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Center of the Elliptic Intersection of a Paraboloid and a Plane

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Abstract. In this paper, the locus of the center of the ellipse formed by cutting a paraboloid with a plane was considered geometrically when the plane tilts. The locus of the center of the ellipse was considered in the cases where the length of the upper line segment is constant, the length of the segment of the parabola is constant, the top surface area is constant, the perimeter of the upper surface is constant, the lateral surface area is constant, and the volume is constant. Here, the behavior of the center of the top surface of the paraboloid cut in a plane can be applicable to describe the motion of conduction electrons with a long scattering relaxation time in solid substances, which has been consistently studied in the field of solid-state physics, since the relationship between an electron's momentum and energy in a material can be often approximated by a paraboloid.

Key Words: Parabola, Paraboloid, Center of ellipse

MSC 2020: 51N20 (primary), 51M04, 51M09

1 Introduction

When a surface formed by a paraboloid is cut by a plane in Euclidean space, the (edge) shape of the cross section can be a circle, an ellipse, or a parabola, depending on the angle of the plane to be cut [1, 3, 6]. In this study, the midpoint of a line segment formed in two dimensions when a straight line intersects a parabola at two points, and the center of an ellipsoid formed in three dimensions when a plane intersects a paraboloid were considered. Specifically, the following loci were considered: In two-dimensional parabola, the midpoint of the upper line segment in the cases where the length of the upper line segment is constant; and the length of the arc of the parabola is constant, respectively. In three-dimensional paraboloid, the top surface area is constant; the perimeter of the upper surface is constant; the lateral surface area is constant; and the volume is constant, respectively. In particular, the locus of the center of the ellipse on the top surface when the volume of the paraboloid is held constant

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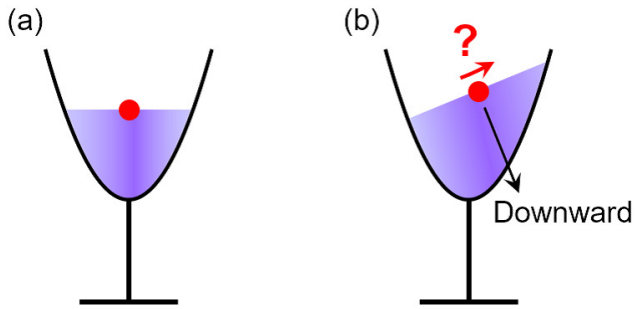


Figure 1: "Liquid" in a "wine glass" when standing upright (a) and when tilted (b). The red point indicates the center of the top surface's ellipse.

corresponds to the locus of the center of the top surface of the "liquid"[1] in the "wine glass" as it tilts (Fig. 1). The red dot represents the center of the ellipse on the top surface. The upper surface of the "liquid" corresponds to the surface when cut by a plane perpendicular to the axis of the paraboloid (Fig. 1 (a)). The shape of the upper surface is a circle. The height of the center depends on the volume of the "liquid" and the shape of the paraboloid "wine glass". The horizontal position of the center is on the axis of the paraboloid. If the normal "down" direction changes while keeping the volume of the "liquid"constant as in case (a), i.e., the plane cutting the paraboloid is not perpendicular to the central axis of paraboloid, the top surface becomes an ellipse. As seen in Fig. 1(b), the height and horizontal position of the center of the top surface is different from that in Fig. 1(a).

Since the relationship between an electron's momentum and energy in a material can be often approximated by a paraboloid, the behavior of the center of the top surface of the paraboloid cut in a plane can be applicable to describe the motion of conduction electrons with a long scattering relaxation time in solid substances, which has been consistently studied in the field of solid-state physics [4]. Therefore, the behavior of the center of the top surface of a paraboloid cut in a plane is applicable not only to the displacement of the center of the top surface of a "liquid" in a "wine glass", but also to approximately describe the motion of conduction electrons in materials such as semiconductors. And also, the behavior of the center of the top surface of a right circular cone cut by a plane, which can be applicable to the motion of conduction electrons in single-layer graphene in which the behavior of conduction electrons is described by a right circular cone (Dirac cone) [2], has been described elsewhere [5].

2 Center of the Ellipse Cut by a Plane at Paraboloid

This section discusses cases where an explicit solution exists.

2.1 Case Where the Length of the Upper Line Segment is Constant

It is considered in two-dimensional geometry as shown in Fig. 2. Determine the locus of point D when the upper line segment $|AB|$ on the parabola.

[1] In this study, a liquid is defined as a substance which moves downward under the force of gravity and changes shape to match the container. The physical concepts of such as surface tension, wettability, and viscosity are ignored.

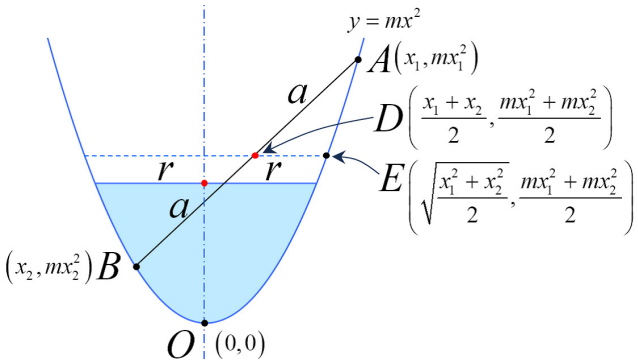


Figure 2: The locus of the center point of the upper side of a parabola.

