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Historical Links to a Lie Theory of Semigroups

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Dedicated to Professor Karl Heinrich Hofmann
on the occasion of his 60th birthday

In the past fifteen years there has emerged a substantial body of mathematics that one might appropriately refer to as a Lie theory of semigroups. In the narrowest sense, such a theory is concerned with the structure of subsemigroups of Lie groups, particularly those subsemigroups that are in an appropriate sense "infinitesimally generated," that is, are determined by a subtangent object in the Lie algebra (the Lie wedge of the semigroup). Such semigroups can then be studied, at least to some extent, by applying the methods of linear algebra and convex analysis to the tangent wedge in the Lie algebra. One also has available the vast theory and methodology of Lie groups to aid one in the study of these semigroups. This combination of factors, and others that have not been mentioned, has made possible the development of a rather substantial theory in a relatively short period of time.

In a broader sense, a Lie theory of semigroups refers to a theory of semigroups that arises in a variety of contexts reminiscent of those in which one encounters Lie groups. Thus one has semigroups naturally arising in various considerations in geometry, in representation theory, in mathematical physics, in harmonic analysis, in probability and in other disciplines with close ties to Lie theory. It is natural to include these areas of overlap or application also in a Lie theory of semigroups.

The purpose of this paper is to point out certain connections between this theory and older mathematical work. It would not quite be appropriate to call this older work the roots of the subject, since the modern Lie theory of semigroups has frequently developed rather independently from this older work, and in many cases in ignorance of it. However, such a historical exercise seems worthwhile for the purpose of properly recognizing and crediting the work of earlier researchers, for developing a more comprehensive view of the subject matter, and simply because it is intrinsicly interesting to see the development process of a subject over a period of decades. Practically, it can also suggest future areas of investigation. I have chosen to focus almost exclusively in this paper on older sources. For a more detailed discussion of what a Lie theory of semigroups is all about and the more recent sources and history of such a theory, I refer the reader to the introduction of [9] and the survey article by K. H. Hofmann [12].

There is some hesitation on my part to write an article such as this one.

I feel very keenly my limited knowledge of the literature, and would be most obliged to others who could point out other works that should be referenced in such a survey. I have greatly benefitted from the survey articles by HOFMANN on topological semigroups in general [11] and on the historical development of the concept of a semigroup [13] in the preparation of this one. There is some overlap in the material presented, but the articles have distinctive foci of investigation.

The Nineteenth Century

In this section we consider very briefly developments up to the time of Sophus Lie. Several early papers of N. H. ABEL in the 1820's dealt with finding solutions to functional equations. In one of these, which appeared in the first issue of *Journal für die Reine und Angewandte Mathematik* (Crelle's Journal), ABEL [1] seeks to find all differentiable functions $f(x,y)$ of two real variables such that the function $F(x,y,z) := f\big(x, f(y,z)\big)$ is symmetric (i.e., unchanged under any permutation of the three variables). ABEL asserts that this assumption forces in addition $f(x,y) = f(y,x)$, and so he considers precisely those differentiable functions f for which (in current terminology) $x{*}y := f(x,y)$ defines a commutative semigroup on some domain in the real line. The principal result of the paper is that there is a real-valued function ψ satisfying for all (x,y) in the domain of f,

$$\psi f(x,y) = \psi(x) + \psi(y).$$

It is clear from context that the function ψ is to be injective although ABEL does not state this in his summary theorem.

There are two interesting interpretations or applications of this result from a modern perspective. If the multiplication f is a group multiplication (e.g. usual multiplication on the positive reals), then ψ is an isomorphism with $(\mathbb{R}, +)$, and thus there is (up to isomorphism) only one differentiable abelian group on the real line (indeed ABEL shows that his function ψ is the log function for the case of the reals under multiplication). Alternatively, if one has a subsemigroup on an open interval (for example, the reals between 0 and 1 under multiplication), then one obtains a group embedding into $(\mathbb{R}, +)$. It should be pointed out that there are certain implicit assumptions made by ABEL about the non-vanishing of certain partials, among other things, that makes his proof go through. His theorem breaks down, for example, for the whole real line under multiplication, although this is an example of a commutative associative differentiable operation. This variance with modern standards of rigor and the apparent omnipresence of implicit assumptions are major barriers that one encounters repeatedly in trying to read, understand, and evaluate the validity of these older works. ABEL's result can be recovered if one assumes cancellation, or even weakened versions thereof. See the survey article on ABEL's equations by J. ACZÉL [2] for further discussion and references.

