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Cartan algebras in symmetric Lie algebras

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Cartan algebras proved to be useful in the classification of semisimple Lie algebras and in the determination of invariant cones in Lie algebras. In the discussion of infinitesimally ordered symmetric spaces, cones of the form $W = \mathfrak{h} \oplus C$ with C pointed and generating in the associated symmetric Lie-algebra $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ which satisfy $e^{\operatorname{ad}\mathfrak{h}} C = C$ play an important role. The idea is to use Cartan algebras also to achieve a classification for those wedges.

In this article the existence of invariant Cartan algebras in a *symmetric* Lie algebra $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ over a field of characteristic zero is established. In the solvable case it is shown, that these Cartan algebras are conjugate under elementary inner automorphisms coming from a fixed element under the involution. We state a general conjugation theorem for an algebraically closed field. In the end we discuss some relations between symmetric Cartan algebras and Levi complements.

1. Some facts on Cartan algebras

We recall some notation and basic facts concerning Cartan algebras in arbitrary Lie algebras. The reader who is not familiar with Cartan algebras is referred to [1] Ch. VII, which is an excellent source of reference on Cartan algebras in general.

In that follows $\mathbb{K}$ denotes an arbitrary field of characteristic zero. Let $\rho\colon \mathfrak{n} \to \operatorname{gl}(V)$ be a finite dimensional representation of a nilpotent Lie-Algebra $\mathfrak{n}$. The *nilspace* of an element $x \in \mathfrak{n}$ is defined by

$$V^0(x) \stackrel{\text{def}}{=} \{v \in V \mid (\exists n \in \mathbb{N})\, \rho(x)^n v = 0\}.$$

For $\lambda \in \mathbb{K}$ we set

$$V^\lambda(x) \stackrel{\text{def}}{=} \{v \in V \mid (\exists n \in \mathbb{N})\, (\rho(x) - \lambda\cdot\mathbf{1})^n v = 0\}.$$

Consider a function $\lambda\colon \mathfrak{n} \to \mathbb{K}$. The *weightspace of* $\mathfrak{n}$ *with weight* λ is given by

$$V^\lambda(\mathfrak{n}) \stackrel{\text{def}}{=} \bigcap_{x\in\mathfrak{n}} V^{\lambda(x)}(x).$$

The space $V^0(\mathfrak{n})$ is called the *nilspace* of $\mathfrak{n}$.

In this article, we consider always a nilpotent subalgebra $\mathfrak{n}$ of a fixed Lie algebra $\mathfrak{g}$, and V is a vector subspace of $\mathfrak{g}$, invariant under the restriction of the adjoint representation $\rho\colon \mathfrak{n} \to \operatorname{gl}(V)$, $\rho(x) = \operatorname{ad} x|_V$. In the case $V = \mathfrak{g}$ we denote by Λ the set of all $\lambda\colon \mathfrak{n} \to \mathbb{K}$, $\lambda \neq 0$ satisfying $\mathfrak{g}^\lambda(\mathfrak{n}) \neq \{0\}$. For $\lambda, \mu \in \Lambda \cup \{0\}$ we have

$$[\mathfrak{g}^\lambda(\mathfrak{n}), \mathfrak{g}^\mu(\mathfrak{n})] \subseteq \mathfrak{g}^{\lambda+\mu}(\mathfrak{n}).$$

In particular, $\mathfrak{g}^0(\mathfrak{n})$ is a subalgebra of $\mathfrak{g}$ containing $\mathfrak{n}$, and $\mathfrak{g}^\lambda(\mathfrak{n})$ is invariant under $\operatorname{ad} \mathfrak{g}^0(\mathfrak{n})$. Further, $\mathfrak{g}^0(\mathfrak{n})$ equals its own normalizer. If all $\operatorname{ad} x$, $x \in \mathfrak{n}$, are triagonizable, we have

$$\mathfrak{g} = \mathfrak{g}^0(\mathfrak{n}) \oplus \mathfrak{g}^+(\mathfrak{n}), \quad \text{where} \quad \mathfrak{g}^+(\mathfrak{n}) \stackrel{\text{def}}{=} \sum_{\lambda \in \Lambda} \mathfrak{g}^\lambda(\mathfrak{n}).$$

There exists an $x \in \mathfrak{n}$ with $\mathfrak{g}^0(\mathfrak{n}) = \mathfrak{g}^0(x)$. If $\mathbb{K}'$ is an extension of $\mathbb{K}$, then we have

$$(\mathfrak{g} \otimes_{\mathbb{K}} \mathbb{K}')^\lambda(\mathfrak{n}) = \mathfrak{g}^\lambda(\mathfrak{n}) \otimes_{\mathbb{K}} \mathbb{K}'.$$

Let $\mathfrak{g}$ be a finite dimensional Lie algebra. The *normalizer* of a subalgebra $\mathfrak{t}$ in $\mathfrak{g}$ is given by

$$N_{\mathfrak{g}}(\mathfrak{t}) \stackrel{\text{def}}{=} \{x \in \mathfrak{g} \mid [x, \mathfrak{t}] \subseteq \mathfrak{t}\}.$$

Definition 1.1. A subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ is called a *Cartan algebra*, if $\mathfrak{t}$ is nilpotent, and equals its own normalizer $N_{\mathfrak{g}}(\mathfrak{t})$.

