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Fuss Relations and Primitive Factorization for the Bicentric 21-gon

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Abstract. We study the Cayley–Poncelet closing condition for the bicentric 21-gon and its factorization in the dimensionless variables $p = (R + d)/r$ and $q = (R - d)/r$. Two explicit tangent constructions, with rotation numbers $k = 1$ and $k = 2$, give representative 21-step closing polynomials in $Q[p, q[$. Each such polynomial has bidegree $(55, 55)$ and contains factors corresponding to closing periods 3 and 7. After these lower-period factors are removed, the two representative Fuss relations $F_{21}^{(1)}$ and $F_{21}^{(2)}$ are the primitive period-21 factors; each has bidegree $(48, 48)$ and 1225 non-zero monomials. The independently derived $k = 2$ full closing polynomial is shown to equal the polynomial obtained from the $k = 1$ full closing polynomial by the substitution $(p, q) \mapsto (-p, -q)$, and the same substitution relates the two primitive factors. We also verify the exact factorization of the Cayley determinant for $n = 21$. Rotation numbers with $\gcd(21, k) > 1$ reduce to the period-3 or period-7 closing conditions, while for the remaining coprime rotation numbers only the representatives $k = 1$ and $k = 2$ are derived symbolically in this paper.

Key Words: Bicentric polygon, Fuss relation, Poncelet polygon, Cayley determinant, primitive factorization, rotation number

MSC 2020: 51M04 (primary), 51M15, 51N20, 14H52, 14Q20

1. Introduction

A polygon that is both cyclic and tangential is called *bicentric*. Let R denote its circumradius, r its inradius, and d the distance between the two centers. For a bicentric quadrilateral these quantities satisfy the classical relation

$$(R^2 - d^2)^2 = 2r^2(R^2 + d^2). \tag{1}$$

Relations of this type go back to Fuss' investigations of polygons that are simultaneously inscribed in and circumscribed about circles [10, 11]. The terminology *Fuss formula* or *Fuss*

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relation is also used in modern work on bicentric polygons and Poncelet porisms [4, 5]. Accordingly, in this paper a *Fuss relation* means an algebraic relation among R, r, d that expresses a closing condition for a bicentric polygon. The underlying Poncelet closing condition can equivalently be formulated by Cayley's determinant criterion; see Cayley's original paper [3] and the modern accounts [1, 2, 7–9, 12].

The purpose of the paper is to derive and factor the representative closing relations for the bicentric 21-gon. The composite case

$$21 = 3 \cdot 7$$

is particularly useful because a 21-step closing polynomial may contain factors already arising from the proper divisors 3 and 7. We therefore make a systematic distinction between a *full* 21*-step closing polynomial*, which contains these lower-period factors, and the *primitive period*-21 *factor*, obtained after the period-3 and period-7 closing factors have been removed. The latter is the new period-21 contribution studied here. The word *primitive* is used only in this divisor-theoretic sense; irreducibility over $\mathbb{Q}[p, q]$ is not asserted unless explicitly stated.

We write the two circles as

$$K:\ x^2 + y^2 = R^2, \qquad C:\ (x-d)^2 + y^2 = r^2, \tag{2}$$

where K is centered at $O = (0, 0)$ and C at $I = (d, 0)$. We use the dimensionless variables

$$p = \frac{R+d}{r}, \qquad q = \frac{R-d}{r}, \tag{3}$$

which are invariant under a common scaling of R, r, d. Conversely, $R = r(p+q)/2$ and $d = r(p-q)/2$.

Two tangent constructions are used throughout the paper. For compactness we occasionally call the construction with rotation number $k = 1$ the *odd representative* and the one with $k = 2$ the *even representative*; these adjectives refer only to the parity of the two representative rotation numbers. We distinguish a *full* 21*-step closing polynomial*, obtained before shorter-period factors are removed, from its *primitive period*-21 *factor*, obtained by exact division by the period-3 and period-7 closing factors. We reserve the notation $F_{21}^{(1)}$ and $F_{21}^{(2)}$ for these two primitive representative Fuss relations. Thus the bidegree $(48, 48)$ and the count of 1225 non-zero monomials refer to $F_{21}^{(1)}$ and $F_{21}^{(2)}$, not to the larger full 21-step closing polynomials. No special terminology is attached to the substitution $(p, q) \mapsto (-p, -q)$; when it is used below, the relevant polynomial identity is stated explicitly.

The main results are as follows. The two representative closing equations are obtained by repeated application of an explicit tangent map and are reduced to rational equations in p, q. After cancellation of common numerator and denominator factors, the relevant numerators are polynomials in $\mathbb{Q}[p, q]$. For $k = 1$ the full 21-step closing polynomial satisfies

$$F_{21}^{\mathrm{odd,full}} = F_3^{\mathrm{odd}} F_7^{\mathrm{odd}} F_{21}^{\mathrm{odd,prim}}, \qquad F_{21}^{(1)} := F_{21}^{\mathrm{odd,prim}},$$

where $F_{21}^{(1)}$ has bidegree $(48, 48)$ and 1225 non-zero monomials. The $k = 2$ construction produces a second full 21-step closing polynomial satisfying

$$F_{21}^{\mathrm{even,full}}(p, q) = F_{21}^{\mathrm{odd,full}}(-p, -q), \qquad F_{21}^{(2)} := F_{21}^{\mathrm{even,prim}}.$$

The full 21-step closing polynomials have bidegree $(55, 55)$, whereas both representative Fuss relations $F_{21}^{(1)}$ and $F_{21}^{(2)}$ have bidegree $(48, 48)$. For the $k = 2$ representative, $F_3^{\mathrm{even}} = pq + p + q$

has no zeros for $p, q > 0$; hence the product of the period-7 factor and $F_{21}^{(2)}$ has bidegree $(54, 54)$.

The Cayley determinant provides an independent description of the total Poncelet closing condition. For $n = 21$ we verify by exact coefficient comparison that it factors, up to the explicit non-zero scalar recorded in Section 3, as the product of the two full representative closing polynomials. This distinguishes the total Cayley condition from the primitive period-21 factors and makes the degree accounting transparent. For rotation numbers with $\gcd(21, k) > 1$, Radić's reduction theorem reduces the problem to the corresponding relation for a 3-gon or 7-gon. For coprime rotation numbers, the present paper derives only $k = 1$ and $k = 2$ symbolically; the predicted identifications for the remaining coprime values are stated explicitly as conjectural.

Whenever computer algebra is used below, the claim needed in the proof is an exact polynomial statement in $\mathbb{Q}[p, q]$: either an exact divisibility check or equality of two polynomials after comparison of all coefficients within a previously established bidegree bound. Thus "coefficient comparison" always means a finite exact comparison of rational coefficients, not a numerical test. Numerical residuals are used only as illustrations for rotation numbers not symbolically derived in the paper.

The necessary background is recalled in Section 2. Section 3 discusses the Cayley condition, while the geometric tangent construction is developed in Sections 5–6. The factorization results are given in Sections 8 and 9; the appendices collect the shorter explicit factors, exact algebraic checks, numerical examples, and figures.

2. Background and previous results

Let

$$A_j = (R\cos\theta_j, R\sin\theta_j), \qquad j = 1, \dots, n,$$

be the successive vertices of an oriented closed Poncelet polygon on the circumcircle K. Choose lifts of the angular parameters so that the successive increments follow the chosen orientation. The *rotation number* k is defined by

$$\theta_{n+1} - \theta_1 = 2\pi k, \qquad A_{n+1} = A_1.$$

Thus k records the number of turns made around the circumcircle during one closed n-step traversal. Reversing the orientation replaces k by $n - k$; for odd n we may therefore restrict attention to

$$1 \le k \le \frac{n-1}{2}.$$

For a convex Poncelet n-gon one has $k = 1$, whereas larger values of k correspond to star-like or self-intersecting traversals. This use of a rotation index for closed Poncelet trajectories is standard in the literature; see, in particular, Radić [19, 20] and the more recent treatments of Fuss formulas for general and self-intersecting Poncelet polygons in [4, 5].

In the tangent-length formulation used by Radić, the same integer is characterized by

$$\sum_{i=1}^{n} \arctan\left(\frac{|A_iT_i|}{r}\right) = k\pi, \tag{4}$$

where T_i is the point of contact of the side A_iA_{i+1} with the incircle. We write

$$F_n^{(k)}(R, r, d) = 0$$

for the reduced relation associated with rotation number k. For the coprime representative cases $k = 1, 2$ when $n = 21$, this means the quotient of the corresponding full 21-step closing polynomial after removal of the proper-divisor period-3 and period-7 factors. For non-coprime k, Radić's reduction theorem below identifies $F_n^{(k)}$ with the corresponding lower-period relation. Thus, in the composite case $n = 21$, the full 21-step closing polynomials, before removal of the lower-period factors, contain period-3 and period-7 factors, whereas $F_{21}^{(1)}$ and $F_{21}^{(2)}$ denote the primitive period-21 quotients of bidegree $(48, 48)$. The larger full closing polynomials of bidegree $(55, 55)$ are always written explicitly as $F_{21}^{\mathrm{odd,full}}$ and $F_{21}^{\mathrm{even,full}}$.