As a passing note, it seems appropriate to mention that the classification of topological semigroups on closed intervals was one of the early undertakings (during the 1950's) in the development of the theory of compact topological semigroups. This theory led in a very natural way to the more general study of compact connected semigroups which are irreducible between an identity and a minimal ideal. The elucidation of the structure of such semigroups is the major contribution of the book *Elements of Compact Semigroups* [16].

A key development of nineteenth-century mathematics was the appearance of group theory. Both GALOIS and ABEL successfully used the permutation groups of the roots of polynomial equations in their treatments of the old problem of finding roots to such equations. (Indeed ABEL's famous paper on the impossiblity of solving general quintic polynomial equations with radicals appeared in the same issue of Crelle's Journal as his previously mentioned work. Another important example where groups appear is CAMILLE JORDAN's treatment of the symmetry groups of crystals by the classification of motion groups of Euclidean space [17]. It was at this time of JORDAN's work on group theory that SOPHUS LIE and FELIX KLEIN visited him in Paris, and JORDAN's influence was a strong stimulus for their interest in group theory. For a detailed treatment of this whole subject of the origins of group theory in the nineteenth century, we refer the reader to the book of I. M. YAGLOM. [41].

Although the work of GALOIS (and later workers) displays deep insight into the nature of permutation groups and significant contributions to the general theory of groups was made during this period, the axiomatic approach to group theory was slow in developing and being recognized. The groups in question were generally permutation groups of finite sets, and in this context of permutations on a finite set, it suffices to require closure with respect to composition (a subsemigroup of a finite group is a group). Thus at this early stage the distinction between a semigroup and a group was quite blurred, and what we today would call a semigroup often went under the name of a group, well into the 20th century (see [13] for an elaboration and documentation of this theme).

Geometry and Felix Klein

Another major line of mathematical investigation in the nineteenth century was in the general area of geometry; indeed this period is sometimes referred to as the "Golden Age of Geometry." In 1872 in a pamphlet distributed on the occasion of his acceptance of a chair at the University of Erlangen, FELIX KLEIN [18] launched what came to be known as his "Erlangen Program". The core of his proposal was that geometric structures should be studied in terms of their automorphism groups:

> *Es ist eine Mannigfaltigkeit und in derselben eine Transformationsgruppe gegeben; man soll die der Mannigfaltigkeit angehörigen Gebilde hinsichtlich solcher Eigenschaften untersuchen, die durch die Transformation der Gruppe nicht geändert werden.*

Interestingly enough, we see even in this proposal the previously mentioned blurring between the notions of a group and a semigroup:

> *Beliebig viele Transformations eines Raumes ergeben zusammengesetzt immer wieder eine Transformation. Hat nun eine gegebene Reihe von Transformationen die Eigenschaft, daß jede Änderung, die aus den ihr zugehörigen durch Zusammensetzung hervorgeht, ihr selbst wieder angehört, so soll die Reihe eine Transformationgruppe genannt werden.*

Although in a later historical footnote KLEIN remarked that the existence of an identity and inverses was implicitly assumed, one could interpret KLEIN's original statement as suggesting the study of geometric structures preserved by semigroups of transformations. We will return to this idea.

Although for SOPHUS LIE the primary focus of his theory of continuous groups was the area of differential equations, the enormously fruitful connections established by H. POINCARÉ, E. CARTAN, and other geometers between the theory of Lie groups and geometry have validated KLEIN's vision and have caused the theory to develop much more in the geometric direction.

