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# A Volume Comparison for Parallel and Orthogonal Projections in a Simplex

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**Abstract.** Let $SA_1 \cdots A_n$ be a nondegenerate simplex in Euclidean $n$-space, where $n \geq 2$, and let $P$ be any point. For each $i$, let $h_i$ be the facet hyperplane opposite $A_i$. Denote by $X_i$ and $U_i$ the projections of $P$ onto $h_i$, respectively parallel to $SA_i$ and orthogonal to $h_i$. We prove that

$$\mathrm{Vol}_n(PX_1 \cdots X_n) \geq \mathrm{Vol}_n(PU_1 \cdots U_n).$$

If $P$ lies on one of the hyperplanes $h_i$, both volumes are zero. Otherwise, equality holds if and only if the edges through $S$ are pairwise orthogonal. The proof uses a reciprocal basis and Hadamard's determinant inequality.

*Key Words:* Simplex, Pedal simplex, Volume comparison, Parallel projection, Orthogonal projection, Reciprocal basis, Hadamard inequality

*MSC 2020:* 51M16 (primary), 51M25, 15A45

## 1 Introduction

We compare two simplices obtained by projecting a point onto the facet hyperplanes through one vertex of a given simplex. One construction uses projections parallel to the edges through that vertex; the other uses orthogonal projections. The first simplex always has at least as large a volume as the second.

For background on higher-dimensional geometry, see Sommerville [17] and Coxeter [7]. Standard references for geometric inequalities include Burago and Zalgaller [6] and, for simplices and tetrahedra, Mitrinović, Pečarić, and Volenec [16].

Examples of higher-dimensional extensions include the Pythagorean theorem [8, 21], van Aubel's theorem [19], the golden section construction in a hypercube [22], and a conjecture of Thébault [23]. Strengthened forms of Klamkin's inequality for an $n$-simplex were obtained in [20].

Related work in the *Journal for Geometry and Graphics* includes Buba-Brzozowa's study of the Monge point and a higher-dimensional analogue of the nine-point sphere [5], volume

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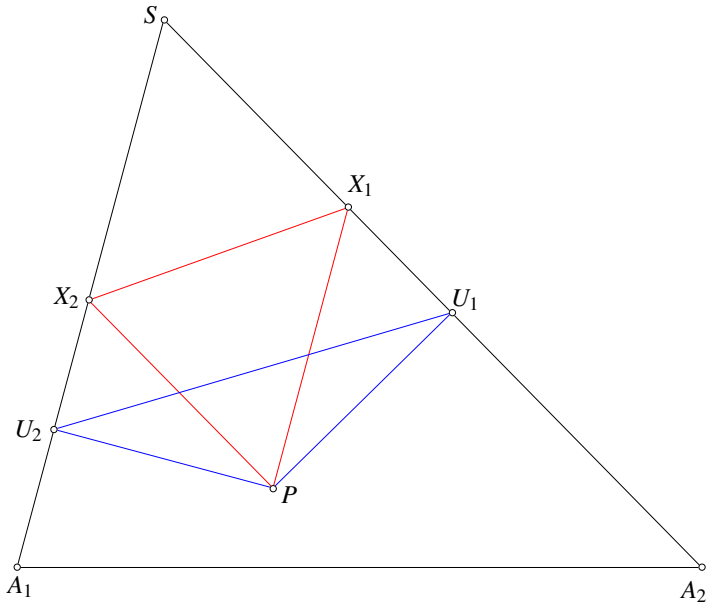



Figure 1: The planar construction.

computations for polyhedra [4], and angle conditions for simplices [14]. For tetrahedra, Bailey and Katsuura [3] proved several geometric inequalities, including an analogue of Weitzenböck's inequality, while Katsuura [13] obtained relations involving the inradius and circumradius.

Inequalities for the volumes of pedal simplices and their subsimplices were studied by Sun Mingbao [18] and Ma and Zhang [15]. Volume maxima for cevian simplices and their parts were studied by Aliyev [1] and Guliyeva and Aliyev [10].

We begin with the planar case.

