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Lie Superalgebras and Lie Supergroups, II

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6. The Hopf Dual.

Let $\mathcal{H} = \mathcal{H}_{\bar{0}} \oplus \mathcal{H}_{\bar{1}}$ denote an affine Hopf superalgebra, i.e. a $\mathbb{Z}_2$-graded commutative, finitely generated Hopf algebra. The set $G(\mathcal{H}) = \hom_{alg}(\mathcal{H}; K)$ of algebra homomorphisms mapping the superalgebra $\mathcal{H}$ onto the base field is a group with respect to convolution, called the *structure group* of $\mathcal{H}$.

The *convolution* is defined by $\gamma_1 * \gamma_2 = (\gamma_1 \otimes \gamma_2)\Delta$. It holds $\gamma_1 * \gamma_2 \in G(\mathcal{H})$ if $\gamma_1, \gamma_2 \in G(\mathcal{H})$, and $\gamma * \varepsilon = \gamma = \varepsilon * \gamma$, and $\gamma * (\gamma\sigma) = \varepsilon = (\gamma\sigma) * \gamma$.

By an easy calculation we have f.i. $\gamma * \varepsilon = (\gamma \otimes \varepsilon)\Delta = \gamma(id_{\mathcal{H}} \otimes \varepsilon)\Delta = \gamma id_{\mathcal{H}} = \gamma$, and $(\gamma\sigma) * \gamma = ((\gamma\sigma) \otimes \gamma)\Delta = (\gamma \otimes \gamma)(\sigma \otimes id_{\mathcal{H}})\Delta = \gamma\mu(\sigma \otimes id_{\mathcal{H}})\Delta = \gamma\iota\varepsilon = \varepsilon$.

Let $\mathcal{H}_\circ = \mathcal{H}/(\mathcal{H}_{\bar{1}})$ denote the quotient of $\mathcal{H}$ by the homogeneous ideal generated by $\mathcal{H}_{\bar{1}}$, which is a coideal too: $\Delta(\mathcal{H}) \subseteq \mathcal{H}_{\bar{1}} \otimes \mathcal{H}_{\bar{0}} + \mathcal{H}_{\bar{0}} \otimes \mathcal{H}_{\bar{1}}$, $\varepsilon(\mathcal{H}_{\bar{1}}) = 0$, and $\sigma(\mathcal{H}_{\bar{1}}) \subseteq \mathcal{H}_{\bar{1}}$.

$\mathcal{H}_\circ$ is an affine Hopf algebra and there is a canonical isomorphism between the structure groups of $\mathcal{H}$ and $\mathcal{H}_\circ$

$$G(\mathcal{H}) \cong G(\mathcal{H}_\circ).$$

Let Λ denote a Grassmann algebra and let $G(\mathcal{H};\Lambda) = \hom_{alg}(\mathcal{H};\Lambda)$ denote the set of algebra homomorphisms mapping $\mathcal{H}$ into Λ, then $G(\mathcal{H};\Lambda)$ is a group with respect to convolution. The equation $\pi_\Lambda(\gamma_\Lambda) = p_0\gamma_\Lambda$ with $\gamma_\Lambda \in G(\mathcal{H};\Lambda)$ defines an epimorphism π_Λ mapping $G(\mathcal{H};\Lambda)$ onto $G(\mathcal{H})$. Moreover, by the inclusion $K \subseteq \Lambda$ we have a canonical embedding ι_Λ of $G(\mathcal{H})$ into $G(\mathcal{H};\Lambda)$. The diagram

$$G(\mathcal{H}) \xrightarrow{\iota_\Lambda} G(\mathcal{H};\Lambda) \xrightarrow{\pi_\Lambda} G(\mathcal{H})$$

implies a semidirect product structure of $G(\mathcal{H};\Lambda)$

$$G(\mathcal{H};\Lambda) \cong G(\mathcal{H}) \odot K(\mathcal{H};\Lambda).$$

The normal subgroup $K(\mathcal{H};\Lambda) = \ker \pi_\Lambda$ is a unipotent group. In the representation by block matrices from $\mathrm{Mat}(m,n;\Lambda)$ the group $K(\mathcal{H};\Lambda)$ consists of matrices of the form

