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A Generalization of Lemma 2 of Archimedes' Book of Lemmas

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Abstract. Lemma 2 of *Archimedes'* book of Lemmas states that if D is a point of a semicircle with diameter AB and the tangents at A and D meet at T, then BT bisects the perpendicular DE to AB. In this work we generalize Lemma 2 to hyperbolas with major axis AB and prove that if D is a point of the branch with vertex A and the tangents at D and A meet at T, then the ray $\overrightarrow{BT}$ bisects the perpendicular DE to AB at F and similarly if D is a point of the branch with vertex B and the tangents at D and B meet at T, then the ray $\overrightarrow{AT}$ bisects the perpendicular DE to BA at F. Also, we prove for parabolas that if D is a point of the parabola with vertex A and the tangents at D and A meet at T, then the ray $\overrightarrow{TF}$ parallel to the axis of symmetry AB of the parabola bisects the perpendicular DE to AB at F. Also it is proved that if the point D moves from a vertex of a hyperbola or a parabola on a branch of a hyperbola or on a parabola, then the locus of the midpoint F of DE is a branch of a hyperbola or is a parabola. Also, this generalization leads to a method to construct a tangent by compass and straight edge at any point of a hyperbola, parabola, or ellipse.

Key Words: hyperbola, parabola, tangent

MSC 2020: 51M04 (primary), 01A20, 01A30

1 Introduction

The *Book of Lemmas* was first introduced in Arabic by *Thābit ibn Qurra* who lived in Baghdad (826 or 836–901) and he attributed the work to *Archimedes.* The Arabic manuscript was first translated into Latin in 1661 see [3]. The Latin work translated into English in [4]. An Arabic manuscript of *Thābit ibn Qurra's* has recently been discovered in Suleymanye Manuscript Library in Istanbul and is translated into English in [2].

"In Remark 2(a) of the author's paper *A Look at the Archimedes' Book of Lemmas* [1], Lemma 2 is generalized to semi-ellipses and proved that if AB is the major axis of a semi-ellipse Ω and the tangents to it at A and B meet the tangent at any other point D of Ω at C and T, respectively, then BC and AT intersect at the midpoint F of the perpendicular DE

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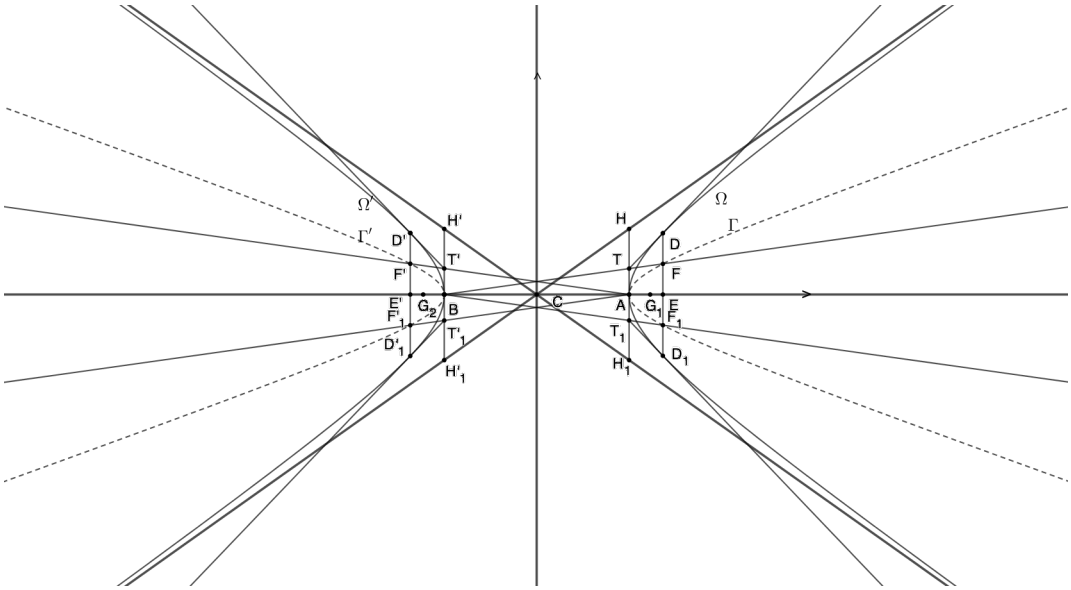