If $\mathfrak{g}'$ is a subalgebra of $\mathfrak{g}$ and $\mathfrak{t}$ is a Cartan algebra of $\mathfrak{g}$ with $\mathfrak{t} \subseteq \mathfrak{g}'$, then $\mathfrak{t}$ is also a Cartan algebra of $\mathfrak{g}'$. Let $\mathbb{K}'$ be an extension of the field $\mathbb{K}$. Then $\mathfrak{t}$ is a Cartan algebra of $\mathfrak{g}$ if and only if $\mathfrak{t} \otimes_{\mathbb{K}} \mathbb{K}'$ is a Cartan algebra of $\mathfrak{g} \otimes_{\mathbb{K}} \mathbb{K}'$.

We consider a Lie algebra $\mathfrak{g}$ with $\dim \mathfrak{g} = n$. For $x \in \mathfrak{g}$,

$$\det(\lambda \cdot \mathbf{1} - \operatorname{ad} x) = \sum_{i=0}^{n} a_i(x) \lambda^i$$

with $a_i(x) \in \mathbb{K}$ is the characteristic polynomial of the endomorphim $\operatorname{ad} x$. Since $a_i(x) = (-1)^{n-i} \operatorname{tr}(\bigwedge^{n-i} \operatorname{ad} x)$, the assignment $x \mapsto a_i(x)$ is a polynomial map into $\mathbb{K}$.

Definition 1.2. The *rank* $\operatorname{rg}(\mathfrak{g})$ of $\mathfrak{g}$ is defined as the minimal number l with $a_l \neq 0$. An element x is called *regular*, if $a_l(x) \neq 0$. We denote by

$$\operatorname{Reg}(\mathfrak{g}) \stackrel{\text{def}}{=} \{x \in \mathfrak{g} \mid x \text{ is regular}\}$$

the set of all regular elements.

For $x \in \mathfrak{g}$ we have $\operatorname{rg}(\mathfrak{g}) \leq \dim \mathfrak{g}^0(x)$. Equality holds if and only if $x \in \operatorname{Reg}(\mathfrak{g})$. The set $\operatorname{Reg}(\mathfrak{g})$ is open and dense with respect to the natural topology and with respect to the Zariski topology. The interplay between Cartan algebras and regular elements is given in the following theorem.

Theorem 1.3. *Let $\mathfrak{g}$ be a finite dimensional Lie algebra. Then the following holds:*

(i) *All Cartan algebras have the same dimensions* $\operatorname{rg}(\mathfrak{g})$ *and the same nilpotent length.*

(ii) *An element x is regular if and only if $\mathfrak{t} \stackrel{\text{def}}{=} \mathfrak{g}^0(x)$ is a Cartan algebra $\mathfrak{g}$. All Cartan algebras of $\mathfrak{g}$ are of this form.*

(iii) *If $\mathfrak{t}$ is a maximal nilpotent subalgebra of $\mathfrak{g}$ and $x \in \mathfrak{t}$ is regular, we have $\mathfrak{t} = \mathfrak{g}^0(x)$.*

(iv) *$\mathfrak{g}$ is the sum of its Cartan algebras.*

■

Finally, we state some results concerning the conjugation of Cartan algebras in the Lie algebra $\mathfrak{g}$. First, we consider a *solvable* Lie algebra $\mathfrak{g}$. We denote by $\mathcal{C}^\infty(\mathfrak{g})$ the intersection of the central descending series, i. e.,

$$\mathcal{C}^\infty(\mathfrak{g}) \stackrel{\text{def}}{=} \bigcap_{n\in\mathbb{N}} \mathfrak{g}^{[n]}.$$

Since $\mathcal{C}^\infty(\mathfrak{g}) \subseteq [\mathfrak{g},\mathfrak{g}]$, this is a nilpotent ideal. In particular, the automorphism $e^{\operatorname{ad} x}$ with $x \in \mathcal{C}^\infty(\mathfrak{g})$ is a polynomial in $\operatorname{ad} x$.