Throughout the paper we use the dimensionless variables introduced in (3). If

$$F(p, q) = \sum_{i,j} c_{ij} p^i q^j \neq 0,$$

then

$$\mathrm{bideg}_{p,q} F = (\alpha, \beta), \qquad \alpha = \max\{i : c_{ij} \neq 0\}, \quad \beta = \max\{j : c_{ij} \neq 0\}.$$

Thus the bidegree records separately the degrees in p and q; it is not the total degree of F.

The coordinate notation follows the earlier work on odd bicentric polygons [17]. A vertex reached from A_1 by the forward tangent construction is denoted by $A_i^{(1)}$, while a vertex reached by the opposite construction from the symmetric starting point $A_{(n+1)/2}$ is denoted by $A_i^{((n+1)/2)}$. Closure requires the two corresponding points on the circumcircle to coincide. In the representative construction used below both candidate points lie on the same specified arc of K; on that arc the abscissa is injective, so equality of the two abscissas is equivalent to equality of the points. This is proved in Lemma 5.2.

A second structural ingredient is the reduction theorem of Radić and Zatezalo [23]. If

$$g = \gcd(n, k), \qquad k = g\ell,$$

then the closing relation for rotation number k reduces to the relation for rotation number ℓ of an (n/g)-gon:

$$F_n^{(k)}(R, r, d) = F_{n/g}^{(\ell)}(R, r, d). \tag{5}$$

Consequently, when $g > 1$ the n-step condition yields no new primitive factor associated with period n. For $n = 21$, this explains why the non-coprime rotation numbers reduce to period-3 or period-7 closing conditions and why the corresponding factors occur inside the full 21-step polynomials.

For coprime rotation numbers, Radić proposed a further statement for odd n: the radius relation should depend only on the parity of k; see [23] and the subsequent discussion in [19, 20]. Accordingly, for $n = 21$ this conjecture predicts that $k = 1$ and $k = 5$ give one relation, while $k = 2, 4, 8, 10$ give the other. The present paper does *not* use this conjecture to derive the two representative polynomials: the cases $k = 1$ and $k = 2$ are obtained independently by the coordinate constructions in Sections 6 and 9. The conjecture is invoked only to state the predicted identifications for the remaining coprime rotation numbers. Numerical checks for these values are reported separately in Appendix D.

For odd n, Chabanyuk, Cieślak, Mozgawa and Naiman define two reduced Fuss relations corresponding to odd and even rotation indices [4]. Their notion of a *reduced Fuss relation* differs from the divisor-theoretic terminology used here: in this paper *primitive* means only that factors inherited from proper divisors of 21 have been removed.

There are several complementary approaches to algebraic relations for bicentric polygons. Radić developed methods based on relations among R, r, d and rotation numbers [19–23]. McAndrew generated such relations by polynomial ideals and Gröbner bases [13], while Cieślak and Mozgawa gave a method for obtaining Fuss formulas for arbitrary n, including self-intersecting Poncelet polygons [5]. Chabanyuk, Cieślak, Mozgawa and Naiman subsequently introduced unified and reduced forms of these relations [4]. Explicit symbolic derivations for higher bicentric polygons also appear in [15, 16]. The tridecagon and pentadecagon are treated in [18], while higher rotation number relations for those cases and the heptadecagon and enneadecagon are treated in [17]. These approaches are complementary to the classical Cayley–Poncelet criterion recalled in the next section. The novelty pursued here is therefore not a general algorithm for generating Fuss relations, but the explicit composite case $21 = 3{\cdot}7$: the exact separation of the period-3 and period-7 factors from the primitive period-21 factors, and the exact comparison of the resulting full 21-step closing condition with the Cayley determinant.

In the normalization (3), the full closing polynomials for the previously computed odd cases $n = 15, 17, 19$ have bidegrees $(28, 28)$, $(36, 36)$ and $(45, 45)$, respectively [17, 18]; these numbers are T_7, T_8, T_9, where $T_m = m(m+1)/2$. For the proper odd divisors of 21, the period-3 and period-7 factors have bidegrees $(1, 1)$ and $(6, 6)$. The exact $n = 21$ Cayley factorization below, together with the explicit closing polynomials derived later, gives bidegree $(55, 55)$ for each full 21-step closing polynomial. Removing the period-3 and period-7 factors therefore leaves the primitive factor

$$(55, 55) - (1, 1) - (6, 6) = (48, 48).$$

This degree accounting is verified directly by the exact factorizations in Sections 8 and 9.

3. The Cayley condition for the bicentric 21-gon

Cayley's criterion gives the classical algebraic closing condition for a Poncelet polygon. We use it here only in the case needed for the present paper, namely $n = 21$. This keeps the role of the determinant separate from the coordinate construction: the Cayley determinant gives the total 21-step closing condition, whereas the iterated tangent maps in later sections identify its two full representative closing polynomials and their lower-period factors. For the classical criterion and its modern treatments, see Cayley's original paper [3], Griffiths–Harris [12], Bos–Kers–Oort–Raven [2], and the accounts in [7–9, 14].

Write the circumcircle and incircle in homogeneous coordinates as

$$K : x^2 + y^2 - R^2 z^2 = 0, \qquad C : (x - dz)^2 + y^2 - r^2 z^2 = 0.$$

With

$$K = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -R^2 \end{pmatrix}, \qquad C = \begin{pmatrix} 1 & 0 & -d \\ 0 & 1 & 0 \\ -d & 0 & d^2 - r^2 \end{pmatrix},$$

the determinant of the pencil is

$$\det(C + \lambda K) = -(1 + \lambda)\left(R^2\lambda^2 + (R^2 - d^2 + r^2)\lambda + r^2\right).$$

After division by the non-zero scalar $-r^2$ and the substitution (3), we use the dimensionless Cayley polynomial

$$\Delta(\lambda) = (1+\lambda)\left(1 + (1+pq)\lambda + \frac{(p+q)^2}{4}\lambda^2\right). \tag{6}$$

Thus

$$\frac{R^2-d^2}{r^2} = pq, \qquad \frac{R^2}{r^2} = \frac{(p+q)^2}{4}.$$

Let

$$\sqrt{\Delta(\lambda)} = \sum_{j=0}^{\infty} c_j(p,q)\lambda^j, \tag{7}$$

where the square root is the formal series with constant term 1. For an odd Poncelet polygon with $n = 2m+1$, Cayley's criterion is expressed by the Hankel determinant formed from these coefficients. In the present case $m = 10$, so we need only

$$\mathcal{C}_{21}(p,q) = \det\Big(c_{i+j}(p,q)\Big)_{i,j=1}^{10}. \tag{8}$$

The equation $\mathcal{C}_{21} = 0$ is the total Cayley closing condition. Here "total" includes parameter values for which the tangent map closes with a proper period dividing 21; consequently the period-3 and period-7 factors occur in the total 21-step condition as well.

The degree bound needed for the exact coefficient comparison below follows directly from the series equation and does not require a separate Hankel-determinant argument. Write $\Delta(\lambda) = \sum_{j=0}^{3} \delta_j \lambda^j$. From $\left(\sum_{j\geq 0} c_j\lambda^j\right)^2 = \Delta(\lambda)$ one obtains, for $j \geq 1$, the recurrence

$$2c_j = \delta_j - \sum_{a=1}^{j-1} c_a c_{j-a}, \qquad \delta_j = 0 \quad (j > 3).$$

Since $\deg_p \delta_j, \deg_q \delta_j \leq j$ for $0 \leq j \leq 3$, induction gives $\deg_p c_j, \deg_q c_j \leq j$. Hence every term $\prod_{i=1}^{10} c_{i+\sigma(i)}$ in the determinant (8) has degree at most

$$\sum_{i=1}^{10}(i+\sigma(i)) = 110$$

in each of p and q. Thus $\operatorname{bideg}_{p,q} \mathcal{C}_{21} \leq (110, 110)$.

Proposition 3.1 (Exact Cayley factorization for $n = 21$)**.** *Let $F_{21}^{\mathrm{odd,full}}$ and $F_{21}^{\mathrm{even,full}}$ denote the full 21-step closing polynomials obtained from the $k = 1$ and $k = 2$ tangent constructions, respectively. With the content-one normalization used in this paper,*

$$\mathcal{C}_{21}(p,q) = 2^{-210} F_{21}^{\mathrm{odd,full}}(p,q) F_{21}^{\mathrm{even,full}}(p,q), \tag{9}$$

and

$$F_{21}^{\mathrm{even,full}}(p,q) = F_{21}^{\mathrm{odd,full}}(-p,-q). \tag{10}$$

Both full 21-step closing polynomials have bidegree $(55, 55)$ and contain 1596 non-zero monomials.

Proof. The two full closing polynomials are obtained independently from the tangent constructions in Sections 8 and 9. The substitution identity (10) is verified there as an exact identity in $\mathbb{Q}[p, q]$.