Theorem 2.1. *The locus of point D is expressed as*

$$Y = mX^2 + \frac{mr^2}{1+4m^2X^2}. \tag{1}$$

Proof. The length of $|AB|$ is expressed as

$$|AB| = \sqrt{(mx_1^2 - mx_2^2)^2 + (x_1 - x_2)^2} = 2r. \tag{2}$$

The coordinate of the midpoint D of A and B is (X, Y), then, X and Y are expressed as

$$X = \frac{x_1 + x_2}{2} \tag{3}$$

and

$$Y = \frac{mx_1^2 + mx_2^2}{2}, \tag{4}$$

respectively. Substituting Eqs. (3) and (4) into Eq. (2), we obtain the following relationship;

$$Y = mX^2 + \frac{mr^2}{1+4m^2X^2}. \tag{5}$$

□

It is symmetric about the vertical axis and asymptotically approaches $Y = mX^2$. If $mr > 1/2$, it takes a "W"-shaped form; if $mr < 1/2$, it takes a "U"-shaped form.

For example, when $m = 1$ and $r = 4$, Eq. (5) becomes

$$Y = X^2 + \frac{16}{1+4X^2}, \tag{6}$$

which is shown in Fig. 3. The dashed curve is the original parabola $Y = X^2$.

2.2 Case Where the Top Surface Area is Constant

As shown in Fig. 4, consider a paraboloid of revolution in three dimensions. This surface is defined by rotating a parabola about the z-axis. The top panel of Fig. 4 shows the top view from the positive z-axis direction, and the bottom panel of Fig. 4shows the side view from the negative y-axis direction. The surface generated by the rotation of parabola around the

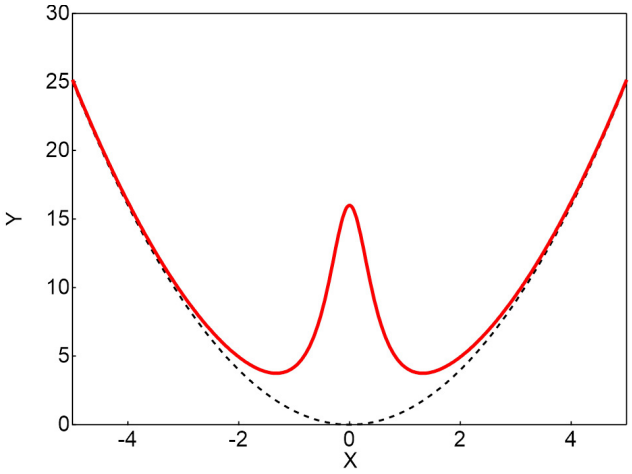


Figure 3: The locus drawn by point D if $m = 1$ and $r = 4$ when the length of the upper line segment is constant.

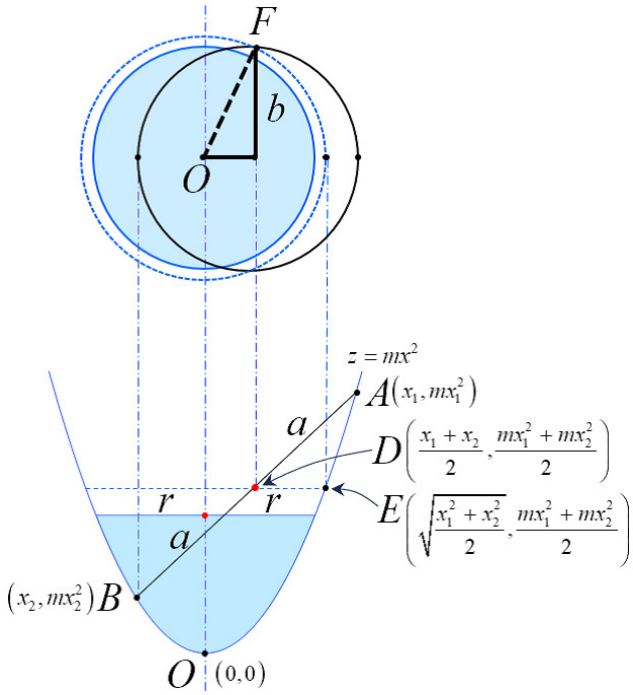


Figure 4: The locus of the center of the top surface and the definition of each point. The drawing emphasizes the changes in the upper surface.

z-axis is a paraboloid. The hatched area in the upper figure of Fig. 4 is the top surface of the parabola with the radius length r, which is cut with a horizontal plane.

