Journal for Geometry and Graphics
Volume 30 (2026), No. 2, 229–236

On the Inscribed Balls of Bisected d-Simplices

Sergey Korotov[1], Michal Křìžek[2]

[1]*Mälardalen University, Västeras, Sweden*
sergey.korotov@mdu.se

[2]*Institute of Mathematics of CAS, Prague, Czech Republic*
krizek@math.cas.cz

Abstract. We bisect a d-dimensional simplex S into two d-subsimplices S_1 and S_2, and prove that $r < r_1 + r_2$, where r, r_1, r_2 are inradii of S, S_1, S_2, respectively. If a given edge is bisected by its midpoint then even two-sided bounds $r_1 < r < 2r_1$ and $r_2 < r < 2r_2$ hold. All these inequalities can be used in the face-to-face longest-edge bisection algorithm for refinement of simplicial partitions of bounded polytopic domains for controlling the shape and size of subsimplices produced.

Key Words: d-simplex, Inscribed ball, Inradius, Bisection

MSC 2020: 52A20 (primary), 52B11, 65N30

1 Introduction

Various bisection-type techniques, when geometric elements like simplices filling the domains of interest are split into two (or more) smaller parts, are very popular in numerical community due to its algorithmic and implementation simplicity and universality. For example, the areas where these methods are widely used include estimation of roots of nonlinear equations (and systems of such equations), mesh generation and adaptivity for finite element methods and computer graphics [2, 5, 6, 9, 10]. In all these applications, it is very important to be able to monitor (or even control) the most critical qualitative characteritics of partitions/meshes generated (size, angles, degeneracy).

A *simplex* S in $\mathbf{R}^d, d \in \{2, 3, \dots\}$, is the convex hull of $d+1$ vertices A_1, A_2, …, A_{d+1} that do not belong to the same $(d-1)$-dimensional hyperplane. The convex hull of d vertices A_1, …, A_{j-1}, A_{j+1}, …, A_{d+1} is called a *facet* of S for any $j \in \{1, \dots, d+1\}$. The union of $d+1$ facets of S (i.e. sides for $d = 2$) is denoted by ∂S. Let $\mathrm{meas}_d\, S$ denote the d-dimensional measure of the simplex S (i.e. area of a triangle $T = S$ in the two-dimensional case), and let $\mathrm{meas}_{d-1}\, \partial S$ stand for the surface measure of S (i.e. perimeter of $T = S$). Then the radius r of the inscribed ball to S is given by the following formula, see e.g. [2, p. 10],

$$r = \frac{d\ \mathrm{meas}_d\ S}{\mathrm{meas}_{d-1}\ \partial S}. \tag{1}$$

ISSN 1433-8157/ © 2026 by the author(s), licensed under CC BY-SA 4.0.

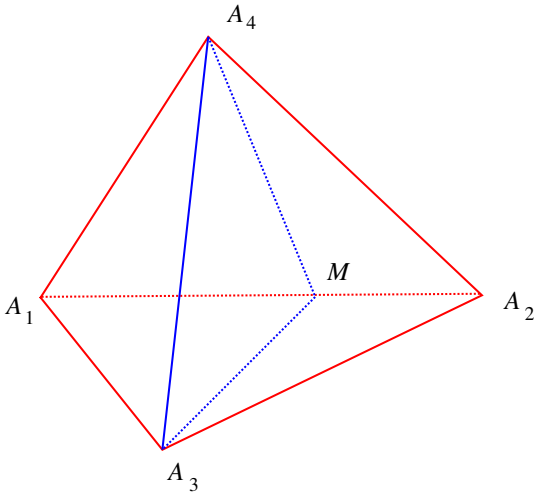


Figure 1: Bisection of a tetrahedron $A_1A_2A_3A_4$ into two subtetrahedra $A_1A_3A_4M$ and $A_2A_3A_4M$, where M lies in the interior of the edge A_1A_2.

We will call r the *inradius*.

The value of $\mathrm{meas}_d\, S$ in (1) can be calculated by the following theorem proved in [1].

