta^2\gamma^2 + b^2\gamma^2 + c^2\beta^2) = (\beta^2 + b^2)(\gamma^2 + c^2) = xy$ and $(b^2c^2 + \beta^2\gamma^2 - b^2\gamma^2 - c^2\beta^2) = (\beta^2 - b^2)(\gamma^2 - c^2) = zw$, the above equation becomes

$$a^2b^2c^2 + a^2\beta^2\gamma^2 + \alpha^2b^2\gamma^2 + \alpha^2c^2\beta^2 = \frac{(x-y)}{2(z+w)(z-w)}\{(z^2+w^2)xy - 2z^2w^2\}. \tag{19}$$

Substituting equations (16)–(19) into Euler's formula, we have

$$\begin{aligned}
&144\mathcal{V}^2 \cdot \tfrac{2(z+w)(z-w)}{x-y} \\
&= (x-y)(-xw^2+yz^2) + (x+z)(x-z)w^2 + (y+w)(y-w)z^2 - (z^2+w^2)xy + 2z^2w^2 \\
&= w^2(-x^2+xy+x^2-z^2-xy) + z^2(xy-y^2+y^2-w^2-xy) + 2z^2w^2 \\
&= -w^2z^2 - z^2w^2 + 2z^2w^2 \\
&= 0.
\end{aligned}$$

Therefore, we have $\mathcal{V} = 0$. This is a contradiction to $\nabla ABCD$ being a non-degenerate tetrahedron. This proves that we must have $z+w=0$ or $z-w=0$.

Case 1: Suppose

$$z+w = (\beta^2-b^2)+(\gamma^2-c^2) = 0. \tag{20}$$

Equations (20) and (11) imply $(\beta^2-b^2)-(\gamma^2-c^2)=0$ or $(\beta^2+b^2)-(\gamma^2+c^2)=0$.

Subcase 1: Suppose

$$(\beta^2-b^2)-(\gamma^2-c^2)=0. \tag{21}$$

Then (20) and (21) gives us $\beta^2=b^2$. From (20), we also have $\gamma^2=c^2$. Since $\beta,\gamma,b,c>0$, we have $\beta=b$ and $\gamma=c$. Hence, $|AB|=|CD|$ and $|AC|=|BD|$.

Subcase 2: Suppose

$$(\beta^2+b^2)-(\gamma^2+c^2)=0. \tag{22}$$

Then (20) and (22) gives us $\beta^2=\gamma^2$. From (22), we also have $b^2=c^2$. From (20), we have $\beta^2=b^2$ and $\gamma^2=c^2$ so that $\beta=b=\gamma=c$. Hence, $|AB|=|CD|=|AC|=|BD|$.

Therefore, $(\beta^2-b^2)=(\gamma^2-c^2)=0$ implies that $|AB|=|CD|$ and $|AC|=|BD|$.

Case 2: Suppose

$$z-w=(\beta^2-b^2)-(\gamma^2-c^2)=0. \tag{23}$$

Equations (23) and (12) imply $(\beta^2-b^2)+(\gamma^2-c^2)=0$ or $(\beta^2+b^2)-(\gamma^2+c^2)=0$. As in Case 1, these imply that $|AB|=|CD|$ and $|AC|=|BD|$.

Therefore, from Cases 1 and 2, $\nabla ABCD$ is reversible. This proves Klain's theorem. $\square$

5 Triangle Inequalities from Tetrahedra

Let $|AB|=c$, $|BC|=a$, $|CA|=b$, $|DA|=\alpha$, $|DB|=\beta$, $|DC|=\gamma$ for a tetrahedron $\nabla ABCD$. Our next Theorem 2 is motivated by the following theorem.