Theorem 1.4. *Let $\mathfrak{g}$ be a solvable finite dimensional Lie algebra. Further, let $\mathfrak{t}$ and $\mathfrak{t}'$ be Cartan algebras of $\mathfrak{g}$. There exists an $x \in \mathcal{C}^\infty(\mathfrak{g})$ with $e^{\operatorname{ad} x}\mathfrak{t} = \mathfrak{t}'$.* ■

We consider now an arbitrary finite dimensional Lie algebra $\mathfrak{g}$ over an *algebraically closed* field $\mathbb{K}$. We have

$$\mathfrak{g} = \mathfrak{g}^0(\mathfrak{t}) \oplus \sum_{\lambda\in\Lambda} \mathfrak{g}^\lambda(\mathfrak{t}).$$

For $x \in \mathfrak{g}^\lambda(\mathfrak{t})$, $\lambda \neq 0$ the endomorphism $\operatorname{ad} x$ is nilpotent, and therefore $e^{\operatorname{ad} x}$ is a polynomial in $\operatorname{ad} x$. We set

$$E(\mathfrak{t}) \stackrel{\text{def}}{=} \langle\{e^{\operatorname{ad} x} \mid (\exists\lambda \in \Lambda) x \in \mathfrak{g}^\lambda(\mathfrak{t})\}\rangle.$$

If $\varphi \in \operatorname{Aut}(\mathfrak{g})$, then obviously $\varphi E(\mathfrak{t})\varphi^{-1} = E(\varphi(\mathfrak{t}))$ holds.

Theorem 1.5. *Let $\mathfrak{g}$ be a finite dimensional Lie algebra over an algebraically closed field $\mathbb{K}$. Let $\mathfrak{t}$ and $\mathfrak{t}'$ be Cartan algebras of $\mathfrak{g}$. Then we have $E(\mathfrak{t}) = E(\mathfrak{t}')$. Defining $E \stackrel{\text{def}}{=} E(\mathfrak{t})$ the following holds: there exists an $\gamma \in E$, such that $\mathfrak{t} = \gamma(\mathfrak{t}')$, i. e., E acts transitively on the set of all Cartan algebras of $\mathfrak{g}$.* ■

To conclude this course on Cartan algebras in general we cite some properties which we need in the following sections.

Proposition 1.6. *Let $\mathfrak{g}$ be a finite dimensional Lie algebra and $\mathfrak{g}'$ a subalgebra of $\mathfrak{g}$. Then the following conditions are equivalent.*

(i) $\mathfrak{g}' \cap \mathrm{Reg}(\mathfrak{g}) \neq \emptyset$ *and* $\mathrm{rg}(\mathfrak{g}') = \mathrm{rg}(\mathfrak{g})$.

(ii) $\mathfrak{g}'$ *contains a Cartan algebra of* $\mathfrak{g}$.

(iii) *Every Cartan algebra of* $\mathfrak{g}'$ *is a Cartan algebra of* $\mathfrak{g}$.

■

Corollary 1.7. *Let $\mathfrak{n}$ be a nilpotent subalgebra of $\mathfrak{g}$. Then $\mathfrak{g}^0(\mathfrak{n})$ satisfies the conditions* (i), (ii), (iii) *of* Proposition 1.6. ■

2. Existence of symmetric Cartan algebras

It is recently proved by SPINDLER, that every symmetric Lie algebra $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ possesses a Cartan Algebra $\mathfrak{t}$ which is *symmetric*, i. e., satisfies $\mathfrak{t} = \mathfrak{t}_{\mathfrak{h}} \oplus \mathfrak{t}_{\mathfrak{q}}$ where $\mathfrak{t}_{\mathfrak{h}} = \mathfrak{t} \cap \mathfrak{h}$ and $\mathfrak{t}_{\mathfrak{q}} = \mathfrak{t} \cap \mathfrak{q}$. For semisimple Lie algebras this is well known. We give here another, more direct, proof in the general case, which does not use the result for semisimple Lie algebras. First, we discuss some preliminaries.

Proposition 2.1. *Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ be a symmetric Lie algebra and $x \in \mathfrak{h}$ or $x \in \mathfrak{q}$. Then $\mathfrak{g}_1 \stackrel{\mathrm{def}}{=} \mathfrak{g}^0(x)$ is a symmetric subalgebra of $\mathfrak{g}$. Every Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}_1$ is a Cartan subalgebra of $\mathfrak{g}$.* ■

We denote the descending central series of a Lie algebra $\mathfrak{g}$ by $\mathfrak{g}^{[0]} = \mathfrak{g},\ldots,\mathfrak{g}^{[m+1]} = [\mathfrak{g}, \mathfrak{g}^{[m]}]$. It is well known, that a Lie algebra $\mathfrak{g}$ is nilpotent of class $\leq m$ if and only if $\mathfrak{g}$ satisfies

$$[m] \qquad [x_1, [\cdots, [x_m, x_{m+1}] \cdots]] = 0 \quad \text{for all } x_1, \ldots, x_{m+1} \in \mathfrak{g}.$$

Lemma 2.2. *Let $\mathfrak{n} \subseteq \mathfrak{g}$ be a nilpotent subalgebra and $y_0 \in N_{\mathfrak{g}}(\mathfrak{n})$ such that $\mathfrak{n} \subseteq \mathfrak{g}^0(y_0)$. Then $\tilde{\mathfrak{n}} \stackrel{\mathrm{def}}{=} \mathfrak{n} \oplus \mathbb{K}\cdot y_0$ is a nilpotent subalgebra.*

