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Cosines of Dihedral Angles of a Tetrahedron

Hidefumi Katsuura

San Jose State University, San Jose, USA
hidefumi.katsuura@sjsu.edu

Abstract. We obtain a new trigonometric equation for a tetrahedron involving dihedral angles that resembles the law of cosine for a triangle. This involves the area of a face $\mathcal{Q}$ of the parallelepiped that inscribes the tetrahedron. Then we find the area of $\mathcal{Q}$ in terms of the edge lengths of the tetrahedron. Finally, we give a new proof of the law of cosines for a tetrahedron using our theorem.

Key Words: Law of cosines, Parallelepiped that inscribes a tetrahedron, Law of cosines for a tetrahedron, Spherical law of cosines, Heron's formula

MSC 2020: 51M04

1 Introduction

The angle of a triangle $\triangle ABC$ at A is denoted by $\sphericalangle BAC$. The interior angle of a tetrahedron $\triangledown ABCD$ between $\triangle ABC$ and $\triangle ABD$ is called the *dihedral angle* at the edge AB and denoted by $\sphericalangle AB$. The laws of cosines for a tetrahedron states as follow.

Law of cosines for a tetrahedron. For a tetrahedron $\triangledown ABCD$, we have

$$\mathcal{A}_A^2 = \mathcal{A}_B^2 + \mathcal{A}_C^2 + \mathcal{A}_D^2 - 2\mathcal{A}_B\mathcal{A}_C\cos\sphericalangle AD - 2\mathcal{A}_C\mathcal{A}_D\cos\sphericalangle AB - 2\mathcal{A}_D\mathcal{A}_B\cos\sphericalangle AC,$$

where A_A is the area of the face $\triangle BCD$.

From this, we can obtain that

$$\mathcal{A}_D^2 + \mathcal{A}_C^2 - 2\mathcal{A}_D\mathcal{A}_C\cos\sphericalangle AB = \mathcal{A}_B^2 + \mathcal{A}_A^2 - 2\mathcal{A}_B\mathcal{A}_A\cos\sphericalangle CD.$$

The resemblance of this to the *Law of cosines for a triangle* (for a triangle $\triangle ABC$, we have $|BC|^2 = |AB|^2 + |AC|^2 - 2|AB||AC|\cos\sphericalangle BAC$) is striking. In Theorem 1 we prove that a face $\mathcal{Q}$ of the parallelepiped $\mathcal{P}$ that inscribes a tetrahedron $\triangledown ABCD$ has a property

$$|\mathcal{Q}|^2 = \mathcal{A}_D^2 + \mathcal{A}_C^2 - 2\mathcal{A}_D\mathcal{A}_C\cos\sphericalangle AB,$$

where $|\mathcal{Q}|$ is the area of $\mathcal{Q}$.

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We will denote this parallelepiped $\mathcal{P}$ that inscribes a tetrahedron $\bigtriangledown ABCD$ by $\mathcal{P}(AB\,CDA^*B^*C^*D^*)$. The vertices A^*, B^*, C^*, D^* forms the twin tetrahedron $\bigtriangledown A^*B^*C^*D^*$ of $\bigtriangledown ABCD$ (see Definition 1 below for detail). This parallelepiped has many interesting properties. See [3] and [4], for example, for additional applications of the parallelepiped $\mathcal{P}(ABCDA^*B^*C^*D^*)$.

Crane and Yetter in [2] mention a theorem like our Theorem 1 due to McConnel, and it states that

> "If P is the area of the projection of $\bigtriangledown ABCD$ on to a plane parallel to lines $\overline{AB}$ and $\overline{CD}$, then $P^2 = \mathcal{A}_D^2 + \mathcal{A}_C^2 - 2\mathcal{A}_D\mathcal{A}_C \cos\sphericalangle AB$."

Here, "the projection of $\bigtriangledown ABCD$" does not mean the orthogonal projection of $\bigtriangledown ABCD$, so it is not easy to interpret "the projection of $\bigtriangledown ABCD$". We interpreted it in Corollary 1.