Crelle's Theorem (See [6] and [11]). *If $\nabla ABCD$ is a tetrahedron, then there is a triangle having edges of lengths $a\alpha$, $b\beta$, and $c\gamma$. Moreover, if $\mathcal{A}$ is the area of the triangle having edges of lengths $a\alpha$, $b\beta$, and $c\gamma$, and if $\mathcal{V}$, R are the volume and circumradius of $\nabla ABCD$, respectively, then $\mathcal{A}=6R\mathcal{V}$.*

Since $2a\alpha \le a^2+\alpha^2$, $2b\beta \le b^2+\beta^2$, and $2c\gamma \le c^2+\gamma^2$, Crelle's theorem suggests the following *question:* Is there a triangle having edges of lengths $a^2+\alpha^2$, $b^2+\beta^2$, and $c^2+\gamma^2$? Equivalently, Euler's formula suggests the following *question:* Are the following inequalities true? $(-a^2+b^2+c^2-\alpha^2+\beta^2+\gamma^2)>0$, $(a^2-b^2+c^2+\alpha^2-\beta^2+\gamma^2)>0$, and $(a^2+b^2-c^2+\alpha^2+\beta^2-\gamma^2)>0$. The answer is *YES*. We prove this in Theorem 2 using the parallelogram law.

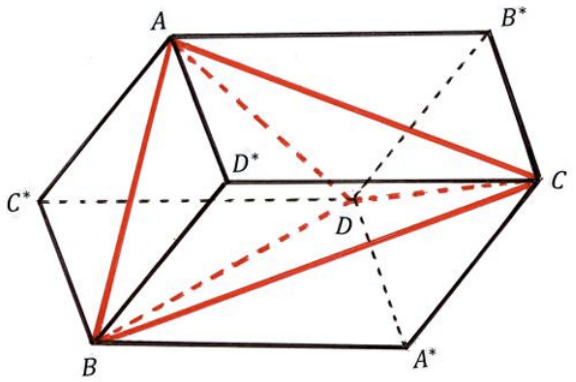

Figure 2: Parallelepiped circumscribing a tetrahedron $\nabla ABCD$ is shown.

Parallelogram Law:. If a quadrilateral $Q(ABCD)$ is a parallelogram, then

$$2|AB|^2 + 2|BC|^2 = |AC|^2 + |BD|^2.$$

Theorem 2. *If $|AB| = c$, $|BC| = a$, $|CA| = b$, $|DA| = \alpha$, $|DB| = \beta$, $|DC| = \gamma$ for a tetrahedron $\nabla ABCD$, then there is a triangle having three edges of lengths $a^2 + \alpha^2$, $b^2 + \beta^2$, and $c^2 + \gamma^2$.*

Proof. We circumscribe a tetrahedron $\nabla ABCD$ by a parallelepiped, call it Π, so that the edges of the tetrahedron are the diagonals of the six parallelogram faces of Π. See Figure 2. So, for example, the face of parallelepiped Π that contains the segment $[AB]$ is parallel to the segment $[CD]$. We label the diagonally opposite vertices of A, B, C, D of the parallelepiped Π by A^*, B^*, C^*, D^*, respectively. ($\nabla A^*B^*C^*D^*$ is called the *twin* of $\nabla ABCD$.) Thus, for example, the parallelogram face $Q(AC^*BD^*)$ of Π is the face containing the segment $[AB]$ parallel to the segment $[CD]$.

Let $x = |AD^*|$, $y = |BD^*|$, $z = |CD^*|$. Then by the parallelogram law, we have $a^2+\alpha^2 = 2y^2 + 2z^2$ from the parallelogram $Q(D^*BA^*C)$, $b^2 + \beta^2 = 2x^2 + 2z^2$ from $Q(D^*CB^*A)$, and $c^2 + \gamma^2 = 2x^2 + 2y^2$ from $Q(D^*BC^*A)$. Hence, we have

$$\begin{aligned}
a^2 + \alpha^2 &= 2y^2 + 2z^2 = (c^2 + \gamma^2 - 2x^2) + (b^2 + \beta^2 - 2x^2) < (c^2 + \gamma^2) + (b^2 + \beta^2),\\
b^2 + \beta^2 &= 2x^2 + 2z^2 = (c^2 + \gamma^2 - 2y^2) + (a^2 + \alpha^2 - 2x^2) < (c^2 + \gamma^2) + (a^2 + \alpha^2), \quad \text{and}\\
c^2 + \gamma^2 &= 2x^2 + 2y^2 = (a^2 + \alpha^2 - 2z^2) + (b^2 + \beta^2 - 2z^2) < (a^2 + \alpha^2) + (b^2 + \beta^2).
\end{aligned}$$