**Theorem 1** (Planar case)**.** *Let $SA_1A_2$ be a triangle, and let $P$ be any point in its plane. Let $X_1$ be the intersection of the line $SA_2$ with the line through $P$ parallel to $SA_1$, and define $X_2$ similarly, with the indices interchanged. Let $U_1$ and $U_2$ be the orthogonal projections of $P$ onto the lines $SA_2$ and $SA_1$, respectively. Then*

$$\text{Area}(PX_1X_2) \geq \text{Area}(PU_1U_2).$$

*If $P$ lies on either line $SA_1$ or $SA_2$, both areas are zero. Otherwise, equality holds if and only if $SA_1 \perp SA_2$. See Figure 1.*

The planar inequality has a direct area proof. Outside the zero-area cases,

$$\text{Area}(PX_1X_2) = \text{Area}(PU_1X_2) \geq \text{Area}(PU_1U_2).$$

The first equality follows because $X_1$ and $U_1$ lie on a line parallel to $PX_2$. For the inequality, use the common base $PU_1$. Since $PX_2 \perp PU_1$, the altitude from $X_2$ has length $PX_2$. The altitude from $U_2$ is at most $PU_2$, and $PU_2 \leq PX_2$ because $U_2$ is the closest point to $P$ on $SA_1$. Equality requires $U_2 = X_2$, which is equivalent to $SA_1 \perp SA_2$; in that case both area comparisons are equalities.

Nguyen Minh Ha [11] gave a tetrahedral analogue with $P$ on the face opposite $S$. The following theorem extends the comparison to every dimension $n \geq 2$ and allows $P$ to be any point of the ambient space.

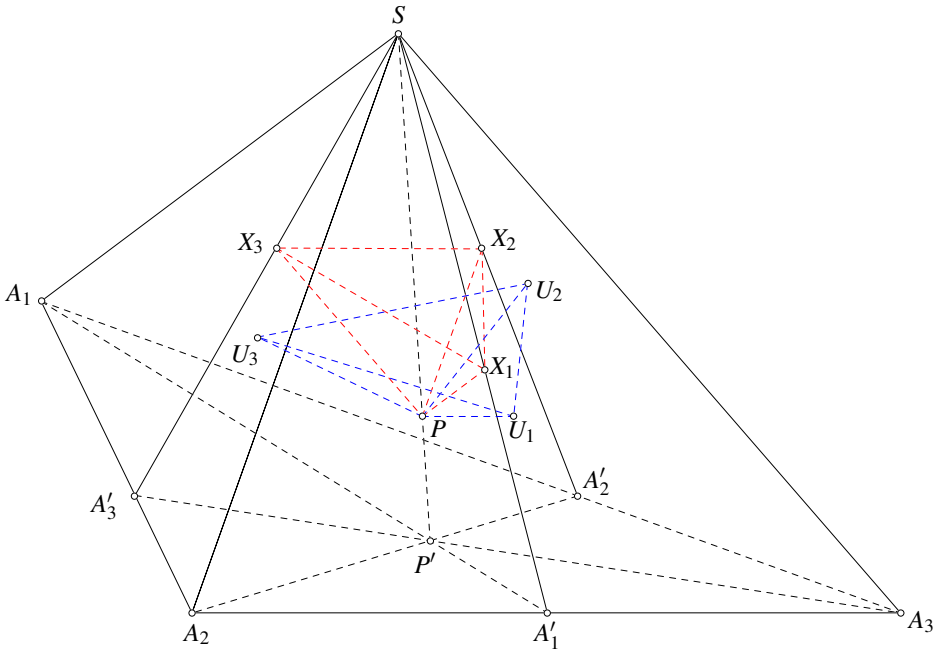



Figure 2: The tetrahedral case. In this illustration, $P' = SP \cap A_1A_2A_3$, and $A_i'$ is the intersection of $A_iP'$ with the opposite edge of $A_1A_2A_3$.

**Theorem 2.** *Let $SA_1 \cdots A_n$ be a nondegenerate $n$-simplex in $\mathbb{R}^n$, where $n \geq 2$, and let $P$ be any point of $\mathbb{R}^n$. For each $i \in \{1, \ldots, n\}$, set*

$$h_i = \mathrm{aff}(S, A_1, \ldots, A_{i-1}, A_{i+1}, \ldots, A_n).$$

*Let $X_i$ be the intersection of $h_i$ with the line through $P$ parallel to $SA_i$, and let $U_i$ be the orthogonal projection of $P$ onto $h_i$. Then*

$$\mathrm{Vol}_n(PX_1X_2 \cdots X_n) \geq \mathrm{Vol}_n(PU_1U_2 \cdots U_n).$$

*If $P \in h_i$ for some $i$, both volumes are zero. Otherwise, equality holds if and only if the edges $SA_1, \ldots, SA_n$ are pairwise orthogonal. See Figure 2.*

If $U_0$ is the orthogonal projection of $P$ onto $\mathrm{aff}(A_1, \ldots, A_n)$, then $U_0U_1 \cdots U_n$ is the pedal simplex of $P$. The simplex $PU_1 \cdots U_n$ is formed by joining $P$ to the face $U_1 \cdots U_n$ of the pedal simplex.