Then we will express $|\mathcal{Q}|$ in terms of the edge lengths of $\bigtriangledown ABCD$ in Theorem 2 using Theorem 1, the spherical law of cosines and Heron's formula. Theorem 1 will be proved twice, and both are independent of the above law of cosine for a tetrahedron. Hence, using Theorem 1, we will give a new proof of the law of cosine for a tetrahedron.

2 Tetrahedral Cosine Formulas

Let us begin by the notations and definitions to describe Theorem 1.

Definition 1. The area of $\triangle ABC$ is denoted by $|\triangle ABC|$. We also denote the area of $\triangle ABC$ by $\mathcal{A}_D$ if we treat it as the face of $\bigtriangledown ABCD$ opposite of the vertex D. A tetrahedron $\bigtriangledown ABCD$ can be inscribed in a parallelepiped $\mathcal{P}$ so that the six edges of $\bigtriangledown ABCD$ are diagonals of the six parallelogram faces of $\mathcal{P}$. We denote the diagonally opposite vertices of A, B, C, D by A^*, B^*, C^*, D^*, respectively, as in Figure 1. We denote this parallelepiped $\mathcal{P}$ by $\mathcal{P}(ABCDA^*B^*C^*D^*)$ so that the quadrilaterals $\mathcal{Q}(AC^*BD^*)$ and $\mathcal{Q}(B^*DA^*C)$ are opposing parallelogram faces. Note that $|A^*B^*| = |AB|$ and $|C^*D^*| = |CD|$. We say that a tetrahedron $\bigtriangledown ABCD$ *is inscribed* in the parallelepiped $\mathcal{P}(ABCDA^*B^*C^*D^*)$. The area of $\mathcal{Q}(AC^*BD^*)$ is denoted by $|\mathcal{Q}(AC^*BD^*)|$. (The tetrahedron $\bigtriangledown A^*B^*C^*D^*$ is the mirror image of $\bigtriangledown ABCD$, and it said to be the *twin* of $\bigtriangledown ABCD$.)

Next is our theorem.

Theorem 1. *Let $\mathcal{P}(ABCDA^*B^*C^*D^*)$ be the parallelepiped that inscribes $\bigtriangledown ABCD$. Then we have*

$$\begin{aligned}|\mathcal{Q}(AC^*BD^*)|^2 &= \mathcal{A}_D^2 + \mathcal{A}_C^2 - 2\mathcal{A}_D\mathcal{A}_C \cos\sphericalangle AB \\ &= \mathcal{A}_B^2 + \mathcal{A}_A^2 - 2\mathcal{A}_B\mathcal{A}_A \cos\sphericalangle CD = |\mathcal{Q}(B^*DA^*C)|^2.\end{aligned}$$

Proof. Let m_C and m_D be lines parallel to the line $\overline{AB}$ through C and D, respectively. Let $C' \in m_C$ and $D' \in m_D$ such that both $\overline{C'A}$ and $\overline{D'A}$ are perpendicular to $\overline{AB}$. Hence, we have $\sphericalangle C'AD' = \sphericalangle AB$. By the law of cosines applied to $\triangle AC'D'$, we have

$$\begin{aligned}|C'D'|^2 &= |C'A|^2 + |D'A|^2 - 2|C'A||D'A| \cos\sphericalangle C'AD' \\ &= |C'A|^2 + |D'A|^2 - 2|C'A||D'A| \cos\sphericalangle AB.\end{aligned}$$