Sophus Lie and Continuous Groups

A driving vision of SOPHUS LIE in his development of his theory of continuous groups was the establishing of a theory that accomplished for differential equations what GALOIS had accomplished for the theory of polynomial equations by introducing finite groups. For his program he needed "continuous groups" as opposed to finite or discrete groups. His concept of a continuous group was a smooth function f defined on an open subset of $\mathbb{R}^n \times \mathbb{R}^r \to \mathbb{R}^n$ so that with appropriate restrictions of the domain $f(f(x,a),b)$ was again defined and given by $f(x,c)$ for some $c \in \mathbb{R}^r$. Thus the variables in $\mathbb{R}^r$ essentially parameterize a transformation semigroup acting on $\mathbb{R}^n$, the operation $(a,b) \to c$ corresponding to composition. (LIE also assumes that the transformations parameterized by $\mathbb{R}^r$ have non-vanishing Jacobians, when regarded as functions from $\mathbb{R}^n \to \mathbb{R}^n$.) However it is not a genuine semigroup, since composition need not always be defined. LIE's system is closer to what today would be called a pseudosemigroup of mappings. One could reasonably conjecture that this setup was at least partially motivated from the fact that Lie wanted to consider groups of fractional transformations on $\mathbb{R}^n$ (he calls them Cremona Transformations) which were not defined at the zeroes of the denominator.

LIE specifically excluded incorporating into his axiom system the assumption of an identity and inverses, although he consistently called the objects he studied "groups". At first he thought he could show that their existence was a necessary consequence of his axioms. One wonders how he could have missed simple examples such as the positive reals acting on the positive reals under addition, but a careful reading indicates that his systems were assumed to be "maximally defined." The first counterexample given by F. ENGELS involved the action of the positive reals on the whole real line, but with the positive reals parameterized by e^t, so the the variables ranged over all of $\mathbb{R} \times \mathbb{R}$ (see the footnote to [22]). Later a holomorphic function on the unit disc which did not extend past the boundary was used to parameterize the complex numbers of absolute value less than 1 to obtain a "continuous group" that did not have an identity nor inverses (see pages 164, 165 of Volume I of [23]).

LIE's primary interest in semigroups that were not groups appears to have been the extent to which their existence affected the axiomatic foundations of his theory and a desire to set his theory in the most appropriate general context. By a change of variables and a process of analytic continuation, LIE eventually claimed to be able to embed any of his transformation semigroups into a larger one containing the identity and, at least locally, inverses (see Theorem 26, page 163 of Volume I of [23]), and hence into one that was infinitesimally generated. This embedding theorem seems to have been his principal contribution to the theory of continuous semigroups as opposed to continuous groups, and after this breakthrough, he seems to have been more-or-less content to restrict himself to the situation where an identity and inverses existed. However, it should be clearly noted that LIE did specifically identify the concept of what we today call a semigroup and was apparently the first researcher on record both to isolate and seriously consider such objects.

The global problem of of determining appropriate conditions for embedding a semigroup into a group, both as an algebraic problem and as a combined topological-algebraic problem, is rather complicated, and has a rather extensive literature. One sees immediately that cancellation is a necessary condition, but it turns out not to be sufficient. The best theorem currently available in the topological setting is that a cancellative topological semigroup on a manifold which is algebraically embeddable in some group is topologically and algebraically embeddable in a Lie group. If the multiplication of the semigroup is differentiable, then this embedding preserves that differentiable structure. However one now has available semigroups that one would want to call Lie semigroups that embed only locally, but not globally, into a Lie group. For a survey of the literature and results available on this question in the topological setting, see [20]. In the differentiable setting one always has available local embeddings, but some care must be exercised to state precisely what this means (see Chapter VII.2 of [9]).

Certainly among LIE's greatest contributions were the results that are are referred to as Lie's Fundamental Theorems. The realization that associated with each Lie group is a uniquely determined Lie algebra, and that this in turn locally determines the Lie group was a profound insight that provided suitable foundations for the whole theory. It made possible on a grand scale the derivation of the Lie group structure from that of the Lie algebra, and in the latter setting one had available the machinery of linear algebra.