Proof. Since $y_0 \subseteq N_{\mathfrak{g}}(\mathfrak{n})$, induction immediately shows that $(\mathrm{ad}\, y_0)\mathfrak{n}^{[m]} \subseteq \mathfrak{n}^{[m]}$. Since $\mathfrak{n} \subseteq \mathfrak{g}^0(y_0)$, there exists $m_1 \in \mathbb{N}$ such that $(\mathrm{ad}\, y_0)^{m_1}\mathfrak{n} = \{0\}$. Assume $\mathfrak{n}$ is nilpotent of class m_2 and set $m_0 \stackrel{\mathrm{def}}{=} m_1 \cdot m_2$. It is enough to show, that $\tilde{\mathfrak{n}}$ satisfies condition $[m_0]$ above. Therefore let $x_1, \ldots, x_{m_0+1} \in \tilde{n}$ and consider $x = [x_1, [\cdots [x_{m_0}, x_{m_0+1}] \cdots]]$. Since the Lie bracket is bilinear, we may assume that for $i = 1, \ldots, m_0 + 1$ we have $x_i \in \mathfrak{n}$ or $x_i = y_0$. If there are at least m_2 elements of $\mathfrak{n}$ in the sequence $x_1, \ldots, x_{m_0+1}$, we have $x = 0$ since $(\mathrm{ad}\, y_0)\mathfrak{n}^{[m]} \subseteq \mathfrak{n}^{[m]}$ and $\mathfrak{n}$ is of class m_2. If this is not the case, then by definition of m_0, the element y_0 occurs at least m_1 times in a row. But then $x = 0$ by definition of m_1. This proves the assertion. ■

Theorem 2.3. (Existence of symmetric Cartan algebras) *Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ be a symmetric Lie algebra. Then there exists a symmetric Cartan algebra $\mathfrak{t}$.*

Proof. We prove the assertion by induction on the dimension of $\mathfrak{g}$. If $\dim \mathfrak{g} = 1$ nothing is to show. Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ be an arbitrary symmetric Lie algebra. If there exists an element $x \in \mathfrak{h}$ or $x \in \mathfrak{q}$ such that $\mathfrak{g}^0(x) \neq \mathfrak{g}$, then $\mathfrak{g}_1 \stackrel{\text{def}}{=} \mathfrak{g}^0(x)$ is a symmetric subalgebra of lower dimension, and the induction hypothesis shows the existence of an symmetric Cartan algebra $\mathfrak{t}$. By Proposition 2.1, $\mathfrak{t}$ is an symmetric Cartan algebra of $\mathfrak{g}$.

Otherwise, for every $x \in \mathfrak{h}$ and $y \in \mathfrak{q}$ the relation $\mathfrak{g}^0(x) = \mathfrak{g}^0(y) = \mathfrak{g}$ holds. In particular, $\mathfrak{h}$ is nilpotent by Engel's theorem. Let $\mathfrak{t}$ be any maximal nilpotent subalgebra with $\mathfrak{h} \subseteq \mathfrak{t}$. We claim that $\mathfrak{t}$ equals its own normalizer, hence $\mathfrak{t}$ is a Cartan algebra which obviously is symmetric. As $\mathfrak{h} \subseteq \mathfrak{t} \subseteq N_{\mathfrak{g}}(\mathfrak{t})$, the normalizer $N_{\mathfrak{g}}(\mathfrak{t})$ is symmetric. Thus we have to show that $N_{\mathfrak{g}}(\mathfrak{t}) \cap \mathfrak{q} \subseteq \mathfrak{t}$. Suppose $y_0 \in N_{\mathfrak{g}}(\mathfrak{t}) \cap \mathfrak{q}$. By assumption, $\mathfrak{g}^0(y_0) = \mathfrak{g}$, hence $\mathfrak{t} \subseteq \mathfrak{g}^0(y_0)$. Thus the assumptions of Lemma 2.2 are satisfied, showing that $\mathfrak{t} \oplus \mathbb{K}{\cdot}y_0$ is nilpotent. Therefore $y_0 \in \mathfrak{t}$ by maximality of $\mathfrak{t}$. This proves the theorem. ■

Theorem 1.3 tells us, that a Lie algebra in general is full of Cartan algebras. Every Cartan algebra is of the form $\mathfrak{g}^0(x)$ with a regular element x.

Definition 2.4. Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ be a symmetric Lie algebra. We set

$$\operatorname{Sym}(\mathfrak{g}) \stackrel{\text{def}}{=} \{x \in \operatorname{Reg}(\mathfrak{g}) \mid \mathfrak{g}^0(x) \text{ is a symmetric Cartan algebra}\}.$$

We call $x \in \operatorname{Reg}(\mathfrak{g})$ *symmetric regular* if $x \in \operatorname{Sym}(\mathfrak{g})$.

As usual we denote by $\tau \colon \mathfrak{g} \to \mathfrak{g}$ the involutive Lie algebra automorphism associated to the decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ of the symmetric Lie algebra. Using the relation $\tau \circ \operatorname{ad} x \circ \tau = \operatorname{ad} \tau(x)$ and the definition of the nilspace of an element $x \in \mathfrak{g}$, we can easily give another characterization of the set of symmetric regular elements.