Therefore, there is a triangle having three edge lengths $a^2 + \alpha^2$, $b^2 + \beta^2$, and $c^2 + \gamma^2$. □

6 Inequality on the Circum-radius of a Tetrahedron

From Mazur's theorem, Crelle's theorem, and Heron's formula, we will find an inequality on the circum-radius for a tetrahedron. Let $|AB| = c$, $|BC| = a$, $|CA| = b$, $|DA| = \alpha$, $|DB| = \beta$, $|DC| = \gamma$ for a tetrahedron $\nabla ABCD$.

Mazur's Theorem (See [11]). *If $\mathcal{V}$ is the volume of $\nabla ABCD$, then*

$$(-a\alpha + b\beta + c\gamma)(a\alpha - b\beta + c\gamma)(a\alpha + b\beta - c\gamma) \geq 72\mathcal{V}^2.$$

The equality holds if, and only if, the tetrahedron is isosceles. In particular, if $\nabla ABCD$ *is an isosceles tetrahedron such that* $|BC| = a = |AD|$, $|CA| = b = |BD|$, $|AB| = c = |CD|$, *and if* $\mathcal{V}$ *is its volume, then* $72\mathcal{V}^2 = (-a^2+b^2+c^2)(a^2-b^2+c^2)(a^2+b^2-c^2)$.

Theorem 3. *If* R *is the circumradius of a tetrahedron* $\nabla ABCD$, *then* $R \geq \frac{1}{2\sqrt{2}}\sqrt{a\alpha + b\beta + c\gamma}$. *The equality holds if, and only if the tetrahedron is isosceles. In particular, if* $\nabla ABCD$ *is an isosceles tetrahedron such that* $|BC| = a = |AD|$, $|CA| = b = |BD|$, $|AB| = c = |CD|$ *with* R *being the circumradius, then* $R^2 = \frac{1}{8}(a^2+b^2+c^2)$.

Proof. Let $\mathcal{A}$ be the area of the triangle having edges of lengths $a\alpha$, $b\beta$, and $c\gamma$. Crelle's Theorems and Heron's formula give us

$$36R^2\mathcal{V}^2 = \mathcal{A}^2 = \frac{1}{16}(a\alpha+b\beta+c\gamma)(-a\alpha+b\beta+c\gamma)(a\alpha-b\beta+c\gamma)(a\alpha+b\beta-c\gamma).$$

By Mazur's Theorem, we have

$$\begin{aligned}36R^2\mathcal{V}^2 &= \frac{1}{16}(a\alpha+b\beta+c\gamma)(-a\alpha+b\beta+c\gamma)(a\alpha-b\beta+c\gamma)(a\alpha+b\beta-c\gamma)\\ &\geq \frac{1}{16}(a\alpha+b\beta+c\gamma)72\mathcal{V}^2.\end{aligned}$$

By cancelling $36\mathcal{V}^2$ from both sides, we have $R^2 \geq \frac{1}{8}(a\alpha+b\beta+c\gamma)$. The equality holds if, and only if the tetrahedron is isosceles. This proves the theorem. □

Next, we will apply Heron's formula and Theorem 3 to obtain an expression for the volume of an isosceles tetrahedron different from the one in Mazur's theorem.

Theorem 4. *Let* $\nabla ABCD$ *be an isosceles tetrahedron such that* $|BC| = a = |AD|$, $|CA| = b = |BD|$, $|AB| = c = |CD|$. *Let* $\mathcal{V}$ *be the volume, and* R *the circumradius of* $\nabla ABCD$. *Let* $\mathcal{A}$ *be the area, and* $\hat{r}$ *the circumradius of* $\triangle ABC$. *Then* $9\mathcal{V}^2 = 16R^2\mathcal{A}^2 - a^2b^2c^2 = 16\mathcal{A}^2(R^2-\hat{r}^2)$.