As a footnote, it is interesting to reflect that the development of the machinery of linear algebra and the notions of an "algebra" or system of "hypercomplex numbers" were practically contemporaneous with LIE, and form another important background source in the nineteenth century mathematics to Lie theory. In some sense his theory arose at practically the earliest moment possible.

LIE's perspective of studying continuous groups in terms of their Lie algebras of "infinitesimal generators" has permeated Lie theory ever since. There is a broader important principle here that other algebraic structures with differentiable operations sometimes can be fruitfully studied by identifying an appropriate set of "infinitesimal generators" with an appropriate structure, and studying the original structures at the infinitesimal level. This view has been championed by HOFMANN in his approach to Lie groups, Lie loops, and Lie semigroups. In each such setting, it is a basic and natural question to seek analogs of Lie's Fundamental Theorems, and this was one of the central concerns in the early current development of a Lie theory of semigroups (see in particular, [14] for a discussion of Lie's Fundamental Theorems and their semigroup formulations). An interesting feature here is that the full generality of Lie's theorems in the semigroup setting holds only in the local setting, as Lie originally proved them in the group setting. (It was only through the later work of E. CARTAN and H. WEYL that they were extended to a global setting.) To what extent there may exist (global) semigroups that one would want to call Lie semigroups associated with every Lie wedge (the appropriate infinitesimal objects in the semigroup setting) remains to this day unsettled.

Hilbert's Fifth Problem

In his celebrated address to the 1900 International Congress of Mathematicians DAVID HILBERT presented his famous list of unsolved problems. The fifth problem was related LIE's theory of continuous groups and asked whether a locally euclidean topological group necessarily admitted a differentiable structure so that multiplication was differentiable. As ACZÉL points out in his article [2], this question was to some extent motivated by ABEL's work. In the lesser known second part of the fifth problem, HILBERT asked if ABEL's functional equations could be solved without necessarily assuming that the solutions were differentiable functions. The topological group component of the fifth problem spanned an enormous amount of research in the next fifty-five years, culminating in a positive solution with the work of MONTGOMERY, ZIPPIN, and GLEASON. Probably of even greater interest than the solution itself was the interesting mathematics that arose along the way, a mathematical phenomenon frequently associated with good problems. In particular, the problem gave strong impetus to an overall theory of (locally compact) topological groups.

HILBERT's problem naturally generalizes to other contexts. In particular, it is natural to ask to what extent the positive solution extends to semigroups, especially since these structures are close to one of ABEL's equations. The positive result, first proved by R. HOUSTON and D. R. BROWN [3], is that cancellative topological semigroups on euclidean manifolds admit an analytic structure (see Chapter VII of [9] and the survey article [20] for this and related results). In some ways such results give added impetus to the development of a Lie theory of semigroups by establishing that in rather general contexts, one is already forced into an analytic setting, and often into considering subsemigroups of groups.

Semigroup Theory at the Turn of the Century

In a variety of ways semigroups made their appearance in the late nineteenth and early twentieth centuries, but consistently within the framework of group theory. Sometimes this resulted from a blurred distinction between groups and semigroups (they both went under the name of "group" at this stage), and sometimes it was because the additional group axioms were simply not needed. A striking example of the latter is an important paper in 1906 of FROBENIUS and SCHUR [6] in which they show that the irreducible representation components of a group represention on a finite dimensional vector space are uniquely determined by the character of the representation, that is, the function which assigns to each element of the group the trace of the linear operator to which it is sent. In the introduction to this paper it is specifically pointed out that in their usage of the word "group" neither the existence of an identity nor of inverses is assumed (and neither do they assume that the representing linear operators are invertible). Interestingly enough, in their proof they use also a semigroup theoretic version of what we today know as Schur's Lemma.

Representation theory has also been an integral part of the modern developments in the Lie theory of semigroups (see, for example, [38]), and continues to be a topic of current research. However, in the modern setting the focus has been representations via operators on a Hilbert space.