Determine the coordinates of the center of the ellipse if the ellipse in the cross section of the parabola moves while the area of this hatched area remains constant ($= \pi r^2$). Let the major axis length of the ellipse be $2a$ and the minor axis length be $2b$. The coordinates of the top (A) and bottom (B) edges of the ellipse on the top surface are expressed as

$$(x_1, mx_1^2) \quad (x_1 > 0) \tag{7}$$

and

$$(x_2, mx_2^2) \quad (x_2 < x_1), \tag{8}$$

respectively. Let the center of the ellipse be point D. A parabolic surface of revolution can be expressed as

$$z = m(x^2 + y^2). \tag{9}$$

Theorem 2.2. *The locus of point D is expressed as*

$$Z = mX^2 + \frac{mr^2}{\sqrt{1 + m^2X^2}}. \tag{10}$$

Proof. Now, consider the lower, side view of Fig. 4, i.e., from symmetry, the value of y is considered as 0. Then, let the coordinate of D be (X, Z), where X and Z are expressed as

$$X = \frac{x_1 + x_2}{2} \tag{11}$$

and

$$Z = \frac{mx_1^2 + mx_2^2}{2}, \tag{12}$$

respectively. Therefore, in the lower in Fig. 4, the length of segment $|AB|$ is expressed as

$$\begin{aligned} |AB| &= \sqrt{(x_1 - x_2)^2 + (mx_1^2 - mx_2^2)^2} \\ &= 2a. \end{aligned} \tag{13}$$

From Eqs. (11) and (12),

$$x_1 = X + \sqrt{\frac{Z}{m} - X^2} \tag{14}$$

and

$$x_2 = X - \sqrt{\frac{Z}{m} - X^2} \tag{15}$$

are obtained. Substituting Eqs. (14) and (15) into Eq. (13), we obtain the following relationship;

$$\begin{aligned} |AB| &= \sqrt{4\left(\frac{Z}{m} - X^2\right) + (mx_1^2 - mx_2^2)^2} \\ &= 2a, \end{aligned} \tag{16}$$

therefore,

$$\sqrt{\left(\frac{Z}{m} - X^2\right)}\sqrt{1 + m^2X^2} = a. \tag{17}$$

In the upper of Fig. 4, F is the point where the minor axis of the ellipse intersects the side of the parabola. The x-coordinate of point F is the same as that of point D, and the z-coordinate of point F is the same as that of D. Therefore, from the theorem of three squares in triangle ODF,

$$\sqrt{\left(\frac{x_1 + x_2}{2}\right)^2 + \left(\frac{mx_1^2 + mx_2^2}{2}\right)^2 + b^2} = |OF| = |OE| = \sqrt{\frac{x_1^2 + x_2^2}{2} + \left(\frac{mx_1^2 + mx_2^2}{2}\right)^2}. \tag{18}$$

Since $b > 0$, Eq. (18) becomes

$$b = \frac{x_1 - x_2}{2}. \tag{19}$$

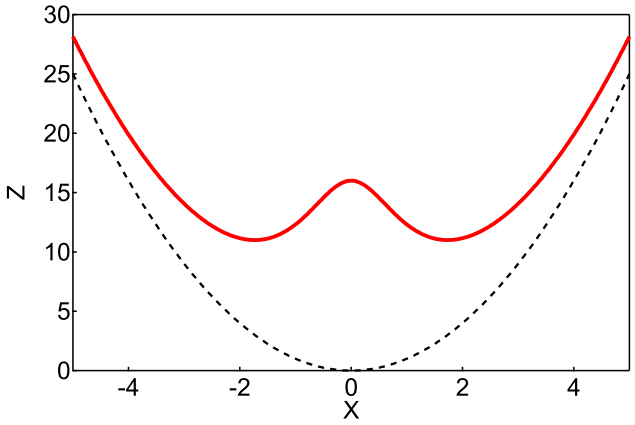


Figure 5: The locus drawn by point D when $m = 1$ and $r = 4$ when the top surface area is constant.

Then, substituting Eqs. (14) and (15) into Eq. (19), we obtain the following relationship;

$$b = \sqrt{\frac{Z}{m} - X^2}. \tag{20}$$

When the area of the ellipse on the top surface is constant;

$$\pi r^2 = \pi ab, \tag{21}$$

from Eqs. (17), (20) and (21), we obtain

$$r^2 = ab = \sqrt{\left(\frac{Z}{m} - X^2\right)}\sqrt{1 + m^2X^2}\sqrt{\left(\frac{Z}{m} - X^2\right)} = \left(\frac{Z}{m} - X^2\right)\sqrt{1 + m^2X^2}. \tag{22}$$

Thus, rearranging the equation, Eq. (10) is obtained. □

It is a symmetrical curve that looks as if only the area near the vertex of the parabola $Z = X^2$ has been raised.

For example, when $m = 1$, Eq. (10) is

$$Z = X^2 + \frac{r^2}{\sqrt{1 + X^2}}. \tag{23}$$

Furthermore, for example, when $r = 4$, Eq. (10) becomes

$$Y = X^2 + \frac{16}{\sqrt{1 + X^2}}. \tag{24}$$

The locus of D is plotted as a solid curve in Fig. 5. The dashed curve is the original parabola $Z = X^2$. It has the shape of a rotated "W", which bulges at the base of the paraboloid and asymptotically approaches the paraboloid in regions where the absolute value of X is large.

2.3 Case Where the Volume is Constant

The diagram is shown in Fig. 4, consider a paraboloid of revolution in three dimensions. Determine the coordinates of the center of the ellipse if the ellipse in the cross section of the paraboloid moves while the volume indicate by hatched area in Fig. 4 remains constant.

