Journal for Geometry and Graphics
Volume 30 (2026), No. 2, 127–155

The Simplest Flexible Cross-Polytope

Hellmuth Stachel

Vienna University of Technology, Austria
stachel@dmg.tuwien.ac.at

Abstract. According to R. Bricard there exist three types of flexible octahedra. The octahedra of Type 3 are unsymmetric and admit two flat poses. With regard to higher-dimensional analogues of octahedra called cross-polytopes, A. A. Gaĭfullin presented in 2014 a complete classification of flexible types in n-dimensional Euclidean, hyperbolic and spherical spaces for $n > 3$. The goal of this presentation is a synthetic approach to a particular family in the Euclidean n-space, the flexible cross-polytopes that admit two poses within hyperplanes. We provide a construction of their flat poses and prove several properties of these higher-dimensional analogues to Bricard's type-3 octahedra. According to Gaĭfullin, they are the simplest from the algebraic point of view.

Key Words: Bricard's octahedra, Flexible cross-polytopes, Twice-flat polytope
MSC 2020: 52C25 (primary), 52B11, 53A17, 70B15

1 Introduction

In 1897, Raoul Bricard [2] proved that in 3-dimensional Euclidean space $\mathbb{E}^3$ there exist three types of flexible octahedra, i.e., polyhedra of the combinatorial type of a regular octahedron with rigid faces and edges functioning as hinges, while self-intersections are ignored. In 1912, G. T. Bennett [1] studied the flexions of these types with regard to their spherical images. A different approach is due to R. Connelly in 1978 [4]. The author classified in [14] the flexible octahedra due to their relation to confocal quadrics and based on a configuration theorem on bipartite graphs. Flexible octahedra can also be seen as Kokotsakis meshes with a triangular basis (note, e.g., [16]). The kinematics of these meshes was recently studied in [10]. In [12], flexible open chains of type-3 octahedra without link interference were presented, where consecutive octahedra share an edge and the adjacent spanned planes.

Bricard's octahedra of Type 3 are unsymmetric and doubly collapsible, i.e., they admit two flat poses. From the geometric point of view, this type is the most complicated. However, from the algebraic point it is the simplest as A. A. Gaĭfullin mentions in [5, p. 90]. Figure 1a shows how a flat pose of a type-3 octahedron can be constructed according to [1, Fig. 8] and [3, Fig. 297, p. 330].

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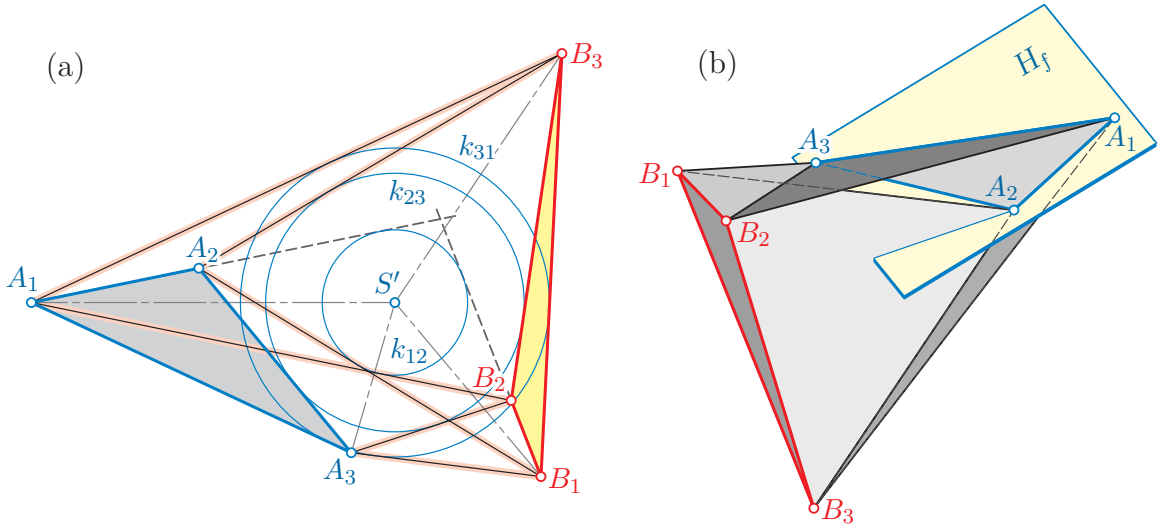


Figure 1: (a) Flat pose of a flexible type-3 octahedron: Two concentric circles k_{12}, k_{31} and two points A_1, B_1 define the remaining vertices $A_2, \dots, B_3$ and a third contacting circle k_{23}. (b) A spatial pose of this octahedron with the fixed triangle $A_1A_2A_3$ in the plane H_f (yellow). At the depicted physical model, the two opposite faces $B_1A_2A_3$ and $A_1B_2B_3$ need to be omitted in order to avoid interpenetrations.

The n-dimensional analogues of octahedra are called *cross-polytopes* $\mathcal{C}^n$. Following the notation in [5], they have $2n$ vertices coupled into pairs of *opposite* vertices (A_i, B_i) for $i = 1, \dots, n$. The 2^n hyperfaces (or *facets*) of $\mathcal{C}^n$ are the simplices $X_1 \dots X_n$ where X_i stands either for A_i or B_i. The $4\binom{n}{2} = 2n(n-1)$ edges of $\mathcal{C}^n$ are X_iX_j for $i \neq j$. More general, the k-faces, $1 < k \leq n-1$ of $\mathcal{C}^n$ are the $2^{k+1}\binom{n}{k+1}$ simplices with $k+1$ vertices $X_{i_0} \dots X_{i_k}$ out of $\{A_1, \dots, A_n, B_1, \dots, B_n\}$ such that the indices $i_0 \dots i_k$ are mutually different. Hence, no k-face contains a pair (A_i, B_i) of opposite vertices. We presuppose that no k-face is of dimension smaller than k, i.e., flat.

In 2014, the young Russian mathematician Alexander A. Gaĭfullin [5] surprised the scientific community with the complete solution of a long-lasting open problem, namely the question for flexible cross-polytopes in the n-dimensional Euclidean, hyperbolic and spherical spaces for $n > 3$. So far, only particular flexible examples in $\mathbb{E}^4$ where known (see [15]). Based on algebraic methods, Gaĭfullin succeeded to classify the flexible types in all spaces, and he even presented parametrizations of the flexions in terms of Jacobian elliptic functions. In several papers Gaĭfullin addressed the problem whether the volumes of flexible cross-polytopes in non-Euclidean spaces remain constant during the flexions (see, e.g., [6, 7]).

According to [5], n-dimensional polytopes are called *flexible*, if they have hinges at their edges and the polytope can deform while each $(n-1)$-dimensional facet remains congruent to itself. In other words, only the dihedral angles between two neighboring facets can vary. We are interested in nontrivial deformations, which means that the deformations are not induced by a motion of a rigid cross-polytope in the ambient space.

When studying the flexibility of cross-polytopes $\mathcal{C}^n$, we follow the convention in [5] and assume that the simplex $\Gamma_f := A_1 \dots A_n$ in the hyperplane H_f is fixed while the opposite simplex $\Gamma_m := B_1 \dots B_n$ in the hyperplane H_m is moving. This means that the vertex B_i traces a circle during its rotation about the $(n-2)$-dimensional *axis* $h_i \subset H_f$ spanned by $A_1, \dots, A_{i-1}, A_{i+1}, \dots, A_n$. Point B_i rotates together with the simplex called *wing*

$\Delta_i := A_1 \dots A_{i-1} B_i A_{i+1} \dots A_n$ relative to the fixed simplex Γ_f (note the flexible octahedron in Figure 2 as an example for $n = 3$).

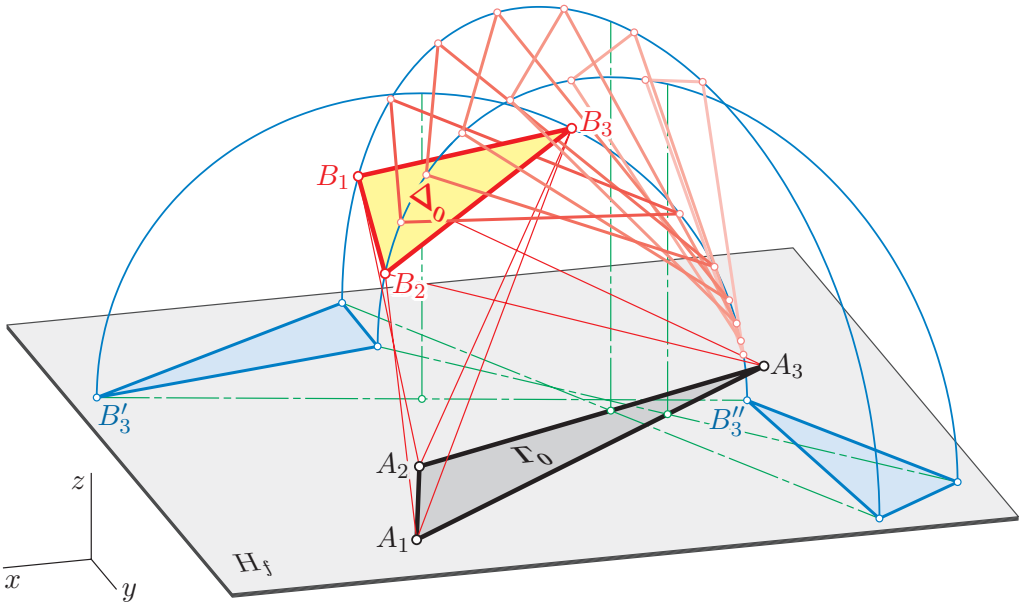


Figure 2: A type-3 octahedron in motion, while the triangle $\Gamma_0 = A_1A_2A_3$ remains fixed in the plane $H_f\colon\ z = 0$.

At flexible cross-polytopes, there exist simultaneous rotations of all n wings $\Delta_1, \dots, \Delta_n$ such that all $\binom{n}{2}$ edges of the moving simplex Γ_m preserve their lengths $\overline{B_iB_j}$.[1] Since the union of the moving simplex and the wings contains all edges of $\mathcal{C}^n$, every other facet remains rigid, too. We call such a continuous self-motion of $\mathcal{C}^n$ a *flexion* and ignore mutual intersections between facets.

Below, we study in the Euclidean n-space $\mathbb{E}^n$ those particular flexible cross-polytopes that pass during their flexion through two flat poses, i.e., poses where the moving simplex Γ_m lies in the hyperplane H_f spanned by the fixed simplex Γ_f.

Definition 1.1. A cross-polytope $\mathcal{C}^n$ is called *twice-flat* if it admits two flat poses such that for each vertex B_i of the moving simplex the two corresponding positions B_i' and B_i'' are different.

According to [5, eq. (3.3)], the constancy of the edge length $\overline{B_iB_j}$, $i \neq j$, during the flexion is equivalent to a particular biquadratic relation between the tangents t_i and t_j of halved dihedral angles ϕ_i, ϕ_j between Γ_f and the respective wings Δ_i, Δ_j (see also [16, eq. (4)]). Since at flexible twice-flat cross-polytopes the values $t_i = t_j = 0$ and $t_i = t_j = \infty$ must satisfy these conditions for each pair (i, j) with $i, j \in \{1, \dots, n\}$ and $i \neq j$, all biquadratic relations split. In this case for each pair of wings (Δ_i, Δ_j) two simultaneous one-parameter rotations about the respective axes h_i, h_j through angles with proportional t_i and t_j are available for preserving the distance $\overline{B_iB_j}$. While Gaĭfullin showed the existence of flexible versions in an algebraic way, the goal of our study is a synthetic approach, a geometric characterization of the flat poses and a geometric analysis of the general poses of flexible twice-flat cross-polytopes in the Euclidean n-space.

[1] Gaĭfullin created the pretty name *butterfly* for the union of the fixed simplex with all wings.

The paper is structured as follows: In the coming section we revisit Type 3 of Bricard's octahedra. Our novel and mainly synthetic approach yields for the flat poses of twice-flat octahedra two geometric conditions that are necessary and sufficient for flexibility. In addition, we disclose several properties of *spatial*, i.e., non-flat poses, namely symmetries between planes spanned by the faces.

Section 3 is devoted to n-dimensional flexible twice-flat cross-polytopes $\mathcal{C}^n$ in $\mathbb{E}^n$. Again, we come up with conditions that characterize flat poses of flexible versions. A construction of flexible twice-flat cross-polytopes is given that generalizes Bricard's construction in [2, p. 144]. Finally, we check which properties of spatial poses at type-3 octahedra can be generalized to higher dimensions.

2 Revisiting Bricard's Type-3 Octahedra

Let $\mathcal{C}^3$ be a cross-polytope in $\mathbb{E}^3$, i.e., an octahedron with the triangles $\Gamma_0 := A_1A_2A_3$ and $\Delta_0 := B_1B_2B_3$ that serve respectively as fixed and moving simplex Γ_f and Γ_m.[2] When $\mathcal{C}^3$ happens to admit two different flat poses, then in these poses the vertices B_1, B_2 and B_3 of Δ_0 will be marked by one or two primes for better distinction from spatial poses.

2.1 Local Symmetries of Twice-flat Octahedra

Theorem 2.1. *A flat octahedron with the fixed triangle $A_1A_2A_3$ and the coplanar opposite triangle $B_1'B_2'B_3'$ admits a second flat pose $B_1''B_2''B_3''$ with $B_i' \neq B_i''$ for each $i \in \{1,2,3\}$ if and only if at each vertex A_i the connecting lines with the remaining two pairs of vertices (A_j, B_j'), $j \neq i$, have common axes of symmetry.*

Proof. Suppose that $\mathcal{C}^3$ admits two flat poses with $B_i' \neq B_i''$ for all i. Then the halfturn in $\mathbb{E}^3$ about the axis $[A_1, A_2]$ sends B_3' to B_3'', and that about $[A_1, A_3]$ sends B_2' to B_2'' (see Figure 3).[3] Thus, within the plane $H_f = [A_1, A_2, A_3]$ the bisectors of the point pairs (B_2', B_2'') and (B_3', B_3'') intersect at A_1, while there are equal distances $\overline{B_2'B_3'} = \overline{B_2''B_3''}$. This implies that A_1 is the center of a rotation within H_f that brings $B_2'B_3'$ to $B_2''B_3''$. One half of the angle of rotation is marked in Figure 3, which reveals that the pairs of lines $([A_1, A_2], [A_1, B_2'])$ and $([A_1, B_3'], [A_1, A_3])$ share the angle bisectors.

The disclosed symmetry at A_1 occurs similarly at A_2 and A_3, which confirms the claim. □

For the existence of a second flat pose it is not relevant which face of the octahedron is fixed. This implies

Corollary 2.1. *Referring to Theorem 2.1, the stated symmetries at A_1, A_2 and A_3 are equivalent to common axes of symmetries at each point B_i', $i \in \{1,2,3\}$, between the connections with the remaining two pairs (A_j, B_j') with $j \neq i$.*

[2] Because of particular symmetries (note Figure 10) it makes sense to use in this chapter the new symbols Γ_0 and Δ_0 instead of Γ_f and Γ_m for the fixed and the moving simplex. Cross-polytopes in $\mathbb{E}^3$ have the particular property that there are no other facets beside the moving and fixed simplex and the wings relative to Γ_f or Γ_m.

[3] Throughout the paper we use brackets $[X, Y, \ldots, Z]$ as the symbol for the affine hull (or span) of the listed elements. For example, $[X, Y]$ stands for the *line* connecting the points X and Y, and $[\Delta_i]$ stands for the hyperplane spanned by the wing Δ_i. On the other hand, the symbol $XY \ldots Z$ denotes the simplex with vertices $X, Y, \ldots, Z$, and in particular XY denotes the *segment* terminated by X and Y.