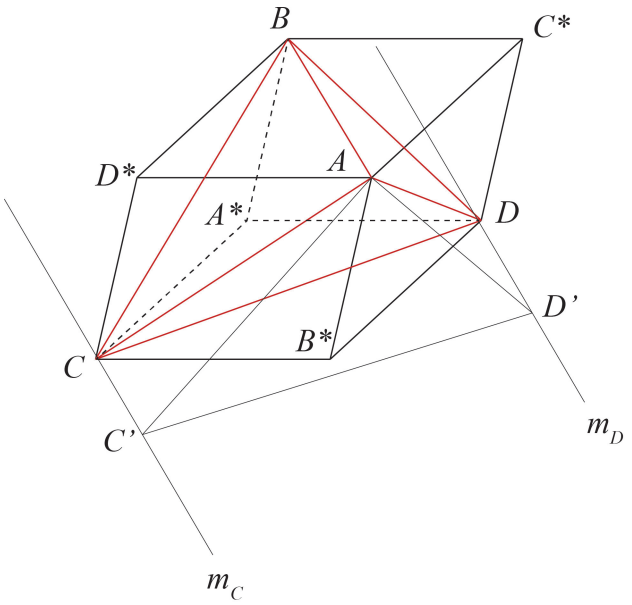


Figure 1: The parallelepiped $\mathcal{P}(ABCDA^*B^*C^*D^*)$, and the notations used in the proof of Theorem 1.

Multiplying both sides by $\frac{1}{4}|AB|^2$, we have

$$\left(\frac{1}{2}|AB||C'D'|\right)^2 = \left(\frac{1}{2}|AB||C'A|\right)^2 + \left(\frac{1}{2}|AB||D'A|\right)^2 - 2\left(\frac{1}{2}|AB||C'A|\right)\left(\frac{1}{2}|AB||D'A|\right)\cos\sphericalangle AB.$$

Since $|C'A|$ and $|D'A|$ are the altitudes of $\triangle ABC$ and $\triangle ABD$ from C and D, respectively, we have

$$\frac{1}{2}|AB||C'A| = \mathcal{A}_D, \qquad \text{and} \qquad \frac{1}{2}|AB||D'A| = \mathcal{A}_C.$$

Hence, this equation becomes

$$\left(\frac{1}{2}|AB||C'D'|\right)^2 = \mathcal{A}_D^2 + \mathcal{A}_C^2 - 2\mathcal{A}_D\mathcal{A}_C\cos\sphericalangle AB.$$

Next, $|AB| = |A^*B^*|$. Since the line $\overline{AB}$ is normal to the plane $AC'D'$, we have $|Q(AC^*BD^*)| = \frac{1}{2}|AB|\,|C'D'| = |\mathcal{Q}(B^*DA^*C)|$. Therefore,

$$|Q(AC^*BD^*)|^2 = \mathcal{A}_D^2 + \mathcal{A}_C^2 - 2\mathcal{A}_D\mathcal{A}_C\cos\sphericalangle AB = |\mathcal{Q}(B^*DA^*C)|^2.$$

Similarly, we have $|\mathcal{Q}(B^*DA^*C)|^2 = \mathcal{A}_B^2 + \mathcal{A}_A^2 - 2\mathcal{A}_B\mathcal{A}_A\cos\sphericalangle CD = |Q(AC^*BD^*)|^2$. This completes the proof. □

Alternate Proof of Theorem 1. Let $\overrightarrow{DA} = \vec{a}$, $\overrightarrow{DB} = \vec{b}$, $\overrightarrow{DC} = \vec{c}$, $\overrightarrow{DA^*} = \vec{\alpha}$, $\overrightarrow{DB^*} = \vec{\beta}$, and $\overrightarrow{DC^*} = \vec{\gamma}$. Then

$$\begin{cases} \vec{\alpha} + \vec{\beta} = \vec{c}, \\ \vec{\alpha} + \vec{\gamma} = \vec{b}, \\ \vec{\beta} + \vec{\gamma} = \vec{a}. \end{cases}$$

Rearranging these equations, we have

$$\begin{cases} \vec{\alpha} = \frac{1}{2}(-\vec{a} + \vec{b} + \vec{c}), \\ \vec{\beta} = \frac{1}{2}(\vec{a} - \vec{b} + \vec{c}), \\ \vec{\gamma} = \frac{1}{2}(\vec{a} + \vec{b} - \vec{c}). \end{cases}$$