Theorem 2.3. *The locus of point D is expressed as*

$$Z = mX^2 + \sqrt{\frac{3mV}{8+3\pi}}, \tag{25}$$

Proof. From Eq. (64), the volume of the region bounded by a paraboloid and a plane is

$$V = \iint_D \Big[(\alpha x + \beta y + \gamma) - m(x^2 + y^2)\Big]\,\mathrm{d}A. \tag{26}$$

Let $u = x - \frac{\alpha}{2m}$ and $v = y - \frac{\beta}{2m}$, then, the integration interval becomes $u^2 + v^2 \leq \frac{\gamma}{m} + \frac{\alpha^2+\beta^2}{4m^2}$, and Eq. (26) becomes

$$V = \iint_D \Big[\gamma + \frac{\alpha^2+\beta^2}{4m} - m(u^2 + v^2)\Big]\,\mathrm{d}A. \tag{27}$$

When expressed in polar coordinates using $u = r\cos\theta$ and $v = r\sin\theta$, the volume is given by

$$\begin{aligned} V &= \int_0^{2\pi}\int_0^R \Big[\gamma + \frac{\alpha^2+\beta^2}{4m} - mr^2\Big] r\,\mathrm{d}r\,\mathrm{d}\theta \\ &= 2\pi\Big[\frac{m}{2}R^4 - \frac{m}{4}R^4\Big] \\ &= \frac{\pi m R^4}{2} \\ &= \frac{\pi}{2m}\left(\gamma + \frac{\alpha^2+\beta^2}{4m}\right)^2. \end{aligned} \tag{28}$$

The center coordinate of the ellipse on the top surface is denoted by

$$\left(\frac{\alpha}{2m}, \frac{\beta}{2m}, \gamma + \frac{\alpha^2+\beta^2}{2m}\right). \tag{29}$$

When the volume is constant, Eq. (28) can be expressed as

$$V = \frac{\pi}{2m}\left(\gamma + \frac{\alpha^2+\beta^2}{4m}\right)^2 \equiv V_0. \tag{30}$$

From $\gamma = \sqrt{\frac{2mV_0}{\pi}} - \frac{\alpha^2+\beta^2}{4m}$, $X = \frac{\alpha}{2m}$, $Y = \frac{\beta}{2m}$, and $Z = \gamma + \frac{\alpha^2+\beta^2}{2m}$,

$$\begin{aligned} Z &= \gamma + \frac{\alpha^2+\beta^2}{2m} \\ &= \sqrt{\frac{2mV_0}{\pi}} - \frac{4m^2X^2 + 4m^2Y^2}{4m} + \frac{4m^2X^2 + 4m^2Y^2}{2m} \\ &= \sqrt{\frac{2mV_0}{\pi}} + m(X^2 + Y^2). \end{aligned} \tag{31}$$

□

When the plane is horizontal ($\alpha = \beta = 0$), Eq. (30) is simplified to

$$V_0 = \frac{\pi}{2m}\gamma^2. \tag{32}$$

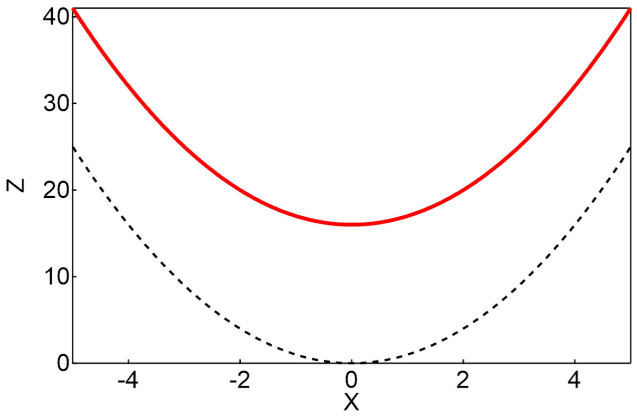


Figure 6: The locus drawn by point D when $m = 1$ and $r = 4$.

Therefore, from Eq. (31),

$$Z = \gamma + m(X^2 + Y^2). \tag{33}$$

When $m = 1$ and $r = 4$, then $\gamma = 16$, Eq. (33) becomes

$$Z = 16 + (X^2 + Y^2). \tag{34}$$

The view the locus from the side ($Y = 0$), it appears as

$$Z = X^2 + 16. \tag{35}$$

Therefore, the locus of D is plotted as a solid curve in Fig. 6. It is a surface obtained by shifting the original paraboloid (dashed curve) by 16 along the positive z-axis.

3 Center of the Ellipse Cut by a Plane at Paraboloid without Explicit Solution

The problems discussed in this section involve elliptic integrals and therefore have no explicit solutions. Consequently, numerical calculations were used in the specific examples.

3.1 Case Where the Length of the Segment of the Parabola is Constant

It is considered in two-dimensional geometry as shown in Fig. 2. Determine the locus of point D when the length of the arc of the parabola AEB is constant.