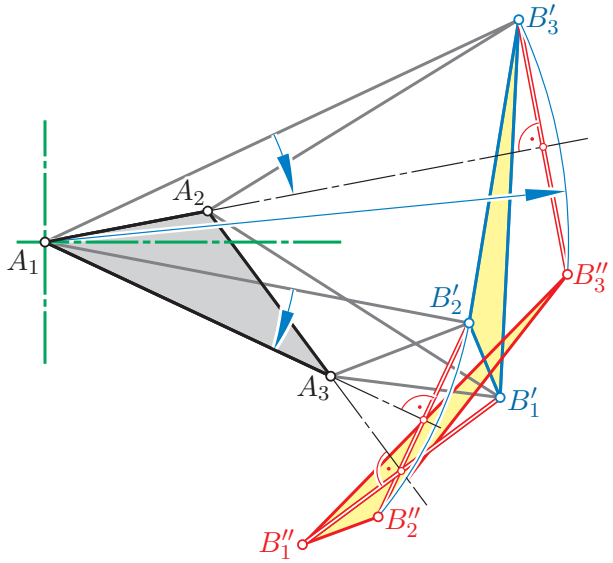


Figure 3: At a twice-flat octahedron the moving triangle Δ_0 varies between $B'_1B'_2B'_3$ and $B''_1B''_2B''_3$ relative to the fixed triangle $\Gamma_0 = A_1A_2A_3$ in the plane H_f. The double lines are orthogonal views of the circular paths traced by the moving vertices in the case of a flexible octahedron.

Definition 2.1. A flat pose $A_1 \dots B'_3$ of an octahedron is called *locally symmetric*, when at each vertex A_i and B'_i the respective connections with the remaining two pairs (A_j, B'_j) for $j \neq i$ have common axes of symmetry.

It will turn out that local symmetry of a flat octahedron is not sufficient for the flexibility of a twice-flat octahedron. Below we summarize a few properties of a locally symmetric pose.

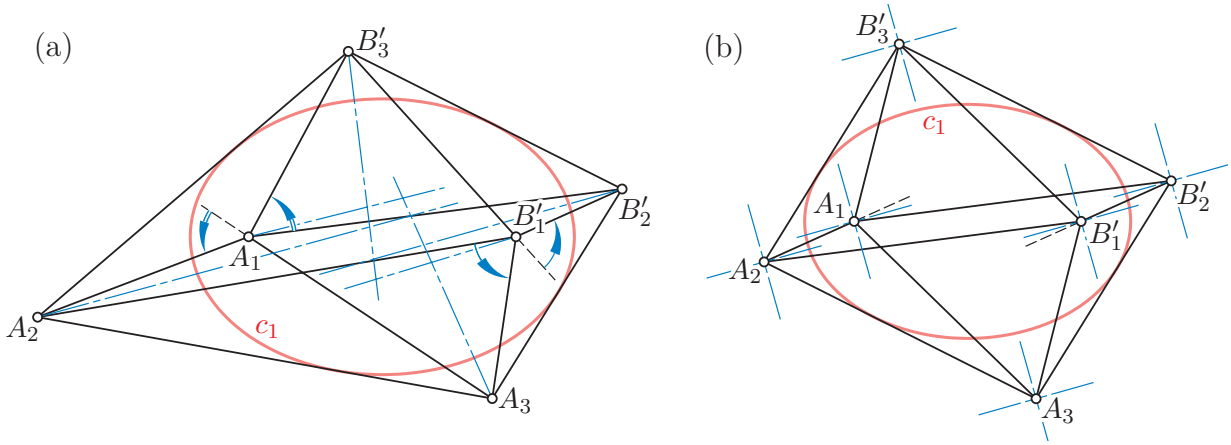


Figure 4: (a) A flat pose $A_1 \dots B'_3$ is locally symmetric if and only if A_1 and B'_1 are focal points of a conic c_1 that contacts the sides of the quadrangle $A_2A_3B'_2B'_3$ (Lemma 2.1).
(b) A locally symmetric flat pose with parallel axes of symmetry between $[A_i, A_j]$ and $[A_i, B'_j]$ as well as between $[B'_i, A_j]$ and $[B'_i, B'_j]$ for all $i \neq j$. In this case all quadrangles $A_j \dots B'_k$ are parallelograms.

Lemma 2.1. *(i) A flat pose $A_1 \dots B'_3$ is locally symmetric if and only if A_1 and B'_1 are focal points of a conic c_1 that contacts the four sides of the quadrangle $A_2A_3B'_2B'_3$ (Figure 4a).*

(ii) At a locally symmetric flat pose the eight pedal points of A_i and B'_i on the sides of the quadrangle $A_jA_kB'_jB'_k$ are concircular for each permutation (i,j,k) of $(1,2,3)$. Moreover, the midpoints of all pairs (A_i, B'_i) are collinear.

(iii) At twice-flat octahedra, the two flat poses $B'_1B'_2B'_3$ and $B''_1B''_2B''_3$ of the moving triangle are indirectly congruent, and the midpoints of (B'_i, B''_i), $i = 1,2,3$, are collinear.

Proof. (i) If a locally symmetric flat pose $A_1 \dots B'_3$ is given, then due to Desargues's involution theorem (see, e.g., [8, p. 336]) the connections of A_1 with the pairs (A_2, B'_2) and (A_3, B'_3) define a symmetric involution. Since its fixed lines are orthogonal, the isotropic lines through A_1 correspond to each other in this involution.

Consequently, there exists a conic c_1 with focal point A_1 that contacts the four sides of the quadrangle $A_2A_3B'_2B'_3$ (note, e.g., [8, p. 288]). Since the tangents from any point X to a conic and the connections with the focal points share the angle bisectors (see, e.g., [8, p. 43]), the symmetries at the vertices A_2, A_3, B'_2, B'_3 of the contacting quadrangle imply that the point B'_1 is the second focal point.

Conversely, let a conic c_1 with focal points A_1, B'_1 and two different points A_2, B'_2 in the exterior of c_1 be given. We specify (A_3, B'_3) as another pair of opposite points of the quadrilateral formed by the tangents drawn from A_2 and B'_2 to c_1. The common angle bisectors between the tangents from any point X to c_1 and the connections with the focal points imply the required symmetries at $A_2, \dots, B'_3$.

The symmetries at A_1 and B'_1 follow from Desargues's involution theorem. The tangents from A_1 or B'_1 to all curves of the range defined by the sides of the quadrangle $A_2 \dots B'_3$ belong to an involution, in particular the connections with (A_2, B'_2) as well as that with (A_3, B'_3).

(ii) The pedal points of the sides of the quadrangle $A_j \dots B'_k$ with respect to (w.r.t. in brief) the focal points A_i and B'_i are located on a circle (see [8, Fig. 2.16]). All conics tangent to the sides of $A_jA_kB'_jB'_j$ belong to a range (dual pencil). Since the poles of a given line w.r.t. the dual conics of a range are collinear, the conics' centers are located on a line that passes also through the midpoints of (A_j, B'_j) and (A_k, B'_k) as the carriers of pairs of line pencils contained in the range.

(iii) The congruence transformation that takes $B'_1B'_2B'_3$ to $B''_1B''_2B''_3$ must be orientation reversing since otherwise it would simultaneously be equal to the three rotations with respective centers A_i, $i = 1,2,3$, that send the single side $B'_jB'_k$ to its respective image $B''_jB''_k$. Thus, it is a glide reflection along an axis that passes through the midpoints of corresponding points. □

Statement (iii) offers another way for obtaining a locally symmetric flat pose: Choose two indirectly congruent triangles $B'_1B'_2B'_3$ and $B''_1B''_2B''_3$ in a common plane. If the bisectors of (B'_i, B''_i) for $i = 1,2,3$ form a triangle, then this serves as the fixed triangle $A_1A_2A_3$.

The flat poses of twice-flat octahedra in Figure 3 show that at the four-sided pyramid connecting A_1 with the sides of the quadrangle $A_2A_3B_2B_3$ opposite interior angles at A_1 are congruent (also called *Voss condition* [11]). Similar congruences appear at all other vertices. However, when the flat pose of the pyramid is not twofold covered (see A_2 in Figure 1a), then opposite angles sum up to π (*anti-Voss condition* by [11]).

Definition 2.2. A four-sided pyramid is called *isogonal* if either opposite interior angles at the apex are congruent or they are supplementary, i.e., they sum up to π. The extended lines form an *isogonal double-pyramid* consisting of four concurrent lines $g_1, \dots, g_4$ in cyclic order where opposite angles are equal, i.e., $\sphericalangle g_1g_2 = \sphericalangle g_3g_4$ and $\sphericalangle g_2g_3 = \sphericalangle g_4g_1$ where $0 \le \sphericalangle g_ig_j \le \frac{\pi}{2}$.

An isogonal four-sided pyramid intersects a sphere centered at the apex along an *isogram*, i.e., a quadrangle with opposite sides of equal length (Figure 5, left). In our case, we obtain a spherical isogram where the straight sides are replaced by arcs on great circles. A spherical isogram is either free of self-intersections or two opposite sides are intersecting quite similar to a planar antiparallelogram. In the latter case we speak of a *crossed* spherical isogram (see $P_1 \dots P_4$ in Figure 5, right). By replacing one edge of the pyramid by its complement, i.e., by the halfline pointing in the opposite direction, supplementary opposite angles become congruent angles, and vice versa. When two adjacent edges of an isogonal pyramid are replaced by their complements, then a crossed isogram is replaced by an uncrossed version, and vice versa (note the isogonal double-pyramid in Figure 5, right, containing three pyramids with congruent opposite angles).

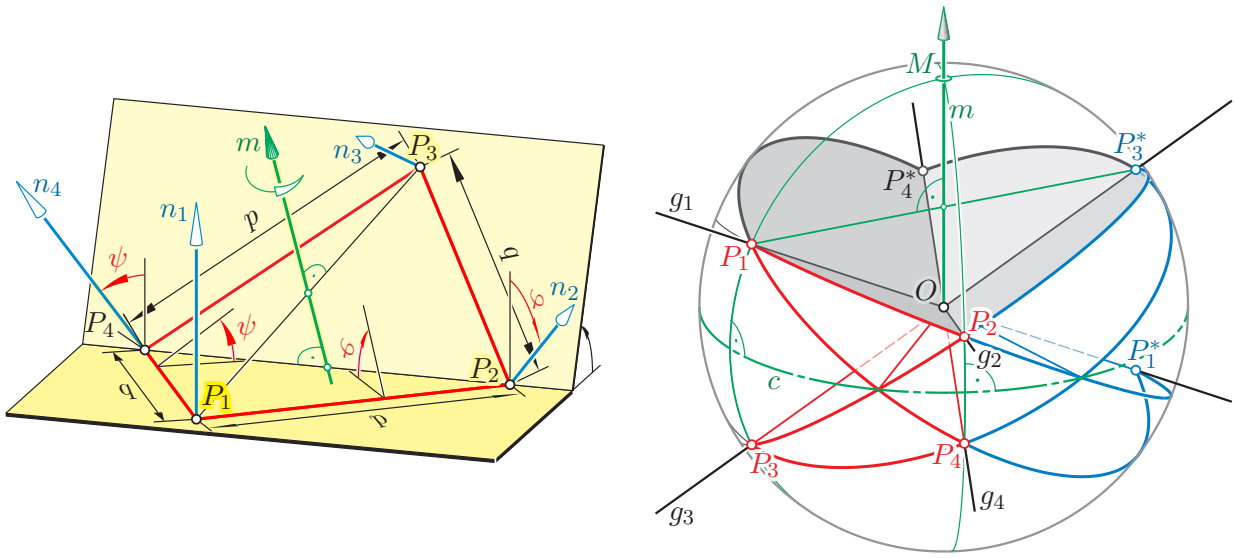


Figure 5: Left: Each skew isogram $P_1P_2P_3P_4$ has an axis of symmetry m, which meets both diagonals orthogonally at their midpoints.
Right: At the depicted isogonal double-pyramid with apex O and generators $g_1, \dots, g_4$ the three isogonal pyramids through $P_1P_2P_3P_4$, $P_1^*P_2P_3^*P_4$ and $P_1P_2P_3^*P_4^*$ are emphasized. Two pyramids are crossed and symmetric w.r.t. to the plane μ of the circle c. The non-crossed pyramid is symmetric w.r.t. the circle's axis m. At each pyramid opposite dihedral angles are congruent.

Lemma 2.2. *If the four lines $g_1, \dots, g_4$ of an isogonal double-pyramid are not coplanar, then the line m of intersection of the two diagonal planes $[g_1, g_3] \cap [g_2, g_4]$ is an axis of symmetry. Dually, the plane μ connecting $[g_1, g_2] \cap [g_3, g_4]$ with $[g_2, g_3] \cap [g_4, g_1]$ is orthogonal to m and a plane of symmetry.*

Proof. We select on the double-pyramid a pyramid with congruent opposite interior angles. Then the resulting isogonal pyramid intersects the unit sphere centered at the apex O along a skew spherical isogram (Figures 5, right, or 7). For each spherical isogram holds that the common perpendicular of the diagonals passes through their midpoints and is the axis of a halfturn with spherical center M that exchanges opposite vertices (see, e.g., [13, p. 555]). This axis passes also through the sphere's center O because of equal distances to the vertices of the isogram. The halfturn about the diameter $[O, M]$ exchanges opposite lines of the original double-pyramid.

After composition with the reflection in the center O, the reflection in the diameter plane perpendicular to $[O, M]$ sends the double-pyramid onto itself, too. This diameter plane μ contains the intersections between corresponding planes $[g_1, g_2] \cap [g_3, g_4]$ and $[g_1, g_4] \cap [g_2, g_4]$. In Figure 5, right, the great circle c serves on the sphere as the axis of symmetry of a crossed spherical isogram. □

Four-sided pyramids (without base quadrangle) are flexible (Figure 7). It is wellknown (see, e.g., [16, eq. (9)]) that during the flexion of an isogonal pyramid the tangents of the halved dihedral angles ϕ_3 along A_1A_2 and ϕ_2 along A_1A_3 are proportional (note Lemma 2.3 below).

2.2 Transmission Between Adjacent Wings

Let $A_1 \dots B_3'$ be a locally symmetric flat pose of an octahedron. Suppose that the octahedron is flexible. For obtaining an exact formula for the transmission between the wings, we assume that the fixed triangle $\Gamma_0 = A_1A_2A_3$ is counter-clockwise oriented, and the directions of the axes of rotations h_1, h_2, h_3 are given, respectively, by the directed segments A_2A_3, A_3A_1 and A_1A_2. The signed interior angles of $A_1A_2A_3$ are denoted by $\alpha_1, \alpha_2, \alpha_3$ (Figure 6).