$$\begin{pmatrix} E_m & B_1(\Lambda) \\ B_0(\Lambda) & E_n \end{pmatrix}.$$

A *differentiation* $\lambda : \mathcal{H} \longrightarrow K$ of $\mathcal{H}$ is a linear mapping satisfying the equation

$$\lambda(XY) = \lambda(X)\varepsilon(Y) + \varepsilon(X)\lambda(Y),\ X, Y \in \mathcal{H}.$$

Let λ_1 and λ_2 denote homogeneous differentiations of $\mathcal{H}$. Then

$$[\lambda_1, \lambda_2] = \lambda_1 * \lambda_2 - (-1)^{|\lambda_1||\lambda_2|}\lambda_2 * \lambda_1$$

is a homogeneous differentiation of $\mathcal{H}$.

Let $L_{\bar{0}}(\mathcal{H})$ and $L_{\bar{1}}(\mathcal{H})$ denote the linear space of even and odd differentiations of $\mathcal{H}$ respectively.

The linear space $L(\mathcal{H}) = L_{\bar{0}}(\mathcal{H}) \oplus L_{\bar{1}}(\mathcal{H})$ *is a* LIE *superalgebra with respect to the brackets defined above.*

$L(\mathcal{H}\ )$ is called the LIE*superalgebra of the* HOPF *superalgebra* $\mathcal{H}$.

Let $\lambda_\Lambda : \mathcal{H} \longrightarrow \Lambda$ denote an even linear mapping satisfying the equation $\lambda_\Lambda(XY) = \lambda_\Lambda(X)\varepsilon(Y) + \varepsilon(X)\lambda_\Lambda(Y),\ X, Y \in \mathcal{H}$. We shall call it a Λ-*differentiation* of $\mathcal{H}$. Then it holds

The linear space $L(\mathcal{H}; \Lambda)$ *of* Λ- *differentiations of* $\mathcal{H}$ *is a* LIE *algebra with respect to the brackets* $[\lambda_{1\Lambda}, \lambda_{2\Lambda}] = \lambda_{1\Lambda} * \lambda_{2\Lambda} - \lambda_{2\Lambda} * \lambda_{1\Lambda}$. *The* LIE *algebra* $L(\mathcal{H}; \Lambda)$ *is isomorphic to the* GRASSMANN-*hull of the* LIE *superalgebra* $L(\mathcal{H})$.

The algebraic dual $\mathcal{H}' = \mathcal{H}'_{\bar{0}} \oplus \mathcal{H}'_{\bar{1}}$ of the HOPF superalgebra $\mathcal{H}$ is an associative superalgebra with unit. The product of $\mathcal{H}'$ is the convolution, i.e. the restriction of the dual mapping $\Delta' : (\mathcal{H} \otimes \mathcal{H})' \longrightarrow \mathcal{H}'$ of the coproduct Δ to the subspace $\mathcal{H}' \otimes \mathcal{H}'$ of $(\mathcal{H} \otimes \mathcal{H})'$.

Let $\mathcal{H}^\circ$ denote the subalgebra of $\mathcal{H}'$ consisting of those linear functionals $\varphi \in \mathcal{H}'$ which vanish on a homogeneous ideal of finite codimension in $\mathcal{H}$. Then it is known, that the restriction of the dual mapping μ' of the product map

μ of $\mathcal{H}$ to $\mathcal{H}^\circ$ maps $\mathcal{H}^\circ$ into $\mathcal{H}^\circ \otimes \mathcal{H}^\circ$ and hence defines a coproduct Δ° in $\mathcal{H}^\circ$. Moreover, the restriction of the dual map σ' of the antipode σ of $\mathcal{H}$ to $\mathcal{H}^\circ$ is an antipode σ° of $\mathcal{H}^\circ$. At last, the counit of $\mathcal{H}^\circ$ is the evaluation of the linear functionals at the unit of $\mathcal{H}$.