Theorem 3.1. *The length of the curve AEB can be expressed in terms of the coordinates*

(X, Y) of point D as

$$L = \frac{1}{2}\left(X + \sqrt{\frac{Y}{m} - X^2}\right)\sqrt{1 + 4m^2\left(X + \sqrt{\frac{Y}{m} - X^2}\right)^2} - \frac{1}{2}\left(X - \sqrt{\frac{Y}{m} - X^2}\right)\sqrt{1 + 4m^2\left(X - \sqrt{\frac{Y}{m} - X^2}\right)^2} + \frac{1}{4m}\log\left(\frac{2m\left(X + \sqrt{\frac{Y}{m} - X^2}\right) + \sqrt{1 + 4m^2\left(X + \sqrt{\frac{Y}{m} - X^2}\right)^2}}{2m\left(X - \sqrt{\frac{Y}{m} - X^2}\right) + \sqrt{1 + 4m^2\left(X - \sqrt{\frac{Y}{m} - X^2}\right)^2}}\right). \quad (36)$$

Proof. The length of curve AEB is expressed as

$$\begin{aligned} L &= \int_{x_2}^{x_1} \sqrt{1 + \left\{\tfrac{\mathrm{d}}{\mathrm{d}x}(mx^2)\right\}^2}\,\mathrm{d}x \\ &= \int_0^{x_1} \sqrt{1 + 4m^2x^2}\,\mathrm{d}x - \int_0^{x_2} \sqrt{1 + 4m^2x^2}\,\mathrm{d}x \\ &= \frac{1}{2}x_1\sqrt{1 + 4m^2x_1^2} - \frac{1}{2}x_2\sqrt{1 + 4m^2x_2^2} \\ &\quad + \frac{1}{4m}\left[\log\left(2mx_1 + \sqrt{1 + 4m^2x_1^2}\right) - \log\left(2mx_2 + \sqrt{1 + 4m^2x_2^2}\right)\right] \\ &= \frac{1}{2}x_1\sqrt{1 + 4m^2x_1^2} - \frac{1}{2}x_2\sqrt{1 + 4m^2x_2^2} + \frac{1}{4m}\log\left(\frac{2mx_1 + \sqrt{1 + 4m^2x_1^2}}{2mx_2 + \sqrt{1 + 4m^2x_2^2}}\right). \end{aligned} \quad (37)$$

Therefore, from Eqs. (3) and (4),

$$L = \frac{1}{2}\left(X + \sqrt{\frac{Y}{m} - X^2}\right)\sqrt{1 + 4m^2\left(X + \sqrt{\frac{Y}{m} - X^2}\right)^2} - \frac{1}{2}\left(X - \sqrt{\frac{Y}{m} - X^2}\right)\sqrt{1 + 4m^2\left(X - \sqrt{\frac{Y}{m} - X^2}\right)^2} + \frac{1}{4m}\log\left(\frac{2m\left(X + \sqrt{\frac{Y}{m} - X^2}\right) + \sqrt{1 + 4m^2\left(X + \sqrt{\frac{Y}{m} - X^2}\right)^2}}{2m\left(X - \sqrt{\frac{Y}{m} - X^2}\right) + \sqrt{1 + 4m^2\left(X - \sqrt{\frac{Y}{m} - X^2}\right)^2}}\right). \quad (38)$$

Here, Y cannot be explicitly expressed as a function of X. Let $D = \sqrt{\frac{Y}{m} - X^2}$, and $x_1 = X + D$ and $x_2 = X - D$, then,

$$L = \frac{1}{2}(X + D)\sqrt{1 + 4m^2(X + D)^2} - \frac{1}{2}(X - D)\sqrt{1 + 4m^2(X - D)^2} + \frac{1}{4m}\log\left(\frac{2m(X + D) + \sqrt{1 + 4m^2(X + D)^2}}{2m(X - D) + \sqrt{1 + 4m^2(X - D)^2}}\right). \quad \square \quad (39)$$

Using Ramanujan's first approximation of the elliptic circumference (first formula) [7]

$$L_{\mathrm{Ram}} = \pi\left\{3(a + b) - \sqrt{(a + 3b)(3a + b)}\right\}, \quad (40)$$

Eq. (39) is expressed as

$$L = \frac{\pi}{2}\Bigg[3\Big(\sqrt{1+4m^2(X+D)^2}+\sqrt{1+4m^2(X-D)^2}\Big) - \sqrt{3\sqrt{1+4m^2(X+D)^2}+\sqrt{1+4m^2(X-D)^2}} \times \sqrt{\sqrt{1+4m^2(X+D)^2}+3\sqrt{1+4m^2(X-D)^2}}\Bigg]. \quad (41)$$

For example, when $m = 1$ and $r = 4$, from Eq. (36),

$$\begin{aligned} L &= \frac{1}{2}\Big(X+\sqrt{Y-X^2}\Big)\sqrt{1+4\Big(X+\sqrt{Y-X^2}\Big)^2} \\ &\quad - \frac{1}{2}\Big(X-\sqrt{Y-X^2}\Big)\sqrt{1+4\Big(X-\sqrt{Y-X^2}\Big)^2} \\ &\quad + \frac{1}{4}\log\left(\frac{2\Big(X+\sqrt{Y-X^2}\Big)+\sqrt{1+4\Big(X+\sqrt{Y-X^2}\Big)^2}}{2\Big(X-\sqrt{Y-X^2}\Big)+\sqrt{1+4\Big(X-\sqrt{Y-X^2}\Big)^2}}\right) \quad (42) \\ &= 2r \\ &= 8. \end{aligned}$$