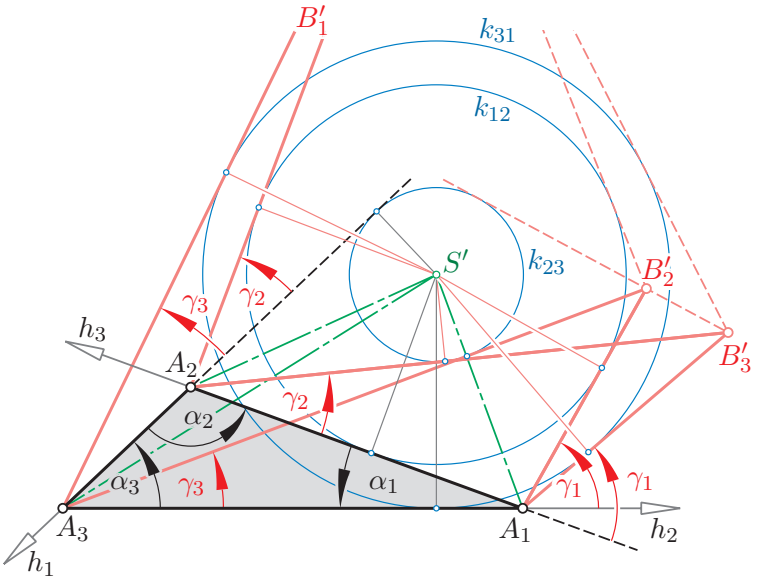


Figure 6: The angles in the fixed face $\Gamma_0 = A_1A_2A_3$ and in the wings $\Delta_1 = A_2A_3B_1$, $\Delta_2 = A_3A_1B_2$, and $\Delta_3 = A_1A_2B_3$ of a twice flat octahedron that is displayed in a flat pose.

With γ_1 we denote the signed angle at A_1 between the lines $[A_1, A_3]$ and $[A_1, B_2']$. It equals the angle between $[A_1, A_2]$ and $[A_1, B_3']$. We use the rotation by γ_1 about A_1 to transfer the orientation of h_2 to the line $[A_1, B_2']$ and denote with p_1 the signed distance A_1B_2'. After transferring the orientation of h_3 to the line $[A_1, B_3']$, the symbol q_1 stands for the signed distance A_1B_3'.[4]

An analogous procedure yields the signed angles γ_2, γ_3 and the signed distances p_2, q_2 and p_3, q_3 at the remaining vertices A_2 and A_3. We assume for all i that $0 < \alpha_i, \gamma_i < \pi$ and $(\alpha_i + \gamma_i), (\alpha_i - \gamma_i) \neq k\pi$ for all $k \in \mathbb{Z}$.

[4]Note that at the example displayed in Figure 6 holds $q_1 < 0$. Vertex B_2' lies on the left hand side of h_2 while B_3' lies on the right hand side of h_3.

Lemma 2.3. *Referring to the previous notation, the angles of rotation ϕ_2 of the wing Δ_2 about h_2 and ϕ_3 of Δ_3 about h_3 relative to Γ_0 with $t_i := \tan\frac{\phi_i}{2}$ for $i = 2, 3$ are related by*

$$t_3 = \frac{-\sin\gamma_1 \pm \sin\alpha_1}{\sin(\gamma_1+\alpha_1)}\, t_2\,, \quad \textit{hence} \quad t_3 = \frac{\sin\frac{\alpha_1-\gamma_1}{2}}{\sin\frac{\alpha_1+\gamma_1}{2}}\, t_2 \quad \textit{or} \quad t_3 = -\frac{\cos\frac{\alpha_1-\gamma_1}{2}}{\cos\frac{\alpha_1+\gamma_1}{2}}\, t_2$$

according to the upper or the lower sign in the first equation.

Proof. We use a right-handed coordinate frame with the flat pose in the plane $z = 0$, with A_1 as origin and h_2 as positive x-axis. From points in the half-space $z > 0$ the triangle $A_1A_2A_3$ appears counter-clockwise oriented, as shown in Figure 6.

Thus, the circular paths of B_2 and B_3 (Figure 8) can be parametrized as

$$B_2 = p_1 \begin{pmatrix} \cos\gamma_1 \\ \sin\gamma_1\cos\phi_2 \\ \sin\gamma_1\sin\phi_2 \end{pmatrix}, \quad B_3 = q_1 \begin{pmatrix} -\cos\alpha_1\cos\gamma_1 - \sin\alpha\sin\gamma_1\cos\phi_3 \\ \sin\alpha_1\cos\gamma_1 - \cos\alpha\sin\gamma_1\cos\phi_3 \\ \sin\gamma_1\sin\phi_3 \end{pmatrix}.$$

After some computations (for details see, e.g., [16]) the claim that the distance $\overline{B_2B_3}$ remains equal to $\overline{B_2'B_3'}$ in the initial flat pose with $\phi_2 = \phi_3 = 0$ is equivalent to

$$\sin(\alpha_1-\gamma_1)\, t_2^2 - \sin(\alpha_1+\gamma_1)\, t_3^2 - 2\sin\gamma_1\, t_2t_3 = 0.$$

This reveals that in the particular case of an isogonal pyramid the biquadratic relation (according to [16, eq. (4)] or [5, eq. (3.3)]) splits into two linear functions

$$t_3 = \frac{-\sin\gamma_1 \pm \sqrt{\sin^2\gamma_1 + \sin^2\alpha_1\cos^2\gamma_1 - \cos^2\alpha_1\sin^2\gamma_1}}{\sin(\alpha_1+\gamma_1)}\, t_2 = \frac{-\sin\gamma_1 \pm \sin\alpha_1}{\sin(\alpha_1+\gamma_1)}\, t_2.$$

Thus we obtain either

$$t_3 = \frac{\sin\frac{\alpha_1-\gamma_1}{2}}{\sin\frac{\alpha_1+\gamma_1}{2}}\, t_2 \quad \text{or} \quad t_3 = -\frac{\cos\frac{\alpha_1-\gamma_1}{2}}{\cos\frac{\alpha_1+\gamma_1}{2}}\, t_2\,. \tag{2.1}$$

Both flat poses allow bifurcations between the two rational movements. Figure 7 shows the spherical images of two poses, one of each movement. □

Lemma 2.4. *Let $A_1 \ldots B_3'$ be a locally symmetric flat pose of an octahedron and (i, j, k) be a cyclic permutation of $(1, 2, 3)$. Then, all points $S \neq A_i$ of one axis of symmetry between the lines $[A_i, A_j], [A_i, B_j']$ and $[A_i, A_k], [A_i, B_k']$ satisfy the relation*

$$t_k = \tan\frac{\phi_k}{2} = \frac{d(S,h_j)}{d(S,h_k)}\, t_j \text{ with } t_j = \tan\frac{\phi_j}{2} \tag{2.2}$$

which guarantees that the distance $\overline{B_jB_k}$ remains constant during the rotations of the wings Δ_j and Δ_k. Here $d(S, h_l)$ for $l = 1, 2, 3$ denotes the signed distance of S to the revolute axis h_l such that for points in the halfplane of h_l enclosing the interior of $A_1A_2A_3$ the distance is positive. In other words, $(d(S, h_1), d(S, h_2), d(S, h_3))$ are the normalized trilinear coordinates of S w.r.t. the fixed triangle $A_1A_2A_3$.

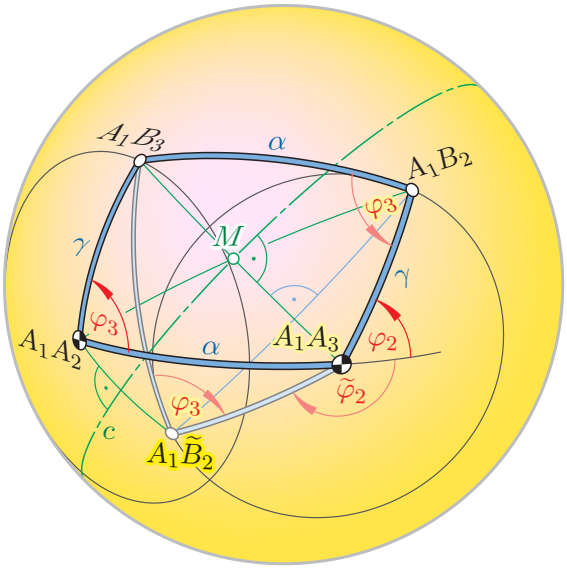


Figure 7: The flexion of the isogonal pyramid with apex A_1 induces on a sphere centered at A_1 two rational spherical motions with bifurcations at the aligned poses. A crossed spherical isogram is met by its spherical axis c of symmetry; otherwise the isogram includes its center M of symmetry.

Proof. If one symmetry axis through A_i meets the opposite side A_jA_k, then it includes with the axes h_k and h_j the angles $\frac{\alpha_i-\gamma_i}{2}$ and $\frac{\alpha_i+\gamma_i}{2}$, respectively (note vertex A_3 in Figure 6). Therefore holds, by virtue of Lemma 2.3,

$$\frac{-\sin\gamma_i+\sin\alpha_i}{\sin(\gamma_i+\alpha_i)}=\frac{\sin\frac{\alpha_i-\gamma_i}{2}}{\sin\frac{\alpha_i+\gamma_i}{2}}=\frac{d(S,h_j)}{d(S,h_k)}\,.$$

Otherwise (note points A_1 or A_2 in Figure 6) one of the axes includes the angles $\frac{\pi+\alpha_i-\gamma_i}{2}$ or $\frac{\pi-\alpha_i+\gamma_i}{2}$ and $\frac{\pi-\alpha_i-\gamma_i}{2}$ with h_j and h_k, and we obtain

$$\frac{-\sin\gamma_i-\sin\alpha_i}{\sin(\gamma_i+\alpha_i)}=-\frac{\cos\frac{\alpha_i-\gamma_i}{2}}{\cos\frac{\alpha_i+\gamma_i}{2}}=\frac{d(S,h_j)}{d(S,h_k)}\,.$$

This confirms the claim that can be rewritten in the form

$$t_1:t_2:t_3=\frac{1}{d(S,h_1)}:\frac{1}{d(S,h_2)}:\frac{1}{d(S,h_3)} \tag{2.3}$$

with $(d(S,h_1),\,d(S,h_2),\,d(S,h_3))$ as normalized *trilinear coordinates* of S w.r.t. $A_1A_2A_3$ (see, e.g., [8, p. 413]). The absolute values of the distances $d(S,h_i)$ are the radii of the concentric circles k_{12},k_{23},k_{31} in Figures 1a and 6. □

2.3 Necessary and Sufficient Condition for Flexibility

The following theorem rephrases a part of Bricard's main result in [2]. The new formulation paves the way to higher-dimensional versions.

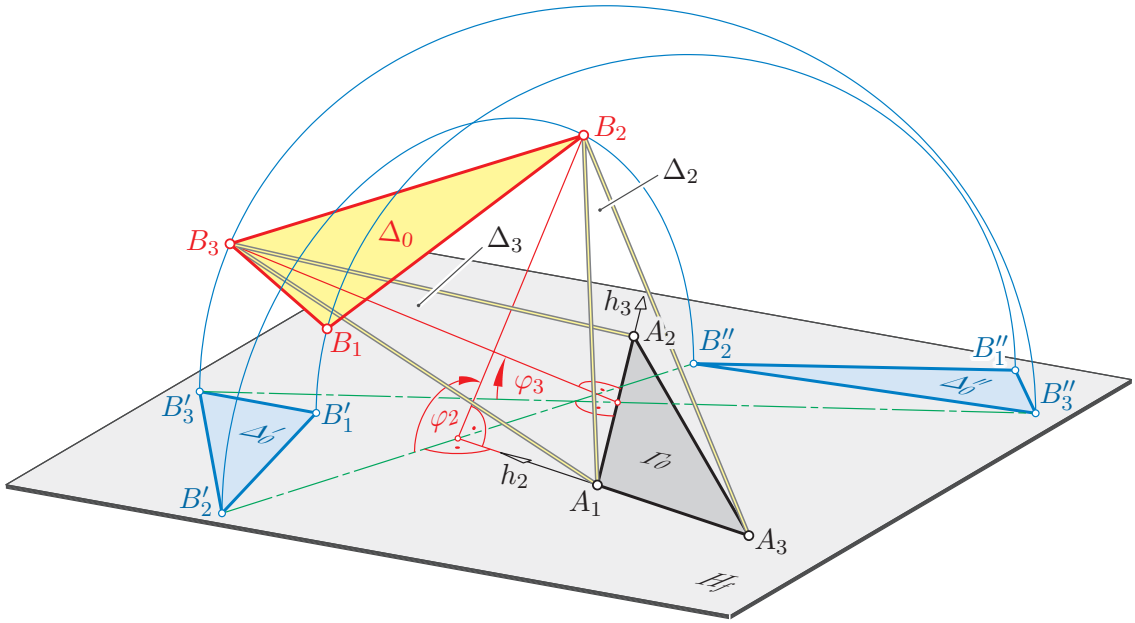


Figure 8: Spatial pose of the moving triangle $B_1B_2B_3$ with the corresponding angles of rotation ϕ_2, ϕ_3 of B_2, B_3 and the wings $\Delta_2 = A_1A_3B_2$ and $\Delta_3 = A_1A_2B_3$ along with the respective axes h_2 and h_3.

Theorem 2.2. *A flat octahedron $A_1 \dots B_3'$ is twice-flat and flexible in $\mathbb{E}^3$ if and only if it is locally symmetric according to Definition 2.1 and at the vertices A_1, A_2 and A_3 one of the axes of symmetry passes through a common (finite or infinite) point S'.*

Proof. (i) We show first that the stated conditions are sufficient for flexibility and begin with the case of a finite S':

By virtue of Lemma 2.4, every side length of the moving triangle is preserved since for the transmissions between the wings' rotations from Δ_1 to Δ_2, from Δ_2 to Δ_3, and finally from Δ_3 to Δ_1 holds

$$\frac{d(S',h_3)}{d(S',h_1)} \cdot \frac{d(S',h_2)}{d(S',h_3)} \cdot \frac{d(S',h_1)}{d(S',h_2)}\, t_1 = t_1. \tag{2.4}$$

This confirms the flexibility.

If S' lies at infinity (see Figure 4b), then we choose for each $i = 1, 2, 3$ a point S_i on the axis through A_i as the image of A_i under a common translation. Thus, for each axis h_k, $k \neq i$, we obtain equal signed distances $d(S_i, h_k) = d(S_j, h_k)$ if $j \neq k$. By Lemma 2.4 follows again, that the product of the ratios $d(S_i, h_j)/d(S_i, h_k)$ equals 1.

(ii) Conversely, if a given twice-flat octahedron is flexible, then the transmission ratios between any two wings define by (2.3) homogeneous trilinear coordinates $(t_1 : t_2 : t_3)$ of a unique point S' in the plane H_f, and by (2.2) S' is located on one of the axes of local symmetry at each vertex A_i of Γ_0. □

From (2.3) follows that at all poses of Δ_0 the vertices B_1, B_2 and B_3 are corresponding points of projective mappings between the circular paths. This holds since $t_i = \tan\frac{\phi_i}{2}$ defines a projective scale on the trajectory of B_i (see, e.g., [8, p. 259]).

Corollary 2.2. *Let $A_1 \dots B_3'$ be a flat pose of a twice-flat octahedron. If at the vertices A_1, A_2 and A_3 one of the axes of symmetry passes through a common point S', then the same is true for the remaining vertices B_1', B_2' and B_3'.*

Proof. Local symmetry and concurrent axes of symmetry at A_1, A_2, A_3 imply flexibility. The latter does not depend on the choice of the fixed triangle. Hence, beside $A_1A_2A_3$ also the triangles $A_1A_2B_3$, $A_1B_2B_3$, and $B_1B_2B_3$ have concurrent axes of symmetry. Since any two consecutive triangles share two axes of symmetry which already define the meeting point uniquely, point S' is common for all, as shown in Figure 6. Note also below the statement (i) in Theorem 2.3. □

Remark 1. The octahedra with S' at infinity are also particular cases of Bricard's line-symmetric Type 1. Their flexions are determined by the flexions of one four-sided pyramid combined with a line reflection in the axis of symmetry of the bounding isogram. These octahedra have the additional property that they admit bifurcations at the flat poses since both axes of symmetry at the vertices belong to families of concurrent axes, which are sufficient for flexibility by Theorem 2.2. Such bifurcations do not exist in the cases with finite S'.