Hence,

$$\begin{aligned} \vec{\beta} \times \vec{\alpha} &= \frac{1}{4}(\vec{a} - \vec{b} + \vec{c}) \times (-\vec{a} + \vec{b} + \vec{c}) \\ &= \frac{1}{4}\Big(\vec{a} \times \vec{b} + \vec{a} \times \vec{c} + \vec{b} \times \vec{a} - \vec{b} \times \vec{c} - \vec{c} \times \vec{a} + \vec{c} \times \vec{b}\Big) = \frac{1}{2}(\vec{a} \times \vec{c} + \vec{c} \times \vec{b}). \end{aligned}$$

Thus, $|\mathcal{Q}(B^*DA^*C)|^2 = |\vec{\beta} \times \vec{\alpha}|^2 = \frac{1}{4}\Big(|\vec{a} \times \vec{c}|^2 + |\vec{c} \times \vec{b}|^2 + 2(\vec{a} \times \vec{c}) \cdot (\vec{c} \times \vec{b})\Big)$. Since vectors $\vec{a} \times \vec{c}$ and $\vec{c} \times \vec{b}$ are both normal vectors to $\triangle ADC$ and $\triangle BDC$, respectively, and since both point outward of $\nabla ABCD$ according to the labeling in Figure 1, the angle between $\vec{a} \times \vec{c}$ and $\vec{c} \times \vec{b}$ is given by $(\pi - \sphericalangle CD)$. Also, since $\mathcal{A}_B = \frac{1}{2}|\vec{a} \times \vec{c}|$, $\mathcal{A}_A = \frac{1}{2}|\vec{c} \times \vec{b}|$, and $\cos(\pi - \sphericalangle CD) = -\cos \sphericalangle CD$, we have

$$\begin{aligned} |\mathcal{Q}(B^*DA^*C)|^2 &= \frac{1}{4}|\vec{a} \times \vec{c}|^2 + \frac{1}{4}|\vec{b} \times \vec{c}|^2 + 2 \cdot \frac{1}{2}|\vec{a} \times \vec{c}| \cdot \frac{1}{2}|\vec{c} \times \vec{b}| \cos(\pi - \sphericalangle CD) \\ &= \mathcal{A}_B^2 + \mathcal{A}_A^2 - 2\mathcal{A}_B\mathcal{A}_A \cos \sphericalangle CD. \end{aligned}$$

Applying this to $\mathcal{Q}(AC^*BD^*)$, we have $|\mathcal{Q}(AC^*BD^*)|^2 = \mathcal{A}_D^2 + \mathcal{A}_C^2 - 2\mathcal{A}_D\mathcal{A}_C \cos \sphericalangle AB$. Since $|\mathcal{Q}(AC^*BD^*)| = |\mathcal{Q}(BD^*AC^*)|$, this proves Theorem 1. □

Corollary 1. *If A', B', C', D' are the orthogonal projection of the vertices A, B, C, D, respectively, of a tetrahedron $\nabla ABCD$ onto a plane parallel to line $\overline{AB}$ and $\overline{CD}$, let $\mathcal{Q}(A'C'B'D')$ be the quadrilateral by joining A', C', B', D' in this order. Let $|\mathcal{Q}(A'C'B'D')|$ be the area of $\mathcal{Q}(A'C'B'D')$. Then $|\mathcal{Q}(A'C'B'D')|^2 = \mathcal{A}_D^2 + \mathcal{A}_C^2 - 2\mathcal{A}_D\mathcal{A}_C \cos \sphericalangle AB$.*

Proof. Without loss of generality, let the plane parallel to lines $\overline{AB}$ and $\overline{CD}$ in this corollary to be the plane Γ containing the line $\overline{AB}$ parallel to the line $\overline{CD}$. Let A', B', C', D' are the orthogonal projection of the vertices A, B, C, D, respectively, of a tetrahedron $\nabla ABCD$ onto Γ. Then $A' = A$ and $B' = B$, and Γ contains the face $Q(AC^*BD^*)$ of the parallelepiped $\mathcal{P}(ABCDA^*B^*C^*D^*)$ that inscribes $\nabla ABCD$. Note that by taking $[AB]$ as the bases of $\triangle C'AB$ and $\triangle D^*AB$, we know that $\triangle C'AB$ and $\triangle D^*AB$ have the same altitude. Hence, $|\triangle C'AB| = |\triangle D^*AB|$. Similarly, $|\triangle D'AB| = |\triangle C^*AB|$. Thus,

$$|\mathcal{Q}(A'C'B'D')| = |\triangle C'AB| + |\triangle D'AB| = |\triangle D^*AB| + |\triangle C^*AB| = |Q(AC^*BD^*)|.$$