The restriction that $P$ lie on the face opposite $S$, used in the earlier tetrahedral formulation [11], is unnecessary for the inequality.

## 2 Preliminaries

### 2.1 Volume Formulas and Reciprocal Bases

Write $\det(\mathbf{v}_1, \ldots, \mathbf{v}_n)$ for the determinant with these columns in a fixed oriented orthonormal basis of $\mathbb{R}^n$. Its value is independent of the choice of such a basis. We use $\langle \cdot, \cdot \rangle$ for the Euclidean inner product.

For a $k$-simplex $Q_0Q_1 \cdots Q_k$ in $\mathbb{R}^n$, put $\mathbf{q}_i = \overrightarrow{Q_0Q_i}$ for $1 \leq i \leq k$. The Gram determinant $\det(\langle \mathbf{q}_i, \mathbf{q}_j \rangle)_{i,j=1}^k$ is the square of the volume of the parallelotope spanned by these vectors.

Dividing its volume by $k!$ gives

$$\mathrm{Vol}_k(Q_0Q_1\cdots Q_k) = \frac{1}{k!}\sqrt{\det\Big(\Big\langle\overrightarrow{Q_0Q_i},\overrightarrow{Q_0Q_j}\Big\rangle\Big)_{i,j=1}^{k}}. \tag{1}$$

See [9, Section 14.1] for a derivation by Gram–Schmidt orthogonalization. We allow degenerate simplices, whose volumes are zero.

In the special case $k = n$, let $B$ be the square matrix whose columns are $\mathbf{q}_1, \ldots, \mathbf{q}_n$. Its Gram matrix is $B^TB$, and hence

$$\det(B^TB) = \det(B^T)\det(B) = \det(B)^2.$$

Substituting this identity into (1) yields

$$\mathrm{Vol}_n(Q_0Q_1\cdots Q_n) = \frac{1}{n!}\Big|\det\Big(\overrightarrow{Q_0Q_1},\ldots,\overrightarrow{Q_0Q_n}\Big)\Big|. \tag{2}$$

For a basis $\mathbf{a}_1, \ldots, \mathbf{a}_n$ of $\mathbb{R}^n$, its *reciprocal basis* is the unique basis $\mathbf{a}_1^*, \ldots, \mathbf{a}_n^*$ satisfying

$$\left\langle \mathbf{a}_i^*, \mathbf{a}_j \right\rangle = \delta_{ij} \tag{3}$$

for $1 \le i, j \le n$, where $\delta_{ij} = 1$ if $i = j$ and 0 otherwise. It is the algebraic dual basis represented by vectors through the Euclidean inner product; see [2, Section 3F]. In particular, $\mathbf{a}_i^*$ is perpendicular to $\mathrm{span}\{\mathbf{a}_j : j \ne i\}$.

## 2.2 Two Determinant Lemmas

We first recall Hadamard's determinant inequality; see Hadamard's original paper [12] or [9, Chapter 14].

**Lemma 1** (Hadamard's inequality)**.** *For arbitrary vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n \in \mathbb{R}^n$,*

$$|\det(\mathbf{v}_1, \ldots, \mathbf{v}_n)| \le \prod_{i=1}^{n} \|\mathbf{v}_i\|.$$

*If all the vectors are nonzero, equality holds if and only if they are pairwise orthogonal.*

*Proof.* Let $\mathbf{w}_1 = \mathbf{v}_1$. For $i \ge 2$, let $\mathbf{w}_i$ be the component of $\mathbf{v}_i$ perpendicular to the span of the preceding vectors. This definition also applies when the vectors are linearly dependent. Gram–Schmidt orthogonalization subtracts linear combinations of preceding columns, so it preserves the determinant. Since the $\mathbf{w}_i$ are mutually orthogonal,

$$|\det(\mathbf{v}_1, \ldots, \mathbf{v}_n)| = |\det(\mathbf{w}_1, \ldots, \mathbf{w}_n)| = \prod_{i=1}^{n} \|\mathbf{w}_i\| \le \prod_{i=1}^{n} \|\mathbf{v}_i\|.$$

If all $\mathbf{v}_i$ are nonzero, equality in the product holds exactly when $\|\mathbf{w}_i\| = \|\mathbf{v}_i\|$ for every $i$. Each $\mathbf{v}_i$ must then be perpendicular to all preceding vectors. This is equivalent to pairwise orthogonality. □