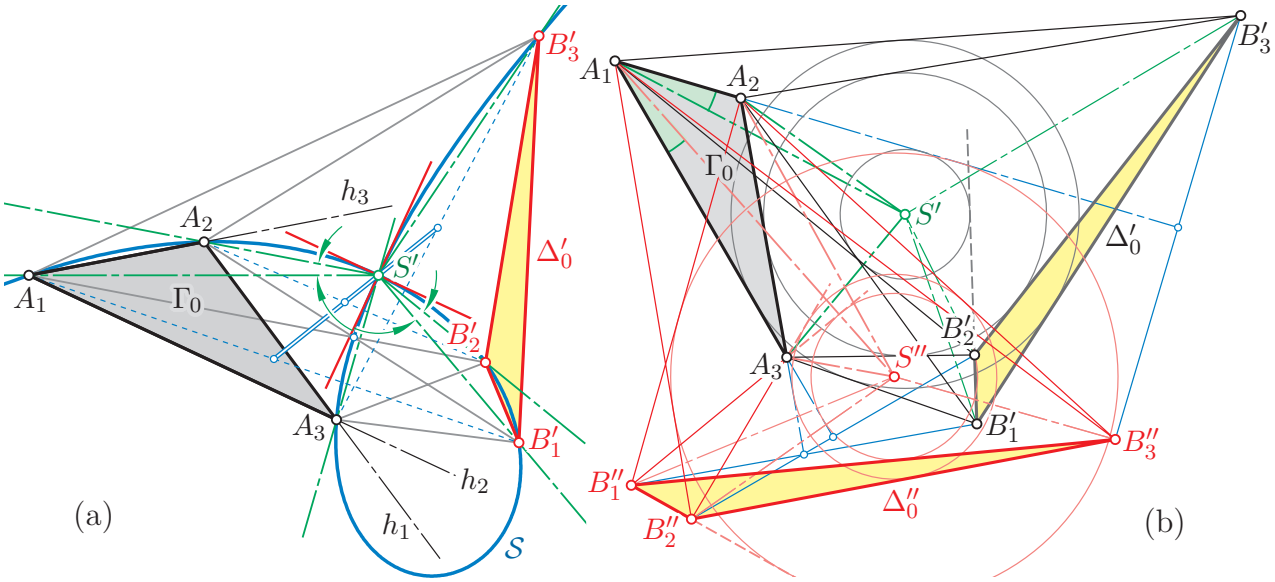


Figure 9: $A_1 \dots B'_3$ and $A_1 \dots B''_3$ are the two flat poses of a flexible twice-flat octahedron:
(a) Opposite vertices (A_i, B'_i) for $i = 1, 2, 3$ are associated points of a strophoid $\mathcal{S}$ with the node S'. The lines $[S', A_i]$, $[S', B'_i]$ are symmetric with respect to the tangents of the strophoid at S'. The points $[A_i, B'_j] \cap [A_j, B'_i]$ for $i \neq j$ belong to $\mathcal{S}$, too.
(b) The meeting points S' and S'' of the concurrent axes of symmetry in the two flat poses are isogonal with respect to the fixed triangle $A_1A_2A_3$.

Theorem 2.3. *The flat poses of a flexible twice-flat octahedron $\mathcal{C}^3$ have the following properties in addition to those listed above in Lemma 2.1, provided that the point S' is finite:*

(i) The quadrangles $A_1A_2B'_1B'_2$, $A_2A_3B'_2B'_3$, and $A_3A_1B'_3B'_1$ have incircles with the common center S' (Figures 1a and 9b). The absolute values of the normalized trilinear coordinates of S' w.r.t. $A_1A_2A_3$ and $B_1B_2B_3$ are equal. The same holds for all other pairs of opposite triangles of $\mathcal{C}^3$.

(ii) All pairs of opposite vertices (A_i, B'_i) are associated points of a strophoid $\mathcal{S}$ with the node S' (Figure 9a).

(iii) The connecting lines $[S', A_i], [S', B'_i]$ for $i = 1, 2, 3$ have the tangents to $\mathcal{S}$ at the node S' as common axes of symmetry. This results in congruent angles at S'.

(iv) Let S'' be the meeting point of the axes of symmetry in the second pose. Then S' and S'' are isogonal w.r.t. the fixed triangle $A_1A_2A_3$ (Figure 9b).

Proof. (i) The three quadrangles have angle bisectors meeting at S'.

(ii) We recall that the strophoid $\mathcal{S}$ is the geometric locus of points X with the property that one angle bisector between the connections with A_1 and B_1' passes through the node S'. The tangents at the node bisect the connections with A_1 and B_1', which are the fixed lines of the symmetric involution. Any two lines passing through S' and symmetric w.r.t. the node tangents intersect $\mathcal{S}$ additionally at a pair of associated points (note [17]). As a consequence, the point of intersection between $[A_i, B_j']$ and $[A_j, B_i']$ for $i \neq j$ is also located on $\mathcal{S}$.

(iii) For each quadrangle, the pairs of line pencils with carriers at opposite vertices and the incircle belong to a dual pencil of conics. The respective tangents through S' define a symmetric involution. Any two quadrangles share a pair of opposite vertices. Therefore, the three involutions are equal.

(iv) We refer to (Figure 9b). Since $[A_1, B_3']$ and $[A_1, B_3'']$ are symmetric w.r.t. $[A_1, A_2]$, the respective angle bisectors with $[A_1, A_3] = h_2$, i.e., the lines $[A_1, S']$ and $[A_1, S'']$ include with h_2 the angles $(\alpha_1 + \gamma_1)/2$ and $(\alpha_1 - \gamma_1)/2$ according to the notation used in Figure 6. When replacing S' by S'', the ratio in Lemma 2.4 becomes reciprocal, i.e., $d(S', h_2) : d(S', h_3) = d(S'', h_3) : d(S'', h_2)$, since the angles ϕ_i of rotations are replaced by $\phi_i - \pi$. □

Remark 2. At the flat pose $A_1 \dots B_3'$ of a twice-flat octahedron, the three pairs of line pencils with carriers (A_i, B_i') span a two-parameter linear set (or net) of dual conics. This net contains the conics mentioned in Lemma 2.1, (i). If the octahedron is flexible, then by virtue of Theorem 2.3, (i) the net contains also the concentric circles centered at S', provided that S' is finite. Then, the net contains the two-fold counted line pencil with the carrier S' as well as the set of isotropic lines and, consequently, with each conic all confocal conics. The strophoid $\mathcal{S}$ is Cayley's curve of this net (see, e.g., [8, p. 334]). Herewith, we obtain a characterization of the configuration of vertices at flat poses of type-3 octahedra.

How to construct a flat pose of a flexible twice-flat octahedron? Bricards first construction in [2, p. 144] is based on Theorem 2.2 and slightly different from that depicted in Figure 1a: It begins with the choice of a triangle $A_1A_2A_3$ and a point S' outside all extended sides $[A_i, A_j]$. Then the line $[A_i, B_j]$, $i \neq j$, is obtained by reflection of $[A_i, A_j]$ in $[A_i, S']$. Thus, point B_j is determined as the intersection of two lines. It will turn out that a similar construction is valid for flexible twice-flat cross-polytopes in $\mathbb{E}^n$.

2.4 Spatial Poses of Flexible Twice-flat Octahedra

The coming theorem enumerates for flexible twice-flat octahedra some properties of their spatial poses, in particular local symmetries. These symmetries do not act on faces but on the planes spanned by faces. It is noteworthy that many of these properties can already be extracted from Figures 5 and 6 in Bennett's paper [1]. Below we use for all permutations (i, j, k) of $(1, 2, 3)$ beside $\Delta_i = B_iA_jA_k$ the symbol $\Gamma_i := A_iB_jB_k$ for the wings of the inverse motion Γ_0/Δ_0.

Theorem 2.4. *Let $A_1 \dots B_3$ be a spatial pose of a flexible twice-flat octahedron $\mathcal{C}^3$ with finite centers S', S'' in the planar poses. Then holds for each permutation (i, j, k) of $(1, 2, 3)$:*

(i) The dihedral angle ψ between the faces $X_iX_jA_k$ and $X_iX_jB_k$ through the edge X_iX_j with $X_i \in \{A_i, B_i\}$ and $X_j \in \{A_j, B_j\}$ is either congruent or supplementary to the angle ϕ_k between the wing Δ_k and the fixed triangle Γ_0. Congruence holds if and only if at the initial flat pose the dihedral angle ψ equals 0.

(ii) The pairs of lines $([A_i, A_j], [A_i, B_j])$ and $([A_i, A_k], [A_i, B_k])$ through A_i have a common bisecting plane μ_i which is also common to the pairs of planes $([A_iA_jA_k], [A_iB_jB_k])$ and $([A_iA_jB_k], [A_iB_jA_k])$. The three bisecting planes μ_1, μ_2, μ_3 share a line s. The analogous planes of symmetry ν_i through B_i pass through s, too.

(iii) The sides of the three skew quadrangles $A_1A_2B_1B_2$, $A_2A_3B_2B_3$ and $A_3A_1B_3B_1$ are located respectively on one-sheeted hyperboloids of revolution $\mathcal{H}_3, \mathcal{H}_1, \mathcal{H}_2$ with the axis s (Figure 11). The three diagonals $[A_1, B_1], \ldots, [A_3, B_3]$ belong to a hyperbolic paraboloid.

(iv) For all i, the pairs of planes (μ_i, ν_i) connecting s with A_i and B_i have common planes of symmetry.

(v) The midpoints of the diagonals A_iB_i are coplanar with s, and the center of each hyperboloid $\mathcal{H}_i$ is aligned with two of these midpoints.

(vi) The vertices $A_1, \ldots, B_3$ belong to a spatial circular cubic $\mathcal{S}$ with s as a bisecant such that the tangents to $\mathcal{S}$ at the meeting points with s lie in orthogonal planes through s.

(vii) The four planes spanned by Γ_0, Γ_1, Γ_2, and Γ_3 are tangent to a cone of revolution with the axis s and apex C.

Similarly, the planes spanned by $\Delta_0, \ldots, \Delta_3$ envelop a coaxial cone with apex D (Figure 11). The absolute values of the trilateral coordinates of C w.r.t. Γ_0 and that of D w.r.t. Δ_0 are proportional to $\frac{1}{|\sin\phi_1|} : \frac{1}{|\sin\phi_2|} : \frac{1}{|\sin\phi_3|}$. The same holds for all other pairs of opposite faces. The rotation about s with $[\Delta_i] \mapsto [\Delta_j]$ takes also $[\Gamma_j]$ to $[\Gamma_i]$ for $i, j \in \{0, \ldots 3\}$. The two quadruples of planes form a pair of Möbius tetrahedra.

Proof. (i) This holds since at each isogonal double-pyramid with apex out from $\{A_1, \ldots, B_3\}$ opposite dihedral angles are congruent (note the octahedron's spherical image in Figure 10). It implies for the double-pyramid connecting the apex A_i with the quadrangle $A_jA_kB_jB_k$ that $\sphericalangle[A_iA_jA_k][A_iA_jB_k] = \sphericalangle[A_iB_jA_k][A_iB_jB_k]$. In other words, replacing A_j by B_j does not change the angle. Hence, after at most two steps follows that $\sphericalangle[X_iX_jA_k][X_iX_jB_k]$ equals $\sphericalangle[A_iA_jA_k][A_iA_jB_k] = \sphericalangle[\Gamma_0][\Delta_k]$.

(ii) By virtue of Lemma 2.2, the isogonal double-pyramid with apex A_i that extends the pyramid through the quadrangle $A_jA_kB_jB_k$ has a plane μ_i of symmetry. The reflection in μ_i preserves the pyramid's diagonal planes $[A_i, A_j, B_j]$, $[A_i, A_k, B_k]$ and exchanges within these planes the lines $[A_iA_j]$ and $[A_iB_j]$ as well as $[A_iA_k]$ and $[A_iB_k]$ (note Figure 5, right, and choose $A_i = O$, $A_j \in g_1$, $A_k \in g_2$, $B_j \in g_3$, and $B_k \in g_4$). Moreover, the reflection in μ_i exchanges opposite planes, i.e.,

$$[\Delta_k] = [A_iA_jB_k] \leftrightarrow [A_iB_jA_k] = [\Delta_j], \quad [\Gamma_0] = [A_iA_jA_k] \leftrightarrow [A_iB_jB_k] = [\Gamma_i].$$

The plane μ_i connects the intersection lines $[\Delta_j] \cap [\Delta_k]$ and $[\Gamma_0] \cap [\Gamma_i]$ (see Figure 10).

Similarly, the opposite double-pyramid with apex B_i has a plane ν_i of symmetry. The reflection in ν_i exchanges $[B_iA_j]$ and $[B_iB_j]$ as well as $[B_iA_k]$ and $[B_iB_k]$, and moreover

$$[\Gamma_j] = [B_iA_jB_k] \leftrightarrow [B_iB_jA_k] = [\Gamma_k], \quad [\Delta_i] = [B_iA_jA_k] \leftrightarrow [B_iB_jB_k] = [\Delta_0].$$

Let s be the line of intersection $\mu_1 \cap \mu_2$ of the planes of symmetry through A_1 and A_2, respectively. The line s must be finite since otherwise the respective mirrors $[\Delta_1]$ and $[\Delta_2]$ of $[\Delta_3]$ would be parallel which contradicts (i).

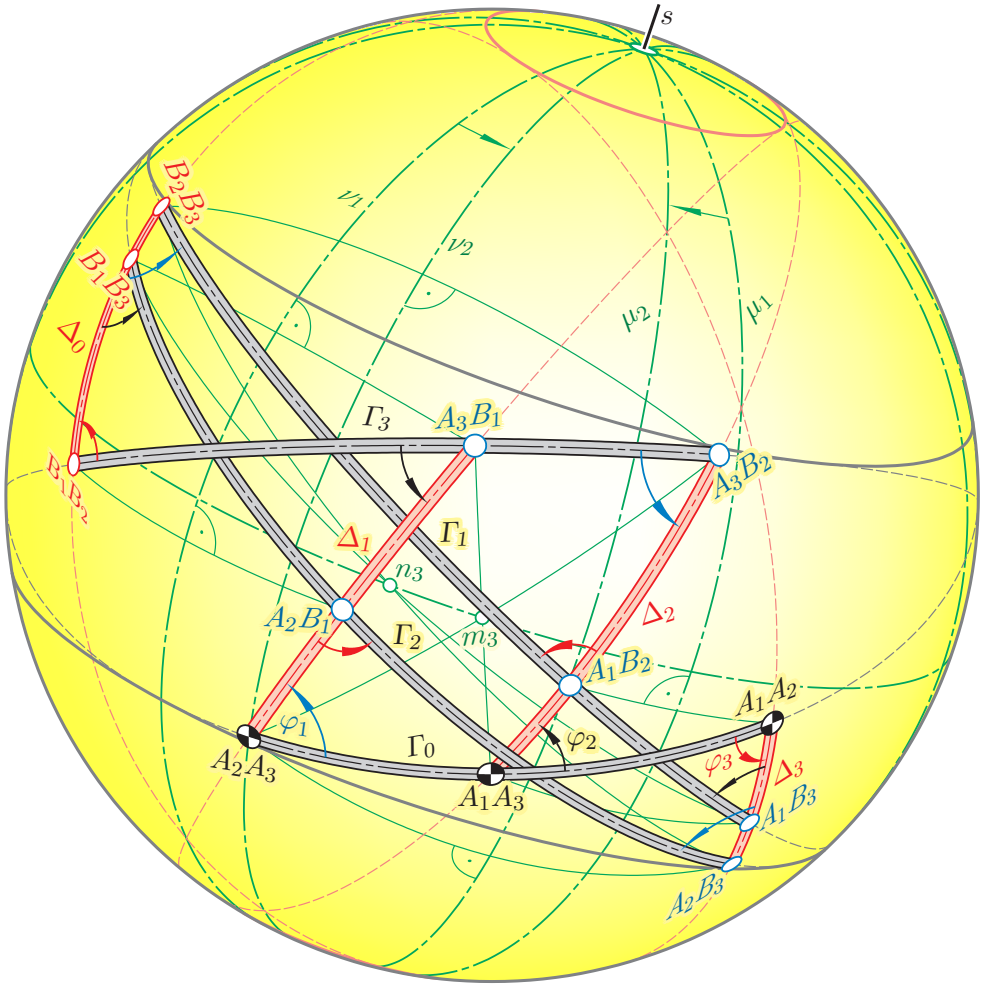


Figure 10: The spherical image of the faces and edges of a type-3 octahedron is an 8-bar linkage with 12 joints [1]. The planes of the wings and $[\Delta_0]$ are equally inclined w.r.t. an axis s. The same holds for the planes spanned by the faces $\Gamma_0 \ldots, \Gamma_3$.