From Eq. (40), Eq. (42) is approximately expressed as

$$\begin{aligned} L = \frac{\pi}{2}\Bigg[&3\Big(\sqrt{1+4\Big(X+\sqrt{Y-X^2}\Big)^2}+\sqrt{1+4\Big(X-\sqrt{Y-X^2}\Big)^2}\Big) \\ &- \sqrt{3\sqrt{1+4\Big(X+\sqrt{Y-X^2}\Big)^2}+\sqrt{1+4\Big(X-\sqrt{Y-X^2}\Big)^2}} \\ &\times \sqrt{\sqrt{1+4\Big(X+\sqrt{Y-X^2}\Big)^2}+3\sqrt{1+4\Big(X-\sqrt{Y-X^2}\Big)^2}}\Bigg] \quad (43) \\ = 8. \end{aligned}$$

When solved numerically, the result is shown in Fig. 7. The dashed curve in the figure is the original parabola. It has the shape of a "W", which bulges at the base of the paraboloid and asymptotically approaches the original parabola (dashed curve) in regions where the absolute value of X is large.

3.2 Case Where the Perimeter of the Upper Surface is Constant

The diagram is shown in Fig. 4 and is the same with Section 2.2, consider a paraboloid of revolution in three dimensions. When the length of the perimeter of the upper surface is constant, determine the locus of the center D of the ellipse as it moves.

Theorem 3.2. *The locus of point D is expressed as*

$$2\pi r = \frac{4}{\sqrt{m}}\sqrt{Z-mX^2}\sqrt{1+m^2X^2}\;E\left(\frac{mX}{\sqrt{1+m^2X^2}}\right), \quad (44)$$

where $E(k)$ $(= \int_0^{\frac{\pi}{2}} \sqrt{1-k^2\sin^2\theta}\,\mathrm{d}\theta)$ is the complete elliptic integral of the second kind.

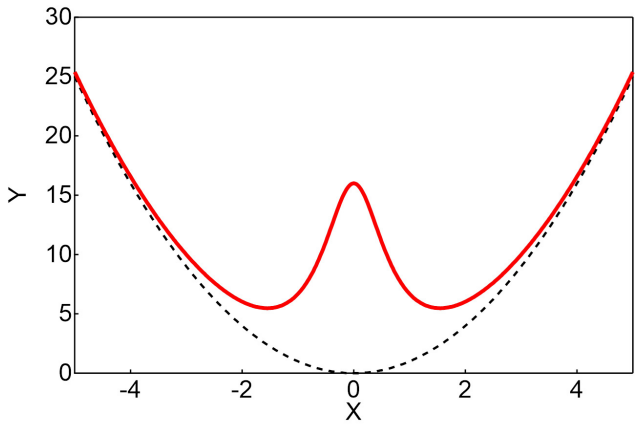


Figure 7: The locus drawn by point D when $m = 1$ and $r = 4$ in the case where the the length of the segment of the parabola is constant.

Proof. Now, consider the lower, side view of Fig. 4. Let the coordinate of D be (X, Z), where X and Z are expressed as

$$X = \frac{x_1 + x_2}{2} \tag{45}$$

and

$$Z = \frac{mx_1^2 + mx_2^2}{2}, \tag{46}$$

respectively. From Eq. (17), the major axis a of the ellipse on the top surface is described as

$$a = \sqrt{\frac{Z}{m} - X^2}\,\sqrt{1 + m^2X^2}. \tag{47}$$

Also, from Eq. (20), the short axis b of the ellipse on the top surface is described as

$$b = \sqrt{\frac{Z}{m} - X^2}. \tag{48}$$

From Eqs. (47) and (48), then, $a = b\sqrt{1 + m^2X^2}$, which means that the ratio of the major axis a to the minor axis b of an ellipse (the aspect ratio) depends only on X and not on Z. Using the complete elliptic integral of the second kind , the length L of the perimeter of ellipse on the top surface is expressed as

$$L = 4aE(e), \tag{49}$$

where e is the eccentricity and is expressed as

$$e = \sqrt{1 - \frac{b^2}{a^2}}. \tag{50}$$

Then, Eq. (49) is re-expressed as

$$L = 4aE\left(\sqrt{1 - \frac{b^2}{a^2}}\right) = 4a\int_0^{\frac{\pi}{2}} \sqrt{1 - \left(1 - \frac{b^2}{a^2}\right)\sin^2\theta}\,\mathrm{d}\theta. \tag{51}$$

Initially, that is, when the top upper segment is horizontal,

$$L_0 = 2\pi r \tag{52}$$

holds. Since the length of the perimeter of the ellipse on the top surface is constant, from Eqs. (51) and (52), we obtain

$$\begin{aligned} 2\pi r &= 4\sqrt{\frac{Z}{m} - X^2}\,\sqrt{1+m^2X^2}\,E\left(\sqrt{1-\frac{b^2}{a^2}}\right) \\ &= 4\sqrt{\frac{Z}{m} - X^2}\,\sqrt{1+m^2X^2}\int_0^{\frac{\pi}{2}}\sqrt{1-\left(1-\frac{b^2}{a^2}\right)\sin^2\theta}\,\mathrm{d}\theta \\ &= \frac{4}{\sqrt{m}}\sqrt{Z-mX^2}\sqrt{1+m^2X^2}\int_0^{\frac{\pi}{2}}\sqrt{1-\frac{m^2X^2}{1+m^2X^2}\sin^2\theta}\,\mathrm{d}\theta \end{aligned} \tag{53}$$