For the Cayley factorization, the coefficients $c_2, \ldots, c_{20}$ are obtained exactly from (7), and the determinant (8) is formed in $\mathbb{Q}[p, q]$. Computing the coefficient of the highest power of p in the determinant gives

$$[p^{110}]\mathcal{C}_{21} = 2^{-210}(q^2 - 1)^{55},$$

whereas

$$[p^{55}]F_{21}^{\text{odd,full}} = (q-1)^{25}(q+1)^{30}, \qquad [p^{55}]F_{21}^{\text{even,full}} = (q-1)^{30}(q+1)^{25}.$$

Hence the scalar in (9) is 2^{-210}. The remaining equality is checked exactly by expanding

$$2^{210}\mathcal{C}_{21} - F_{21}^{\text{odd,full}} F_{21}^{\text{even,full}}$$

and verifying that every coefficient is zero. Since both sides have bidegree at most $(110, 110)$, this is a finite polynomial identity in $\mathbb{Q}[p, q]$. □

The factorization in Proposition 3.1 also clarifies the degree distinction used throughout the paper. The bidegree $(55, 55)$ belongs to each *full* 21*-step closing polynomial*. These polynomials still contain the closing factors inherited from periods 3 and 7. As proved in Sections 8 and 9, removing the inherited factors of bidegrees $(1, 1)$ and $(6, 6)$ leaves the primitive period-21 factor of bidegree

$$(55, 55) - (1, 1) - (6, 6) = (48, 48),$$

with 1225 non-zero monomials. Thus the primitive period-21 relation and the full 21-step closing polynomial are different objects: the former is the new primitive period-21 factor, while the latter is the factor occurring in the total Cayley 21-step condition.

4. A brief elliptic interpretation of the coprime cases

This short elliptic interpretation is included only to explain the torsion viewpoint behind the coprime cases. It is not needed for the symbolic derivations in Sections 8 and 9, and it is not used as a proof of Radić's conjecture that, for coprime rotation numbers, the relation depends only on the parity of k. In the standard treatment of Poncelet's porism, the step map on a nonsingular Cayley curve can be represented as translation by a point on an elliptic curve; see, for example, [7–9, 14]. For the present pair of circles the corresponding curve is

$$E_{p,q} : y^2 = \Delta(\lambda),$$

with Δ defined in (6).

If a translation point Q has exact order n, then the associated translation closes after n steps and not before. Moreover, if $\gcd(k, n) = 1$, then kQ again has exact order n: indeed, $s(kQ) = O$ implies $n \mid sk$, and hence $n \mid s$. This elementary property of torsion points (see also [24]) shows that multiplication by a unit modulo n preserves exact order. It does *not*, by itself, imply that two rotation numbers yield the same polynomial relation in p, q, nor does it determine which real tangent construction realizes a given rotation number.

For $n = 21$, the coprime rotation numbers in the range $1 \le k \le 10$ are

$$1, 2, 4, 5, 8, 10.$$

Radić's conjecture groups $k = 1, 5$ together and $k = 2, 4, 8, 10$ together. In this paper only the representatives $k = 1$ and $k = 2$ are derived symbolically. The predicted identifications for $k = 5$ and $k = 4, 8, 10$ are therefore stated only as consequences of Radić's conjecture and are accompanied by the numerical checks in Appendix D. The non-coprime cases do not rely on this elliptic interpretation: they are reduced rigorously by Radić's reduction theorem to the period-3 or period-7 closing relations, as summarized in Section 7.

5. Geometric construction for the representative $k = 1$

We use the two circles (2) in the nonconcentric strictly nested range

$$R > d + r, \qquad d > 0, \qquad r > 0. \tag{11}$$

Equivalently, the normalized variables satisfy $p > q > 1$. The coordinate construction below is carried out for $d > 0$. The concentric specialization $d = 0$ corresponds to $p = q = R/r$ and may be considered separately by substituting $p = q$ in the resulting polynomial identities; no separate concentric coordinate chart is needed for the factorization proved here.

Remark 5.1 (Rotation number labels for the two real representatives). On any open set on which the prescribed tangent choices are real and nondegenerate, the prescribed tangent steps vary continuously with the circle parameters. For a closed configuration the lifted total angular increment is $2\pi k$; consequently, k is constant on each connected component of the closed real locus as long as no degeneracy is crossed. We use this fact to label the two real components before algebraic elimination: the component whose oriented traversal makes one turn is the $k = 1$ representative, and the component whose traversal makes two turns is the $k = 2$ representative. The choices of the roots in the tangent formulas below are fixed from these real tangent components before the rational expressions are simplified. Thus the labels $k = 1$ and $k = 2$ are geometric labels of the prescribed real tangent constructions; they are not inferred from the subsequent polynomial factorization.

For the representative with rotation number $k = 1$, the initial side is the vertical tangent to C at the point of contact $P_+ = (d + r, 0)$. Its upper intersection with K is

$$A_1^{(1)} = \left(d + r, \sqrt{R^2 - (d + r)^2}\right). \tag{12}$$

For the symmetric construction from the opposite side we start from $A_{11}^{(11)} = (-R, 0)$. The two half-chains are

$$A_1^{(1)}, A_2^{(1)}, \ldots, A_6^{(1)}, \qquad A_{11}^{(11)}, A_{10}^{(11)}, \ldots, A_6^{(11)},$$

and closure of this representative is equivalent to

$$A_6^{(1)} = A_6^{(11)}. \tag{13}$$

The next lemma explains why one coordinate is sufficient once the real tangent choices have been specified.

Lemma 5.2 (Abscissa criterion)**.** *Let*

$$\theta_1 = \arccos\frac{d+r}{R}, \qquad \Gamma_+ = \{(R\cos\theta, R\sin\theta) : \theta_1 \le \theta \le \pi\}.$$

For the $k = 1$ construction used below, the two five-step half-chains are chosen so that the two points denoted by $A_6^{(1)}$ and $A_6^{(11)}$ lie on Γ_+. Then

$$A_6^{(1)} = A_6^{(11)} \quad \Longleftrightarrow \quad x\Big(A_6^{(1)}\Big) = x\Big(A_6^{(11)}\Big).$$

Proof. On Γ_+ the parametrization by θ uses an interval contained in $[0,\pi]$, on which $\cos\theta$ is strictly decreasing. Hence the projection $(R\cos\theta, R\sin\theta) \mapsto R\cos\theta$ is injective on this arc. Equality of the abscissas therefore implies equality of the two points, and the converse is immediate. □

Consequently, for every admissible parameter pair for which the prescribed tangent steps are defined and remain on this arc, the geometric closing condition can be written as the scalar equation

$$x_6^{(1)} - x_6^{(11)} = 0. \tag{14}$$

This equation is imposed before any denominators are cleared. Section 6 now writes the two abscissas explicitly by iterating one tangent-step map and then reduces their difference to a rational function of p and q.

6. Coordinate construction for the representative $k = 1$

Let $A = (x_0, y_0) \in K$. A non-vertical line through A has the form $y = ax+b$, with $b = y_0 - ax_0$. It is tangent to C precisely when the distance from $I = (d, 0)$ to the line equals r, which gives

$$r^2(1+a^2) - (ad+b)^2 = 0. \tag{15}$$

Equivalently,

$$\Big(r^2 - (d-x_0)^2\Big)a^2 - 2(d-x_0)y_0 a + \Big(r^2 - y_0^2\Big) = 0. \tag{16}$$

Its two roots are

$$a_\varepsilon(A) = \frac{(d-x_0)y_0 + \varepsilon r\sqrt{(x_0-d)^2 + y_0^2 - r^2}}{r^2 - (d-x_0)^2}, \qquad \varepsilon \in \{+1, -1\}. \tag{17}$$

For either finite root, the second intersection of this tangent line with K is

$$x' = -\frac{2ab}{1+a^2} - x_0, \qquad y' = ax' + b. \tag{18}$$

We denote the resulting tangent step by $T_\varepsilon(A) = (x', y')$. Thus a repeated tangent construction means simply successive application of the explicitly defined maps T_+ or T_-.

By scale invariance, we may set $r = 1$; then (3) gives $R = (p+q)/2$ and $d = (p-q)/2$. The $k = 1$ representative is obtained by applying five $+$ steps from $A_1^{(1)}$ and five $-$ steps from $A_{11}^{(11)}$. With $\pi_x(x, y) = x$, define

$$X_+ = \pi_x\Big(T_+^5(A_1^{(1)})\Big), \qquad X_- = \pi_x\Big(T_-^5(A_{11}^{(11)})\Big). \tag{19}$$

By Lemma 5.2, the closing equation for this representative is

$$X_+ - X_- = 0. \tag{20}$$

After the prescribed tangent choices and the substitution (3), the symbolic expression $X_+ - X_-$ reduces to a rational function of p and q. Indeed, with $r = 1$ the starting ordinate satisfies

$$\left(y_1^{(1)}\right)^2 = (p+1)(q-1).$$

In the exact symbolic simplification of the two five-step abscissas, the algebraic square roots are eliminated using their defining quadratic relations, and the final difference contains no radical. Hence $X_+ - X_- \in \mathbb{Q}(p,q)$. We write it in lowest terms as

$$X_+(p,q) - X_-(p,q) = \frac{H_{21}^{\mathrm{odd}}(p,q)}{D_{21}^{\mathrm{odd}}(p,q)}, \qquad \gcd\left(H_{21}^{\mathrm{odd}}, D_{21}^{\mathrm{odd}}\right) = 1, \tag{21}$$

where H_{21}^{odd} and D_{21}^{odd} denote the corresponding coprime numerator and denominator of this lowest-term representation. They are scaled by a common non-zero rational constant to lie in $\mathbb{Z}[p,q]$, and the numerator is chosen with content one. Thus H_{21}^{odd} means simply the reduced numerator of the $k = 1$ closing equation; no additional geometric object is attached to this notation. The later factorization is carried out entirely in $\mathbb{Q}[p,q]$.