Each point $S \in s$ has equal distances to $[\Gamma_0]$ and $[\Gamma_1]$ as well as to $[\Gamma_0]$ and $[\Gamma_2]$. The same holds for the distances to $[\Delta_3]$ and $[\Delta_2]$ as well as to $[\Delta_3]$ and $[\Delta_1]$.

From equal distances to $[\Gamma_1] = [A_1B_2B_3]$ and $[\Gamma_2] = [B_1A_2B_3]$, which are opposite planes through B_3, follows that s belongs either to ν_3 or to the second bisecting plane of these two planes. The latter can be excluded since in the initial flat pose s coincides with the perpendicular s' to H_f through S' or S'' and all μ_i and ν_i pass through s' and continuity guarantees that the bisecting planes cannot switch during the flexion. Thus, the plane ν_3 passes through s, too. Similarly, equal distances to the opposite planes $[\Delta_1] = [B_1A_2A_3]$ and $[\Delta_2] = [A_1B_2A_3]$ through A_3 imply $s \subset \mu_3$. Iteration by cyclically increasing the involved subscripts confirms the statement.

(iii) Concerning the skew quadrangle $A_1A_2B_1B_2$, the consecutive reflections in the planes μ_2, ν_1, ν_2, and μ_1 take $[A_1, A_2]$ via $[A_2, B_1]$ and $[B_1, B_2]$ to $[B_2, A_1]$. This confirms the existence of the hyperboloid $\mathcal{H}_3$ passing through the four lines. It agrees with the fact that due to the existing incircles in the flat poses, an alternating sum of the side lengths vanishes, which guarantees the location of the skew quadrangle on a hyperboloid of revolution (note [13, Theorem 2.2.2]).

As noted in (ii), the diagonal planes of the double-pyramid with apex A_i are orthogonal to μ_i. Hence, also the line of intersection $m_i := [A_i, A_j, B_j] \cap [A_i, A_k, B_k]$ through A_i is orthogonal to μ_i and, consequently, also orthogonal to s. Note that m_i is the transversal drawn from A_i to the remaining diagonals $[A_j, B_j]$ and $[A_k, B_k]$. The lines $n_i \perp \nu_i$ through B_i play a similar role. Common transversals of the three diagonals $[A_i, B_i]$ for $i = 1, 2, 3$ like m_i and n_i belong to the complementary regulus of the quadric defined by the diagonals. Since $m_1, \ldots, n_3$ are orthogonal to s, the quadric through the three diagonals is a hyperbolic paraboloid.

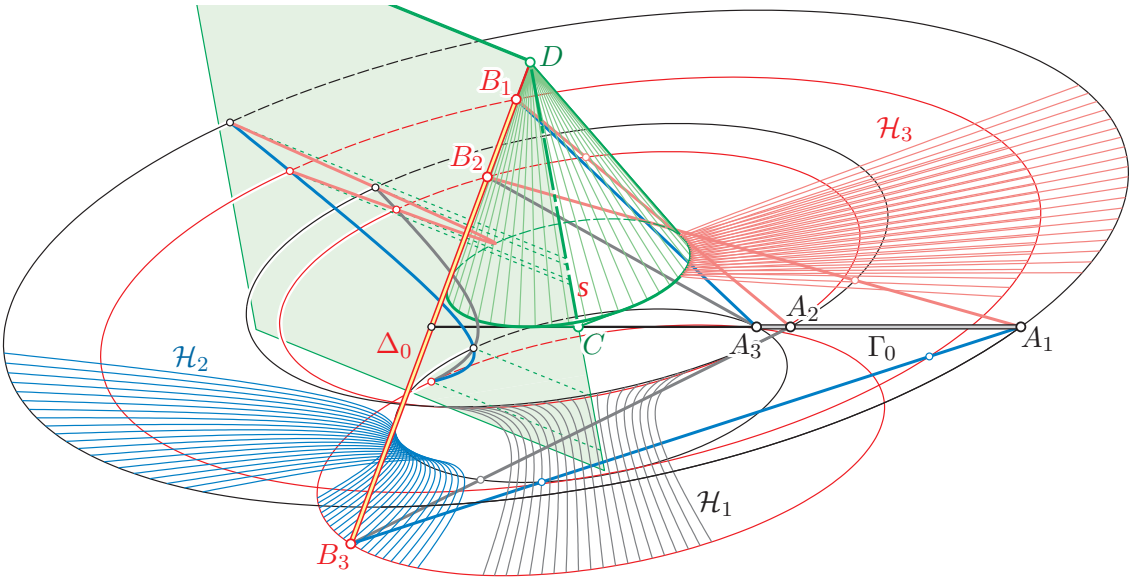


Figure 11: Spatial pose of a flexible twice-flat octahedron showing both triangles $\Gamma_0 = A_1A_2A_3$ and $\Delta_0 = B_1B_2B_3$ in edge views. The three quadrangles $A_2A_3B_2B_3$ (grey), $A_3A_1B_3B_1$ (blue) and $A_1A_2B_1B_2$ (red) are located on hyperboloids of revolution $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3$ with the common axis s and displayed meridians. The planes $[\Gamma_0], \ldots, [\Gamma_3]$ envelop a cone of revolution with axis s and apex $C \in [\Gamma_0]$. Similarly, $[\Delta_0]$ and the planes spanned by the wings envelop a cone with axis s and apex $D \in [\Delta_0]$.

(iv) The quadrics through the sides of the skew quadrangle $A_iA_jB_iB_j$ belong to a dual pencil (range). This range includes the pairs of bundels of planes with carriers (A_i, B_i) and (A_j, B_j). According to Desargues's involution theorem, each dual quadric of the range sends through s two planes that define an involution. Since the hyperboloid of revolution $\mathcal{H}_k$ passing through $A_iA_jB_iB_j$ sends two isotropic tangent planes through s, the involution is symmetric. This symmetric involution is the same for all three quadrangles, as any two share a pair of opposite vertices.

(v) The quadrics of the range mentioned before have their centers on a line passing through the midpoints of the diagonals A_iB_i and A_jB_j. This center line intersects s at the center of hyperboloid $\mathcal{H}_k$. Since any two of the three quadrangles share a pair of opposite vertices, the center lines of the three ranges must be coplanar with the axis s.

(vi) The cubic is defined as locus of points X with the property that the connecting lines with two given points A_1, B_1 have a plane of symmetry passing through s (see [3]).[5]

(vii) Referring to (ii), the reflection σ_i in the plane μ_i through s exchanges $[\Gamma_0]$ with $[\Gamma_i]$ for $i = 1, 2, 3$. Consequently, the planes $[\Gamma_0], \ldots, [\Gamma_3]$ enclose the same angle γ with s and share

[5]Points $X \in \mathcal{S}$ can be found as follows: For any given plane ε through s, reflect A_1 in ε to A_1^*. Then, $X = [A_1^*, B_1] \cap \varepsilon$. The mirror points A_1^* are located on a circle through A_1 with axis s.

a point $C \in s$. Thus, they envelop a cone of revolution with axis s. Analogously, the planes $[\Delta_0], \ldots, [\Delta_3]$ envelop a second cone with axis s, apex D and apex angle δ (Figure 11).

The planes $[\Gamma_0]$ through C and $[\Delta_k]$ through D meet at the hinge h_k and enclose the dihedral angle $|\phi_k|$. Thus, for the angles γ between H_f and s and δ between $[\Delta_k]$ and s holds, when the overline symbolizes shortest distances,

$$\overline{CD}\sin\gamma = \overline{D[\Gamma_0]} = \overline{Dh_k}\,|\sin\phi_k|,\ \overline{CD}\sin\delta = \overline{C[\Delta_k]} = \overline{Ch_k}\,|\sin\phi_k|,$$

and consequently

$$\overline{Ch_k} : \overline{Dh_k} : \frac{1}{|\sin\phi_k|} = \overline{CD}\,\sin\delta : \overline{CD}\,\sin\gamma : 1, \tag{2.5}$$

independent of the index k. The same is true for all other pairs of faces that by (i) enclose the same dihedral angle like Γ_0 and Δ_k.

Moreover, the planes $[\Gamma_0], [\Delta_1], [\Delta_2], [\Delta_3]$ bound the tetrahedron $A_1A_2A_3D$ that is in- and circumscribed the tetrahedron $B_1B_2B_3C$ bounded by $[\Delta_0], [\Gamma_1], [\Gamma_2], [\Gamma_3]$.

By the way, at each spatial pose the infinitesimal screw motion of Δ_0 relative to Γ_0 is associated with a null polarity that maps B_i to the orthogonal plane $[\Delta_i]$ of its trajectory, while D is the null point of Δ_0 (see, e.g., [13, p. 485-486]). This null polarity interchanges the two tetrahedra. □

Remark 3. The properties listed in Theorem 2.4 reveal, that the orthogonal projection of a spatial pose in direction of the axis s yields an image that has all properties of the flat poses as listed in Theorem 2.3. The spatial cubic $\mathcal{S}$ from (vi) is sent to the strophoid from Theorem 2.3, (ii). The negative pedal curve of the strophoid w.r.t. its node is a parabola (see [8, Fig. 2.27]) that equals the visual contour of the paraboloid through the diagonals $[A_i, B_i]$ and the lines $m_1, \ldots, n_3$.

At the spherical image of a spatial pose the octahedron's faces and edges are represented as bars and joints of a spherical 8-bar linkage with 12 revolute joints [1, 18]. It is the union of six isogonal spherical four-bar linkages. Three four-bars have their bases on the image of Γ_0, while their respective couplers are aligned with arms of other three four-bars with bases on the spherical image of Δ_0 (Figure 10).

3 Flexible Twice-flat Cross-polytopes in $\mathbb{E}^n$

As reported in Section 1, a cross-polytope $\mathcal{C}^n$ in $\mathbb{E}^n$ has n pairs of opposite vertices (A_i, B_i), $i = 1, \ldots, n$. We return to the original denotation and assume that the simplex $\Gamma_f \coloneqq A_1 \ldots A_n$ in the hyperplane H_f is fixed while the opposite simplex $\Gamma_m \coloneqq B_1 \ldots B_n$ in the hyperplane H_m is moving. Hence, during the rotation of the simplex called wing $\Delta_i \coloneqq A_1 \ldots A_{i-1}B_iA_{i+1} \ldots A_n$ about the $(n-2)$-dimensional hinge $h_i = [A_1 \ldots A_{i-1}A_{i+1} \ldots A_n]$ the vertex B_i traces a circle (note case $n = 3$ in Figs. 2 and 8 and case $n = 4$ in Figure 14).

Two facets of $\mathcal{C}^n$ like $X_1 \ldots A_i \ldots X_n$ and $X_1 \ldots B_i \ldots X_n$ with $X_k \in \{A_k, B_k\}$ are called *neighboring*, if they share an $(n-2)$-face. In the case of a flexible cross-polytope, the relative motion of two neighboring facets is a rotation about the hinge spanned by the common $(n–2)$-face. It will turn out that the $(n-3)$-faces of $\mathcal{C}^n$ serve as apices of the higher-dimensional versions of the isogonal pyramids at the type-3 octahedra. Each $(n-3)$-face of $\mathcal{C}^n$ is the meet of four facets where each two consecutive facets in cyclic order are neighboring. For example, the facets through the $(n-3)$-face $X_3 \ldots X_n$ are $A_1A_2X_3 \ldots X_n$, $A_1B_2X_3 \ldots X_n$, $B_1B_2X_3 \ldots X_n$, and $B_1A_2X_3 \ldots X_n$.

3.1 Local Symmetries at Flat Poses of $\mathcal{C}^n$

Let $\mathcal{C}^n$ be a twice-flat cross-polytope in $\mathbb{E}^n$ for $n > 3$. This means by Definition 1.1 that $\mathcal{C}^n$ admits beside a given flat pose $A_1 \dots A_n B'_1 \dots B'_n$ a second flat pose $A_1 \dots A_n B''_1 \dots B''_n$ with $B'_i \neq B''_i$ for all $i \in \{1, \dots, n\}$.

The second pose B''_1 of the vertex B_1 arises in $\mathbb{E}^n$ from the first pose B'_1 by a halfturn about h_1. This halfturn acts within the fixed hyperplane H_f like a reflection in h_1. Consequently, within H_f the $(n-2)$-dimensional space h_1 is the bisecting hyperplane between B'_1 and B''_1. Similarly, h_2 is the bisecting hyperplane between B'_2 and B''_2. Because of equal distances $\overline{B'_1 B'_2} = \overline{B''_1 B''_2}$ we obtain congruent simplices $B'_1 B'_2 A_3 \dots A_n$ and $B''_1 B''_2 A_3 \dots A_n$. Hence, there exists in H_f a rotation about the $(n-3)$-dimensional space $v_{12} := h_1 \cap h_2 = [A_3 \dots A_n]$ with $B'_1 \mapsto B''_1$ and $B'_2 \mapsto B''_2$ (Figure 12).

Within H_f, the rotation about v_{12} through the halved angle takes $[A_3 \dots A_n B'_1] = [v_{12}, B'_1]$ to the bisecting hyperplane $h_1 = [v_{12}, A_2]$ and $[v_{12}, B'_2]$ to $h_2 = [v_{12}, A_1]$. Thus, within H_f the pairs of hyperplanes $([v_{12}, A_1], [v_{12}, B'_1])$ and $([v_{12}, A_2], [v_{12}, B'_2])$ have a common bisecting hyperplane.

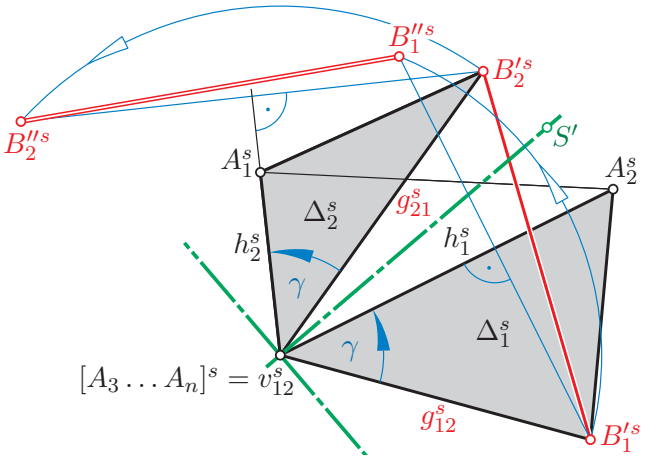


Figure 12: Two-dimensional view of the two wings Δ_1, Δ_2 in a flat pose within H_f after an orthogonal projection of H_f parallel to $v_{12} = [A_3 \dots A_n]$ into a 2-plane.