**Lemma 2** (Determinant of the reciprocal basis)**.** *Let $\mathbf{a}_1, \ldots, \mathbf{a}_n$ be a basis of $\mathbb{R}^n$, and let $\mathbf{a}_1^*, \ldots, \mathbf{a}_n^*$ be its reciprocal basis. Then*

$$\det(\mathbf{a}_1^*, \ldots, \mathbf{a}_n^*) = \frac{1}{\det(\mathbf{a}_1, \ldots, \mathbf{a}_n)}.$$

*Proof.* Let $A$ and $A^*$ be the matrices with columns $\mathbf{a}_i$ and $\mathbf{a}_i^*$, respectively. By (3),

$$(A^*)^T A = I_n.$$

Since $A$ is invertible, $(A^*)^T = A^{-1}$, and hence $A^* = (A^{-1})^T$. Taking determinants gives

$$\det A^* = \det\Big((A^{-1})^T\Big) = \det(A^{-1}) = \frac{1}{\det A}. \qquad \square$$

## 3 Proof of the Main Theorem

*Proof of Theorem 2.* Take $S$ as the origin and write $\mathbf{a}_i = \overrightarrow{SA_i}$ for $1 \le i \le n$. Since the simplex is nondegenerate, these vectors form a basis. Write

$$\overrightarrow{SP} = \sum_{i=1}^{n} \mu_i \mathbf{a}_i, \tag{4}$$

where the coefficients $\mu_i$ are real.

Each $h_i$ passes through $S$, so

$$h_i = \operatorname{span}\{\mathbf{a}_j : j \neq i\}. \tag{5}$$

The line defining $X_i$ has direction $\mathbf{a}_i$, so $\overrightarrow{PX_i} = t_i \mathbf{a}_i$ for some scalar $t_i$. Then

$$\overrightarrow{SX_i} = \overrightarrow{SP} + t_i \mathbf{a}_i = \sum_{j \neq i} \mu_j \mathbf{a}_j + (\mu_i + t_i)\mathbf{a}_i.$$

The condition $X_i \in h_i$ forces the coefficient of $\mathbf{a}_i$ to vanish. Thus $t_i = -\mu_i$, and

$$\overrightarrow{PX_i} = -\mu_i \mathbf{a}_i. \tag{6}$$

By (2) and multilinearity,

$$\begin{aligned} n!\operatorname{Vol}_n(PX_1 \cdots X_n) &= |\det(-\mu_1 \mathbf{a}_1, \ldots, -\mu_n \mathbf{a}_n)| \\ &= \left(\prod_{i=1}^{n} |\mu_i|\right) |\det(\mathbf{a}_1, \ldots, \mathbf{a}_n)|. \end{aligned} \tag{7}$$

To find $\overrightarrow{PU_i}$, let $F_i$ be the orthogonal projection of $A_i$ onto $h_i$, and let $\pi_i : \mathbb{R}^n \to h_i$ be the orthogonal projection map. Since $h_i$ is a linear subspace, $\pi_i$ is linear. It fixes $\mathbf{a}_j$ for $j \neq i$ and sends $\mathbf{a}_i$ to $\overrightarrow{SF_i}$. Applying it to (4) gives

$$\overrightarrow{SU_i} = \pi_i\left(\sum_{j=1}^{n} \mu_j \mathbf{a}_j\right) = \mu_i \overrightarrow{SF_i} + \sum_{j \neq i} \mu_j \mathbf{a}_j.$$

Subtracting (4), the terms with $j \neq i$ cancel, leaving

$$\overrightarrow{PU_i} = \overrightarrow{SU_i} - \overrightarrow{SP} = \mu_i(\overrightarrow{SF_i} - \mathbf{a}_i) = \mu_i \overrightarrow{A_i F_i}. \tag{8}$$

Let $\mathbf{a}_1^*, \ldots, \mathbf{a}_n^*$ be the reciprocal basis. By (3) and (5), $\mathbf{a}_i^*$ is normal to $h_i$. The vector $\overrightarrow{A_i F_i}$ is also normal to $h_i$, so

$$\overrightarrow{A_i F_i} = c_i \mathbf{a}_i^* \tag{9}$$

for some scalar $c_i$. Since $F_i \in h_i$ and $\overrightarrow{SF_i} = \mathbf{a}_i + c_i\mathbf{a}_i^*$,

$$0 = \left\langle \mathbf{a}_i^*, \overrightarrow{SF_i} \right\rangle = \langle \mathbf{a}_i^*, \mathbf{a}_i + c_i\mathbf{a}_i^* \rangle = 1 + c_i\|\mathbf{a}_i^*\|^2 .$$