Theorem 1. *Let S be a d-simplex with vertices A_1, A_2, $\dots A_{d+1} \in \mathbf{R}^d$. Then*

$$\mathrm{meas}_d\, S = \frac{1}{d!}|\det(A_1 - A_{d+1}, A_2 - A_{d+1}, \dots, A_d - A_{d+1})|.$$

If $d = 3$, then $\mathrm{meas}_3\, S$ stands for the classical volume of a tetrahedron S. Its surface measure $\mathrm{meas}_2\, \partial S$ is just the sum of areas of four triangular faces on the surface of S. For an arbitrary spece dimension $d > 1$ see Remark 3.

Choose a point M in the interior of any edge of S. Without loss of generality, we can assume that M lies in the interior of the edge A_1A_2. The division of S by the $(d-1)$-dimensional hyperplane containing the points M and A_3, $\dots$, A_{d+1} is called the *bisection* of S into two subsimplices A_1, A_3, $\dots$, $A_{d+1}M$ and A_2, A_3, $\dots$, $A_{d+1}M$, see Figure 1 for $d = 3$. Their d-dimensional measures do not have to be the same, in general. We will assume that they are the same only in Lemma 1 and Theorem 4.

2 Two-dimensional Case

To show the main idea of this paper, we first deal with case of $d = 2$.

Theorem 2. *Let T be an arbitrary triangle with inradius r. Bisect T into two subtriangles T_1 and T_2 (whose interiors are disjoint) such that $T = T_1 \cup T_2$. Then*

$$r < r_1 + r_2, \tag{2}$$

where r_i is the inradius of T_i for $i = 1, 2$.

Proof. Let $V = \mathrm{meas}_2\, T$, $V_i = \mathrm{meas}_2\, T_i$ for $i = 1, 2$, and let a, b, c be sides of T. Then $\mathrm{meas}_1\, \partial T = a + b + c$. Suppose that the side $a = a_1 + a_2$ is bisected by a straight line segment

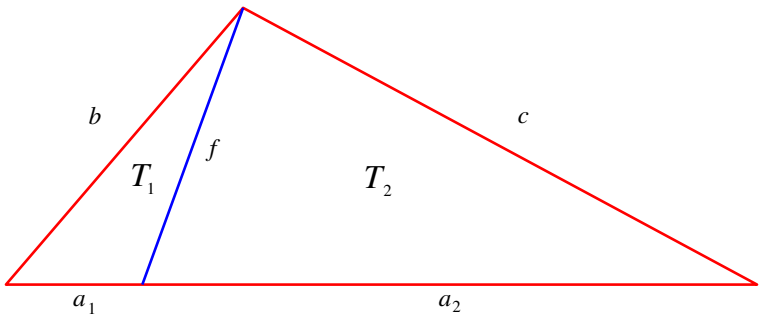


Figure 2: Denotation for the triangle T.

f (see Figure 2). By the triangle inequalities $f < a_2 + c$ and $f < a_1 + b$, we get

$$\frac{1}{a+b+c} = \frac{1}{a_1+b+a_2+c} < \frac{1}{a_1+b+f} \quad \text{and} \quad \frac{1}{a+b+c} < \frac{1}{a_2+c+f}.$$

From this we immediately find that

$$\frac{V_1}{a+b+c} < \frac{V_1}{a_1+b+f}, \tag{3}$$

$$\frac{V_2}{a+b+c} < \frac{V_2}{a_2+c+f}. \tag{4}$$

Summing (3)–(4) and multiplying by $d = 2$, we arrive by (1) at

$$r = \frac{2V}{a+b+c} < \frac{2V_1}{a_1+b+f} + \frac{2V_2}{a_2+c+f} = r_1 + r_2.$$

Hence, (2) is valid. □

Remark 1. In fact, we have a two-sided estimate for $i = 1, 2$

$$r_i < r < r_1 + r_2,$$

where the both lower and upper bounds are attainable for $r_{3-i} \to 0$.

If the point M is the midpoint of A_1A_2, then we can get estimates sharper than (2):

Lemma 1. *Let the side a of a triangle T be halved. Then*

$$\frac{1}{2}r < r_1 \quad \textit{and} \quad \frac{1}{2}r < r_2.$$

Proof. From the relation $V_1 = V_2 = \frac{1}{2}V$, (1), and (3) we obtain

$$\frac{1}{2}r = \frac{2V_1}{a+b+c} < \frac{2V_1}{a_1+b+f} = r_1.$$

Similarily, from (4), we come to $\frac{1}{2}r < r_2$. □

Remark 2. For n-section of a triangle (i.e. the splitting into n subtriangles, see e.g. [8]) we can use the induction to get $r < r_1 + r_2 + \cdots + r_n$.