By Theorem 1, we have this corollary. □

3 Area of the Parallelogram $Q(AC^*BD^*)$

By solving for $\cos \sphericalangle AB$ in Theorem 1, we have

$$\cos \sphericalangle AB = \frac{\mathcal{A}_D^2 + \mathcal{A}_C^2 - |\mathcal{Q}(AC^*BD^*)|^2}{2\mathcal{A}_D\mathcal{A}_C}.$$

There is another way to find $\cos \sphericalangle AB$ by the spherical law of cosines.

Spherical law of cosines: For a tetrahedron $\nabla ABCD$, we have

$$\cos \sphericalangle AB = \frac{\cos \sphericalangle CAD - \cos \sphericalangle BAC \cos \sphericalangle BAD}{\sin \sphericalangle BAC \sin \sphericalangle BAD}. \quad \text{(See Lemma 5 in [5] for a proof.)}$$

Heron's formula: If $a = |BC|, b = |CA|, c = |AB|$ for a triangle $\triangle ABC$, then the area $|\triangle ABC|$ of $\triangle ABC$ is given by $16|\triangle ABC|^2 = (2bc)^2 - (b^2 + c^2 - a^2)^2$.

Theorem 2. *Let $a = |BC|$, $b = |CA|$, $c = |AB|$, $x = |DA|$, $y = |DB|$, and $z = |DC|$ be the edge lengths of a tetrahedron $\nabla ABCD$. If $\mathcal{P}(ABCDA^*B^*C^*D^*)$ is the parallelepiped that inscribe $\nabla ABCD$, then $16|\mathcal{Q}(AC^*BD^*)|^2 = 4c^2z^2 - \Big[(a^2 + x^2) - (b^2 + y^2)\Big]^2$.*

Proof. By the law of cosines for a triangle applied to the faces of $\nabla ABCD$, we have

$$\cos \sphericalangle CAD = \frac{x^2 + b^2 - z^2}{2xb}, \qquad \cos \sphericalangle BAC = \frac{c^2 + b^2 - a^2}{2cb}, \qquad \cos \sphericalangle BAD = \frac{x^2 + c^2 - y^2}{2xc}.$$

Hence,

$$\cos \sphericalangle CAD - \cos \sphericalangle BAC \cos \sphericalangle BAD = \frac{2c^2(x^2 + b^2 - z^2) - (c^2 + b^2 - a^2)(x^2 + c^2 - y^2)}{4xbc^2},$$

and

$$\begin{aligned}
\sin \sphericalangle BAC \sin \sphericalangle BAD &= \sqrt{(1 - \cos^2 \sphericalangle BAC)(1 - \cos^2 \sphericalangle BAD)} \\
&= \frac{\sqrt{\Big\{(2cb)^2 - (c^2 + b^2 - a^2)^2\Big\}\Big\{(2xc)^2 - (x^2 + c^2 - y^2)^2\Big\}}}{4xbc^2} \\
&= \frac{16\mathcal{A}_D\mathcal{A}_C}{4xbc^2}
\end{aligned}$$

by Heron's formula. By the spherical law of cosines, we have

$$\begin{aligned}
\cos \sphericalangle AB &= \frac{\cos \sphericalangle CAD - \cos \sphericalangle BAC \cos \sphericalangle BAD}{\sin \sphericalangle BAC \sin \sphericalangle BAD} \\
&= \frac{2c^2(x^2 + b^2 - z^2) - (c^2 + b^2 - a^2)(x^2 + c^2 - y^2)}{16\mathcal{A}_D\mathcal{A}_C}.
\end{aligned}$$