Thus, Eq. (44) is obtained. □

Using Ramanujan's approximation of the elliptic circumference (first formula) [7]

$$L_{\mathrm{Ram}} = \pi\left\{3(a+b) - \sqrt{(a+3b)(3a+b)}\right\}, \tag{54}$$

Eq. (53) is expressed as

$$\begin{aligned} 2r = {} & 3\left(\sqrt{\frac{Z}{m} - X^2}\,\sqrt{1+m^2X^2} + \sqrt{\frac{Z}{m} - X^2}\right) \\ & - \sqrt{\left(\sqrt{\frac{Z}{m} - X^2}\,\sqrt{1+m^2X^2} + 3\sqrt{\frac{Z}{m} - X^2}\right)\left(3\sqrt{\frac{Z}{m} - X^2}\,\sqrt{1+m^2X^2} + \sqrt{\frac{Z}{m} - X^2}\right)} \end{aligned} \tag{55}$$

As the absolute value of X increases, Z also increases.

For example, when $m = 1$, then Eq. (53) becomes

$$\begin{aligned} 2\pi r &= 4X\sqrt{1+\left(\frac{Z}{X}\right)^2}\,E\left(\sqrt{\frac{2}{1+\left(\frac{Z}{X}\right)^2}}\right) \\ &= 4X\int_0^{\frac{\pi}{2}}\sqrt{1+\left(\frac{Z}{X}\right)^2 - 2\sin^2\theta}\,\mathrm{d}\theta \end{aligned} \tag{56}$$

And, Eq. (55) becomes

$$2r = \sqrt{Z-X^2}\left\{3\left(\sqrt{1+X^2}+1\right) - \sqrt{\left(3\sqrt{1+X^2}+1\right)\left(\sqrt{1+X^2}+3\right)}\right\}. \tag{57}$$

Furthermore, for example, when $r = 4$, Eqs. (56) and (57) become

$$\begin{aligned} 2\pi &= X\sqrt{1+\left(\frac{Z}{X}\right)^2}E\left(\sqrt{\frac{2}{1+\left(\frac{Z}{X}\right)^2}}\right) \\ &= X\int_0^{\frac{\pi}{2}}\sqrt{1+\left(\frac{Z}{X}\right)^2 - 2\sin^2\theta}\,\mathrm{d}\theta \end{aligned} \tag{58}$$

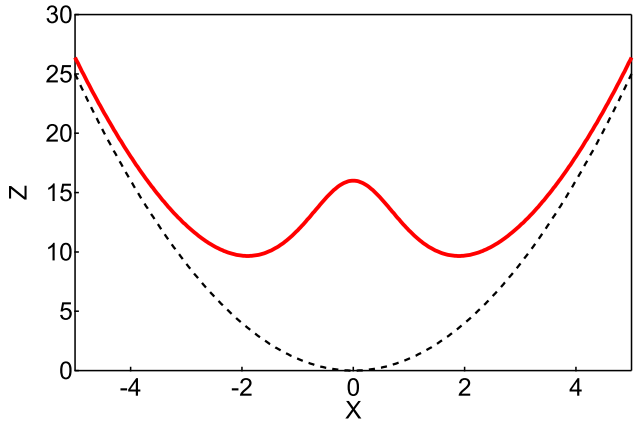


Figure 8: The locus drawn by point D when $m = 1$ and $r = 4$ in the case where the length of the perimeter of the ellipse on the top surface is constant.

and

$$8 = \sqrt{Z - X^2}\left\{3\left(\sqrt{1+X^2}+1\right) - \sqrt{\left(3\sqrt{1+X^2}+1\right)\left(\sqrt{1+X^2}+3\right)}\right\}, \tag{59}$$

respectively. Equation (59) can be solved for Z;

$$Z = X^2 + \frac{64}{\left[3\left(\sqrt{1+X^2}+1\right) - \sqrt{\left(3\sqrt{1+X^2}+1\right)\left(\sqrt{1+X^2}+3\right)}\right]^2}. \tag{60}$$

It is shown in Fig. 8. The dashed curve in the figure is the original parabola $Z = X^2$. It has the shape of a rotated "W", which bulges at the base of the paraboloid and asymptotically approaches the paraboloid in regions where the absolute value of X is large.

3.3 Case Where the Lateral Surface Area of Parabola is Constant

The diagram is shown in Fig. 4 and is the same with Section 2.2 and Section 3.2, consider a paraboloid of revolution in three dimensions. When the lateral surface area of parabola is constant, determine the locus of the center D of the ellipse as it moves. In this case, the paraboloid surface is represented as

$$z = m(x^2 + y^2). \tag{61}$$

Let the plane intersecting the paraboloid be

$$z = \alpha x + \beta y + \gamma, \tag{62}$$

then, those intersecting curve is given by

$$z = m(x^2 + y^2) = \alpha x + \beta y + \gamma. \tag{63}$$

Therefore, the following relationship holds;

$$\left(x - \frac{\alpha}{2m}\right)^2 + \left(y - \frac{\beta}{2m}\right)^2 = \frac{\gamma}{m} + \frac{\alpha^2 + \beta^2}{4m^2}, \tag{64}$$