$\mathcal{H}^\circ = \mathcal{H}^\circ_{\bar{0}} \oplus \mathcal{H}^\circ_{\bar{1}}$ *is a cocommutative* HOPF *superalgebra.* It is called the HOPF*dual* of $\mathcal{H}$.

The structure group $G(\mathcal{H})$ consists of the *group-like* elements of $\mathcal{H}^\circ : \gamma \in G(\mathcal{H})$ iff $\Delta^\circ(\gamma) = \gamma \otimes \gamma$.

The structure group of $\mathcal{H}$ *is the group of units in* $\mathcal{H}^\circ$.

The LIE superalgebra $L(\mathcal{H})$ consists of the *primitive* or LIE*algebra-like* elements of $\mathcal{H}^\circ : \lambda \in L(\mathcal{H})$ iff $\Delta^\circ(\lambda) = \lambda \otimes \varepsilon + \varepsilon \otimes \lambda$.

It is easy to verify the following equations :

$$\begin{aligned} < \Delta^\circ(\gamma), X \otimes Y > &= < \gamma, XY > \\ = < \gamma, X >< \gamma, Y > &= < \gamma \otimes \gamma, X \otimes Y >, \\ \text{and} < \Delta^\circ(\lambda), X \otimes Y > &= < \lambda, XY > \\ = < \lambda, X >< \varepsilon, Y > &+ < \varepsilon, X >< \lambda, Y > \\ = < \lambda \otimes \varepsilon &+ \varepsilon \otimes \lambda, X \otimes Y >, X, Y \in \mathcal{H}. \end{aligned}$$

Structure Theorem of cocommutative HOPF **superalgebras** . (SWEEDLER,KOSTANT) *Let* $\mathcal{H}$ *denote a cocommutative* HOPF *superalgebra,* G *its group of group-like elements and* $\mathcal{L}$ *its* LIE *superalgebra of primitive elements. Let* $K(G)$ *denote the group algebra of* G *and* $U(\mathcal{L})$ *the enveloping superalgebra of* $\mathcal{L}$. *Then* $\mathcal{H}$ *is a smashed product of* $K(G)$ *and* $U(\mathcal{L})$.

As a corollary from this theorem we mention

The HOPF *dual* $\mathcal{H}^\circ$ *of a* HOPF *superalgebra* $\mathcal{H}$ *is generated by its structure group* $G(\mathcal{H})$ *and its* LIE *superalgebra* $L(\mathcal{H})$.

7. Affine Algebraic Supergroups.

Let G denote a group. A representative function on G is a K-valued function f on G with the property that $\mathrm{span}\{f_g; g \in G\}$ (or $\mathrm{span}\{{}_gf; g \in G\}$) is finite dimensional. It is $f_g(g') = f(g'g)$ and ${}_gf(g') = f(g^{-1}g')$.

Proposition.(HOCHSCHILD) *The representative functions of* G *make up an affine* HOPF *algebra* $\mathcal{R}_\circ(G)$. *The coproduct* $\Delta_\circ$ *is defined by the equation* $\widetilde{f}(g_1, g_2) = f(g_1g_2)$ *using the isomorphism of* $\mathcal{R}_\circ(G \times G)$ *with* $\mathcal{R}_\circ(G) \otimes \mathcal{R}_\circ(G)$. *The counit* $\varepsilon_\circ$ *is the evaluation at the identity :* $\varepsilon_\circ(f) = f(e)$ *and the antipode* $\sigma_\circ$ *is defined by* $\sigma_\circ(f)(g) = f(g^{-1})$.