Figure 1: Illustrating the proof of Theorem 1

to AB and the locus of point F is a semi-ellipse as D moves on the semi-ellipse Ω from A to B". So, this also leads to the following method to construct tangents to ellipses. Draw the tangents to the ellipse at A and B that are perpendicular to the major axis AB and from any other point D of the ellipse draw the perpendicular DE to AB that meets AB at E and construct the midpoint F of DE and extend AF and BF to meet the tangents at B and A at T and C, respectively. Then it follows from *Remark 2(a)* in [1] that TC is tangent to the ellipse at D.

The scope of this article is to generalize Lemma 2 to the other conic sections, the hyperbolas in Theorem 1 and the parabolas in Theorem 2. In Corollaries 1(a) and 2(a) we show how to construct a tangent at a point of a hyperbola and at a point of a parabola.

2 Main Theorems

Theorem 1. *Let AB be the major axis and the axis of symmetry of a hyperbola with two branches Ω and Ω' and with two vertices A and B. Let the tangent to the hyperbola at A meet the tangents at any other points D and D_1 of the upper and lower semi-branches of Ω at T and T_1, respectively. Also, let the tangent at B meet the tangents at the points D' and D'_1 of the upper and lower semi-branches of Ω' at T' and T'_1, respectively, as shown in Figure 1. Let DD_1 and $D'D'_1$ be perpendicular to the axis of symmetry BA and intersect it at E and E', respectively. Then the rays $\overrightarrow{BT}$ and $\overrightarrow{BT_1}$ intersect DE and D_1E at the midpoints F of DE and F_1 of D_1E, respectively. Also, F and F_1 move from A on the upper and lower semi-branches of the branch Γ of the hyperbola with two branches Γ and Γ', as D and D_1 move from A on the upper and lower semi-branches of Ω, respectively. Also, the rays $\overrightarrow{AT'}$ and $\overrightarrow{AT'_1}$ intersect $D'E'$ and D'_1E' at the midpoints F' of $D'E'$ and F'_1 of D'_1E', respectively. Also, F' and F'_1 move from B on the upper and lower semi-branches of the branch Γ' of the hyperbola with two branches Γ and Γ', as D' and D'_1 move from B on the upper and lower semi-branches of Ω', respectively.*