Similar symmetries exist at all other pairs of vertices (A_i, B'_i) and (A_j, B'_j), $i \neq j$, for their $(n-2)$-dimensional connections with the $(n-3)$-dimensional space v_{ij} spanned by the remaining vertices A_k of Γ_f with $k \neq i, j$.

Theorem 3.1. *A flat cross-polytope $\mathcal{C}^n$ with the simplex $\Gamma_f = A_1 \dots A_n$ and the coplanar opposite simplex $\Gamma'_m = B'_1 \dots B'_n$ is twice-flat if and only if for all $i, j \in \{1, \dots, n\}$ the connecting $(n-2)$-spaces of the pairs (A_i, B'_i) and (A_j, B'_j) with the $(n-3)$-face of Γ_f opposite to $A_i A_j$ have common angle bisecting $(n-2)$-spaces.*

Proof. Let from now on v_{ij} and v_{ji} denote the span of $\{A_k \,|\, k \in \{1 \dots n\} \setminus \{i, j\}\}$. Then, the stated condition for being twice-flat is necessary and sufficient in H_f as common angle bisectors of the pairs of hyperplanes $([v_{ij}, A_i], [v_{ij}, B'_i])$ and $([v_{ij}, A_j], [v_{ij}, B'_j])$ characterize equal distances $\overline{B'_i B'_j} = \overline{B''_i B''_j}$. □

Since for the existence of a second flat pose it is not relevant which facet of the cross-polytope $\mathcal{C}^n$ is fixed, the symmetries stated above result in similar symmetries at all $(n-3)$-faces of $\mathcal{C}^n$.

Definition 3.1. A flat cross-polytope $A_1 \dots B'_n$ in a hyperplane of $\mathbb{E}^n$ is called *locally symmetric* when it satisfies the condition mentioned in Theorem 3.1. In this case similar symmetries hold for each $(n-3)$-face of the flat cross-polytope.

3.2 Transmission Between two Wings of a Twice-flat Cross-polytope

Now we investigate the relation between the angles of rotation ϕ_1, ϕ_2 of the wings $\Delta_1 = A_2A_3 \dots A_nB_1$ and $\Delta_2 = A_1A_3 \dots A_nB_2$ relative to the fixed simplex $\Gamma_f = A_1 \dots A_n$ when the moving edge B_1B_2 preserves its length.

The wing Δ_1 with B_1 rotates about $h_1 = [A_2A_3 \dots A_n]$, while Δ_2 with B_2 rotates about $h_2 = [A_1A_3 \dots A_n]$. The two wings are connected by the facet $\Delta_{12} \coloneqq A_3 \dots A_nB_1B_2$. In non-flat poses, the four facets Δ_1, Δ_{12}, Δ_2, and Γ_f form a four-sided pyramid in $\mathbb{E}^n$ with the $(n-3)$-dimensional apex space $v_{12} = [A_3 \dots A_n] = h_1 \cap h_2$ and the quadrangular base $B_1B_2A_1A_2$. The 'edges' of the pyramid are the intersections between adjacent facets, i.e., the $(n-2)$-faces $\Delta_1 \cap \Delta_{12} = A_3 \dots A_nB_1$, $\Delta_{12} \cap \Delta_2 = A_3 \dots A_nB_2$, $A_3 \dots A_nA_1$ spanning h_2, and $A_3 \dots A_nA_2$ spanning h_1.

For the investigation of the pyramid's flexion in $\mathbb{E}^n$, we recall from Linear Algebra that two (linear) subspaces U, V in $\mathbb{E}^n$ are called *orthogonal* (or total-orthogonal) if each direction in U is orthogonal to all directions in V. If the sum of dimensions of the two spaces equals n, then conversely each direction that is orthogonal to all directions in V is parallel to U.

Lemma 3.1. *Let (U_i, V_i) for $i = 1, 2$ be two pairs of orthogonal subspaces in $\mathbb{E}^n$ where in both cases the dimensions sum up to n. Then $U_1 \subset U_2$ implies that each direction in V_2 is parallel to V_1.*

Our analysis utilizes a projection p_{12} of $\mathbb{E}^n$ in direction of the $(n-3)$-dimensional apex space v_{12} into a 3-space Π_{12}, which is orthogonal to v_{12}. We call the image of this orthogonal projection briefly a *side view* and mark images with the superscript s. Thus, the side view of the pyramid in $\mathbb{E}^n$ is a four-sided pyramid in Π_{12} with the point v_{12}^s as apex and the quadrangular base $B_1^sB_2^sA_1^sA_2^s$ (Figure 13). The restriction of p_{12} to the hyperplane $H_f = [\Gamma_f]$ sends the pyramid's flat pose in a 2-plane $H_f^s \subset \Pi_{12}$ (Figure 12).

Lemma 3.2. *The orthogonal projection $p_{12} \colon \mathbb{E}^n \to \Pi_{12}$ in direction of the $(n-3)$-dimensional space $v_{12} = [A_3 \dots A_n]$ maps the four-sided pyramid through $B_1B_2A_1A_2$ with the apex space v_{12} to a three-dimensional pyramid while dihedral angles between neighboring facets and interior angles at the apex v_{12}^s are preserved. The projection p_{12} preserves also the distances of points $X \in H_f$ to the hinges h_1 and h_2.*

Proof. (i) The dihedral angle ϕ_1 between the wing Δ_1 and the fixed facet Γ_f is measured in a 2-plane orthogonal to the extended edge $[\Delta_1 \cap \Gamma_f] = h_1$. From $v_{12} \subset h_1$ follows by Lemma 3.1 that the 2-plane with the angle ϕ_1 lies parallel to the image space Π_{12}. Consequently, the side view shows the dihedral angle ϕ_1 in true size. The same holds for the other dihedral angles of the pyramid in $\mathbb{E}^n$ (Figure 13).

(ii) Within the wing Δ_1, the dihedral angle between the two extended edges $[v_{12}, A_2] = h_1$ and $[v_{12}, B_1]$ is measured in a 2-plane that is orthogonal to v_{12} and therefore parallel to Π_{12}. Since the same holds for the other three facets of the four-sided pyramid in $\mathbb{E}^n$, the orthogonal projection p_{12} preserves the pyramid's interior angles at the apex (Figure 13).

(iii) Within H_f, the distance of any point X to the $(n–2)$-dimensional hinge h_1 is measured along a line that is orthogonal to h_1. From $v_{12} \subset h_1$ follows within H_f by Lemma 3.1 that this line is parallel to $H_f^s \subset \Pi_{12}$. Therefore, the side view shows for each point $X \in H_f$ the distance to h_1 in true size as well that to h_2. □

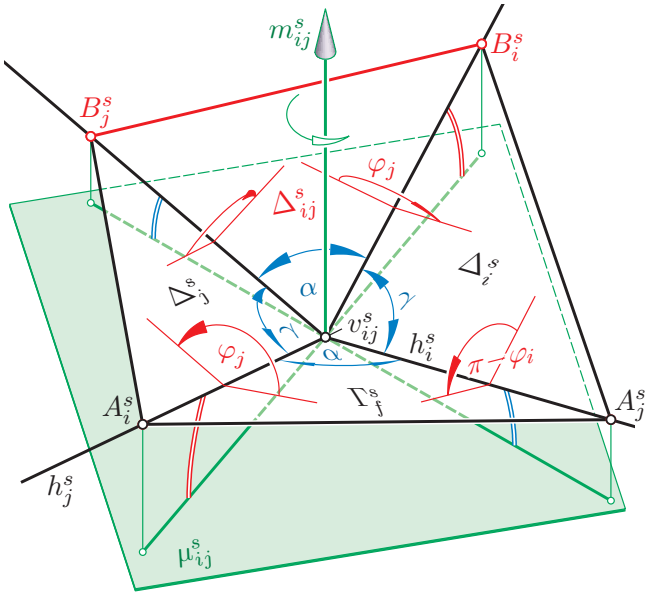


Figure 13: The orthogonal projection p_{ij} of $\mathbb{E}^n$ parallel to $v_{ij} = h_i \cap h_j$ into a 3-space Π_{ij} yields this view of the isogonal pyramid with the $(n-3)$-dimensional apex space v_{ij} and the two wings Δ_i, Δ_j in a spatial pose. The plane μ_{ij}^s is the image of a hyperplane μ_{ij} of symmetry of the extended double-pyramid. The preimage of m_{ij}^s is the $(n-2)$-dimensional axis m_{ij} of a halfturn that exchanges opposite elements of the double-pyramid in $\mathbb{E}^n$, where m_{ij} and μ_{ij} are orthogonal spaces in the bundle with carrier v_{ij}.

Remark 4. Lemma 3.2 reveals that the four-sided pyramid in $\mathbb{E}^n$ with the apex space v_{12} can also be seen as a prism with generators parallel to v_{12} and the three-dimensional pyramid in Π_{12} as an orthogonal cross-section.

From Lemma 3.2 follows that the transmission from wing Δ_1 to Δ_2 in $\mathbb{E}^n$ satisfies the same formula like that in 3-space. This is a result that can already be found in [5, Proposition 3.3].

By virtue of Theorem 3.1, the four-sided pyramid in Π_{12} is isogonal, i.e., opposite angles at the apex v_{12}^s are congruent or supplementary (Figure 12). The same holds, due to Lemma 3.2, for the preimage in $\mathbb{E}^n$, and the relation between the revolute angles ϕ_1 and ϕ_2 matches (2.1), where α_1 and γ_1 stand for interior angles at the n-dimensional pyramid with apex space v_{12}. In addition, due to Lemma 3.2, also eq. (2.2) is still valid for the distances from h_1 and h_2 within H_f.

Similar orthogonal projections for all other $(n-3)$-faces of the fixed simplex Γ_f yield analogous results. Thus we can state:

Lemma 3.3. *At a twice-flat cross-polytope $\mathcal{C}^n$ in $\mathbb{E}^n$ the distance between two vertices $B_i, B_j \in \Gamma_m$ remains constant if and only if the angles of rotation ϕ_i, ϕ_j of the wings Δ_i and Δ_j satisfy*

$$t_i : t_j = \tan\frac{\phi_i}{2} : \tan\frac{\phi_j}{2} = \frac{1}{d(S', h_i)} : \frac{1}{d(S', h_j)}, \tag{3.1}$$

provided that, in a flat pose $A_1, \dots, B'_n$ of $\mathcal{C}^n$, the point $S' \in H_f$ lies in a common bisecting $(n-2)$-space through $h_i \cap h_j = v_{ij}$ between the $(n-2)$-dimensional spaces $[v_{ij}, A_i]$ and $[v_{ij}, B'_i]$ as well as between $[v_{ij}, A_j]$ and $[v_{ij}, B'_j]$ according to Theorem 3.1.

The symbol $d(S', h_k)$ with $k = 1, \dots, n$ stands for the signed distance of the point $S' \in H_f$ to the hinge h_k of the wing Δ_k, such that for each hinge in H_f the distance is positive if and only if S' belongs to the halfspace of h_k containing the interior of the fixed simplex $\Gamma_f = A_1 \dots A_n$.

3.3 Necessary and Sufficient Condition for Flexibility of $\mathcal{C}^n$

Theorem 3.2. *A flat cross-polytope $A_1 \dots A_n B'_1 \dots B'_n$ in the hyperplane H_f is twice-flat and flexible in $\mathbb{E}^n$, $n > 3$, if and only if it is locally symmetric according to Definition 3.1 and at every $(n-3)$-face of the fixed simplex $A_0 \dots A_n$ one of the $(n-2)$-dimensional symmetry spaces according to Theorem 3.1 passes through a common (finite or infinite) point S'.*

Proof. A cross-polytope $\mathcal{C}^n$ is flexible if the simultaneous continuous rotations of the wings preserve all distances $\overline{B_i B_j}$ with $i \neq j$. For twice-flat $\mathcal{C}^n$ this means by Lemma 3.3 that for each (i, j) the tangents t_i, t_j of the halved angles of rotation ϕ_i, ϕ_j satisfy (3.1) for any point S'_{ij} in one of the $(n-2)$-planes of symmetry of the pairs $\big([v_{ij}, A_i], [v_{ij}, B'_i]\big)$ or $\big([v_{ij}, A_j], [v_{ij}, B'_j]\big)$.

(i) If there exists a common center S' for all (i, j), then for given ϕ_1 the choice

$$t_i := \frac{d(S', h_1)}{d(S', h_i)}\, t_1 \quad \text{for } i = 2, \dots, n$$

guarantees flexibility since for every two t_i, t_j eq. (3.1) holds true.

(ii) Conversely, if a given twice-flat cross-polytope $\mathcal{C}^n$ is flexible, then we use the vertices $A_1, \dots, A_n$ of the fixed simplex Γ_f as base points of a projective coordinate frame in the projective extension of H_f and the center of the insphere of Γ_f as its unit point.[6] Then, the homogeneous coordinates of any finite point in H_f are proportional to the signed distances from the facets of Γ_f.

The transmissions between the rotations of any two wings Δ_i, Δ_j of the given cross-polytope $\mathcal{C}^n$ define by (3.1) homogeneous coordinates $(t_1 : \dots : t_n)$ of a unique point $S' \in H_f$, and by (2.1), (2.2) and Lemma 3.3 the point S' is located on one of the axes of local symmetry at each $(n-3)$-face of Γ_f. □

Like in $\mathbb{E}^3$, the motion of the moving simplex Γ_m relative to Γ_f defines a projective mapping between the circular trajectories of every two moving points B_i, B_j.

Corollary 3.1. *If in H_f at a flat pose $A_1 \dots B'_n$ of a twice-flat cross-polytope $\mathcal{C}^n$ through each $(n-3)$-face of the fixed simplex $\Gamma_f = A_1 \dots A_n$ one of the $(n-2)$-dimensional symmetry spaces in the sense of Theorem 3.1 passes through a common point S', then the same is true for all $(n-3)$-faces in the flat pose $A_1 \dots B'_n$ of $\mathcal{C}^n$.*

Proof. By virtue of Theorem 3.2 the existence of S' characterizes flexible twice-flat cross-polytopes. However, for the flexibility it is not relevant which facet of $\mathcal{C}^n$ is supposed to be fixed. When we proceed step by step from Γ_f to other facets by iteratively replacing one vertex A_i by B_i so that we finally reach Γ_m, then each pair of consecutive facets shares at least $n-1$ symmetry spaces which already define the meeting point uniquely. Thus, S' is common for $(n-2)$-dimensional symmetry spaces of all $(n-3)$-faces in H_f. □

[6] This choice is equivalent to the usage of signed Hessian normal forms for the n bounding hyperplanes of the simplex Γ_f in order to define the coordinates.

Due to Theorem 3.2, a flat pose of a flexible twice-flat cross-polytope $\mathcal{C}^n$ can be constructed as given below. The three-dimensional version dates back to Bricard in [2, p. 144].

Construction. For obtaining a flat pose of a flexible twice-flat cross-polytope in $\mathbb{E}^n$, $n \geq 3$,

1. choose within a hyperplane H_f of $\mathbb{E}^n$ a simplex $\Gamma_f = A_1 \dots A_n$ and a point S' off from the bounding hyperplanes and from hyperplanes that bisect any dihedral angle of Γ_f.
2. For all (i, j) with $i, j \in \{1, \dots, n\}$ and $i \neq j$, let v_{ij} denote the span of the $(n-3)$-face of Γ_f opposite to A_iA_j. Then, reflect $h_j = [v_{ij}, A_i]$ in the connection $[v_{ij}, S']$ to obtain the $(n-2)$-dimensional space g_{ij} passing through B'_i (Figure 12).
3. Within the hyperplane H_f, the point B'_i is the intersection of the $(n-2)$-planes g_{ij} for $j \in \{1, \dots, n\} \setminus \{i\}$ (see the choices $(i, j) = (1, 2)$ or $(2, 1)$ in Figure 12).