Definition.(HOCHSCHILD) The *structure of an affine algebraic group* is a pair $(G, \mathcal{P}_\circ)$ consisting of a group G and a sub HOPF algebra $\mathcal{P}_\circ$ of $\mathcal{R}_\circ(G)$ satisfying the following properties

i.) $\mathcal{P}_\circ$ separates the points of G ;

ii.) every algebra homomorphism $\gamma_\circ : \mathcal{P}_\circ \longrightarrow K$ is the evaluation at a group element.

The algebra $\mathcal{P}_\circ$ is called the *algebra of polynomial functions* on G.

The properties i.) and ii.) imply a canonical isomorphism $G \cong G(\mathcal{P}_\circ)$ and $\mathcal{L}_{\bar{0}} = L(\mathcal{P}_\circ)$ is the LIE algebra of the affine algebraic group $(G, \mathcal{P}_\circ)$.

Proposition.(HOCHSCHILD) *Let* $\mathcal{H}_\circ$ *denote an affine* HOPF *algebra, then* $(G(\mathcal{H}_\circ), \mathcal{H}_\circ)$ *is an affine algebraic group structure.*

Definition. The *structure of an affine algebraic supergroup* is a pair $(G, \mathcal{P})$ consisting of a group G and an affine HOPF superalgebra $\mathcal{P}$ satisfying the following property:.

There is a sub HOPF algebra $\mathcal{P}_\circ$ of $\mathcal{R}_\circ(G)$ and a finite dimensional vector space W such that it holds

i.) $\mathcal{P} \cong \mathcal{P}_\circ \otimes \Lambda(W)$ is an isomorphism of associative superalgebras;

ii.) the canonical projection $p : \mathcal{P} \longrightarrow \mathcal{P}_\circ$ is a morphism of supercoalgebras compatible with the antipodes;

iii.) $(G, \mathcal{P}_\circ)$ is the structure of an affine algebraic group.

Property ii.) is equivalent to the equations : $(p \otimes p)\Delta = \Delta_\circ p$, $\varepsilon = \varepsilon_\circ p$, and $p\sigma = \sigma_\circ p$. Property iii.) implies canonical isomorphisms $G(\mathcal{P}) \cong G(\mathcal{P}_\circ) \cong G$. Every algebra homomorphism $\gamma : \mathcal{P} \longrightarrow K$ is of the following type: projection by p and evaluation at a group element.

$\mathcal{P} = \mathcal{P}_{\bar{0}} \oplus \mathcal{P}_{\bar{1}}$ is called the *superalgebra of polynomial functions in commuting and anticommuting variables on* G .

$L(\mathcal{P}) = \mathcal{L} = \mathcal{L}_{\bar{0}} \oplus \mathcal{L}_{\bar{1}}$ is called the LIE *superalgebra of the affine algebraic supergroup* $(G, \mathcal{P})$.

Evidently holds: $\dim \mathcal{L}_{\bar{0}} = \dim(G, \mathcal{P}_\circ)$ = degree of transcendency of $Q(\mathcal{P}_\circ)/K$; $Q(\mathcal{P}_\circ)$ denotes the quotient field of $\mathcal{P}_\circ$ provided the algebraic group structure $(G, \mathcal{P}_\circ)$ is irreducible, i.e. $\mathcal{P}_\circ$ is an integral domain; and $\dim \mathcal{L}_{\bar{1}} = \dim W$.

Proposition. *Let $\mathcal{H} = \mathcal{H}_{\bar{0}} \oplus \mathcal{H}_{\bar{1}}$ denote an affine* HOPF *superalgebra, then $(G(\mathcal{H}), \mathcal{H})$ is the structure of an affine algebraic supergroup.*

If $\mathcal{H} \cong \mathcal{H}_\circ \otimes \Lambda(W)$, then it holds $G(\mathcal{H}) \cong G(\mathcal{H}_\circ)$ canonically and $(G(\mathcal{H}), \mathcal{H}_\circ)$ is the structure of an affine algebraic group, the *underlying algebraic group* of the algebraic supergroup $(G(\mathcal{H}), \mathcal{H})$.