Proof. Let the origin of the algebraic coordinate system be the midpoint C of AB and the x-axis be the major axis AB of the hyperbola with two branches Ω and Ω' and the y-axis be the perpendicular to AB at C as shown in Figure 1. Let the tangents HH_1 and $H'H'_1$ at the vertices A and B, meet the asymptotes CH and CH' at H, H'_1 and at H', H_1, respectively. Let the semi-major axis $CA = CB = a$ and its semi-minor axis be equal to $AH = AH_1 = BH' = BH'_1 = b$. Then the algebraic equation of the hyperbola with two branches Ω and Ω' is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, $a > b > 0$, $|x| \geq a$ and the equation of the branch Ω is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, $a > b > 0$, $x \geq a$ and the slope of the tangent TD at $D = (x, y)$ is equal $\frac{b^2 x}{a^2 y}$. Thus if $D = (c, d)$ is a point of the upper semi-branch of Ω, then $d = \frac{b}{a}\sqrt{c^2 - a^2}$ and the equation of the tangent line TD is given by $\frac{y-d}{x-c} = \frac{b^2 c}{a^2 d}$ and hence the coordinates of the point T is given by $T = \left(a, \frac{a^2d^2 - b^2c^2 + acb^2}{a^2 d}\right)$. But $d = \frac{b}{a}\sqrt{c^2 - a^2}$. Therefore $T = \left(a, b\sqrt{\frac{c-a}{c+a}}\right)$. Since $B = (-a, 0)$ and slope of $BT = \frac{b}{2a}\sqrt{\frac{c-a}{c+a}}$, it follows that the equation of BT is given by $y = \frac{b(x+a)}{2a}\sqrt{\frac{c-a}{c+a}}$. Thus the ray $\overrightarrow{BT}$ intersects DE when $x = c$ and hence $y = \frac{b}{2a}\sqrt{c^2 - a^2} = \frac{d}{2}$ and the ray $\overrightarrow{BT}$ intersects DE at the midpoint $F = (c, \frac{b}{2a}\sqrt{c^2 - a^2})$ of DE. Also, by symmetry $D_1 = (c, -d)$ and $T_1 = \left(a, -b\sqrt{\frac{c-a}{c+a}}\right)$ and the ray $\overrightarrow{BT_1}$ intersects D_1E at the midpoint $F_1 = (c, -\frac{b}{2a}\sqrt{c^2 - a^2})$ of D_1E as required. Also, by symmetry the equation of the branch Ω' is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, $a > b > 0$, $x \leq -a$. So, if $c > a$, $D' = (-c, d)$ and $D'_1 = (-c, -d)$ are points of the upper and lower semi-branches of Ω', respectively, and $D'D'_1$ is perpendicular to the axis of symmetry AB at the point $E' = (-c, 0)$, then the tangents to Ω' at D' and D'_1 meet the tangent $H'H'_1$ at $T' = (-a, b\sqrt{\frac{c-a}{c+a}})$ and at $T'_1 = (-a, -b\sqrt{\frac{c-a}{c+a}})$, respectively. Thus as in the case of the branch Ω, the ray $\overrightarrow{AT'}$ intersects $D'E'$ when $x = -c$ and hence $y = \frac{b}{2a}\sqrt{c^2 - a^2} = \frac{d}{2}$ and the ray $\overrightarrow{AT'}$ intersects $D'E'$ at the midpoint $F' = (-c, \frac{b}{2a}\sqrt{c^2 - a^2})$ of $D'E'$. Also, by symmetry $D'_1 = (-c, -d)$ and $T'_1 = \left(-a, -b\sqrt{\frac{c-a}{c+a}}\right)$ and the ray $\overrightarrow{AT'_1}$ intersects D'_1E' at the midpoint $F'_1 = (-c, -\frac{b}{2a}\sqrt{c^2 - a^2})$ of D'_1E' as required.

Next, we prove that the locus of the points F and F_1 is the branch Γ as D and D_1 move from A on the branch Ω. Since the algebraic equation of the branch Ω is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, $x \geq a$ and the coordinates of $D = (x, y)$ and $D_1 = (x, -y)$, it follows that the coordinates of $F = (x, \frac{y}{2})$. Thus, when $D = (a, 0) = D_1$, we have $F = (a, 0) = F_1$. Therefore the equation of the branch Γ of the hyperbola with two branches Γ and Γ' and with semi-major axis a and semi-minor axis $\frac{b}{2}$, is $\frac{x^2}{a^2} - \frac{4y^2}{b^2} = 1, x \geq a$. Thus if $x = c$, D and D_1 are points of the upper and lower semi-branches of Ω, respectively, and DD_1 is perpendicular to the axis of symmetry of Ω at E, then $D = (c, \frac{b}{a}\sqrt{c^2 - a^2})$ and $D_1 = (c, -\frac{b}{a}\sqrt{c^2 - a^2})$ are points of the branch Ω and the midpoints $F = (c, \frac{b}{2a}\sqrt{c^2 - a^2})$ of DE and $F_1 = (c, -\frac{b}{2a}\sqrt{c^2 - a^2})$ of D_1E satisfy the equation of Γ and hence Γ is the locus of F and F_1 as required.