3 Generalization to a d-simplex

Now we will generalize Theorem 2 to an arbitrary space dimension $d > 1$. First, we present and prove an analogue of the triangle inequality for the case of d-simplices.

Lemma 2. *Let S be an arbitrary d-simplex whose measures of $(d-1)$-dimensional facets are $b_1, b_2, \ldots, b_{d+1}$. Then for any $j \in \{1, 2, \ldots, d+1\}$ we have*

$$b_j < \sum_{k \neq j} b_k. \tag{5}$$

Proof. Let S be a d-simplex and let $j \in \{1, 2, \ldots, d+1\}$ be arbitrary. Let v_j be the altitude of S corresponding to the base with measure b_j, see [2, p. 9]. Then the d-dimensional measure of a d-simplex S is

$$V = \operatorname{meas}_d S = \frac{v_j}{d} b_j. \tag{6}$$

From this and (1) we get

$$\sum_{k=1}^{d+1} b_k = \frac{d}{r} V = \frac{v_j}{r} b_j,$$

where r is the inradius of S. Hence,

$$\sum_{k \neq j}^{d+1} b_k = \Big(\frac{v_j}{r} - 1 \Big) b_j > b_j,$$

since the altitude v_j is greater than the diameter of the inscribed ball, i.e. $v_j > 2r$. □

Remark 3. Let $j \in \{1, 2, \ldots, d+1\}$ be fixed and let the facet opposite to the vertex A_j of S be contained in the hyperplane $\alpha_1 x_1 + \cdots + \alpha_d x_d + \beta = 0$. Denote by $\|\alpha\|$ the Euclidean norm of the vector $\alpha = (\alpha_1, \ldots, \alpha_d)$ and let $\cdot$ denote the scalar product. Then the distance of A_j from this hyperplane is given by

$$v_j = \frac{\alpha \cdot A_j + \beta}{\|\alpha\|}.$$

Now the surface measure b_j can be calculated from (1) and (6).

Theorem 3. *Let S be an arbitrary d-simplex with inradius r. Bisect S into two d-subsimplices S_1 and S_2 (whose interiors are disjoint) such that $S = S_1 \cup S_2$. Then*

$$r < r_1 + r_2, \tag{7}$$

where r_i is the inradius of S_i for $i = 1, 2$.

Proof. Let $d > 1$ be an arbitrary integer. (The case $d = 2$ was already proved in Theorem 2 to illustrate the main idea.) Suppose that we bisect a d-simplex S into two d-subsimplices S_1 and S_2 by an interior $(d-1)$-dimensional simplicial facet whose measure is f. Let $V = \operatorname{meas}_d S$, $V_i = \operatorname{meas}_d S_i$ for $i = 1, 2$, and let $b_1, b_2, \ldots, b_{d+1}$ be $(d-1)$-dimensional measures of $d+1$ facets of S. Then $\operatorname{meas}_{d-1} \partial S = b_1 + b_2 + \cdots + b_{d+1}$. Without loss of generality, we may assume that the two d-subsimplices S_1 and S_2 have facets with measures $b_1, f, p_3, \ldots, p_{d+1}$ and $b_2, f, q_3, \ldots, q_{d+1}$, respectively, where

$$b_j = p_j + q_j \quad \text{for } j > 2$$

(see Figure 3 for $d = 3$). From this and Lemma 2 we get

$$f < b_2 + \sum_{k>2} q_k \qquad \text{and} \qquad f < b_1 + \sum_{k>2} p_k,$$

which yields

$$\Big(\sum_{k=1}^{d+1} b_k\Big)^{-1} = \Big(b_1 + b_2 + \sum_{k>2}(p_k + q_k)\Big)^{-1} < \Big(b_1 + f + \sum_{k>2} p_k\Big)^{-1},$$

and similarly

$$\Big(\sum_{k=1}^{d+1} b_k\Big)^{-1} = \Big(b_2 + f + \sum_{k>2} q_k\Big)^{-1}.$$