Proof. Heron's formula gives us $16\mathcal{A}^2 = 2a^2b^2+2b^2c^2+2c^2a^2-a^4-b^4-c^4$. Theorem 3 gives us $8R^2 = (a^2+b^2+c^2)$. Thus,

$$\begin{aligned}8R^2\cdot 16\mathcal{A}^2 &= (a^2+b^2+c^2)(2a^2b^2+2b^2c^2+2c^2a^2-a^4-b^4-c^4)\\ &= a^4b^2+a^4c^2+a^2b^4+b^4c^2+a^2c^4+b^2c^4-a^6-b^6-c^6+6a^2b^2c^2\\ &= (-a^2+b^2+c^2)(a^2-b^2+c^2)(a^2+b^2-c^2)+8a^2b^2c^2\\ &= 72\mathcal{V}^2+8a^2b^2c^2 \quad \text{by Mazur's theorem.}\end{aligned}$$

Hence, by rearranging this equation, we have $9\mathcal{V}^2 = 16R^2\mathcal{A}^2 - a^2b^2c^2$. Since $4\mathcal{A}\hat{r} = abc$ by Lemma 1, we have $9\mathcal{V}^2 = 16R^2\mathcal{A}^2 - a^2b^2c^2 = 16\mathcal{A}^2(R^2-\hat{r}^2)$.

Alternately, let Q be the circumcenter of $\triangle ABC$, and P the circumcenter of $\nabla ABCD$. Since the circumcenter and the incenter are the same for an isosceles tetrahedron[1], $\triangle PQA$ is a right triangle. Since $|PA| = R$, $r = |PQ|$, and $|QA| = \hat{r}$, we have $R^2 = r^2 + \hat{r}^2$ by Pythagorean theorem. Since $\mathcal{V} = 4\cdot\frac{1}{3}r\mathcal{A}$, we have $9\mathcal{V}^2 = 16r^2\mathcal{A}^2 = 16\mathcal{A}^2(R^2-\hat{r}^2)$. □

Remark 4. Let A, B, C, D be four distinct points in $\mathbb{R}^3$, no three of them are collinear. Let $|AB| = c$, $|BC| = a$, $|CA| = b$, $|DA| = \alpha$, $|DB| = \beta$, $|DC| = \gamma$. Then a weak form of Lemma 3 says that the four points A, B, C, D forms a cyclic quadrilateral $Q(ABCD)$, $Q(ACBD)$, or $Q(ACDB)$ if and only if $(-a\alpha+b\beta+c\gamma)(a\alpha-b\beta+c\gamma)(a\alpha+b\beta-c\gamma) = 0$. This may be an interesting comparison to the left side of the inequality in Mazur's theorem.

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[1] A tetrahedron is isosceles if, and only if, its circumcenter and the in-center are the same.

Remark 5. The volume formula for an isosceles tetrahedron in Mazur's theorem is derived in [8] without Euler's formula. Similarly, a volume formula for a reversible tetrahedron is given in [8]. These formulas are derived in [10] using Euler's formula.

7 A Theorem Motivated by the Parallelogram Law

Theorem 1 and Lemma 3 give two methods to determine four points in the space being co-planar. Theorem 1 of [6] is another way to determine four points being co-planar in terms of angles. The next theorem is another one motivated by the parallelogram law.

Theorem 5. *Suppose A, B, C, D are four distinct points in the space $\mathbb{R}^3$, no three of them are collinear. If $|AB| = |CD|$ and $|AD| = |BC|$, and if $2|AB|^2 + 2|BC|^2 = |AC|^2 + |BC|^2$, then the four points A, B, C, D form a parallelogram $Q(ABCD)$.*