It is commonly agreed that the word "semigroup" appears for the first time during this period in a 1904 book by DE SÉGUIER [4], where he uses the word for what we today call cancellative semigroups. But this terminology found acceptance only gradually. At this stage, however, there did begin to emerge what one might refer to as a genuine semigroup theory, distinct from group theory.

I believe it to be a fair assessment to say that the modern algebraic theory of semigroups has been more affected by the multiplicative theory of rings and algebras than it has by group theory (although certainly both are important). However, the historical roots of the theory appear to lie closer to group theory, despite the fact that algebras and rings were studied extensively in the nineteenth century.

Charles Loewner, Semigroups, and the Bieberbach Conjecture

KAREL (Czech) or KARL (German) LÖWNER was a Czech mathematician, who americanized his name to CHARLES LOEWNER after moving to the United States in 1939. He is a key figure in this survey, because in his work semigroups move out from under the shadow of groups and into the foreground (the word "semigroup" appears in six of LOEWNER's papers, and semigroups play an important role in several of his other papers). But the approach that LOEWNER used in his study of semigroups was consistent with that of Lie, namely the study of semigroups via their set of infinitesimal generators. He thus appears to be deserving of being called the father of the Lie theory of semigroups. His works are now most readily available in the volume of his collected works edited by LIPMAN BERS [36].

The most famous of LOEWNER's papers is the last of a cycle of papers on geometric function theory [25], written in 1923. It treats complex power series of the form $z + a_2 z^2 + a_3 z^3 + \dots$ which converge on the open unit disk D and define a schlicht, that is, injective function from D into the complex numbers. LOEWNER's result that $|a_3| \leq 3$ made the Bieberbach conjecture that $|a_n| \leq n$ the famous problem that it was for decades.

In this paper LOEWNER assumes that the power series defines a bounded function and then multiplies the series by a positive constant small enough so that the function defined carries the open unit disc D into itself. He thus considers the set of all injective analytic functions from D into itself which carry 0 to 0 and are defined by a power series having leading z-coefficient a positive real number less than or equal to 1. This set of functions forms a semigroup S under composition, which is clearly far from being a group. LOEWNER then considers continuous curves $t \mapsto f_t$, $t \geq 0$ into S such that f_0 is the identity, $f_t \in S$ for $t \geq 0$, and the curve is sufficiently differentiable so that one can form a continuous vector field

$$X(z) := \lim_{t \to 0^+} \frac{f_t(z) - f_0(z)}{t}, \quad |z| < 1.$$

The vector field X is called an *infinitesimal generator* for the semigroup S. Except for modification that $t \geq 0$, this rather closely parallels LIE's method of defining the infinitesimal generators that made up the members of the Lie algebra, and indeed LIE had also carried out similar infinite-dimensional constructions.

Conversely given a set of vector fields (the infinitesimal generators), one can consider the semigroup they infinitesimally generate. LOEWNER carries out this generation as follows. Let $F\colon D \times [0, t_0] \to \mathbb{C}$ denote a continuous function such that for each fixed t the vector field $z \mapsto F(t, z)$ is one of the given vector fields. One then considers the differential equation

$$\frac{dw}{dt} = F(w, t).$$

A solution of this differential equation is a function $f(z, t)$ satisfying $f(z, 0) = z$ and $(\partial f/\partial t)(z, t) = F(z, t)$. All functions of the form $w_t(z) := f(z, t)$ form a parameterized curve of functions, and this curve is the functional solution starting at the identity of the differential equation. All the functions w_t arising in this way together with their limits in the topology of uniform convergence on compact sets are said to be infinitesimally generated. By concatenating the functions F to form a third differential equation from any two, one sees easily that the set of all such w_t, and hence their closure, forms a semigroup.

LOEWNER shows that the previously defined semigroup S of schlicht functions is infinitesimally generated by its infinitesimal generators. Indeed he shows that it suffices to consider only differential equations of the form

$$\frac{dw}{dt} = -w\frac{1+\chi(t)w}{1-\chi(t)w}$$

where $\chi(t)$ ranges over continuous functions of absolute value 1, and that the solutions of this differential equation are dense in the semigroup S. This is the celebrated Loewner differential equation which became an indispensable tool in the theory of schlicht functions and was again used by DE BRANGES in his proof of the Bieberbach conjecture. By means of this computation the overall problem was reduced to considering those functions satisfying this differential equation.