6.1. Finite-slope formula and exceptional cases

The representation $y = ax + b$ does not include a vertical line. Accordingly, the formula (17) cannot be used when

$$S(A) = r^2 - (d - x(A))^2 = 0. \tag{22}$$

This is a limitation of the finite-slope formula, not an additional closing condition. When $S(A) = 0$, one of the two tangents is vertical. If $y(A) \neq 0$, the other tangent is finite and follows directly from the linear equation that remains in (16):

$$a_{\mathrm{fin}}(A) = \frac{r^2 - y(A)^2}{2(d - x(A))y(A)}. \tag{23}$$

At the first vertex $A_1^{(1)}$ this is exactly what happens: the vertical tangent is the initial side, while the finite tangent continues the forward half-chain.

For real finite slopes, the factor $1 + a^2$ in (18) is non-zero. Other denominator factors can also appear when the iterated formulas are simplified. They occur in the chosen finite-slope representation and mark parameter values at which that particular rational formula is not valid. The tangent step itself can instead be described by an implicit line equation, with a vertical tangent treated separately when necessary.

For the factorization below we use only the reduced rational expression (21). Since this expression is written in lowest terms,

$$\gcd\left(H_{21}^{\mathrm{odd}}, D_{21}^{\mathrm{odd}}\right) = 1 \quad \text{in } \mathbb{Q}[p,q]. \tag{24}$$

Consequently, no factor of the reduced denominator is also a factor of the closing numerator. Thus clearing the reduced denominator does not introduce an additional factor into the polynomial closing equation. Conversely, a geometric configuration is not discarded merely

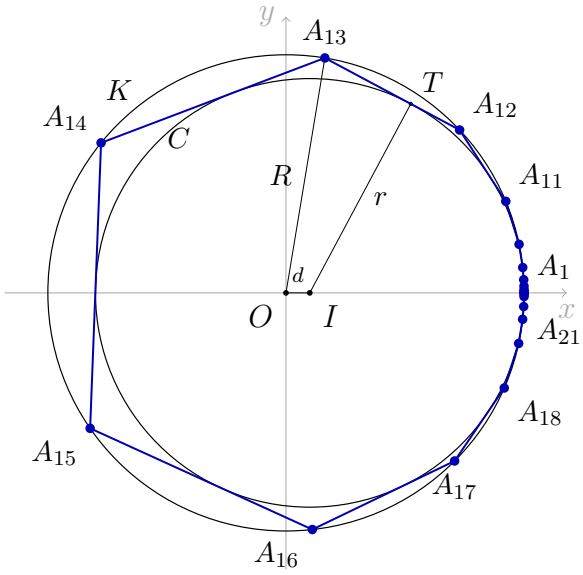


Figure 1: Numerical configuration for the representative $k = 1$ construction, with $R = 1$, $d = 0.1$, and $r = r_{21,1} \approx 0.899996$. Since r is close to $R - d = 0.9$, the initial vertical tangent is close to a common tangent of the two circles, and several consecutive vertices are visually clustered near its point of contact with C. The circumcircle K and incircle C have centres O and I, respectively. The indicated lengths satisfy $|OA_{13}| = R$ and $|IT| = r$, where T is the point of contact of C with the side $A_{12}A_{13}$. Only some vertices are labelled.

because one denominator in the displayed finite-slope chart vanishes: whenever the tangent itself is well-defined, the same step can be described by the implicit tangent equation or, for a vertical tangent, by the separate case above. The polynomial numerator is therefore the algebraic continuation of the reduced closing equation from the open set on which the displayed rational chart is valid. A zero lying on an exceptional chart set is interpreted geometrically only after checking the original tangent construction; the polynomial equation by itself does not assert nondegeneracy there.

The present computation uses explicit substitution and reduction of rational functions rather than a Gröbner-basis elimination; for general elimination methods see [6], and for a Gröbner-basis approach to relations of bicentric polygons see [13]. The $k = 2$ construction is treated analogously in Section 9.

Figure 1 fixes the notation for the two circles and illustrates the representative with rotation number $k = 1$. The numerical example is used only as a geometric reference for the coordinate construction.

7. Rotation number reductions for the 21-gon

The reduction theorem recalled in (5) gives an exact classification of all non-coprime rotation numbers for $n = 21$. If $g = \gcd(21, k)$ and $k = g\ell$, then

$$F_{21}^{(k)}(R, r, d) = F_{21/g}^{(\ell)}(R, r, d).$$

Thus $k = 3, 6, 9$ reduce to period-7 closing relations and $k = 7$ reduces to the period-3 closing relation. These reductions are direct consequences of the reduction theorem in [23]; no conjectural identification is involved.

For the six coprime values $k = 1, 2, 4, 5, 8, 10$, the situation is different. The present paper derives the closing polynomials only for $k = 1$ and $k = 2$. Radić's conjecture for odd n [23] predicts that the radius relation depends only on the parity of the coprime rotation number. For $n = 21$ this predicts that $k = 5$ has the same relation as $k = 1$, and that $k = 4, 8, 10$ have the same relation as $k = 2$. These four equalities are *not proved in this paper*. Table 1 distinguishes the proved statements from these conjectural identifications.

k	$\gcd(21, k)$	closing relation	status in this paper
1	1	$F_{21}^{(1)} = F_{21}^{\text{odd,prim}}$	derived from the $k = 1$ tangent construction after removal of periods 3 and 7
2	1	$F_{21}^{(2)} = F_{21}^{\text{even,prim}}$	derived from the $k = 2$ tangent construction after removal of periods 3 and 7
3	3	$F_{21}^{(3)} = F_{7}^{(1)}$	exact reduction $(21, 3) \to (7, 1)$
4	1	predicted: $F_{21}^{(4)} = F_{21}^{(2)}$	Radić's conjecture; not proved here
5	1	predicted: $F_{21}^{(5)} = F_{21}^{(1)}$	Radić's conjecture; not proved here
6	3	$F_{21}^{(6)} = F_{7}^{(2)}$	exact reduction $(21, 6) \to (7, 2)$
7	7	$F_{21}^{(7)} = F_{3}^{(1)}$	exact reduction $(21, 7) \to (3, 1)$
8	1	predicted: $F_{21}^{(8)} = F_{21}^{(2)}$	Radić's conjecture; not proved here
9	3	$F_{21}^{(9)} = F_{7}^{(3)}$	exact reduction $(21, 9) \to (7, 3)$
10	1	predicted: $F_{21}^{(10)} = F_{21}^{(2)}$	Radić's conjecture; not proved here

Table 1: Rotation number statements used for $n = 21$. The four coprime identifications marked as conjectural are included only to record Radić's prediction; they are not used in the symbolic derivation of the $k = 1$ and $k = 2$ polynomials.

For the reduced period-7 cases, the notation in Table 1 is compatible with the short factors used below. For $n = 7$, the reduced-Fuss formalism gives one relation for odd rotation indices and one for even rotation indices [4]. With the content-one normalization used in this paper, the odd relation is F_7^{odd} and the even relation is F_7^{even}. Consequently,

$$F_7^{(1)} = F_7^{(3)} = F_7^{\text{odd}}, \qquad F_7^{(2)} = F_7^{\text{even}}.$$

In particular, the $k = 9$ reduction $(21, 9) \to (7, 3)$ justifies the use of F_7^{odd} in the numerical table of Appendix D.

The numerical examples in Appendix D serve only as checks of individual closing configurations for the coprime values not derived symbolically. They do not turn the conjectural equalities in Table 1 into proofs.

For the two derived representatives, the degree bookkeeping is now straightforward. Each full 21-step closing polynomial has bidegree $(55, 55)$. It contains the lower-period factors associated with the proper divisors 3 and 7, of bidegrees $(1, 1)$ and $(6, 6)$. After those factors are removed, the remaining primitive period-21 factor has bidegree

$$(55, 55) - (1, 1) - (6, 6) = (48, 48).$$

This distinction between a full 21-step closing polynomial and its primitive factor is used explicitly in the two factorizations below.

8. Factorization of the $k = 1$ closing polynomial

The numerator H_{21}^{odd} was defined in (19)–(21) from the two five-step half-chains of the $k = 1$ tangent construction. The factorization is carried out in two stages. First we remove a factor that has no zeros in the admissible bicentric range. We then separate the factors corresponding to shorter periods 3 and 7. The remaining quotient is the primitive period-21 factor. All divisions below are exact divisions in $\mathbb{Q}[p, q]$.

Removal of the factor $1 + p$

The reduced numerator of the coordinate closing equation contains the linear factor $1 + p$. Since $1 + p > 0$ in the admissible range $p > q > 1$, removing this factor does not change the geometric zero set there.