Remark 5. The choice of S' in a hyperplane that bisects the dihedral angle of Γ_f at v_{ij} yields flat wings Δ_i and Δ_j (see Figure 12), which has been excluded from the beginning.

Theorem 3.3. *Let $A_1 \dots B'_n$ in the hyperplane H_f of $\mathbb{E}^n$, $n \geq 4$, be a flat pose of a flexible twice-flat cross-polytopes $\mathcal{C}^n$ with a finite center S' of local symmetries. Then holds:*

(i) For each $i \in \{1, \dots, n\}$, the spans of all $(n-2)$-faces with vertices different from A_i and B'_i contact the same hypersphere $\mathcal{S}_i$ centered at S'. Consequently, each $(n-2)$-face of the given flat pose of $\mathcal{C}^n$ has a span which is tangent to one of in total n concentric hyperspheres $\mathcal{S}_1, \dots, \mathcal{S}_n$ in H_f.

(ii) The n dual quadrics in H_f of rank 2 consisting of two bundels of hyperplanes with respective carriers A_i and B'_i for $i = 1, \dots, n$ span a linear set that contains the dual rank-1 quadric of hyperplanes through S' and the family of isotropic hyperplanes.

(iii) The midpoints of the diagonals $A_iB'_i$ for $i = 1, \dots, n$ are coplanar with the center S' of local symmetries.

(iv) If n is even, then the second flat pose $B''_1 \dots B''_n$ of the moving simplex Γ_m is directly congruent to the first $B'_1 \dots B'_n$ within H_f. Otherwise the congruence transformation in H_f from Γ'_m to Γ''_m is orientation reversing.

(v) Let S'' be the center of local symmetry in the second flat pose and $v_{ij} = h_i \cap h_j$ be the span of any $(n-3)$-face of the fixed simplex Γ_f. Then, the connections $[v_{ij}, S']$ and $[v_{ij}, S'']$ as well as the facets h_i and h_j of Γ_f share the angle bisecting hyperplanes within H_f. In this sense, the points S' and S'' are isogonal w.r.t. $A_1 \dots A_n$.

Proof. (i) By virtue of Corollary 3.1, the local symmetry of the given flat pose of $\mathcal{C}^n$ implies within the $(n-1)$-dimensional H_f that the connection of S' with any $(n-3)$-face of $\mathcal{C}^n$ is a hyperplane of local symmetry. This means for example, that the reflection in $[S', X_3 \dots X_n]$ with $X_k \in \{A_k, B_k\}$ exchanges $[A_2X_3 \dots X_n]$ with $[B_2X_3 \dots X_n]$. Consequently, these two $(n-2)$-spaces have equal distances to S'. Iteration shows that $[B_2A_3 \dots X_n]$ and $[B_2B_3 \dots X_n]$ have equal distances to S' due to their symmetry w.r.t. $[S', B_2X_4 \dots X_n]$, and so on. Thus, we obtain 2^{n-1} different $(n-2)$-faces of the flat cross-polytope in H_f with spans at equal distances to S'. These $(n-2)$-faces are characterized by containing no vertex with the subscript 1.

Statement (i) is apparently the n-dimensional generalization of Figure 1(a); the families of 2^{n-1} $(n-2)$-faces of $\mathcal{C}^n$ tangent to the same hypersphere are the counterparts of the quadrangles on octahedra.

(ii) We prove that the hyperspheres $\mathcal{S}_1, \dots, \mathcal{S}_n$ belong to the addressed linear system of quadrics in H_f. As a consequence, also the dual pencil of concentric hyperspheres is included and consequently the mentioned rank-1 quadric and the family of isotropic hyperplanes in H_f.

It is sufficient to show that $\mathcal{S}_1$ is contained in the subsystem spanned by the rank-2 quadrics with carriers (A_2, B_2'), ..., (A_n, B_n'), which are the vertices of a full-dimensional cross-polytope in H_f:

Each quadric of this subsystem in H_f contains the hyperplanes $[X_2 \dots X_n]$ since they are common to all spanning rank-2 quadrics. Thus, the subsystem is a subset of the dual quadrics containing the 2^{n-1} hyperplanes.

Conversely, the dual quadrics in H_f through these hyperplanes form a linear system, since the entries in the related symmetric coefficient matrices are the solutions of a linear system of equations. For each quadric $\mathcal{Q}$ out of this linear system holds for each $j \in \{2, \dots, n\}$: If a hyperplane of $\mathcal{Q}$ passes through one point of the pairs (A_i, B_i') for all $i \in \{2, \dots, n\}$ with $i \neq j$, then it also passes through A_j or B_j'. Passing through A_i or B_i' is for hyperplanes equivalent to be contained in the rank-2 quadric with carriers (A_i, B_i'). Thus, $\mathcal{Q}$ belongs to the $(n-2)$-dimensional subsystem. With other words, each full-dimensional cross-polytope $\mathcal{C}^{n-1}$ in H_f defines an $(n-2)$-dimensional linear system of quadrics that contain the spans of all facets of $\mathcal{C}^{n-1}$.

Due to (i), the hypersphere $\mathcal{S}_1$ with center S' contacts all extended facets of the cross-polytope with vertices $A_2 \dots A_n B_2' \dots B_n'$ and is therefore contained in the subsystem defined above. Similar properties of the other hyperspheres $\mathcal{S}_2 \dots \mathcal{S}_n$ confirm the statement (ii).

(iii) In H_f, the dual quadrics contained in a $(n-1)$-parametric linear system have their centers located in a hyperplane. This is true since the symmetric coefficient matrices of the involved quadrics are linear combinations of n base matrices. Consequently, any given hyperplane has its poles w.r.t. the dual quadrics of this $(n-1)$-parametric linear system within a hyperplane.

(iv) Below there is a sequence of $n+1$ facets of $\mathcal{C}^n$ beginning with Γ_m and ending with Γ_f. Every two consecutive facets are neighboring, i.e., they share an $(n-2)$-face that spans the hinge between the facets.

$$\begin{array}{ll} \text{facets:} & B_1B_2 \dots B_n, \quad A_1B_2 \dots B_n, \quad A_1A_2B_3 \dots B_n, \quad \dots, \quad A_1 \dots A_{n-1}B_n, \quad A_1A_2 \dots A_n \\ \text{hinges:} & \qquad [B_2 \dots B_n], \quad [A_1B_3 \dots B_n], \quad [A_1A_2B_4 \dots B_n], \quad \dots, \quad [A_1A_2 \dots A_{n-1}]. \end{array} \tag{3.2}$$

We use this sequence to explain how the first flat pose $\Gamma_m' = B_1' \dots B_n'$ of Γ_m can be transferred in the second flat pose $\Gamma_m'' = B_1'' \dots B_n''$ by n consecutive halfturns:

A first halfturn is carried out about the hinge $[B_2' \dots B_n']$ between Γ_m' and $A_1B_2'B_3' \dots B_n'$ in order to achieve the correct relative position of these two facets according to the second flat pose. Afterwards, the moving simplex is attached to $A_1B_2' \dots B_n'$ and performs the halfturn about the hinge $[A_1B_3' \dots B_n']$ between $A_1B_2' \dots B_n'$ and $A_1A_2B_3' \dots B_n'$ to achieve the correct positions of the first two facets relative to the third one according to the other flat pose. Afterwards, Γ_m is attached to $A_1A_2B_3' \dots B_n'$ during the halfturn about the hinge $[A_1A_2B_4' \dots B_n']$ with $A_1A_2A_3B_4' \dots B_n'$, and so on. The final halfturn uses the hinge $h_n = [A_1 \dots A_{n-1}]$ between $A_1 \dots A_{n-1}B_n' = \Delta_n$ and $A_1 \dots A_n = \Gamma_f$.

Within the hyperplane H_f, which contains both flat poses, the halfturns act like reflections in $(n-2)$-dimensional spaces, i.e., hyperplanes of H_f, and each of them reverses the orientation. In Theorem 3.5, (iii) we show that there exist $n!$ different sequences with the same property like that in (3.2).

(v) This is a consequence of the local symmetries of the two flat poses $A_1 \dots B_n'$ and $A_1 \dots B_n''$ and the symmetry of B_i' and B_i'' w.r.t. the hinge h_i (note Figure 12): The product of the reflections in $[v_{12}, S']$ and h_1 takes $[v_{12}, A_1] = h_2$ via $[v_{12}, B_1']$ to $[v_{12}, B_1'']$. Therefore, $\frac{1}{2}\sphericalangle[v_{12}, A_1][v_{12}, B_1''] = \sphericalangle[v_{12}, A_1][v_{12}, S''] = \sphericalangle[v_{12}, S']\, h_1 = \sphericalangle[v_{12}, S'][v_{12}, A_2]$. □

The following theorem is an extension of the statement (ii) in Theorem 3.3. Here we use the symbol $\mathbf{M}_{XY}$ for the symmetric coefficient matrix of the rank-2 dual quadric of hyperplanes passing through the points X or Y in the hyperplane H_f of $\mathbb{E}^n$. This means, if $(1 : x_1 : \cdots : x_{n-1})$ and $(1 : y_1 : \cdots : y_{n-1})$ are homogeneous Cartesian coordinates of X and Y, that

$$2\,\mathbf{M}_{XY} \coloneqq (1\,x_1 \ldots x_{n-1})^\top (1\,y_1 \ldots y_{n-1}) + (1\,y_1 \ldots y_{n-1})^\top (1\,x_1 \ldots x_{n-1}). \tag{3.3}$$

Theorem 3.4. *The cross-polytope $\mathcal{C}^n$ given by the flat pose $A_1 \ldots B'_n$ is twice-flat and flexible in $\mathbb{E}^n$, $n \geq 3$, if and only if the set of linear combinations of the $(n \times n)$-matrices*

$$\mathbf{M}_{A_1B'_1},\ \mathbf{M}_{A_2B'_2},\ \ldots,\ \mathbf{M}_{A_nB'_n}$$

contains a rank-1 matrix of type $\mathbf{M}_{S'S'}$ and the diagonal matrix $\operatorname{diag}(0, 1, \ldots, 1)$.

Proof. The dual quadrics of the linear system addressed in Theorem 3.3 (ii) have coefficient matrices that are linear combinations of $\mathbf{M}_{A_1B'_1}, \ldots, \mathbf{M}_{A_nB'_n}$. In order to prove the converse of Theorem 3.3 (ii), we choose any $(n-3)$-face of Γ_f and confirm the local symmetry within H_f. It means no restriction of generality to assume that this is the face $A_3 \ldots A_n$ with the span v_{12}.

From the viewpoint of Projective Geometry, for each dual quadric out from the linear system, the hyperplanes passing through the subspace v_{12} form again a quadric in the bundle except that the whole bundle is part of the quadric.[7] The latter holds for all rank-2 quadrics with carriers (A_i, B'_i) for $i \geq 3$. Thus, the restriction of our linear system to the bundle with carrier v_{12} is spanned by the restrictions of the two rank-2 quadrics with carriers (A_1, B'_1) and (A_2, B'_2). The orthogonal view in Figure 12 with the point v^s_{12} as image of v_{12} visualizes the restriction of the linear system as the dual pencil of conics tangent to the sides of the quadrangle $A^s_1A^s_2B'^s_1B'^s_2$. Due to Desargues's involution theorem (see, e.g., [8, Sect. 7.4]), the tangents from v^s_{12} to the conics form an involution where the connection with S'^s is one fixed line. Since in the orthogonal view also the isotropic lines through v^s_{12} are a pair of corresponding lines, the two fixed lines are orthogonal and the involution with the particular pairs $([v^s_{12}, A^s_1], [v^s_{12}, B'^s_1])$ and $([v^s_{12}, A^s_2], [v^s_{12}, B'^s_2])$ is symmetric w.r.t. $[v^s_{12}, S'^s]$.

The same holds for all $(n-3)$-faces of Γ_f, and this confirms the converse direction of the stated equivalence. □

Note that the first row of the matrices in Theorem 3.4 confirms again the statement (iii) in Theorem 3.3.

3.4 Spatial Poses of Flexible Twice-flat Cross-polytopes

Theorem 3.5. *Let $A_1 \ldots B_n$ be a spatial pose of a flexible twice-flat cross-polytope $\mathcal{C}^n$ in $\mathbb{E}^n$, $n \geq 4$, with a finite center S' of local symmetries in a flat pose. Then holds for each permutation $(i_1, \ldots, i_n)$ of $(1, \ldots, n)$:*

(i) The dihedral angle along the $(n-2)$-dimensional hinge $[X_{i_2} \ldots X_{i_n}]$ between the neighboring facets $A_{i_1}X_{i_2} \ldots X_{i_n}$ and $B_{i_1}X_{i_2} \ldots X_{i_n}$ of $\mathcal{C}^n$, where $X_k \in \{A_k, B_k\}$, is either congruent or supplementary to the dihedral angle ϕ_{i_1} between the wing Δ_{i_1} and the fixed

[7] This becomes apparent when for hyperplanes homogeneous projective coordinates $(x_0, \ldots, x_n)$ are used such that v_{12} is given by $x_k = x_{k+1} = \cdots = x_n = 0$ for any $k > 0$. In the excluded cases the quadratic form representing the quadric turns to a nullform when substituting $x_k = \cdots = x_n = 0$.

simplex Γ_f. *Congruence holds if and only if at the initial flat pose the corresponding dihedral angle equals* 0. *Thus, the* $2^{n-1}\cdot n$ *dihedral angles of* $\mathcal{C}^n$ *can be subdivided into* n *classes of congruent or supplementary angles.*

(ii) The one-parameter motion of the simplex Γ_m *relative to* Γ_f *can be expressed as a product of* n *rotations through respective angles* $\phi_1, \dots, \phi_n$ *about fixed hinges spanned by* $(n-2)$*-faces of the flat pose* $A_1 \dots B'_n$ *in the hyperplane* H_f. *There exist* $n!$ *different decompositions of this kind. The induced motion of a facet with* m *different* B*-vertices* $B_{i_1}, \dots, B_{i_m}$, $m < n$, *is the product of* m *rotations through respective angles* $\phi_{i_1}, \dots, \phi_{i_m}$ *about hinges in* H_f.

(iii) Among the four $(n-2)$*-faces of* $\mathcal{C}^n$ *passing through the* $(n-3)$*-face* $X_{i_3} \dots X_{i_n}$, *the connections with* A_{i_1} *and* B_{i_1} *as well as that with* A_{i_2} *and* B_{i_2} *have a common angle bisecting hyperplane* μ. *This hyperplane* μ *through* $v := [X_{i_3} \dots X_{i_n}]$ *bisects also the angle between the pairs of opposite hyperplanes* $([A_{i_1}A_{i_2}, v],\ [B_{i_1}B_{i_2}, v])$ *and of* $([A_{i_1}B_{i_2}, v],\ [B_{i_1}A_{i_2}, v])$, *which extend facets of* $\mathcal{C}^n$.

(iv) All these angle-bisecting hyperplanes μ *of* $\mathcal{C}^n$ *share a line* s *in* $\mathbb{E}^n$. *All hinges of type* $[X_{i_2} \dots X_{i_n}]$ *have the same distance to* s *and enclose the same angle* ψ_{i_1} *with* s.