Proof. We will first prove that the volume of $\nabla ABCD$ is zero. Let $|AB| = c$, $|BC| = a$, $|CA| = b$, $|DA| = \alpha$, $|DB| = \beta$, $|DC| = \gamma$ for a tetrahedron $\nabla ABCD$. Since $c = |AB| = |CD| = \gamma$ and $\alpha = |AD| = |BC| = a$, the volume $\mathcal{V}$ of $\nabla ABCD$ is given by Euler's formula as

$$\begin{aligned}144\mathcal{V}^2 &= a^4[-2a^2 + 2c^2 + (b^2 + \beta^2)] + b^2\beta^2[2a^2 + 2c^2 - (b^2 + \beta^2)] \\ &\qquad + c^4[2a^2 - 2c^2 + (b^2 + \beta^2)] - 2a^2c^2(b^2 + \beta^2).\end{aligned}$$

The equation $2|AB|^2 + 2|BC|^2 = |AC|^2 + |BD|^2$ becomes $b^2 + \beta^2 = 2a^2 + 2c^2$. Substituting this into the above equation, we have

$$\begin{aligned}144\mathcal{V}^2 &= a^4[-2a^2 + 2c^2 + (2a^2 + 2c^2)] + b^2\beta^2[2a^2 + 2c^2 - (2a^2 + 2c^2)] \\ &\qquad + c^4[2a^2 - 2c^2 + (2a^2 + 2c^2)] - 2a^2c^2(2a^2 + 2c^2) \\ &= 4a^4c^2 + 4a^2c^4 - 2a^2c^2(2a^2 + 2c^2) = 0.\end{aligned}$$

Therefore, $\mathcal{V} = 0$ and $\nabla ABCD$ is degenerate. This implies that A, B, C, D are coplanar.

Next, we must prove that A, B, C, D form a quadrilateral $Q(ABCD)$ by showing $[AB] \cap [CD] = \varnothing$ and $[AD] \cap [BC] = \varnothing$. On the contrary, suppose that $[AB] \cap [CD] \neq \varnothing$. Since no three points of A, B, C, D are collinear, $[AB] \cap [CD]$ must be a singleton set, say $[AB] \cap [CD] = \{E\}$ and E is different from the points A, B, C, D. Let $\sphericalangle ADC = \sphericalangle ADE = \theta$, and $\sphericalangle DAB = \sphericalangle DAE = \mu$. Hence, $0 < \theta$, $\mu < \pi$. (See the footnote[2]) By the law of cosines, we have

$$\begin{aligned}|AC|^2 &= |AD|^2 + |CD|^2 - 2|AD||CD|\cos\theta \\ &= |AB|^2 + |BC|^2 - 2|AB||BC|\cos\theta, \qquad \text{and} \\ |BD|^2 &= |AD|^2 + |AB|^2 - 2|AD||AB|\cos\mu \\ &= |AB|^2 + |BC|^2 - 2|AB||BC|\cos\mu \qquad \text{since } |AB| = |CD| \text{ and } |AD| = |BC|.\end{aligned}$$

Hence, $2|AB|^2 + 2|BC|^2 = |AC|^2 + |BD|^2$

$$= |AB|^2 + |BC|^2 - 2|AB||BC|\cos\theta + |BC|^2 + |AB|^2 - 2|BC||AB|\cos\mu.$$

This simplifies to $2|AB||BC|(\cos\theta + \cos\mu) = 0$. Thus, $\cos\mu = -\cos\theta$. Since $0 < \theta$, $\mu < \pi$ and since $\cos(\pi - \theta) = -\cos\theta = \cos\mu$, we must have $\mu = \pi - \theta$. Hence, $\theta + \mu = \pi$. But this is

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[2] The angle $\sphericalangle ADC$, for example, is the angle of the triangle $\triangle ADC$. Therefore, $0 < \theta < \pi$. Note that an interior angle of a quadrilateral can exceed π.

a contradiction to $\theta + \mu < \pi$ since $\sphericalangle ADE = \theta$ and $\sphericalangle DAE = \mu$ are angles of $\triangle ADE$. Hence, $[AB] \cap [CD] = \varnothing$. Similarly, we can show that $[AD] \cap [BC] = \varnothing$. Therefore, A, B, C, D form a planar quadrilateral $Q(ABCD)$. Since $|AB| = |CD|$ and $|AD| = |BC|$, the quadrilateral $Q(ABCD)$ is a parallelogram. ☐

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