Lemma 8.1 (Removal of the factor $1 + p$)**.** *The reduced numerator satisfies*

$$H_{21}^{\mathrm{odd}}(p, q) = (1 + p)F_{21}^{\mathrm{odd,full}}(p, q). \tag{25}$$

Since the admissible range satisfies $p > q > 1$, the factor $1+p$ is strictly positive there. Hence, on this range, the equation $H_{21}^{\mathrm{odd}} = 0$ is equivalent to $F_{21}^{\mathrm{odd,full}} = 0$.

Proof. Exact polynomial division in $\mathbb{Q}[p, q]$ gives

$$H_{21}^{\mathrm{odd}} = (1 + p)Q_{21}^{\mathrm{odd}}.$$

We normalize the quotient by requiring integer coefficients with content one and by choosing the sign so that

$$[p^{55}q^{55}]\, Q_{21}^{\mathrm{odd}} = 1.$$

With this normalization, $Q_{21}^{\mathrm{odd}} = F_{21}^{\mathrm{odd,full}}$. Moreover, H_{21}^{odd} and D_{21}^{odd} are coprime by (24), so $1 + p$ is not a common factor of the numerator and denominator. Finally, $1 + p > 0$ for $p > q > 1$. □

Thus $F_{21}^{\mathrm{odd,full}}$ is the full 21-step closing polynomial for the $k = 1$ construction. It should not yet be interpreted as the primitive period-21 relation, because it still contains factors corresponding to Poncelet orbits that close after 3 or 7 steps. Since $\mathrm{bideg}_{p,q} F_{21}^{\mathrm{odd,full}} = (55, 55)$, (25) also gives $\mathrm{bideg}_{p,q} H_{21}^{\mathrm{odd}} = (56, 55)$.

Lower-period factors

The factors associated with periods 3 and 7 occur because a shorter-period Poncelet orbit also satisfies the 21-step return equation. More precisely, let Q denote the translation point associated with the chosen Poncelet step on the elliptic curve. If the orbit has period dividing an odd divisor s of 21, then $sQ = O$, and therefore

$$21Q = \frac{21}{s}(sQ) = O.$$

Thus every parameter value for which the chosen Poncelet orbit has period 3 or 7 also satisfies the 21-step closing equation. The corresponding polynomial factors therefore describe shorter-period zeros of the full 21-step relation, not orbits of exact period 21. This is the standard translation interpretation of Poncelet closure; see, for example, [7, 14].

For the present polynomial, the corresponding algebraic statements are exact:

$$F_3^{\mathrm{odd}} \mid F_{21}^{\mathrm{odd,full}}, \qquad F_7^{\mathrm{odd}} \mid F_{21}^{\mathrm{odd,full}} \quad \text{in } \mathbb{Q}[p,q]. \tag{26}$$

The explicit period-3 and period-7 factors are listed in Appendix A.

The normalization of the period-7 factor is compatible with the same Cayley construction used for $n = 21$. For $n = 7$, the 3×3 Cayley determinant formed from the coefficients in (7) satisfies

$$\mathcal{C}_7(p,q) = -2^{-21} F_7^{\mathrm{odd}}(p,q) F_7^{\mathrm{even}}(p,q), \qquad F_7^{\mathrm{even}}(p,q) = F_7^{\mathrm{odd}}(-p,-q). \tag{27}$$

Both period-7 factors have bidegree $(6,6)$. Equation (27) is included only to identify the normalization of the period-7 factors appearing in (26); no separate $n = 7$ degree argument is needed here.

The primitive period-21 factor

After the lower-period factors have been removed, the remaining quotient is

$$F_{21}^{\mathrm{odd,prim}}(p,q) = \frac{F_{21}^{\mathrm{odd,full}}(p,q)}{F_3^{\mathrm{odd}}(p,q) F_7^{\mathrm{odd}}(p,q)}. \tag{28}$$

The exact divisions above show that this quotient belongs to $\mathbb{Z}[p,q]$ after normalization to content one. An additional exact coprimality check gives

$$\gcd\Big(F_{21}^{\mathrm{odd,prim}}, F_3^{\mathrm{odd}} F_7^{\mathrm{odd}}\Big) = 1 \quad \text{in } \mathbb{Q}[p,q], \tag{29}$$

so neither lower-period factor remains in the quotient. This primitive quotient is the representative Fuss relation for rotation number $k = 1$; from this point on we also write

$$F_{21}^{(1)} := F_{21}^{\mathrm{odd,prim}}. \tag{30}$$

We choose its sign by requiring

$$[p^{48} q^{48}]\, F_{21}^{\mathrm{odd,prim}} = 1.$$

Hence

$$F_{21}^{\mathrm{odd,full}} = F_3^{\mathrm{odd}} F_7^{\mathrm{odd}} F_{21}^{\mathrm{odd,prim}} \quad \text{in } \mathbb{Q}[p,q], \tag{31}$$

and, equivalently,

$$H_{21}^{\mathrm{odd}} = (1+p)F_3^{\mathrm{odd}}F_7^{\mathrm{odd}}F_{21}^{\mathrm{odd,prim}}. \tag{32}$$

The coefficient of the highest power of p in the primitive factor is

$$[p^{48}]F_{21}^{\mathrm{odd,prim}} = (q-1)^{20}(q+1)^{28}. \tag{33}$$

Its bidegree is $(48, 48)$ and it contains 1225 non-zero monomials. By contrast, the full 21-step polynomial has bidegree $(55, 55)$ and 1596 non-zero monomials, because it also contains the period-3 and period-7 factors.

factor	$\deg_p$	$\deg_q$	non-zero monomials
F_3^{odd}	1	1	3
F_7^{odd}	6	6	22
$F_{21}^{\mathrm{odd,prim}}$	48	48	1225
$F_{21}^{\mathrm{odd,full}}$	55	55	1596

Table 2: Bidegrees and monomial counts for the factors in the $k = 1$ closing polynomial.

Theorem 8.2 (Factorization of the $k = 1$ closing polynomial)**.** *For the $k = 1$ tangent construction, the full 21-step closing polynomial factors as*

$$F_{21}^{\mathrm{odd,full}} = F_3^{\mathrm{odd}}F_7^{\mathrm{odd}}F_{21}^{\mathrm{odd,prim}}. \tag{34}$$

The factors have bidegrees

$$\mathrm{bideg}_{p,q} F_3^{\mathrm{odd}} = (1,1), \qquad \mathrm{bideg}_{p,q} F_7^{\mathrm{odd}} = (6,6), \qquad \mathrm{bideg}_{p,q} F_{21}^{\mathrm{odd,prim}} = (48,48).$$

Consequently,

$$\mathrm{bideg}_{p,q} F_{21}^{\mathrm{odd,full}} = (55,55).$$

The factor $F_{21}^{\mathrm{odd,prim}}$ is the new primitive period-21 contribution; F_3^{odd} and F_7^{odd} are shorter-period factors of the same 21-step return equation.

Here, as in the Introduction, the word *primitive* refers only to the removal of factors inherited from the proper divisor periods 3 and 7. No irreducibility of $F_{21}^{\mathrm{odd,prim}}$ over $\mathbb{Q}[p, q]$ is asserted.

Proof. Equation (25) follows by exact division of the reduced numerator of the coordinate closing equation. The two exact divisibilities in (26) then give the quotient (28). The normalized quotient has bidegree $(48, 48)$ and 1225 non-zero monomials, so (34) is an identity in $\mathbb{Q}[p, q]$. Adding the bidegrees gives

$$(1,1) + (6,6) + (48,48) = (55,55).$$

□

The symbolic derivation in this section concerns the representative rotation number $k = 1$. Radić's conjecture predicts that the coprime value $k = 5$ has the same closing relation; this identification is not proved here. The numerical example for $k = 5$ in Appendix D is included only as an independent consistency check.

9. The $k = 2$ representative

We now turn to the second representative, obtained directly from the tangent construction for $k = 2$. The rotation number label is used in the sense of Remark 5.1: the tangent choices are fixed on the real closing component whose lifted angular increment is 4π before any algebraic elimination is performed. This calculation first produces the full 21-step closing polynomial $F_{21}^{\text{even,full}}$; after the period-3 and period-7 factors are removed, the representative Fuss relation is

$$F_{21}^{(2)} = F_{21}^{\text{even,prim}}.$$

The other coprime values $k = 4, 8, 10$ are not derived symbolically here; their identification with the same relation is only predicted by Radić's conjecture, as stated in Section 7.

This section records the direct $k = 2$ coordinate calculation and compares its resulting polynomial with the one already obtained for $k = 1$. Since the two expanded polynomials are related by a simple substitution in the normalized variables, the factorization analysis of Section 8 need not be repeated.

By the same scale invariance used in Section 6, we set $r = 1$; then (3) gives $R = (p+q)/2$ and $d = (p-q)/2$. The $k = 2$ construction starts from

$$A_1^{(2)} = \left(d - 1, \sqrt{R^2 - (d-1)^2}\right), \qquad A_{11}^{(11)} = (-R, 0),$$

Using the same tangent map as in Section 6, define

$$Y_+ = \pi_x\Big(T_+^5(A_1^{(2)})\Big), \qquad Y_- = \pi_x\Big(T_-^5(A_{11}^{(11)})\Big). \tag{35}$$

The choices of tangents are fixed before the resulting coordinates are simplified. For the prescribed real $k = 2$ construction, the terminal points of the two five-step half-chains are chosen on the upper semicircle

$$\Gamma_+^{(2)} = \{(R\cos\theta, R\sin\theta) : 0 \le \theta \le \pi\}.$$

On this arc the function $\theta \mapsto R\cos\theta$ is strictly decreasing, so projection to the x-axis is injective. Therefore $Y_+ = Y_-$ is equivalent to equality of the two terminal points and hence to closure of the chosen $k = 2$ half-chain.