Lemma 2.5. *Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ be a symmetric Lie algebra. Then*

$$\begin{aligned} \operatorname{Sym}(\mathfrak{g}) &= \{x \in \operatorname{Reg}(\mathfrak{g}) \mid \mathfrak{g}^0(x) = \mathfrak{g}^0(\tau(x))\} \\ &= \{x \in \operatorname{Reg}(\mathfrak{g}) \mid \tau(x) \in \mathfrak{g}^0(x)\}. \end{aligned}$$

■

The set $\operatorname{Sym}(\mathfrak{g})$ is very thin in the topological point of view. We denote by $\Sigma(\mathfrak{g})$ the set of all subalgebras of $\mathfrak{g}$. We endow $\Sigma(\mathfrak{g})$ with the topology of the Grassmann manifold of all vector subspaces of $\mathfrak{g}$. On $\operatorname{Reg}(\mathfrak{g})$ we have the natural topology and the Zariski topology. One can show, that the map $\operatorname{Reg}(\mathfrak{g}) \to \Sigma(\mathfrak{g}) \colon x \mapsto \mathfrak{g}^0(x)$ is continuous with respect to the natural topology on $\operatorname{Reg}(\mathfrak{g})$.

Proposition 2.6. *The set of symmetric regular elements* $\operatorname{Sym}(\mathfrak{g})$ *is closed in* $\operatorname{Reg}(\mathfrak{g})$ *with respect to the natural topology and the Zariski topology. In particular,* $\operatorname{Sym}(\mathfrak{g})$ *is nowhere dense or equals* $\operatorname{Reg}(\mathfrak{g})$. ■

We denote by $\mathfrak{c}$ the span of all symmetric Cartan algebras and define $\mathfrak{a} \stackrel{\text{def}}{=} \mathfrak{h} + \mathfrak{c}$.

Lemma 2.7. *If* $\varphi \in \mathrm{Aut}(\mathfrak{g})$ *satisfies* $\varphi \circ \tau = \tau \circ \varphi$, *then* $\varphi(\mathrm{Sym}(\mathfrak{g})) = \mathrm{Sym}(\mathfrak{g})$. *In particular, the set of symmetric regular elements* $\mathrm{Sym}(\mathfrak{g})$ *is invariant under* $e^{\mathrm{ad}\,\mathfrak{h}}$. *Further, we have:*

(i) $\mathfrak{c} = \mathfrak{c}_{\mathfrak{h}} \oplus \mathfrak{c}_{\mathfrak{q}}$ *is a symmetric* $\mathfrak{h}$*-module.*

(ii) $\mathfrak{a} = \mathfrak{h} \oplus \mathfrak{c}_q$ *is a symmetric subalgebra with* $N_{\mathfrak{g}}(\mathfrak{a}) = \mathfrak{a}$.

■

3. Conjugation in Lie algebras over an algebraically closed field

In this section $\mathbb{K}$ is supposed to be an algebraically closed field. Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ be a symmetric Lie algebra over $\mathbb{K}$, and $\mathfrak{t}$ a symmetric Cartan algebra of $\mathfrak{g}$. Hence the action of the nilpotent subalgebra $\mathfrak{t}$ gives a weight space decomposition

$$\mathfrak{g} = \mathfrak{t} \oplus \sum_{\lambda \in \Lambda} \mathfrak{g}^{\lambda}(\mathfrak{t}),$$

where $\Lambda \subseteq \widehat{\mathfrak{t}}$ as usual is the set of weights. One easily verifies that, if $\lambda \in \Lambda$, then $\lambda \circ \tau \in \Lambda$. More precisely,

$$\tau(\mathfrak{g}^{\lambda}(\mathfrak{t})) = \mathfrak{g}^{\lambda \circ \tau}(\mathfrak{t}).$$

Lemma 3.1. *Let* $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ *be a symmetric Lie algebra,* $\mathfrak{t}$ *a symmetric Cartan algebra and* $\lambda \in \Lambda$. *Then the following conditions are equivalent.*

(1) $\lambda|_{\mathfrak{t} \cap \mathfrak{h}} \equiv 0$.

(2) $\lambda \circ \tau = -\lambda$.

■

This motivates the following definition.

Definition 3.2. Let $\mathfrak{t}$ be a symmetric Cartan algebra in the symmetric Lie algebra $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$. Then we set

$$\Lambda_0 \stackrel{\text{def}}{=} \{\lambda \in \Lambda \mid \lambda \circ \tau = -\lambda\} \quad \text{and} \quad \Lambda_+ \stackrel{\text{def}}{=} \Lambda \setminus \Lambda_0.$$

Let $\lambda \in \Lambda_+$ and $x \in \mathfrak{g}^{\lambda}(\mathfrak{t})$. Obviously, $\mathrm{ad}(x + \tau(x))$ is a nilpotent linear operator. In particular, $e^{\mathrm{ad}(x+\tau(x))}$ is a polynomial in $\mathrm{ad}(x + \tau(x))$.