Noteworthy features about this article of LOEWNER from the perspective of the current survey are the (of necessity) decisively semigroup orientation, the application of Lie theoretic methods, and the fact that the semigroup theory is developed as a tool toward an end. One wonders whether or not there are other close connections between semigroups and differential equations of a similar nature that would be fruitful to pursue and develop.

An interesting historical observation, especially in light of earlier remarks, is that LOEWNER continued to use the word "group" for this semigroup which he studied. So apparently the terminology of "semigroup" had not yet gained acceptance in Berlin, one of the mathematical centers of the day, by 1923. (LOEWNER was located in Berlin during that period of time.)

Higher Order Monotone Functions and Pseudosemigroups

In several of LOEWNER's papers one encounters the notion of a pseudosemigroup, which LOEWNER consistently shortens to semigroup (see [27, 29, 32, 33] and implicitly in [26]). Let M be a manifold and let $\mathcal{S}$ be a set of local homeomorphisms between open connected subsets of M. The set $\mathcal{S}$ is a *pseudosemigroup* if it satisfies (a) through (d).

(a) If $f, g \in \mathcal{S}$, $f\colon O_1 \to O_2$, and $g\colon O_2 \to O_3$, then $g \circ f\colon O_1 \to O_3$ is in $\mathcal{S}$.
(b) The restriction of any $f \in \mathcal{S}$ to a smaller connected open set is again in $\mathcal{S}$.
(c) The identity mapping on any open connected set belongs to $\mathcal{S}$.
(d) If a sequence of mappings f_n in $\mathcal{S}$ are all defined on the same domain and converge uniformly to f on any compact subset of the domain, then $f \in \mathcal{S}$.

Sometimes it is convenient to add a completeness condition of the form

(e) If the domain U is the union $\bigcup U_\alpha$ and a homeomorphism f with domain U when restricted to each U_α belongs to $\mathcal{S}$, then $f \in \mathcal{S}$.

If the inverse of every element of $\mathcal{S}$ is again in $\mathcal{S}$, then $\mathcal{S}$ is called a *pseudogroup*.

If one applies the Cayley transform to the open unit complex disk, then one obtains the open upper half plane. Thus the semigroup of functions that LOEWNER considered in his 1923 work [25] transforms to the semigroup of (schlicht) holomorphic functions carrying the upper half plane into itself. One could also consider those real-valued functions defined on open intervals that arose as analytic continuations of these functions. Since the original set was a semigroup, these extensions form a pseudosemigroup of mappings on the real line. In another masterpiece [26] LOEWNER set out to study this and related pseudosemigroups. These results were refined and elaborated on in [27], [30], [32], and [34].

Let f be a strictly increasing continuous function defined on an open interval I on the real line. Then by the functional calculus, one can define $f(A)$ for any $n \times n$ symmetric linear operator on $\mathbb{R}^n$ (one way of doing this is by picking a basis for which A is diagonal and applying f to the diagonal elements). The function f is said to be *monotone of order* n if $A \leq B$ always yields $f(A) \leq f(B)$, where $A \leq B$ means that $B - A$ is positive semidefinite for the symmetric linear transformations A, B.

LOEWNER showed that the functions monotone of order n formed with increasing n a decreasing family of pseudosemigroups and derived alternate characterizations and various properties of these classes of functions. He showed that their intersection is again a pseudosemigroup on the real line, and that the functions in this pseudosemigroup are precisely those that admit an analytic extension to a Pick function, that is, to a function analytic on the open upper half plane which carries the open upper half-plane into itself (but this function need not be schlicht).