which corresponds to a circle on the x–y plane with center at $x = \frac{\alpha}{2m}$ and $y = \frac{\beta}{2m}$, and radius $R = \sqrt{\frac{\gamma}{m} + \frac{\alpha^2+\beta^2}{4m^2}}$. Here, from $f_x = 2mx$ and $f_y = 2my$, the area term is

$$\mathrm{d}S = \sqrt{1 + 4m^2(x^2 + y^2)}\,\mathrm{d}x\,\mathrm{d}y. \tag{65}$$

In Eq. (65), $u = x - \frac{\alpha}{2m}$ and $v = y - \frac{\beta}{2m}$ are substituted integration interval becomes $u^2 + v^2 \leq \frac{\gamma}{m} + \frac{\alpha^2+\beta^2}{4m^2}$. Therefore,

$$\mathrm{d}S = \sqrt{1 + 4m^2(u^2 + v^2) + 4m\alpha u + 4m\beta v + (\alpha^2 + \beta^2)}\,\mathrm{d}u\,\mathrm{d}v. \tag{66}$$

When expressed in polar coordinates using $u = r\cos\theta$ and $v = r\sin\theta$, the area is given by

$$\begin{aligned} S &= \int_0^R \int_0^{2\pi} \sqrt{1 + \alpha^2 + \beta^2 + 4m^2r^2 + 4mr(\alpha\cos\theta + \beta\sin\theta)}r\,\mathrm{d}r\,\mathrm{d}\theta \\ &= 4\int_0^R r\sqrt{1 + \alpha^2 + \beta^2 + 4m^2r^2 + 4mr\sqrt{\alpha^2 + \beta^2}} \\ &\qquad E\left(\sqrt{\frac{8mr\sqrt{\alpha^2 + \beta^2}}{1 + \alpha^2 + \beta^2 + 4m^2r^2 + 4mr\sqrt{\alpha^2 + \beta^2}}}\right)\mathrm{d}r, \end{aligned} \tag{67}$$

where $E(k)$ is the complete elliptic integral of the second kind. Then, using Ramanujan's approximation of the elliptic circumference (first formula) [7], Eq. (67) is expressed as

$$\begin{aligned} S = \pi\int_0^R r\Bigg[&3\left(\sqrt{1 + \left(2mr + \sqrt{\alpha^2 + \beta^2}\right)^2} + \sqrt{1 + \left(2mr - \sqrt{\alpha^2 + \beta^2}\right)^2}\right) \\ &- \sqrt{3\sqrt{1 + \left(2mr + \sqrt{\alpha^2 + \beta^2}\right)^2} + \sqrt{1 + \left(2mr - \sqrt{\alpha^2 + \beta^2}\right)^2}} \\ &\times \sqrt{\sqrt{1 + \left(2mr + \sqrt{\alpha^2 + \beta^2}\right)^2} + 3\sqrt{1 + \left(2mr - \sqrt{\alpha^2 + \beta^2}\right)^2}}\Bigg]\mathrm{d}r. \end{aligned} \tag{68}$$

When the plane intersecting the paraboloid is horizontal, Eq. (68) simplifies to

$$S_0 = \frac{\pi}{6m^2}\left[(1 + 4m\gamma)^{\frac{3}{2}} - 1\right]. \tag{69}$$

Then, the lateral surface area is constant, the position of the center of the top-surface ellipse is denoted by $X = \frac{\alpha}{2m}$ and $Y = \frac{\beta}{2m}$,

$$Z = 2m(X^2 + Y^2) + \frac{(1 + 6m^2S_0)^{\frac{2}{3}} - 1}{4m}. \tag{70}$$

As the absolute value of X increases, Z also increases.

For example, when $m = 1$, then Eq. (69) becomes

$$S_0 = \frac{\pi}{6}[(1 + 4\gamma)^{\frac{3}{2}} - 1]. \tag{71}$$

Furthermore, when $r = 4$, $\gamma = 16$, then, Eq. (71) becomes

$$S_0 = \frac{\pi}{6}[65\sqrt{65} - 1]. \tag{72}$$

Therefore, when the lateral area is constant, if $m = 1$ and $Y = 0$ are fixed in Eq. (70), the locus of Z is expressed as

$$Z = 2X^2 + 16, \tag{73}$$

which is plotted as a solid curve in Fig. 9.

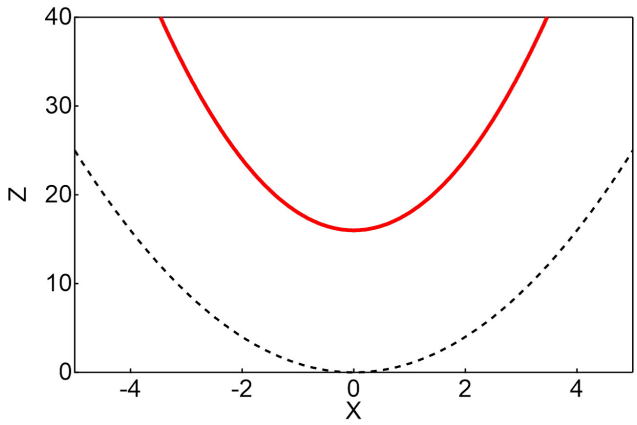


Figure 9: The locus drawn by point D when $m = 1$ and $r = 4$ in the case where the lateral surface area of parabola is constant.

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