Definition 3.3. If $\mathfrak{t}$ is a symmetric Cartan algebra, we write

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_+$$

where $\mathfrak{g}_0 \stackrel{\text{def}}{=} \mathfrak{t} \oplus \sum_{\lambda \in \Lambda_0} \mathfrak{g}^{\lambda}(\mathfrak{t})$ and $\mathfrak{g}_+ \stackrel{\text{def}}{=} \sum_{\lambda \in \Lambda_+} \mathfrak{g}^{\lambda}(\mathfrak{t})$.

We remark that both $\mathfrak{g}_0$ and $\mathfrak{g}_+$ depend on the Cartan algebra $\mathfrak{t}$.

Proposition 3.4. *Let* $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ *be a symmetric Lie algebra and* $\mathfrak{t}$ *a symmetric Cartan algebra. Then the following holds:*

(i) $\mathfrak{g}_0$ *is a symmetric subalgebra of* $\mathfrak{g}$ *with* $N_{\mathfrak{g}}(\mathfrak{g}_0) = \mathfrak{g}_0$.

(ii) $\mathfrak{g}_+$ *is a* $\mathfrak{g}_0$*-module, i. e., we have* $[\mathfrak{g}_0, \mathfrak{g}_+] \subseteq \mathfrak{g}_+$.

■

The following proposition describes the structure of the symmetric subalgebra $\mathfrak{g}_0$.

Proposition 3.5. *Let* $\mathfrak{t}$ *be a symmetric Cartan algebra of* $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$, *and* $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_+$ *according* Definition 3.3. *Denote by* $\mathfrak{g} = \mathfrak{g}^0(\mathfrak{t}_{\mathfrak{h}}) \oplus \sum_{\mu \in M} \mathfrak{g}^{\mu}(\mathfrak{t}_{\mathfrak{h}})$ *the weight decomposition of* $\mathfrak{g}$ *given by the action of the nilpotent subalgebra* $\mathfrak{t}_{\mathfrak{h}} = \mathfrak{t} \cap \mathfrak{h}$. *Then the following holds:*

(i) $\mathfrak{g}_0 = \mathfrak{g}^0(\mathfrak{t}_{\mathfrak{h}})$, *and* $\mathfrak{g}^0(\mathfrak{t}_{\mathfrak{h}}) = \mathfrak{h}^0(\mathfrak{t}_{\mathfrak{h}}) \oplus \mathfrak{q}^0(\mathfrak{t}_{\mathfrak{h}})$.

(ii) *Set* $\Lambda_\mu \stackrel{\text{def}}{=} \{\lambda \in \Lambda_+ \mid \lambda|_{\mathfrak{t}_{\mathfrak{h}}} \equiv \mu\}$. *Then* $\mathfrak{g}_+ = \sum_{\mu \in M} \mathfrak{g}^{\mu}(\mathfrak{t}_{\mathfrak{h}})$, *where* $\mathfrak{g}^{\mu}(\mathfrak{t}_{\mathfrak{h}}) = \sum_{\lambda \in \Lambda_\mu} \mathfrak{g}^{\lambda}(\mathfrak{t})$.

(iii) *There exists* $x \in \mathfrak{t}_{\mathfrak{h}}$ *such that* $\mathfrak{g}_0 = \mathfrak{g}^0(x)$. *Further,* $\mathfrak{h}^0(\mathfrak{t}_{\mathfrak{h}}) = \mathfrak{h}^0(x)$ *and* $\mathfrak{q}^0(\mathfrak{t}_{\mathfrak{h}}) = \mathfrak{q}^0(x)$.

(iv)
$$\mathfrak{h}^0(\mathfrak{t}_{\mathfrak{h}}) = \mathfrak{t}_{\mathfrak{h}} \oplus \sum_{\lambda \in \Lambda_0} (\mathfrak{g}^{\lambda}(\mathfrak{t}) \oplus \mathfrak{g}^{-\lambda}(\mathfrak{t})) \cap \mathfrak{h}$$
and
$$\mathfrak{q}^0(\mathfrak{t}_{\mathfrak{h}}) = \mathfrak{t}_{\mathfrak{q}} \oplus \sum_{\lambda \in \Lambda_0} (\mathfrak{g}^{\lambda}(\mathfrak{t}) \oplus \mathfrak{g}^{-\lambda}(\mathfrak{t})) \cap \mathfrak{q}.$$

(v) *For* $\mu \in M$, *we have* $\mathfrak{g}^{\mu}(\mathfrak{t}_{\mathfrak{h}}) = \mathfrak{h}^{\mu}(\mathfrak{t}_{\mathfrak{h}}) \oplus \mathfrak{q}^{\mu}(\mathfrak{t}_{\mathfrak{h}})$ *with*
$$\mathfrak{h}^{\mu}(\mathfrak{t}_{\mathfrak{h}}) = \sum_{\lambda \in \Lambda_\mu} (\mathfrak{g}^{\lambda}(\mathfrak{t}) \oplus \mathfrak{g}^{\lambda \circ \tau}(\mathfrak{t})) \cap \mathfrak{h}$$
and
$$\mathfrak{q}^{\mu}(\mathfrak{t}_{\mathfrak{h}}) = \sum_{\lambda \in \Lambda_\mu} (\mathfrak{g}^{\lambda}(\mathfrak{t}) \oplus \mathfrak{g}^{\lambda \circ \tau}(\mathfrak{t})) \cap \mathfrak{q}.$$