Similarly, by symmetry if the algebraic equation of the branch Ω' is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, x \leq -a$ and $D' = (-c, d)$ and $D'_1 = (-c, -d)$ are point of the upper and lower semi-branches of Ω', $D'D'_1$ is perpendicular to the axis of symmetry AB at E', then $\frac{c^2}{a^2} - \frac{4d^2}{4b^2} = \frac{c^2}{a^2} - \frac{d^2}{b^2} = 1$, and hence the midpoints $F' = (-c, \frac{d}{2})$ of $D'E'$ and $F'_1 = (-c, -\frac{d}{2})$ of D'_1E' satisfy the equation $\frac{x^2}{a^2} - \frac{4y^2}{b^2} = 1, x \leq -a$ of the branch Γ' of the hyperbola with two branches Γ and Γ' and the locus of the points F' and F'_1, as D' and D'_1 move from B on the upper and lower semi-branches of Ω', is the branch Γ' of the hyperbola with two branches Γ and Γ' as required. □

Corollary 1(a). *From Theorem 1 we conclude that to construct, by compass and straight edge, tangents to a hyperbola at a point D of a semi-branch of Ω, draw the perpendicular*

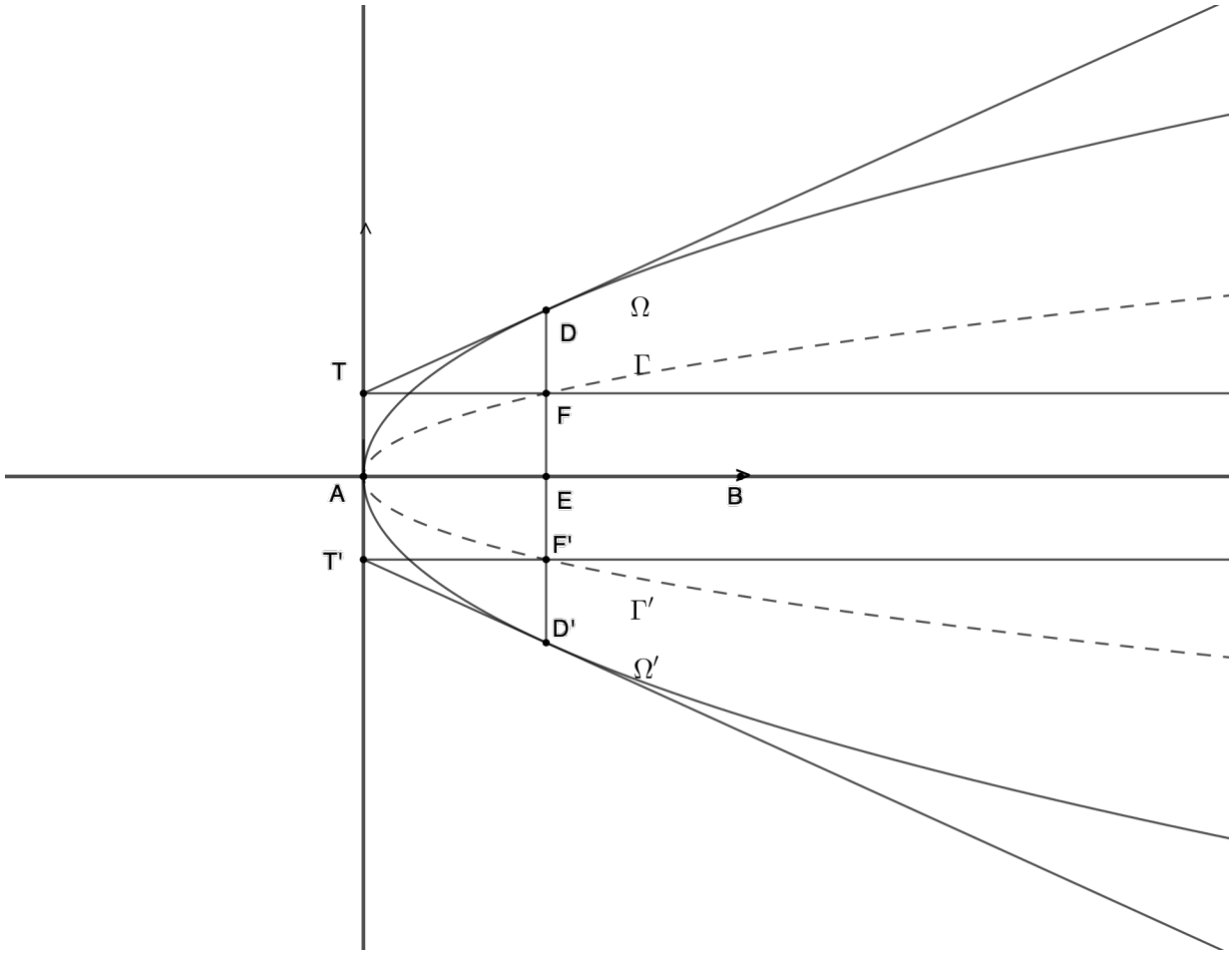


Figure 2: Illustrating the proof of Theorem 2

DD_1 to AB that meets AB at E and the other semi-branch at D_1, construct the midpoints F of DE and F_1 of D_1E and join BF and BF_1 that intersect the tangent HH', which is perpendicular to AB, at T and T_1, then TD and T_1D_1 are tangents to the branch Ω at D and D_1. Also, by symmetry the tangents $T'D'$ and $T'_1D'_1$ to the branch Ω' at D' and D'_1 can be constructed by drawing the perpendicular $D'D'_1$ to AB that intersects AB at E' and by drawing the midpoints F' and F'_1 of $D'E'$ and D'_1E', respectively. Then by joining AF' and AF'_1 that intersect the tangent $H'H'_1$ at T' and at T'_1 we conclude that $T'D'$ and $T'_1D'_1$ are tangents to Ω' at D' and D'_1.