From this we find that

$$V_1\Big(\sum_{k=1}^{d+1} b_k\Big)^{-1} < V_1\Big(b_1 + f + \sum_{k>2} p_k\Big)^{-1}, \tag{8}$$

$$V_2\Big(\sum_{k=1}^{d+1} b_k\Big)^{-1} < V_2\Big(b_2 + f + \sum_{k>2} q_k\Big)^{-1}. \tag{9}$$

Summing (8)–(9) and multiplying by d, we arrive by (1) at

$$r = dV\Big(\sum_{k=1}^{d+1} b_k\Big)^{-1} < dV_1\Big(b_1 + f + \sum_{k>2} p_k\Big)^{-1} + dV_2\Big(b_2 + f + \sum_{k>2} q_k\Big)^{-1} = r_1 + r_2,$$

where $V = V_1 + V_2$. Hence, (7) is valid. □

Remark 4. From (1) and (6) we also see that

$$\frac{1}{r} = \frac{1}{v_1} + \frac{1}{v_2} + \cdots + \frac{1}{v_{d+1}}.$$

4 Applications

First, we generalize Lemma 1 to an arbitrary space dimension $d > 1$.

Theorem 4. *Let M be the midpoint of the edge A_1A_2 of a d-simplex S. Suppose that S was bisected by a hyperplane passing through M and $A_3, A_4, \ldots, A_{d+1}$. Then for $S = S_1 \cup S_2$ we get*

$$\frac{1}{2}r < r_1 \quad \textit{and} \quad \frac{1}{2}r < r_2.$$

Proof. From the relation $V_1 = V_2 = \frac{1}{2}V$, (1), and (8), we obtain

$$\frac{1}{2}r = dV_1\Big(\sum_{k=1}^{d+1} b_k\Big)^{-1} < dV_1\Big(b_1 + f + \sum_{k>2} p_k\Big)^{-1} = r_1.$$

The inequality $\frac{1}{2}r < r_2$ can be proved similarly from (9). □

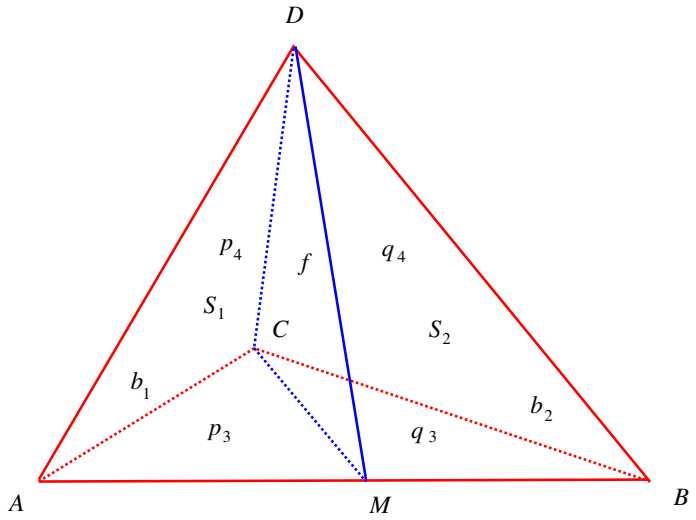


Figure 3: Denotation for areas of triangles $b_1 = |ACD|$, $b_2 = |BCD|$, $f = |CDM|$, $p_3 = |ACM|$, $q_3 = |BCM|$, $p_4 = |ADM|$, $q_4 = |BDM|$, where $|\cdot|$ stands for the two-dimensional measure. Since $b_3 = p_3 + q_3$ and $b_4 = p_4 + q_4$, the inequality (5) holds.

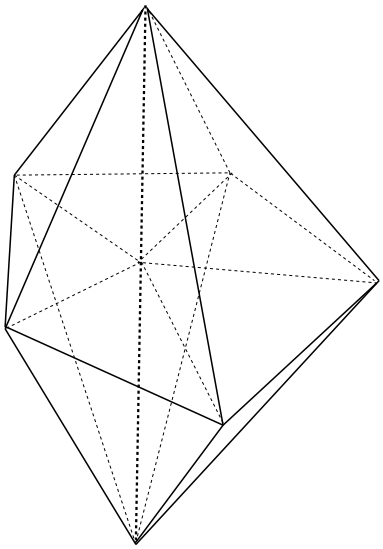


Figure 4: Face-to-face bisection of tetrahedra sharing the longest (vertical) edge.