Thus $c_i = -1/\|\mathbf{a}_i^*\|^2$, and (9) becomes

$$\overrightarrow{A_iF_i} = -\frac{\mathbf{a}_i^*}{\|\mathbf{a}_i^*\|^2}. \tag{10}$$

Combining (8) and (10), we obtain

$$\overrightarrow{PU_i} = -\mu_i\frac{\mathbf{a}_i^*}{\|\mathbf{a}_i^*\|^2}. \tag{11}$$

The same determinant calculation gives

$$\begin{aligned} n!\,\mathrm{Vol}_n(PU_1\cdots U_n) &= \left|\det\left(-\mu_1\frac{\mathbf{a}_1^*}{\|\mathbf{a}_1^*\|^2}, \ldots, -\mu_n\frac{\mathbf{a}_n^*}{\|\mathbf{a}_n^*\|^2}\right)\right| \\ &= \left(\prod_{i=1}^n |\mu_i|\right)\frac{|\det(\mathbf{a}_1^*, \ldots, \mathbf{a}_n^*)|}{\prod_{i=1}^n\|\mathbf{a}_i^*\|^2}. \end{aligned} \tag{12}$$

Hadamard's inequality applied to the reciprocal basis gives

$$|\det(\mathbf{a}_1^*, \ldots, \mathbf{a}_n^*)|^2 \le \prod_{i=1}^n\|\mathbf{a}_i^*\|^2 . \tag{13}$$

Since the reciprocal vectors form a basis, the quantity

$$|\det(\mathbf{a}_1^*, \ldots, \mathbf{a}_n^*)| \prod_{i=1}^n\|\mathbf{a}_i^*\|^2$$

is positive. Dividing (13) by it and using Lemma 2, we obtain

$$\frac{|\det(\mathbf{a}_1^*, \ldots, \mathbf{a}_n^*)|}{\prod_{i=1}^n\|\mathbf{a}_i^*\|^2} \le \frac{1}{|\det(\mathbf{a}_1^*, \ldots, \mathbf{a}_n^*)|} = |\det(\mathbf{a}_1, \ldots, \mathbf{a}_n)|.$$

Multiplying by the nonnegative factor $\frac{1}{n!}\prod_{i=1}^n |\mu_i|$ and using (7) and (12) proves the desired volume inequality.

For equality, first suppose $\mu_i = 0$ for some $i$. By (4) and (5), this is equivalent to $P \in h_i$. Equations (6) and (11) then give $X_i = U_i = P$, so both volumes are zero.

If every $\mu_i$ is nonzero, the common factor $\prod_{i=1}^n |\mu_i|$ in (7) and (12) is positive. Equality of the volumes is therefore equivalent to equality in (13), which holds exactly when the reciprocal vectors are pairwise orthogonal.

To express this condition in terms of the original edges, let $A$ and $A^*$ again denote the matrices with columns $\mathbf{a}_i$ and $\mathbf{a}_i^*$. Their Gram matrices are

$$\begin{aligned} G &= A^TA = (\left\langle \mathbf{a}_i, \mathbf{a}_j \right\rangle)_{i,j=1}^n, \\ G^* &= (A^*)^TA^* = (\left\langle \mathbf{a}_i^*, \mathbf{a}_j^* \right\rangle)_{i,j=1}^n. \end{aligned}$$

Since $A^* = (A^{-1})^T$,

$$G^* = A^{-1}(A^{-1})^T = (A^TA)^{-1} = G^{-1}.$$

The reciprocal vectors are pairwise orthogonal exactly when $G^*$ is diagonal. An invertible matrix is diagonal if and only if its inverse is diagonal, so this is equivalent to $G$ being diagonal, or to the original edges being pairwise orthogonal.

Since $\mathbf{a}_i^*$ is normal to $h_i$, this is also equivalent to the hyperplanes $h_1, \ldots, h_n$ being pairwise orthogonal, meaning that their normal vectors are pairwise perpendicular. □

**Corollary 1.** *The cases $n = 2$ and $n = 3$ of Theorem 2 give Theorem 1 and the tetrahedral inequality, respectively. In the latter case, $P$ may be any point of $\mathbb{R}^3$.*

## 4 Conclusion

For a nondegenerate simplex $SA_1 \cdots A_n$ and any point $P$, projections parallel to the edges $SA_i$ give a simplex of volume at least that obtained by orthogonal projections onto the same facet hyperplanes. The result holds in every dimension $n \geq 2$.

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