With $r = 1$, the starting ordinate satisfies

$$\left(y_1^{(2)}\right)^2 = (p-1)(q+1).$$

As in the $k = 1$ calculation, exact simplification using the defining quadratic relations eliminates the algebraic square roots from $Y_+ - Y_-$, so the difference belongs to $\mathbb{Q}(p, q)$. In lowest terms we write

$$Y_+(p,q) - Y_-(p,q) = \frac{H_{21}^{\text{even}}(p,q)}{D_{21}^{\text{even}}(p,q)}, \qquad \gcd\Big(H_{21}^{\text{even}}, D_{21}^{\text{even}}\Big) = 1, \tag{36}$$

where H_{21}^{even} and D_{21}^{even} denote the corresponding coprime numerator and denominator of this lowest-term representation. The reduced numerator obtained from this calculation satisfies the exact identity

$$H_{21}^{\text{even}}(p,q) = (p-1)F_{21}^{\text{even,full}}(p,q). \tag{37}$$

Since $p > q > 1$ in the admissible range, the factor $p - 1$ is strictly positive there. Removing it therefore does not change the geometric closing condition in the admissible range. The quotient $F_{21}^{\mathrm{even,full}}$ is chosen as a content-one polynomial in $\mathbb{Z}[p, q]$, with the same normalization convention as $F_{21}^{\mathrm{odd,full}}$.

The independently computed $k = 2$ polynomial is related to the $k = 1$ polynomial as follows.

Proposition 9.1 (Substitution identity for the two representatives)**.** *With the normalization used throughout the paper,*

$$F_{21}^{\mathrm{even,full}}(p, q) = F_{21}^{\mathrm{odd,full}}(-p, -q) \quad \textit{in } \mathbb{Q}[p, q]. \tag{38}$$

Proof. The left-hand side is obtained independently from the iterative $k = 2$ tangent construction through (35)–(37). The right-hand side is obtained from the $k = 1$ polynomial by the indicated substitution. Their equality is verified coefficientwise by comparing the coefficients of $p^i q^j$ for $0 \le i, j \le 55$. Thus (38) is an identity in $\mathbb{Q}[p, q]$; the substitution $(p, q) \mapsto (-p, -q)$ is not being interpreted as a geometric transformation of the positive parameter domain. □

Applying (38) to the factorization from Theorem 8.2 gives

$$F_{21}^{\mathrm{even,full}} = F_3^{\mathrm{even}} F_7^{\mathrm{even}} F_{21}^{\mathrm{even,prim}}, \tag{39}$$

where

$$\begin{aligned} F_3^{\mathrm{even}}(p, q) &= F_3^{\mathrm{odd}}(-p, -q) = pq + p + q, \\ F_7^{\mathrm{even}}(p, q) &= F_7^{\mathrm{odd}}(-p, -q), \\ F_{21}^{\mathrm{even,prim}}(p, q) &= F_{21}^{\mathrm{odd,prim}}(-p, -q). \end{aligned}$$

Equivalently, the primitive factor is the exact quotient

$$F_{21}^{\mathrm{even,prim}}(p, q) = \frac{F_{21}^{\mathrm{even,full}}(p, q)}{F_3^{\mathrm{even}}(p, q) F_7^{\mathrm{even}}(p, q)}. \tag{40}$$

Accordingly,

$$F_{21}^{(2)} := F_{21}^{\mathrm{even,prim}}. \tag{41}$$

The substitution identities and (29) also give

$$\gcd\Big(F_{21}^{\mathrm{even,prim}}, F_3^{\mathrm{even}} F_7^{\mathrm{even}}\Big) = 1 \quad \text{in } \mathbb{Q}[p, q].$$

Consequently,

$$\mathrm{bideg}_{p,q} F_7^{\mathrm{even}} = (6, 6), \qquad \mathrm{bideg}_{p,q} F_{21}^{\mathrm{even,prim}} = (48, 48),$$

and $F_{21}^{\mathrm{even,full}}$ has bidegree $(55, 55)$. The primitive factor has 1225 non-zero monomials. It is the primitive period-21 factor corresponding to the second representative, whereas F_3^{even} and F_7^{even} are inherited lower-period factors of the 21-step return condition.

For positive p and q,

$$F_3^{\mathrm{even}}(p, q) = pq + p + q > 0.$$

Hence this period-3 factor cannot vanish for an admissible pair of nested circles. The product of the factors that can vanish in the admissible positive parameter range is therefore

$$F_7^{\mathrm{even}}(p, q) F_{21}^{\mathrm{even,prim}}(p, q), \tag{42}$$

which has bidegree $(54, 54)$.

We emphasize that (38) is established here only for the two symbolically derived representatives $k = 1$ and $k = 2$. Extending these identifications to the remaining coprime rotation numbers $k = 4, 5, 8, 10$ requires the additional assertions predicted by Radić's conjecture; those assertions are not proved in this paper.

10. Conclusion

We have studied the 21-step Cayley–Poncelet closing condition for two nested circles and the factorization of the two full 21-step polynomials obtained from the tangent constructions in the dimensionless variables $p = (R + d)/r$ and $q = (R - d)/r$. The classical Cayley determinant gives the total 21-step condition, while the explicit tangent constructions for the representatives $k = 1$ and $k = 2$ identify the two full representative closing polynomials occurring in that determinant factorization.

For the $k = 1$ representative, the full 21-step closing polynomial has the exact factorization

$$F_{21}^{\mathrm{odd,full}} = F_3^{\mathrm{odd}} F_7^{\mathrm{odd}} F_{21}^{\mathrm{odd,prim}}.$$

The factors of periods 3 and 7 are inherited lower-period factors of the 21-step return condition. After they are removed, the remaining primitive period-21 factor has bidegree $(48, 48)$ and 1225 non-zero monomials. Thus the bidegree $(55, 55)$ of the full 21-step closing polynomial should not be confused with the bidegree of the primitive period-21 factor. For the $k = 2$ representative, an independent iterative tangent construction gives the second full 21-step closing polynomial, and the exact polynomial identity

$$F_{21}^{\mathrm{even,full}}(p, q) = F_{21}^{\mathrm{odd,full}}(-p, -q)$$

transfers the corresponding factorization. Its primitive factor again has bidegree $(48, 48)$ and 1225 non-zero monomials. The period-3 factor $F_3^{\mathrm{even}} = pq + p + q$ has no zeros for positive p, q, so the product of the factors that can vanish for positive p, q has bidegree $(54, 54)$.

The statements concerning rotation numbers are separated according to their logical status. Radić's reduction theorem rigorously accounts for the non-coprime values $k = 3, 6, 7, 9$ by reduction to the period-3 or period-7 relation. Among the coprime values, only $k = 1$ and $k = 2$ are derived symbolically here. The identifications for $k = 4, 5, 8, 10$ remain consequences of Radić's conjecture that the relation depends only on the parity of a coprime rotation number and are supported in this paper only by auxiliary high-precision numerical checks; they are not used in the proofs of the two representative factorizations.

The main new algebraic objects obtained in the paper are therefore the two primitive representative Fuss relations $F_{21}^{(1)}$ and $F_{21}^{(2)}$, each of bidegree $(48, 48)$, obtained after the lower-period factors have been separated from the corresponding full 21-step closing polynomial. The case $21 = 3 \cdot 7$ makes this distinction explicit and provides a composite example in which the Cayley closing condition, divisor-induced factors and the new primitive period-21 factors can all be compared exactly in $\mathbb{Q}[p, q]$.

Acknowledgments

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A. Explicit lower-period factors

This appendix records only the short factors that are needed in the two period-21 factorizations. The rotation number reductions themselves are already summarized in Table 1; repeating that table in an appendix would add no new information. The factors below are written in the normalized variables (3). They represent the period-3 and period-7 factors contained in the full 21-step closing polynomials, while the quotients left after their removal are the primitive period-21 factors defined in (28) and (40).

Period 3

With the normalization used throughout the paper,

$$F_3^{\mathrm{odd}}(p,q) = pq - p - q, \tag{43}$$
$$F_3^{\mathrm{even}}(p,q) = pq + p + q. \tag{44}$$

The two factors are related by

$$F_3^{\mathrm{even}}(p,q) = F_3^{\mathrm{odd}}(-p,-q).$$

Moreover, $F_3^{\mathrm{even}}(p,q) > 0$ for $p,q > 0$. Hence this algebraic period-3 factor occurs in the full $k = 2$ closing polynomial, but it has no zero in the admissible range $p > q > 1$.