One can easily modify the definitions of infinitesimal generator and infinitesimal generation of the previous section to pseudogroups. LOEWNER showed that for $n > 2$, the pseudosemigroups of functions monotone of order n were not infinitesimally generated, and the same held for their intersection, the semigroup of mappings monotone of every order. Indeed if one took the infinitesimal generators of this latter semigroup, and considered the pseudosemigroup that it generated, one obtained precisely the pseudosemigroup of functions that admitted an extension to a Pick function that was also schlicht. Furthermore, any pseudosemigroup that contained properly the proper projective pseudogroup of all mappings $(ax+b)/(cx+d)$ with positive determinant on the real line must also contain this latter subsemigroup, or its inverse. (There was an additional mild restriction that the pseudosemigroup must contain a three-times differentiable member not in the projective pseudogroup.) LOEWNER's functions monotone of every order turned out to be of importance for electrical engineering and quantum physics (see [40]). For a detailed treatment of these monotone functions, many of LOEWNER's results regarding them, and other developments in their theory, see the book of DONOGHUE [5].

There is another interesting connection here between the work of LOEWNER and that of LIE. One of LIE's important theorems was that the real line only admitted the faithful action (or pseudoaction) of three Lie groups, the real line by translation, the non-abelian two-dimensional Lie group by affine mappings, and the projective group by projective transformations. (A modern formulation of this theorem, proved independently by J. TITS and K. H. HOFMANN in the 1960's, is that a hyperplane subalgebra of a Lie algebra must contain an ideal so that the quotient algebra is the Lie algebra of one of these three groups.) If one considers the Lie algebra of vector fields making up the infinitesimal generators for the projective action, then Lie showed that any Lie algebra of vector fields properly containing it is infinite dimensional. Loewner's theorem sharpens this result by actually characterizing the smallest pseudosemigroups containing the pseudogroup of projective mappings, namely the ones admitting a schlicht analytic extension to the upper half plane, or the inverse of this semigroup.

Loewner's work on monotone functions sparked his interest in finding and studying other interesting classes of pseudosemigroups. In particular, he was interested in characterizing those which were minimal with respect to containing some given pseudogroup of transformations. We call this a Loewner problem of Type I and formalize it as follows:

> PROBLEM I. Let G be a finite-dimensional pseudogroup which admits no extension to a larger finite-dimensional pseudogroup. Characterize those pseudosemigroups which are minimal with respect to properly containing G.

See [33] for his most complete discussion of these matters and problems. Loewner never published proofs of several of his results in that survey paper, and it would be of interest to have these verified and the proofs on record and available.

It seems that it would be a worthwhile project to try to develop a fuller Lie theory of pseudosemigroups in light of what has been learned from the theory of Lie semigroups. Some considerations of this nature may be found in the work of the modern school of geometric control theory, see, for example, the article of Kupka and Jurdjevic [19].

The Lie-Loewner Theory of Semigroups

There are several papers in which Loewner considered finite dimensional semigroups of transformations and in which certain of his results anticipated or overlapped the more recent theory. However, like Lie, Loewner consistently worked in the context of transformation semigroups, and this was perhaps partially the cause for an undue neglect of his work by those who focused on the structure of the semigroup itself.

In 1950 in [27] Loewner formally introduced what is today called a Lie wedge:

> *The investigation of subgroups of a Lie group is essentially facilitated by the introduction of the infinitesimal viewpoint. It is a well known fact that a closed connected subgroup of a Lie group g is completely determined by its infinitesimal elements. It is, therefore, appropriate to ask whether a* subsemigroup *s of g can be connected in a similar way with a system of infinitesimal elements and also how far the subsemigroup is determined by them.*

Loewner then proceeded to define the infinitesimal elements of a closed subsemigroup s of a Lie group as limiting directions in the tangent space at the identity (the Lie algebra) of directions from which the identity e can be approached by members of s. He showed that the set of infinitesimal elements form a closed convex cone (that is, a wedge) in the Lie algebra, that this cone is invariant under the adjoint action of the group of units of the semigroup, and that the semigroup that they infinitesimally generate is contained in the original semigroup. All of this was carried out some thirty years before being independently rediscovered by G. I. OL'SHANSKII [38], I. KUPKA and V. JURJEVIC [19], and HOFMANN and LAWSON [14]. In light of this, it appears that a more appropriate name for these sets of infinitesimal generators of semigroups would have been a Loewner wedge, or a Lie-Loewner wedge, rather than the terminology of a Lie wedge that has been adopted. We remark that the reverse implication that associated with each Lie-Loewner wedge, there is a local semigroup for which the wedge is the set of infinitesimal generators is true, but is much more difficult to establish. This was first successfully done in the 1980's by OL'SHANSKII [38] and by HOFMANN and HILGERT [8].