(vi) $\mathfrak{h} = \mathfrak{h}^0(\mathfrak{t}_{\mathfrak{h}}) \oplus \sum_{\mu \in M} \mathfrak{h}^{\mu}(\mathfrak{t}_{\mathfrak{h}})$ *and* $\mathfrak{q} = \mathfrak{q}^0(\mathfrak{t}_{\mathfrak{h}}) \oplus \sum_{\mu \in M} \mathfrak{q}^{\mu}(\mathfrak{t}_{\mathfrak{h}})$.

■

Remark 3.6. We see in Lemma 3.5 that $\mathfrak{g}_0 = \mathfrak{g}^0(\mathfrak{t}_{\mathfrak{h}})$ is also defined over an arbitrary field $\mathbb{K}$ of characteristic zero. Also, $\mathfrak{g}_+$ is the uniquely determined $\mathfrak{t}_{\mathfrak{h}}$-module complement of $\mathfrak{g}_0$ in $\mathfrak{g}$.

Suppose $\mathfrak{t}$ and $\mathfrak{t}'$ are different symmetric Cartan algebras of $\mathfrak{g}$. What is the relation between $\mathfrak{g}_0 = \mathfrak{g}^0(\mathfrak{t}_{\mathfrak{h}})$ and $\mathfrak{g}_0' = \mathfrak{g}^0(\mathfrak{t}_{\mathfrak{h}}')$?

Lemma 3.7. *Let $\mathfrak{t}$ be a symmetric Cartan algebra and $\varphi \in \mathrm{Aut}(\mathfrak{g})$ such that $\varphi \circ \tau = \tau \circ \varphi$. Then the following holds:*

(i) $\mathfrak{t}' \stackrel{\text{def}}{=} \varphi(\mathfrak{t})$ *is a symmetric Cartan algebra of* $\mathfrak{g}$.

(ii) φ *induces a bijection* $\Lambda \to \Lambda' \colon \lambda \mapsto \lambda \circ \varphi^{-1}$, *and* $\varphi(\mathfrak{g}^{\lambda}(\mathfrak{t})) = \mathfrak{g}^{\lambda\circ\varphi^{-1}}(\mathfrak{t}')$.

(iii) $\lambda \in \Lambda_0$ *if and only if* $\lambda \circ \varphi^{-1} \in \Lambda_0'$. *Further,* φ *induces isomorphisms* $\varphi|_{\mathfrak{g}_0} \colon \mathfrak{g}_0 \to \mathfrak{g}_0'$ *and* $\varphi|_{\mathfrak{g}_+} \colon \mathfrak{g}_+ \to \mathfrak{g}_+'$.

■

The group of of automorphisms

$$E_\tau \stackrel{\text{def}}{=} \langle \{ e^{\operatorname{ad} h} \mid h \in \mathfrak{h},\ \operatorname{ad} h \text{ nilpotent} \} \rangle$$

is uniquely determined by the involution τ. Also the group of automorphisms

$$E_0 \stackrel{\text{def}}{=} \langle \{ e^{\operatorname{ad} x} \mid (\exists \lambda \in \Lambda_0)\, x \in \mathfrak{g}^{\lambda}(\mathfrak{t}) \} \rangle$$

depends only on the associated subalgebra $\mathfrak{g}_0$. With these preparations we can state our main conjugation theorem.

Theorem 3.8. *If $\mathfrak{t}$ and $\mathfrak{t}'$ are symmetric Cartan algebras of a symmetric Lie algebra $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$, then there exists an element $\gamma \in E_\tau$ such that $\mathfrak{g}_0' = \gamma(\mathfrak{g}_0)$ and $\mathfrak{g}_+' = \gamma(\mathfrak{g}_+)$. Further, there exists an $\alpha \in E_0$, such that with $\varphi \stackrel{\text{def}}{=} \gamma \circ \alpha$ the relations $\mathfrak{t}' = \varphi(\mathfrak{t})$ and $\varphi(\mathfrak{g}^{\lambda}(\mathfrak{t})) = \mathfrak{g}^{\lambda\circ\varphi^{-1}}(\mathfrak{t}')$ hold.* ■

Corollary 3.9. *Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ be a symmetric Lie algebra, $\mathfrak{t}$ a symmetric Cartan algebra and $\mathfrak{g}_0 = \mathfrak{g}^0(\mathfrak{t}_{\mathfrak{h}})$. Then the following holds:*