Period 7

The two period-7 factors used in Sections 8 and 9 are

$$\begin{aligned} F_7^{\mathrm{odd}}(p,q) = {} & p^6 + 2p^5q - 2p^6q - p^4q^2 + 2p^5q^2 - p^6q^2 - 4p^3q^3 + 4p^6q^3 \\ & - p^2q^4 + 2p^4q^4 - p^6q^4 + 2pq^5 + 2p^2q^5 - 2p^5q^5 - 2p^6q^5 \\ & + q^6 - 2pq^6 - p^2q^6 + 4p^3q^6 - p^4q^6 - 2p^5q^6 + p^6q^6, \end{aligned} \tag{45}$$

$$\begin{aligned} F_7^{\mathrm{even}}(p,q) = {} & p^6 + 2p^5q + 2p^6q - p^4q^2 - 2p^5q^2 - p^6q^2 - 4p^3q^3 - 4p^6q^3 \\ & - p^2q^4 + 2p^4q^4 - p^6q^4 + 2pq^5 - 2p^2q^5 - 2p^5q^5 + 2p^6q^5 \\ & + q^6 + 2pq^6 - p^2q^6 - 4p^3q^6 - p^4q^6 + 2p^5q^6 + p^6q^6. \end{aligned} \tag{46}$$

Again,

$$F_7^{\mathrm{even}}(p,q) = F_7^{\mathrm{odd}}(-p,-q), \tag{47}$$

and both factors have bidegree $(6,6)$. Their normalization agrees with the Cayley factorization for $n = 7$ (27). These are lower-period factors of the 21-step equations; they are not part of the new primitive period-21 contribution.

B. Polynomial data for the representative 21-step factors

The expanded full closing polynomials are very large: each full 21-step polynomial has 1596 non-zero monomials. Printing both expansions would therefore obscure the factorization that is the main point of the paper, and it would also be redundant because the second polynomial is obtained from the first by the exact substitution (38). We consequently record here the exact defining identities and the structural coefficient data needed to identify the two full

closing polynomials and the two primitive period-21 factors. No proof in the paper depends on a printed term-by-term expansion. For a polynomial $F(p,q)$, write

$$\operatorname{supp} F = \{(i,j) : [p^i q^j]F \neq 0\};$$

thus $|\operatorname{supp} F|$ is the number of non-zero monomials of F.

The $k=1$ full closing polynomial is the normalized quotient determined by the coordinate construction,

$$F_{21}^{\mathrm{odd,full}}(p,q) = \frac{H_{21}^{\mathrm{odd}}(p,q)}{1+p}, \qquad [p^{55}q^{55}]\, F_{21}^{\mathrm{odd,full}} = 1, \tag{48}$$

where the division is exact in $\mathbb{Q}[p,q]$. It has

$$\operatorname{bideg}_{p,q} F_{21}^{\mathrm{odd,full}} = (55,55), \qquad |\operatorname{supp} F_{21}^{\mathrm{odd,full}}| = 1596, \tag{49}$$

and its coefficient of the highest power of p is

$$[p^{55}]\, F_{21}^{\mathrm{odd,full}} = (q-1)^{25}(q+1)^{30}. \tag{50}$$

The lower-period factors from Appendix A divide this polynomial exactly, and the primitive quotient is

$$F_{21}^{\mathrm{odd,prim}}(p,q) = \frac{F_{21}^{\mathrm{odd,full}}(p,q)}{F_3^{\mathrm{odd}}(p,q)F_7^{\mathrm{odd}}(p,q)}. \tag{51}$$

Thus

$$\operatorname{bideg}_{p,q} F_{21}^{\mathrm{odd,prim}} = (48,48), \qquad |\operatorname{supp} F_{21}^{\mathrm{odd,prim}}| = 1225, \qquad [p^{48}q^{48}]\, F_{21}^{\mathrm{odd,prim}} = 1, \tag{52}$$

with

$$[p^{48}]\, F_{21}^{\mathrm{odd,prim}} = (q-1)^{20}(q+1)^{28}. \tag{53}$$

In particular, $F_{21}^{(1)} = F_{21}^{\mathrm{odd,prim}}$ has bidegree $(48,48)$ and 1225 non-zero monomials, whereas the bidegree $(55,55)$ and the 1596 monomials refer to the full $k=1$ 21-step closing polynomial that still contains the period-3 and period-7 factors.

The independently obtained $k=2$ closing polynomial satisfies

$$F_{21}^{\mathrm{even,full}}(p,q) = F_{21}^{\mathrm{odd,full}}(-p,-q), \tag{54}$$

so a second 1596-term expansion carries no additional information. It has

$$\operatorname{bideg}_{p,q} F_{21}^{\mathrm{even,full}} = (55,55), \qquad |\operatorname{supp} F_{21}^{\mathrm{even,full}}| = 1596, \qquad [p^{55}]\, F_{21}^{\mathrm{even,full}} = (q-1)^{30}(q+1)^{25}. \tag{55}$$

Likewise,

$$F_{21}^{(2)}(p,q) = F_{21}^{\mathrm{even,prim}}(p,q) = F_{21}^{\mathrm{odd,prim}}(-p,-q) = \frac{F_{21}^{\mathrm{even,full}}(p,q)}{F_3^{\mathrm{even}}(p,q)F_7^{\mathrm{even}}(p,q)}, \tag{56}$$

with

$$\begin{gathered}\operatorname{bideg}_{p,q} F_{21}^{\mathrm{even,prim}} = (48,48), \qquad |\operatorname{supp} F_{21}^{\mathrm{even,prim}}| = 1225, \\ [p^{48}]\, F_{21}^{\mathrm{even,prim}} = (q-1)^{28}(q+1)^{20}.\end{gathered} \tag{57}$$

For convenient comparison, the four principal polynomials are summarized in Table B.1.

No term-by-term expansion is required for the statements or proofs: the polynomials are determined in the manuscript by the iterated tangent maps, the exact quotient identities, and the Cayley determinant. The exact algebraic checks used for these identities are summarized in Appendix C.

polynomial	bidegree	monomials	definition
$F_{21}^{\mathrm{odd,full}}$	$(55, 55)$	1596	(48)
$F_{21}^{\mathrm{odd,prim}}$	$(48, 48)$	1225	(51)
$F_{21}^{\mathrm{even,full}}$	$(55, 55)$	1596	(54)
$F_{21}^{\mathrm{even,prim}}$	$(48, 48)$	1225	(56)

Table B.1: Structural data for the two full 21-step closing polynomials and their primitive period-21 factors.

C. Exact algebraic checks

All symbolic checks in the paper are carried out with exact rational arithmetic in $\mathbb{Q}[p, q]$. No numerical tolerance is involved. The exact algebraic statements recorded below consist of polynomial identities, exact-division tests, and lowest-term reductions.

check	exact statement
lowest-term rational forms	The two rational closing equations are written in lowest terms in $\mathbb{Q}(p, q)$; hence their numerator–denominator pairs satisfy $\gcd(H_{21}^{\mathrm{odd}}, D_{21}^{\mathrm{odd}}) = \gcd(H_{21}^{\mathrm{even}}, D_{21}^{\mathrm{even}}) = 1$.
removal of $1 + p$ and $p - 1$	The divisions $H_{21}^{\mathrm{odd}}/(1+p)$ and $H_{21}^{\mathrm{even}}/(p-1)$ are exact in $\mathbb{Q}[p, q]$ and give the content-one full 21-step closing polynomials $F_{21}^{\mathrm{odd,full}}$ and $F_{21}^{\mathrm{even,full}}$.
lower-period factors	Exact division gives $F_{21}^{\mathrm{odd,full}} = F_3^{\mathrm{odd}} F_7^{\mathrm{odd}} F_{21}^{\mathrm{odd,prim}}$ and $F_{21}^{\mathrm{even,full}} = F_3^{\mathrm{even}} F_7^{\mathrm{even}} F_{21}^{\mathrm{even,prim}}$.
primitive coprimality	Exact gcd computation gives $\gcd(F_{21}^{\mathrm{odd,prim}}, F_3^{\mathrm{odd}} F_7^{\mathrm{odd}}) = 1$; the corresponding even identity follows by the substitution $(p, q) \mapsto (-p, -q)$.
substitution identity	Every coefficient of $F_{21}^{\mathrm{even,full}}(p, q) - F_{21}^{\mathrm{odd,full}}(-p, -q)$ is zero.
Cayley factorization	Every coefficient of $2^{210}\mathcal{C}_{21} - F_{21}^{\mathrm{odd,full}} F_{21}^{\mathrm{even,full}}$ is zero.
structural data	Expansion in the monomial basis $p^i q^j$ gives bidegree $(55, 55)$ and 1596 monomials for each full 21-step closing polynomial, and bidegree $(48, 48)$ and 1225 monomials for each primitive factor; the leading coefficients are those recorded in Appendix B.

Table C.1: Exact algebraic statements and symbolic checks used for the two representatives.

For the substitution identity it is enough to compare coefficients in the rectangle $0 \leq i, j \leq 55$; for the Cayley factorization the corresponding rectangle is $0 \leq i, j \leq 110$. These bounds follow from the bidegrees of the polynomials already stated in the manuscript. Thus both assertions are finite identities in $\mathbb{Q}[p, q]$, rather than numerical tests.