Assume $\mathcal{L} = \mathcal{L}_{\bar{0}} \oplus \mathcal{L}_{\bar{1}}$ to be a finite dimensional Lie superalgebra. Denote by $U(\mathcal{L})$ its enveloping algebra, $U(\mathcal{L})$ is a cocommutative HOPF superalgebra: $\Delta(\lambda) = \lambda \otimes \varepsilon + \varepsilon \otimes \lambda$; $\lambda \in \mathcal{L}$, ε denotes the unit element of $U(\mathcal{L})$. Its HOPF dual $U^\circ(\mathcal{L})$ is a commutative HOPF superalgebra.

It holds the following isomorphism of associative, commutative superalgebras:

$$U^\circ(\mathcal{L}) \cong U^\circ(\mathcal{L}_{\bar{0}}) \otimes \Lambda(\mathcal{L}'_{\bar{1}}).$$

Proposition. *The following statements are equivalent*

(i) *$U^\circ(\mathcal{L})$ is an affine* HOPF *superalgebra;*

(ii) $[\mathcal{L}_{\bar{0}}, \mathcal{L}_{\bar{0}}] = \mathcal{L}_{\bar{0}}$;

(iii) $L(U^\circ(\mathcal{L})) \cong \mathcal{L}$.

If one of the statements (i) - (iii) holds, then $\widetilde{G} = G(U^\circ(\mathcal{L})) \cong G(U^\circ(\mathcal{L}_{\bar{0}}))$ is the connectend and simply connected algebraic LIE group associated to $\mathcal{L}_{\bar{0}}$ and $(\widetilde{G}, U^\circ(\mathcal{L}))$ is the structure of an affine algebraic supergroup.

Corollary. *Let $\mathcal{L}$ denote a finite dimensional* LIE *superalgebra with semisimple even part $\mathcal{L}_{\bar{0}}$, and let $\widetilde{G}$ denote the connected, simply connected, semisimple* LIE *group corresponding to $\mathcal{L}_{\bar{0}}$, then $(\widetilde{G}, U^\circ(\mathcal{L}_{\bar{0}}))$ is the structure of an affine algebraic supergroup.*

8. Representations.

Let us start with a motivation.

Let $\mathcal{L} = \mathcal{L}_{\bar{0}} \oplus \mathcal{L}_{\bar{1}}$ denote a LIE superalgebra, and let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ denote a finite dimensional linear superspace. Assume that $\rho : \mathcal{L} \longrightarrow L_L(V)$ is a representation of $\mathcal{L}$ and $U(\rho) : U(\mathcal{L}) \longrightarrow L(V)$ its lift to the enveloping algebra $U(\mathcal{L})$. Put $\delta(v)(u) = (-1)^{|u||v|} U(\rho)(u)v$, with $u \in U(\mathcal{L}), v \in V$ and homogeneous. $\delta(v)$ is a V-valued representative function of $\mathcal{L} : \delta(v) \in U^\circ(\mathcal{L}) \otimes V$. The map $\delta : V \longrightarrow U^\circ(\mathcal{L})$ makes V an $U^\circ(\mathcal{L})$-left supercomodule.

Every representation of the LIE *superalgebra $\mathcal{L}$ defines an $U^\circ(\mathcal{L})$-left supercomodule structure on the representation superspace V.*

Definition. Let $\mathcal{H} = \mathcal{H}_{\bar{0}} \oplus \mathcal{H}_{\bar{1}}$ denote a HOPF superalgebra. A *$\mathcal{H}$-left supercomodule* is a linear superspace $V = V_{\bar{0}} \oplus V_{\bar{1}}$ endowed with an

even linear mapping $\delta : V \longrightarrow \mathcal{H} \otimes V$

satisfying the following equations:

$$(\Delta \otimes id_V)\delta = (id_{\mathcal{H}} \otimes \delta)\delta \; ; \; (\varepsilon \otimes id_V)\delta = id_V.$$

Example 1 $^\diamond$. Let $\mathcal{H} = \mathcal{H}_{\bar{0}} \oplus \mathcal{H}_{\bar{1}}$ denote a HOPF superalgebra. Put $V = \mathcal{H}$ and $\delta = \Delta$.