Moreover,

$$\begin{aligned}
&2c^2(x^2 + b^2 - z^2) - (c^2 + b^2 - a^2)(x^2 + c^2 - y^2) \\
&= \frac{1}{2}(2b^2c^2 + 2c^2a^2 - c^4) + \frac{1}{2}(2y^2c^2 + 2c^2x^2 - c^4) + b^2y^2 + a^2x^2 - 2c^2z^2 - a^2y^2 - b^2x^2 \\
&= \frac{1}{2}(2a^2b^2 + 2b^2c^2 + 2c^2a^2 - a^4 - b^4 - c^4) + \frac{1}{2}(2x^2y^2 + 2y^2c^2 + 2c^2x^2 - x^4 - y^4 - c^4) \\
&\qquad - a^2b^2 - x^2y^2 + \frac{1}{2}a^4 + \frac{1}{2}b^4 + \frac{1}{2}x^4 + \frac{1}{2}y^4 + b^2y^2 + a^2x^2 - 2c^2z^2 - a^2y^2 - b^2x^2 \\
&= \frac{16}{2}\Big\{\mathcal{A}_D^2 + \mathcal{A}_C^2 - \frac{1}{16}\Big[4c^2z^2 + 2(a^2y^2 + b^2x^2 + a^2b^2 + x^2y^2) - (a^2 + x^2)^2 - (b^2 + y^2)^2\Big]\Big\} \\
&= \frac{16}{2}\Big\{\mathcal{A}_D^2 + \mathcal{A}_C^2 - \frac{1}{16}\Big[4c^2z^2 + 2(a^2 + x^2)(b^2 + y^2) - (a^2 + x^2)^2 - (b^2 + y^2)^2\Big]\Big\} \\
&= \frac{16}{2}\Big\{\mathcal{A}_D^2 + \mathcal{A}_C^2 - \frac{1}{16}\Big[4c^2z^2 - \Big[(a^2 + x^2) - (b^2 + y^2)\Big]^2\Big]\Big\}.
\end{aligned}$$

Substituting this into the above equation, we have

$$\cos \sphericalangle AB = \frac{\mathcal{A}_D^2 + \mathcal{A}_C^2 - \frac{1}{16}\{4c^2z^2 - [(a^2+x^2)-(b^2+y^2)]^2\}}{2\mathcal{A}_D\mathcal{A}_C}.$$

Since $\cos \sphericalangle AB = \dfrac{\mathcal{A}_D^2 + \mathcal{A}_C^2 - |\mathcal{Q}(AC^*BD^*)|^2}{2\mathcal{A}_D\mathcal{A}_C}$ by Theorem 1, we have

$$16|\mathcal{Q}(AC^*BD^*)|^2 = 4c^2z^2 - [(a^2+x^2)-(b^2+y^2)]^2. \qquad \square$$

Remark 1. Like the alternate proof of Theorem 1, Theorem 2 can also be proved by vectors.

4 Proving the Law of Cosines for a Tetrahedron

We will give a new proof of the law of cosine for a tetrahedron using Theorem 1 and the next lemma. Even though the next lemma was proved in [1] and [7], we give alternate proof using Theorem 2.

Lemma 1 (Yetter)**.** *Let $\mathcal{P}(ABCDA^*B^*C^*D^*)$ be the parallelepiped that inscribes $\nabla ABCD$. Then we have $\mathcal{A}_D^2 + \mathcal{A}_C^2 + \mathcal{A}_B^2 + \mathcal{A}_A^2 = |Q(AC^*BD^*)|^2 + |Q(AD^*CB^*)|^2 + |Q(AB^*DC^*)|^2$.*