Theorem 2. *Let the ray $\overrightarrow{AB}$ be the axis of symmetry of a parabola with vertex A and two semi-parabolas Ω and Ω'. Let D be a point of Ω and D' be a point of Ω' and DD' be perpendicular to AB and intersect AB at E and let the tangent at A meet the tangent at D at T and the tangent at D' at T' as shown in Figure 2. Let the rays $\overrightarrow{TF}$ and $\overrightarrow{T'F'}$ be parallel to the axis of symmetry AB and intersect DE and $D'E$ at F and F', respectively. Then F is the midpoint of DE and F' is the midpoint of $D'E$. Also F and F' move from A on the semi-parabolas Γ and Γ' as D and D' move from A on the semi-parabolas Ω and Ω', respectively.*

Proof. Let the origin of the coordinate system be the vertex A and the x-axis be the axis of symmetry AB of the parabola with two semi-parabolas Ω and Ω'. Let $y^2 = 4ax$ be the algebraic equation of the parabola with the semi-parabolas Ω and Ω', vertex A, and axis of

symmetry AB as shown in Figure 2. Let the coordinates of $D = (c, d)$, $d = 2\sqrt{ac}$ and the slope of the tangent DT be equal to $\frac{2a}{y} = \frac{a}{\sqrt{ac}}$. Then the equation of DT is given by $y - 2\sqrt{ac} = \frac{a}{\sqrt{ac}}(x - c)$. But the x-coordinate of $T = 0$. So the y-coordinate of $T = 2\sqrt{ac} - \frac{ac}{\sqrt{ac}} = \sqrt{ac}$ and $T = (0, \sqrt{ac})$. Thus the equation of the ray $\overrightarrow{TF}$ which is parallel to AB is $y = \sqrt{ac}$. Therefore TF intersects DE at the point $F = (c, \sqrt{ac})$ which is the midpoint of DE where $D = (c, 2\sqrt{ac})$ and $E = (c, 0)$ as required. Also, it follows by symmetry that if D' is a point of the semi-parabola Ω' and the tangent at D' meets the tangent AT at T', then the ray $T'F'$ parallel to AB intersects the perpendicular $D'E$ to AB at F' and F' is the midpoint of $D'E$ as required. So, if $E = (c, 0)$, then $D = (c, 2\sqrt{ac})$ and $D' = (c, -2\sqrt{ac})$. Thus $F = (c, \sqrt{ac})$ and $F' = (c, -\sqrt{ac})$ and it is clear that as D and D' move from A on the semi-parabolas Ω and Ω', the points F and F' move from A on the semi-parabolas Γ and Γ' of the parabola with algebraic equation $y^2 = ax$, respectively. □