$\mathcal{H}$ is a $\mathcal{H}$-left (-right) supercomodul.

Example 2 $^\diamond$. Assume $\mathcal{H} = \mathcal{SP}(m,n) \cong K[X,X']/(detX - detX') \otimes \Lambda(Y,Y')$

$$\mathcal{X} = \begin{pmatrix} X & Y' \\ Y & X' \end{pmatrix} \ ; \ \mathcal{L} = \mathfrak{sl}(m,n).$$

Take $V = K^{m,n} = K^{m,0} \oplus K^{0,n}$, and let $e_1, ..., e_m$ and $f_1, ..., f_n$ denote the canonical basis of $K^{m,0}$ and $K^{0,n}$ respectively. Write

$$\mathcal{E} = \begin{pmatrix} e_1 \\ \vdots \\ e_m \\ f_1 \\ \vdots \\ f_n \end{pmatrix} \text{ and } \delta(\mathcal{E}) = \mathcal{X} \otimes \mathcal{E}.$$

In more detail

$$\begin{aligned} \delta(e_\mu) &= \sum_{j=1}^{m} X_{\mu j} \otimes e_j + \sum_{k=1}^{n} Y'_{\mu k} \otimes f_k, \\ \delta(f_\nu) &= \sum_{j=1}^{m} Y_{\nu j} \otimes e_j + \sum_{k=1}^{n} X'_{\nu k} \otimes f_k. \end{aligned}$$

It holds $(\Delta \otimes id_V)\delta(\mathcal{E}) = (\Delta \otimes id_V)(\mathcal{X} \otimes \mathcal{E}) = \Delta\mathcal{X} \otimes \mathcal{E} = (\mathcal{X} \otimes \mathcal{X}) \otimes \mathcal{E} = \mathcal{X} \otimes (\mathcal{X} \otimes \mathcal{E}) = (id_\mathcal{H} \otimes \delta)\delta(\mathcal{E}); (\varepsilon \otimes id_V)\delta(\mathcal{E}) = (\varepsilon \otimes id_V)(\mathcal{X} \otimes \mathcal{E}) = \varepsilon(\mathcal{X}) \otimes \mathcal{E} = E\mathcal{E}$. $K^{m,n}$ *is a* $\mathcal{SP}(m,n)$*-left supercomodule* .

Let $\mathcal{H} = \mathcal{H}_{\bar{0}} \oplus \mathcal{H}_{\bar{1}}$ denote a HOPF superalgebra. A block matrix $\mathcal{X} \in \mathrm{Mat}(m,n;\mathcal{H})$ is called *multiplicative*, if it holds

$$\Delta(\mathcal{X}) = \mathcal{X} \otimes \mathcal{X}.$$

Proposition.(MANIN) *The $\mathcal{H}$-left supercomodules $(K^{m,n}, \delta)$ are in one-to-one correspondence to the multiplicative block matrices* $\mathcal{X} \in \mathrm{Mat}(m,n;\mathcal{H})$.

Given $\delta : K^{m,n} \longrightarrow \mathcal{H} \otimes K^{m,n}$ we may compute $\mathcal{X}$ from

$$\delta(\mathcal{E}) = \mathcal{X} \otimes \mathcal{E},$$

and given a multiplicative $\mathcal{X} \in \mathrm{Mat}(m,n;\mathcal{H})$, δ is defined by the same equation.

A linear mapping $f : K^{m,n} \longrightarrow K^{m',n'}$ is a morphism of the supercomodules $(K^{m,n}, \delta)$ and $(K^{m',n'}, \delta')$, if there is a block matrix $\mathcal{F}$ interchanging the corresponding multiplicative block matrices $\mathcal{X}$ and $\mathcal{X}'$: $\mathcal{F}\mathcal{X}' = \mathcal{X}\mathcal{F}$ and $f(\mathcal{E}) = \mathcal{F}\mathcal{E}'$.