Proof. By Heron's formula and Theorem 2, we have

$$\begin{aligned}
&16\mathcal{A}_D^2 + 16\mathcal{A}_C^2 + 16\mathcal{A}_B^2 + 16\mathcal{A}_A^2 \\
&= \{2a^2b^2 + 2b^2c^2 + 2c^2a^2 - a^4 - b^4 - c^4\} + \{2x^2y^2 + 2y^2c^2 + 2c^2x^2 - x^4 - y^4 - c^4\} \\
&\quad + \{2x^2z^2 + 2b^2x^2 + 2z^2b^2 - x^4 - b^4 - z^4\} + \{2a^2y^2 + 2y^2z^2 + 2z^2a^2 - a^4 - y^4 - z^4\} \\
&= \{4c^2z^2 + 2(a^2y^2 + b^2x^2 + a^2b^2 + x^2y^2) - a^4 - x^4 - b^4 - y^4 - 2a^2x^2 - 2b^2y^2\} \\
&\quad + \{4b^2y^2 + 2(a^2z^2 + c^2x^2 + a^2c^2 + x^2z^2) - a^4 - x^4 - c^4 - z^4 - 2a^2x^2 - 2c^2z^2\} \\
&\quad + \{4a^2x^2 + 2(c^2y^2 + b^2z^2 + b^2c^2 + y^2z^2) - b^4 - z^4 - c^4 - y^4 - 2c^2z^2 - 2b^2y^2\} \\
&= \left\{4c^2z^2 - [(a^2+x^2)-(b^2+y^2)]^2\right\} + \left\{4b^2y^2 - [(a^2+x^2)-(c^2+z^2)]^2\right\} \\
&\quad + \left\{4a^2x^2 - [(c^2+z^2)-(b^2+y^2)]^2\right\} \\
&= 16|Q(AC^*BD^*)|^2 + 16|Q(AD^*CB^*)|^2 + 16|Q(AB^*DC^*)|^2.
\end{aligned}$$

This proves this lemma. $\square$

Law of cosines for a tetrahedron. For a tetrahedron $\nabla ABCD$, we have

$$\mathcal{A}_A^2 = \mathcal{A}_B^2 + \mathcal{A}_C^2 + \mathcal{A}_D^2 - 2\mathcal{A}_B\mathcal{A}_C \cos \sphericalangle AD - 2\mathcal{A}_C\mathcal{A}_D \cos \sphericalangle AB - 2\mathcal{A}_D\mathcal{A}_B \cos \sphericalangle AC.$$

Proof. By Theorem 1, we have $|Q(AC^*BD^*)|^2 = \mathcal{A}_D^2 + \mathcal{A}_C^2 - 2\mathcal{A}_D\mathcal{A}_C \cos \sphericalangle AB$, $|Q(AD^*CB^*)|^2 = \mathcal{A}_D^2 + \mathcal{A}_B^2 - 2\mathcal{A}_D\mathcal{A}_B \cos \sphericalangle AC$, and $|Q(AB^*DC^*)|^2 = \mathcal{A}_B^2 + \mathcal{A}_C^2 - 2\mathcal{A}_B\mathcal{A}_C \cos \sphericalangle AD$. Substitution these into the equation in Lemma 1, we have

$$\begin{aligned}
&\mathcal{A}_D^2 + \mathcal{A}_C^2 + \mathcal{A}_B^2 + \mathcal{A}_A^2 \\
&\qquad = 2\mathcal{A}_D^2 + 2\mathcal{A}_B^2 + 2\mathcal{A}_C^2 - 2\mathcal{A}_D\mathcal{A}_C \cos \sphericalangle AB - 2\mathcal{A}_D\mathcal{A}_B \cos \sphericalangle AC - 2\mathcal{A}_B\mathcal{A}_C \cos \sphericalangle AD.
\end{aligned}$$

Simplifying this equation, we have the law of cosine for a tetrahedra. $\square$

5 Epilogue

For your information, there is a theorem for a tetrahedron that resembles to the Law of sines for a triangle.

Theorem 3 (See Problem 300 and its solution in [6])**.** *Let $\mathcal{V}$ be the volume of a tetrahedron $\bigtriangledown ABCD$. Then*

$$\frac{|AB||CD|}{\sin\sphericalangle AB \sin\sphericalangle CD} = \frac{|AC||BD|}{\sin\sphericalangle AC \sin\sphericalangle BD} = \frac{|AD||BC|}{\sin\sphericalangle AD \sin\sphericalangle BC} = \frac{4\mathcal{A}_A\mathcal{A}_B\mathcal{A}_C\mathcal{A}_D}{9\mathcal{V}^2}.$$

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