(i) *If $\mathfrak{t} = \mathfrak{g}_0$, then there exists an $x \in \mathrm{Reg}(\mathfrak{g}) \cap \mathfrak{h}$ such that $\mathfrak{t} = \mathfrak{g}^0(x)$. All symmetric Cartan algebras are conjugate under E_τ.*

(ii) *If $\mathfrak{g} = \mathfrak{g}_0$, then there exists an $x \in \mathrm{Reg}(\mathfrak{g}) \cap \mathfrak{q}$ such that $\mathfrak{t} = \mathfrak{g}^0(x)$.*

■

4. Conjugation in solvable symmetric Lie algebras

In this section let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ be a *solvable* symmetric Lie algebra. Suppose $\mathfrak{n}$ is a symmetric ideal. Then the quotient space $\mathfrak{g}/\mathfrak{n}$ becomes a symmetric Lie algebra in a canonical way. We have $\mathfrak{g}/\mathfrak{n} = (\mathfrak{h} + \mathfrak{n}/\mathfrak{n}) \oplus (\mathfrak{q} + \mathfrak{n}/\mathfrak{n})$. We denote by $\mathcal{C}^\infty(\mathfrak{g})$ the intersection of the descending central series. That is, $\mathcal{C}^\infty(\mathfrak{g}) \stackrel{\text{def}}{=} \bigcap_{n\in\mathbb{N}} \mathfrak{g}^{[n]}$. Since all $\mathfrak{g}^{[n]}$ are symmetric, and $\mathcal{C}^\infty(\mathfrak{g}) \subseteq [\mathfrak{g}, \mathfrak{g}]$, the set $\mathcal{C}^\infty(\mathfrak{g})$ is a nilpotent symmetric ideal. The set $\{e^{\operatorname{ad} x} \mid x \in \mathcal{C}^\infty(\mathfrak{g})\}$ is a subgroup of $\mathrm{Aut}(\mathfrak{g})$, such that every $e^{\operatorname{ad} x}$ is a polynomial in $\operatorname{ad} x$.

Theorem 4.1. *Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ be a solvable symmetric Lie algebra und $\mathfrak{t}, \mathfrak{t}'$ symmetric Cartan algebras of $\mathfrak{g}$. Then there exists an $x \in \mathcal{C}^\infty(\mathfrak{g}) \cap \mathfrak{h}$ such that $e^{\operatorname{ad} x}\mathfrak{t} = \mathfrak{t}'$.* ■

The proof of this theorem preceeds in an analogous way as the proof of Theorem 1.4. It remains to ensure that the element x, under which the Cartan algebras are conjugated, is contained in $\mathfrak{h}$.

5. Symmetric Cartan algebras and Levi complements

Let $\mathfrak{r}$ be the radical of a symmetric Lie algebra $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$. Obviously, $\mathfrak{r}$ is symmetric. In [3], Appendix 9, it is shown, that there exists a symmetric Levi subalgebra $\mathfrak{s}$. A Levi decomposition $\mathfrak{g} = \mathfrak{r} \oplus \mathfrak{s}$ and a Cartan algebra $\mathfrak{t}$ are called *adapted*, if $\mathfrak{t} = (\mathfrak{t} \cap \mathfrak{r}) \oplus (\mathfrak{t} \cap \mathfrak{s})$. If this is the case, we write $\mathfrak{t}_{\mathfrak{r}} \stackrel{\text{def}}{=} \mathfrak{t} \cap \mathfrak{r}$ and $\mathfrak{t}_{\mathfrak{s}} \stackrel{\text{def}}{=} \mathfrak{t} \cap \mathfrak{s}$. In [4] the following result is proved. There is also a proof of (i) in [2], which is much shorter and more algebraic than the original one.

Theorem 5.1. *Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ be a symmetric Lie algebra with radical $\mathfrak{r}$. Then the following holds:*

(i) *Suppose $\mathfrak{s}$ is a symmetric Levi complement and $\mathfrak{t}_{\mathfrak{s}}$ a symmetric Cartan algebra of $\mathfrak{s}$. Then there exists a symmetric subalgebra $\mathfrak{t}_{\mathfrak{r}}$ of $\mathfrak{r}$ such that $\mathfrak{t} \stackrel{\text{def}}{=} \mathfrak{t}_{\mathfrak{r}} \oplus \mathfrak{t}_{\mathfrak{s}}$ is a symmetric Cartan algebra of $\mathfrak{g}$ adapted to the decomposition $\mathfrak{g} = \mathfrak{r} \oplus \mathfrak{s}$.*

(ii) *Conversely, suppose $\mathfrak{t}$ is a symmetric Cartan algebra of $\mathfrak{g}$. Then there exists a symmetric Levi complement $\mathfrak{s}$, a symmetric Cartan algebra $\mathfrak{t}_{\mathfrak{s}}$ of $\mathfrak{s}$, and a symmetric subalgebra $\mathfrak{t}_{\mathfrak{r}}$ of $\mathfrak{r}$ such that $\mathfrak{t} = \mathfrak{t}_{\mathfrak{r}} \oplus \mathfrak{t}_{\mathfrak{s}}$.*

■

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