(v) The facets of $\mathcal{C}^n$ *with an even number of* B*-vertices span hyperplanes that share a point* $C \in s$, *and in each facet the distances of* C *to the hinges in an appropriate order are proportional to* $\frac{1}{|\sin\phi_1|} : \dots : \frac{1}{|\sin\phi_n|}$. *Similarly, all hyperplanes spanned by facets with an odd number of* B*-points pass through another common point* $D \in s$, *and also in these hyperplanes the distances of* D *to the hinges satisfy this proportion.*

(vi) All 2^n *facets of* $\mathcal{C}^n$ *are placed in hyperplanes that contact a hypersphere centered on* s. *The midpoints of the diagonals* A_iB_i *belong to a hyperplane passing through* s.

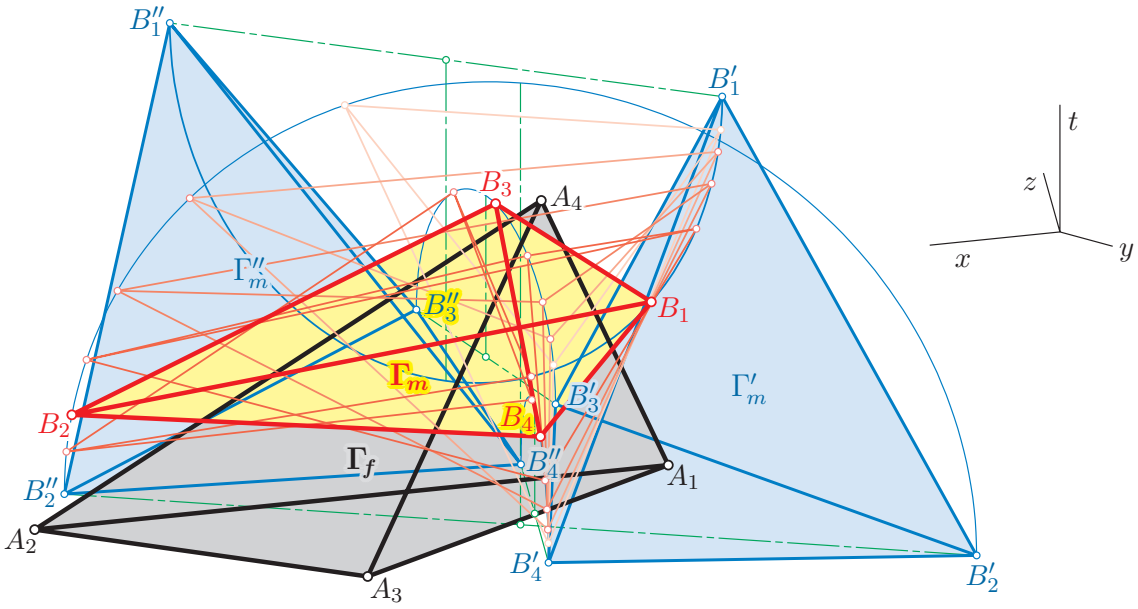


Figure 14: Two-dimensional orthogonal view of a four-dimensional flexible twice-flat cross-polytope $\mathcal{C}^4$ with different poses of the moving tetrahedron $\Gamma_m = B_1 \dots B_4$ in $\mathbb{E}^4$ relative to the fixed tetrahedron $\Gamma_f = A_1 \dots A_4$ in the hyperplane $t = 0$.

Proof. (i) The four facets of $\mathcal{C}^n$ through the $(n-3)$-face $X_{i_3} \dots X_{i_n}$ form an isogonal pyramid where opposite dihedral angles are congruent or supplementary (Figure 13). Hence, the

dihedral angle between the neighboring faces $A_{i_1}A_{i_2}X_{i_3}\dots X_{i_n}$ and $B_{i_1}A_{i_2}X_{i_3}\dots X_{i_n}$ along the hinge $[A_{i_2}X_{i_3}\dots X_{i_n}]$ is congruent or supplementary to that between $B_{i_1}B_{i_2}X_{i_3}\dots X_{i_n}$ and $A_{i_1}B_{i_2}X_{i_3}\dots X_{i_n}$ along the hinge $[B_{i_2}X_{i_3}\dots X_{i_n}]$; in both involved hinges the vertices with index i_1 are missing. In other words, a dihedral angle at $\mathcal{C}^n$ remains congruent or becomes supplementary if at a hinge one vertex is replaced by its opposite. Iteration confirms the stated property.

(ii) We refer to the sequence of neighboring facets in (3.2), $\Gamma_m = B_1\dots B_n$, $A_1B_2\dots B_n$, $A_1A_2B_3\dots B_n$, …, $A_1\dots A_n = \Gamma_f$ with n hinges between as $[B_2\dots B_n]$, $[A_1B_3\dots B_n]$, …, $[A_1\dots A_{n-1}] = h_n$. According to (i), at the spatial pose of $\mathcal{C}^n$ the dihedral angles along these hinges are congruent or supplementary to $\phi_1, \phi_2, \dots, \phi_n$, respectively. Hence, we obtain the general pose stepwise by appropriate rotations about the hinges in the flat pose in order to establish these angles between the neighboring faces.

Here we face the problem that the angles $\phi_1, \dots, \phi_n$ of rotations about the hinges in H_f are oriented. How to orientate an $(n-2)$-dimensional hinge?

We follow the example of octahedra (see Figure 6) and use the orthogonal projection along an $(n-3)$-dimensional subspace of the hinge h similar to that in Figure 12. Under the condition that the projection yields an orientation-preserving view of a predefined side of H_f, we orientate the view h^s of the hinge h such that the interior of Γ_f lies left. Now we define that, when looking in this direction, a rotation through a positive angle is carried out counter-clockwise.

The sequence of facets in (3.2) leads to the natural order $\phi_1, \phi_2, \dots, \phi_n$ of the angles. And this order corresponds to that of the consecutive rotations. However, we can also start by predefining any order of the angles. Then, conversely, this determines the sequence of the facets according to (i) and consequently the sequence of the hinges and the rotations. This yields $n!$ different representations. In a similar way we proceed when we represent the movement of any other facet with B-vertices $B_{i_1}, \dots, B_{i_m}$, $m < n$, against Γ_f.

The 3d-versions of these representations are equivalent to the different decompositions of the motion polynomial of a type-3 Bricard octahedron as discussed in [9].

(iii) This follows from the symmetry of the isogonal double-pyramid with the (n–3)-dimensional apex space $v = [X_{i_3}\dots X_{i_n}]$ and 'edges' through the vertices $A_{i_1}, A_{i_2}, B_{i_1}, B_{i_2}$ (note the orthogonal view in Figure 13 with $(i_1, i_2) = (i, j)$). Obviously, this property of $\mathcal{C}^n$ is the spatial analogue of the local symmetry of flat poses in H_f as stated in Theorem 3.1: From the viewpoint of $\mathbb{E}^n$, at the flat pose all angle bisecting hyperplanes of the connections of v with (A_{i_1}, B'_{i_1}) and with (A_{i_2}, B'_{i_2}) share the perpendicular s' to H_f through the center S' (note Figure 13).

(iv) The symmetries of the double-pyramids assign to each $(n-3)$-face of $\mathcal{C}^n$ a unique hyperplane μ of symmetry between the spans of the pyramid's opposite facets. In addition, this angle-bisecting plane μ is already uniquely defined by one of the two pairs of opposite hyperplanes, as it can be continuously transferred into the corresponding hyperplane through s' in the flat pose. Opposite facets of such a pyramid are those which differ at exactly two vertices while the number of A-points either changes by 2 or remains the same. Of course, the same holds for the number of B-points. Why do all $2^{n-3}n(n-1)$ hyperplanes μ, that are assigned to the $(n-3)$-faces of $\mathcal{C}^n$, share a line s?

At the begin, we focus on the $(n-3)$-faces of $\Gamma_f = A_1\dots A_n$ and refer to the previously introduced notation v_{ij} for the span of all A-vertices different from A_i and A_j. The assigned bisecting hyperplane will be denoted by μ_{ij}, and we define the line s as the intersection of

$n-1$ hyperplanes, namely

$$s := \mu_{12} \cap \mu_{23} \cap \mu_{34} \cap \mu_{45} \cap \cdots \cap \mu_{n-1\,n}. \tag{3.4}$$

Hence, each point $S \in s$ has equal distances to the pairs of hyperplanes $([A_iA_{i+1}, v_{i\,i+1}]$, $[B_iB_{i+1}, v_{i\,i+1}])$ and $([A_iB_{i+1}, v_{i\,i+1}], [B_iA_{i+1}, v_{i\,i+1},])$ for $i = 1, \ldots, n-1$. This defines an equivalence relation between facets for which we introduce the symbol '$\sim$'. For consecutive $v_{i\,i+1}$ follow by (3.4) the equivalences

$$\begin{aligned} &\Gamma_f \sim B_1B_2A_3\ldots \sim A_1B_2B_3A_4\ldots \sim A_1A_2B_3B_4A_5\ldots \sim \cdots \sim A_1\ldots A_{n-2}B_{n-1}B_n,\\ &A_1B_2A_3\ldots A_n \sim B_1A_2A_3\ldots A_n,\ A_1A_2B_3A_4\ldots A_n \sim A_1B_2A_3\ldots A_n,\\ &\ldots,\ A_1\ldots A_{n-1}B_n \sim A_1\ldots A_{n-2}B_{n-1}A_n. \end{aligned} \tag{3.5}$$

Our goal is to extend the list of the involved $(n-3)$-faces step by step so that finally all $2^{n-3}\cdot n(n-1)$ apex-spaces of pyramids of $\mathcal{C}^n$ are involved.

1. The first line in (3.5) shows that opposite facets through the (n–3)-dimensional apex-spaces

$$[B_2A_4A_5\ldots],\ [A_1B_3A_5\ldots],\ \ldots,\ [A_1\ldots A_{n-3}B_{n-1}] \tag{3.6}$$

are equivalent. The first equivalence in the second line of (3.5) along with the first (n–3)-face in (3.6) yield

$$B_1A_2A_3A_4\ldots \sim A_1B_2A_3A_4\ldots \sim B_1B_2B_3A_4\ldots \tag{3.7}$$

as opposite facets w.r.t. $[B_1A_4\ldots]$. Together with the first equivalence in (3.5) follows

$$\Gamma_f \sim B_1B_2A_3A_4\ldots \sim B_1A_2B_3A_4\ldots \tag{3.8}$$

as opposite facets w.r.t. $[A_2A_4A_5\ldots] = v_{13}$.

When (under $n > 4$) we iteratively increase the involved subscripts by 1 as long as none of them exceeds n, then we obtain a sequence of $(n-3)$-dimensional apex-spaces v_{13}, v_{24}, …, $v_{n-2\,n}$ with associated hyperplanes $\mu_{13}, \mu_{24}, \mu_{35}, \ldots, \mu_{n-2\,n}$ passing through s.

2. The definition of s remains valid if we exchange in (3.5) the indices 3 and 4. This implies that the planes $\mu_{14}, \mu_{25}, \mu_{36}, \ldots, \mu_{n-3\,n}$ pass through s, too. We repeat by exchanging 4 and 5 and get $s \subset \mu_{15}, \mu_{26}, \ldots, \mu_{n-4\,n}$. Iteration yields finally $s \subset \mu_{1n}$ so that all planes of symmetry μ_{ij} assigned to the $(n-3)$-faces of Γ_f are proved to contain the line s as defined in (3.4).

3. We continue with other facets of $\mathcal{C}^n$. Each facet $\Delta_{ij} = A_1\ldots B_i\ldots B_j\ldots A_n$ that contains two B-points is opposite to Γ_f w.r.t. v_{ij}. Each facet $\Delta_i = A_1\ldots B_i\ldots A_n$ with exactly one B-point is opposite to Δ_1 w.r.t. v_{1i}. Thus, all facets with one or two B-points can be inserted in our two equivalence classes. Moreover, the $(n-3)$-faces of Δ_i are either part of Γ_f or they contain one B-point. If it contains B_i but not A_j and A_k, $j \neq k$, then at the corresponding pyramid the equivalent facets $\ldots B_i\ldots A_j\ldots B_k\ldots$ and $\ldots B_i\ldots B_j\ldots A_k\ldots$ with two B-points are opposite so that the plane of symmetry assigned to this $(n-3)$-face of Δ_i passes again through s.

Hence, we can replace Γ_f by Δ_i and start the procedure again. As a result, all facets containing B_i and two further B-points are equivalent to Δ_i. Moreover, we can prove like before that all $(n-3)$-faces of Δ_{ij} which contain two B-points are the meet of two opposite and equivalent facets as they contain three B-points. Thus, for all apex-spaces of Δ_{ij} the assigned hyperplanes μ pass through s. Now, we iterate and replace Γ_f by Δ_{ij}. This allows

to extend our equivalence classes to facets of $\mathcal{C}^n$ with four B-points, and so on, until we reach Γ_m with n B-points.

4. Let μ be the plane of symmetry assigned to the $(n-3)$-face $X_{i_3} \dots X_{i_n}$. The reflection in μ exchanges opposite hinges of the double-pyramid and preserves $s \subset \mu$ (see Figure 13). Thus, the angles between s and opposite hinges $[A_{i_2} X_{i_3} \dots X_{i_n}]$ and $[B_{i_2} X_{i_3} \dots X_{i_n}]$ are congruent, and the respective distances from any given point $S \in s$ are equal. By iterated transition to opposite hinges one obtains only hinges that contain neither A_{i_1} or B_{i_1}. These are exactly the hinges of $\mathcal{C}^n$ with dihedral angles congruent or supplementary to ϕ_{i_1} as mentioned in (i).

(v) Each point $S \in s$ has equal distances to the spans of equivalent facets. According to the two equivalence classes, the distances define two linear functions on s. The zeros of these functions show that the hyperplanes of each class have a point on s in common, and these two points $C, D \in s$ must be different since otherwise all hinges of $\mathcal{C}^n$ would share a point. We cannot exclude that one of these points lies at infinity.

(vi) The fixed simplex Γ_f and the wing Δ_k are representatives of the two equivalence classes. An orthogonal projection of $\mathbb{E}^n$ along the hinge h_k into a 2-plane shows H_f and $[\Delta_k]$ as two lines enclosing the angle ϕ_k. Distances to H_f, $[\Delta_k]$ and h_k appear in true size. Therefore, the arguments in the proof of Theorem 2.4 (vii) and (2.5) are still valid.

The two angle bisecting hyperplanes of H_f and $[\Delta_k]$ intersect line s at two points S_1, S_2 in harmonic position w.r.t. C and D. Each S_i has equal distances to H_f and $[\Delta_k]$ and, consequently, to all other hyperplanes, provided that S_i is finite.

The proof of Theorem 3.3, (ii) revealed that cross-polytopes in Γ_f define a $(n–2)$-parametric linear system of dual quadrics containing all hyperplanes spanned by the cross-polytope's facets. The same holds for $\mathcal{C}^n$ in $\mathbb{E}^n$. The $(n-1)$-parametric linear system is spanned by the rank-2 quadrics with carriers (A_i, B_i) for $i = 1, \dots, n$ and contains also the two hyperspheres with centers $S_1, S_2 \in s$. All dual quadrics of this linear system have their centers in a common hyperplane which must pass through s. Continuity guarantees this also when one S_i tends to infinity. □

Remark 6. The comment in Remark 3 can be generalized to $\mathbb{E}^n$ due to Theorem 3.5 (iii): An orthogonal projection of any spatial pose of a flexible twice-flat cross-polytope $\mathcal{C}^n$ in direction of the line s yields an $(n-1)$-dimensional image that has the properties of flat poses as listed in Theorem 3.3.

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Received June 8, 2026; final form September 23, 2026.