Corollary 2(a). *From Theorem 2 we conclude that to construct, by compass and straight edge, tangents to a parabola at a point D of the semi-parabola Ω, draw the perpendicular DD' to AB that meets AB at E and the semi-parabola Ω' at D'. Construct the midpoints F of DE and F' of $D'E$ and draw FT and $F'T'$ parallel to the axis of symmetry AB that intersect the tangent at A, which is perpendicular to AB, at T and T'. Then we conclude that TD and $T'D'$ are tangents to the semi-parabolas Ω at D and Ω' at D' of the given parabola.*

In the next section we summarize the results related to Lemma 2 and the generalizations of Lemma 2 and to see how they are related.

3 Conclusions

This paper is a continuation of the ideas developed in the author's paper *A look at Archimedes' Book of Lemmas* [1] and in this paper Lemma 2 is generalized from circles to other conic sections the hyperbolas and parabolas.

For convenience, let Ω stand for a conic section either circle, ellipse, hyperbola, or parabola and AB stand for the axis of Ω which is either a diameter of a circle with endpoints A and B, the major axis of an ellipse or a hyperbola with vertices A and B, or the axis of symmetry of a parabola with vertex A. Also, let semi-conic section stand for either a semi-circle, semi-ellipse, semi-hyperbola, or a semi-parabola.

So, if D is a point of the semi-conic section of Ω and $DE \perp AB$ and meets AB at E and with midpoint F and the tangents to the semi-section of Ω at D and A meet at T, then it is proved that BT intersects DE at the midpoint F for the semi-circle of Ω in Lemma 2 and for semi-ellipse of Ω in Remark 2(a) in [1]. Also it is proved in Theorem 1 of this paper that the ray $\overrightarrow{BT}$ intersects DE at the midpoint F for the semi-branch of a hyperbola Ω and in Theorem 2 it is proved that the line $TF \parallel AB$ intersects DE at the midpoint F of the semi-parabola Ω.

It is worth mentioning here that the lines BT that bisect DE for semi-conic sections have something in common for circles, ellipses, and hyperbolas where T is the point of intersection of the two tangents from D and from A and also the point B is either the other endpoint of a diameter, the other endpoint of the major axis of ellipse, or the second vertex of a hyperbola. But there is no other endpoint for the axis of symmetry AB of a parabola and that is why TF parallel to AB is chosen for parabolas.

Also, it is proved that, for the four conic sections, the locus of the midpoint F of DE as D moves on the semi-conic section is also a semi-ellipse for circles and ellipses, semi-hyperbola for hyperbolas, and semi-parabola for parabolas.

Finally, finding the line that bisects DE leads to find a method for the construction of a tangent at any point D of the conic section Ω as shown in Corollaries 1 and 2. This is done by finding the point T where $\overrightarrow{DT}$ is the required tangent. The point T is found to be the point of intersection of the ray $\overrightarrow{BF}$ with the tangent at A for circles, ellipses, and hyperbolas and the intersection of the line through F and parallel to the axis of symmetry of a parabola.

Notice that the lines BT and TF for semi-conic sections intersect the locus Γ of the midpoint F of DE as D moves on the semi-conic section of Ω.

So, we conclude from the fact that all the fifteen Lemmas of Archimedes are related to circular arcs, semi-circles, or circles and from the generalizations of Lemma 2 and other Lemmas in [1] that all of them open the way for researchers, high school and college instructors and students to look for generalizations of other Lemmas to other conic sections or to a higher dimension.

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