Assume now that $\overline{\Omega} \subset \mathbf{R}^d$ is a closed domain (i.e. the closure of a domain). If its boundary $\partial\overline{\Omega}$ is contained in a finite number of $(d-1)$-dimensional hyperplanes, we say that $\overline{\Omega}$ is *polytopic*.

Next we define a *simplicial partition* $\mathcal{T}_h$ over a bounded closed polytopic domain $\overline{\Omega} \subset \mathbf{R}^d$ as follows. We subdivide $\overline{\Omega}$ into a finite number of simplices (called *elements*), so that their union is $\overline{\Omega}$, any two simplices have disjoint interiors and any facet of any simplex is a facet of another simplex from the partition or belongs to the boundary $\partial\overline{\Omega}$. The maximal diameter of all elements $S \in \mathcal{T}_h$ is the so-called *discretization parameter* h.

The set $\mathcal{F} = \{\mathcal{T}_h\}_{h\to 0}$ is called a *family of partitions* if for any $\varepsilon > 0$ there exists $\mathcal{T}_h \in \mathcal{F}$ with $h < \varepsilon$.

One step of the face-to-face longest-edge bisection algorithm (see [7]) is defined as follows:

(I) choose the longest edge ℓ in a given face-to-face simplicial partition;
(II) bisect each simplex S sharing ℓ by a hyperplane passing through the midpoint of ℓ and the vertices of S that do not belong to ℓ (see Figure 4).

Theorem 4, in fact, allows to estimate the size of d-simplices at each bisection step. Kearfot [6] proved that the discretization parameter h tends to zero for the bisection algorithm. However, the simplices produced can degenerate which may lead to various numerical problems, see e.g. [2]. In [3, 7], we proved a regularity of triangulations produced by the longest-edge bisection algorithm. This means that the triangles do not degenerate to needles when $h \to 0$. A numerical regularity of tetrahedral partitions generated by the longest-edge algorithm is illustrated in [4].

Acknowledgement

This paper was supported by grant no. 24-10586S of the Grant Agency of the Czech Republic and project RVO 67985840 of the Czech Republic.

References

[1] A. Ben-Israel: *A volume associated with $m \times n$ matrices*. Linear Algebra Appl. **167**, 87–111, 1992. doi: 10.1016/0024-3795(92)90340-G.

[2] J. Brandts, S. Korotov, and M. Křížek: *Simplicial Partitions with Applications to the Finite Element Method*. Springer-Verlag, Berlin, 2020.

[3] A. Hannukainen, S. Korotov, and M. Křížek: *On global and local mesh refinements by a generalized conforming bisection algorithm*. J. Comput. Appl. Math. **235**(2), 419–436, 2010. doi: 10.1016/j.cam.2010.05.046.

[4] A. Hannukainen, S. Korotov, and M. Křížek: *On numerical regularity of the face-to-face longest-edge bisection algorithm for tetrahedral partitions*. Sci. Comput. Programming **90**, 34–41, 2014. doi: 10.1016/j.scico.2013.05.002.

[5] R. Horst: *On generalized bisection of n-simplices*. Math. Comp. **66**, 691–698, 1997. doi: 10.1090/S0025-5718-97-00809-0.

[6] R. B. Kearfott: *A proof of convergence and an error bound for the method of bisection in R^n*. Math. Comp. **32**(144), 1147–1153, 1978. doi: 10.2307/2006341.

[7] S. Korotov, M. Křížek, and A. Kropáč: *Strong regularity of a family of face-to-face partitions generated by the longest-edge bisection algorithm*. Comput. Math. Math. Phys. **48**, 1687–1698, 2008. doi: 10.1134/S0965542508090170.

[8] S. Korotov, Á. Plaza, and J. Suárez: *Longest-edge n-section algorithms: Properties and open problems*. J. Comput. Appl. Math. **293**, 139–146, 2016. doi: 10.1016/j.cam.2015.03.046.

[9] M. C. Rivara: *Algorithms for refining triangular grids suitable for adaptive and multigrid techniques*. Internat. J. Numer. Methods Engrg. **20**(4), 745–756, 1984. doi: 10.1002/nme.1620200412.

[10] M. Stynes: *Subdivision of simplexes—is bisection best?* Irish Math. Soc. Newslett. **8**, 38–44, 1983.

Received August 9, 2026; final form September 3, 2026.