Let $\mathcal{H} = \mathcal{H}_{\bar{0}} \oplus \mathcal{H}_{\bar{1}}$ denote a HOPF superalgebra and let (V, δ) denote a $\mathcal{H}$-left supercomodule (not necessarily finite dimensional). If $\varphi \in \mathcal{H}' = \mathcal{H}'_{\bar{0}} \oplus \mathcal{H}'_{\bar{1}}$ then the equation

$$\rho'(\varphi) = (\varphi \otimes id_V)\delta$$

defines a representation ρ' of the associative superalgebra $\mathcal{H}'$ over V.

By suitable restrictions of ρ' we get representations of the superalgebra $\mathcal{H}^\circ$, of the structure group $G(\mathcal{H})$, and of the LIE superalgebra $L(\mathcal{H})$:

$$\begin{aligned} \rho^\circ(\varphi) &= (\varphi \otimes id_V)\delta\ ,\ \varphi \in \mathcal{H}^\circ; \\ \rho_G(\gamma) &= (\gamma \otimes id_V)\delta\ ,\ \gamma \in G(\mathcal{H}); \\ \rho_L(\lambda) &= (\lambda \otimes id_V)\delta\ ,\ \lambda \in L(\mathcal{H}). \end{aligned}$$

Let $(G, \mathcal{P})$ denote an affine algebraic supergroup and assume $V = \mathcal{P}$ as $\mathcal{P}$-left supercomodule $(\mathcal{P}, \Delta)$. The corresponding representation ρ_L of the LIE superalgebra $\mathcal{L} = L(\mathcal{P})$ is called the *left regular representation* of $\mathcal{L}$.

Proposition. *The left regular representation of the* LIE *superalgebra* $\mathcal{L} = L(\mathcal{P})$ *of the affine algebraic supergroup* $(G, \mathcal{P})$ *is an isomorphism of* $\mathcal{L}$ *onto the* LIE *superalgebra* $\mathrm{Der}^{(\mathrm{r})}\mathcal{P}$ *of right invariant derivations on* $\mathcal{P} : L(\mathcal{P}) \cong \mathrm{Der}^{(r)}\mathcal{P}$.

Let us start with an affine algebraic supergroup $(G, \mathcal{P})$. If $(K^{m,n}, \delta)$ is a $\mathcal{P}$-left supercomodule , and X denotes the corresponding multiplicative block matrix, then we may get matrix representations of the group G and its LIE superalgebra $\mathcal{L} = L(\mathcal{P})$ as well, in the following way:

$$\begin{aligned} \rho_G(\gamma) = \gamma(\mathcal{X}) &= \begin{pmatrix} \gamma(X) & 0 \\ 0 & \gamma(X') \end{pmatrix}; \\ \rho_L(\lambda) = \Lambda(\mathcal{X}) &= \begin{pmatrix} \lambda(X) & \lambda(Y') \\ \lambda(Y) & \Lambda(X') \end{pmatrix}. \end{aligned}$$

We may also get representations of the group $G_\Lambda = G(\mathcal{P}; \Lambda)$ and of the LIE algebra $\mathcal{L}_\Lambda = L(\mathcal{P}; \Lambda)$ by Λ matrices :

$$\begin{aligned} \rho_G(\gamma_\Lambda) = \gamma_\Lambda(\mathcal{X}) &= \begin{pmatrix} \gamma_\Lambda(X) & \gamma_\Lambda(Y') \\ \gamma_\Lambda(Y) & \gamma_\Lambda(X') \end{pmatrix}; \\ \rho_L(\lambda_\Lambda) = \lambda_\Lambda(\mathcal{X}) &= \begin{pmatrix} \lambda_\Lambda(X) & \lambda_\Lambda(Y') \\ \lambda_\Lambda(Y) & \lambda_\Lambda(X') \end{pmatrix}. \end{aligned}$$

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