D. Numerical closing checks

The exact polynomial identities used in the proofs are recorded in the preceding appendix. The computations below have a different and strictly auxiliary purpose: they verify the closing behavior and rotation number of the chosen positive roots. In particular, the rows $k =$

$4, 5, 8, 10$ provide numerical evidence for the identifications predicted by Radić's conjecture, but no numerical computation in this appendix is used to prove an identity in $\mathbb{Q}[p, q]$.

We use the test pair

$$R = 1, \qquad d = 0.1, \qquad p = \frac{1.1}{r}, \qquad q = \frac{0.9}{r}.$$

For the coprime rotation numbers we solve the representative Fuss relations $F_{21}^{(1)} = F_{21}^{\mathrm{odd,prim}} = 0$ or $F_{21}^{(2)} = F_{21}^{\mathrm{even,prim}} = 0$, rather than the corresponding full 21-step polynomials. This removes the period-3 and period-7 zeros before the numerical root search. For the non-coprime rotation numbers we use the reduced period-3 or period-7 factor dictated by Table 1. In particular, the $k = 9$ row uses $F_7^{\mathrm{odd}} = F_7^{(3)}$, as explained after that table.

For each chosen positive root, the oriented tangent construction is then run independently for 21 steps. If A_1 is the initial point and A_{22}^{num} is the point obtained after the numerical iteration, we report the closing residual

$$\rho_{\mathrm{cl}} = \left\| A_{22}^{\mathrm{num}} - A_1 \right\|_2.$$

At the same time the polar angle is lifted continuously along the orbit and its total increment is checked to be $2\pi k$. This angular check verifies the rotation number. We do not tabulate a polynomial residual: because r is obtained by solving the corresponding polynomial equation, such a residual is only an accuracy diagnostic and is not independent evidence for the identification of the rotation number.

The roots were computed with 100 decimal digits, and the closing residuals in Table D.1 were evaluated before rounding the displayed values of $r_{21,k}$. In every row the lifted angular increment recovers the listed rotation number k.

k	polynomial used	status	$r_{21,k}$	ρ_{cl}
1	$F_{21}^{\mathrm{odd,prim}} = 0$	derived	0.899996214987	$3.9 \cdot 10^{-28}$
2	$F_{21}^{\mathrm{even,prim}} = 0$	derived	0.896426092706	$4.0 \cdot 10^{-28}$
3	$F_7^{\mathrm{odd}} = 0$	reduced	0.867948377133	$3.2 \cdot 10^{-29}$
4	$F_{21}^{\mathrm{even,prim}} = 0$	conjectural	0.807181182896	$1.8 \cdot 10^{-30}$
5	$F_{21}^{\mathrm{odd,prim}} = 0$	conjectural	0.721260053614	$2.5 \cdot 10^{-29}$
6	$F_7^{\mathrm{even}} = 0$	reduced	0.615856292562	$1.4 \cdot 10^{-30}$
7	$F_3^{\mathrm{odd}} = 0$	reduced	0.495000000000	$1.8 \cdot 10^{-100}$
8	$F_{21}^{\mathrm{even,prim}} = 0$	conjectural	0.362176034299	$2.3 \cdot 10^{-29}$
9	$F_7^{\mathrm{odd}} = 0$	reduced	0.220762803668	$9.6 \cdot 10^{-30}$
10	$F_{21}^{\mathrm{even,prim}} = 0$	conjectural	0.074165654449	$2.0 \cdot 10^{-30}$

Table D.1: Numerical closing checks for $R = 1$ and $d = 0.1$. "Derived" refers to the two representatives obtained symbolically in this paper; "reduced" refers to the non-coprime cases covered by Radić's reduction theorem; and "conjectural" marks the four coprime identifications that are not proved here. The displayed values of $r_{21,k}$ are rounded, whereas the tangent iteration uses the unrounded high-precision roots.

The rows $k = 1, 2, 3, 6, 7, 9$ are numerical consistency checks for results already established symbolically or by Radić's reduction theorem. The only rows that bear on the conjectural coprime classification are $k = 4, 5, 8, 10$. In each of those four cases, a positive zero of the

predicted primitive factor produces a numerically closed 21-step orbit whose lifted angular increment has the stated rotation number. This is compatible with Radić's conjecture, but a finite collection of numerical examples cannot establish equality of the corresponding polynomial relations for all admissible parameters.

For $k = 1$ the value $r_{21,1}$ lies close to the limiting value $R - d = 0.9$, which explains the clustering visible in Figure 1. Only one test pair is reported because these calculations are illustrative; the algebraic conclusions of the paper are independent of the numerical data.

E. Figures

Figure 2 illustrates the remaining rotation number cases summarized in Table 1. All configurations use $R = 1$ and $d = 0.1$, with the corresponding values $r = r_{21,k}$ from Table D.1. The figure is illustrative only: the symbolic status of each case is the one stated in Table 1.

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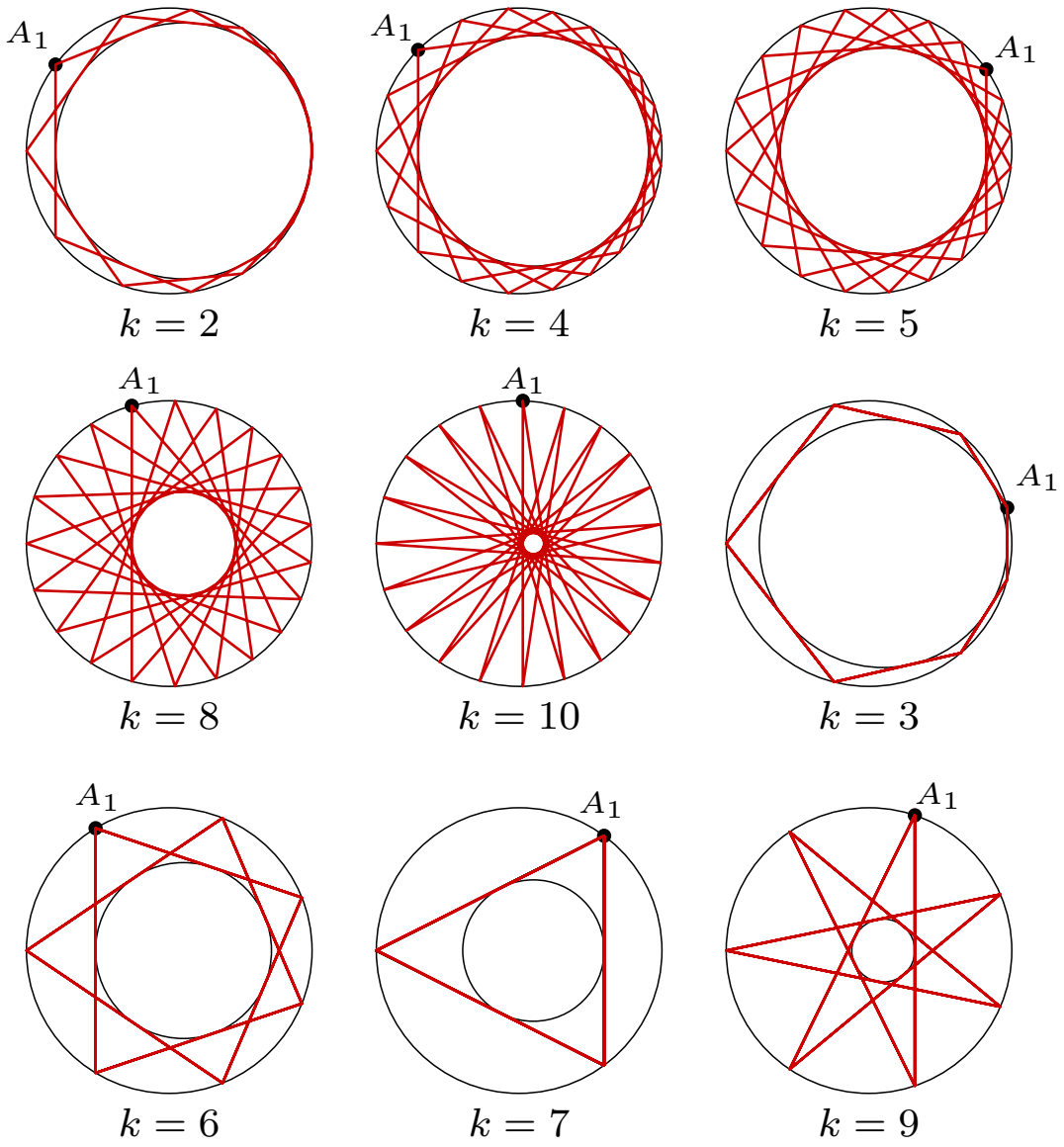


Figure 2: Numerical configurations for the remaining rotation numbers, with the outer and inner circles given by K and C in (2). The coprime panels $k = 2, 4, 5, 8, 10$ display 21-step cycles; among them, $k = 2$ is the second representative derived symbolically in this paper, whereas $k = 4, 5, 8, 10$ illustrate the conjectural identifications described in Section 7. For $k = 3, 6, 9$, the 21-step sequence traverses the corresponding 7-gon cycle three times, and for $k = 7$ it traverses the 3-gon cycle seven times; hence only the distinct geometric edges are visible in these four panels. The drawings use the unrounded numerical roots underlying Table D.1.

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