In the work under consideration [27] LOEWNER applies this machinery to certain finite and infinite dimensional groups arising from composition of power series, and certain distinguished subsemigroups thereof. These semigroups turned out to be closely related to his earlier work on pseudosemigroups of monotone functions of higher order. See [37] for a fuller discussion of these matters.

In [28] LOEWNER applied the ideas of infinitesimal generation to a specific class of semigroups, namely the semigroup of totally positive matrices of order $n \times n$ for each n. (A matrix is *totally positive* if all of its minors or subdeterminants are non-negative.) He computed the Lie-Loewner wedge of infinitesimal generators to consist of all matrices with arbitrary entries on the diagonal, non-negative entries on the two off-diagonals, and 0's elsewhere. He then used a theorem of A. M. WHITNEY to establish the more difficult converse, that the semigroup of totally positive matrices is indeed infinitesimally generated. (In general this is a difficult computation, because connectedness of the semigroup does not suffice to insure infinitesimal generation as it does in the group case.)

LOEWNER's other transformation semigroup examples were geometric in nature, and we turn to these in the next section.

Before closing this section it seems appropriate to mention that E. HILLE also developed certain prototypes of a Lie theory of semigroups during the 1940's and 50's (see the chapter on Lie semigroups in [10]). His idea appears to have been to extend the highly successful theory of one-parameter semigroups of operators to multiparameters. Thus his semigroups were differentiable and were defined on a cone with the apex point of the cone being an identity for the semigroup. The infinitesimal generators of the semigroup were identified with the one-parameter semigroups, and the differentiability and substantial calculations were used to show that the same cone parameterized the infinitesimal generators. Today it is known that the differentiability gives local embeddability into a Lie group, which in turn makes many of these calculations superfluous (see Chapter VII.2 of [9]).

The Extended Erlangen Program

KLEIN had suggested in his *Erlangen Program* studying geometric structures via the group of isometries. LOEWNER carried this program one step further and extended it to semigroups. In [29] LOEWNER introduced the important idea of fixing an open convex domain in a geometric structure and looking at the "compression semigroup" of all isometries in the isometry group that carried the domain into itself and corresponding geometric properties. For example if D is the interior of the unit sphere in projective space, then the set of isometries carrying D onto D gives the Klein model of Bolyai-Lobachevski geometry, but the semigroup of compressions is larger. In a modification of a problem of Type I (mentioned earlier), LOEWNER determines the smallest infinitesimally generated semigroup strictly containing the group of isometries of D.

In [31] LOEWNER points out that the compression semigroup for hyperbolic geometry contains a minimal normal infinitesimally generated subsemigroup, which may be considered as an analog of the translation group for Euclidean geometry. This suggests the following Loewner problem of Type II, which he considers in that paper:

> PROBLEM II. Let V be a finite-dimesional geometry on a manifold, and let G be the group of isometries. Find all systems of vector fields σ on V which form the Lie-Loewner wedge for an infinitesimally generated semigroup of transformations, such that σ is invariant under G, σ is finite-dimensional, and σ is minimal with respect to these properties.

LOEWNER provides solutions to this problem in certain cases. The case of bounded symmetric complex domains was solved by L. ROTHKRANTZ in [39].

In [35] LOEWNER shows how compression semigroups may be used to carry over notions of parallelism to appropriately modified variants when one has to replace the translation group by only a semigroup of mappings.

Compression semigroups have also been important in the current Lie theory of semigroups and are closely related to an important class of semigroups called Ol'shanskii semigroups. Loewner Problems of Type II are also not that distant from the problem of finding and classifying invariant cones in Lie algebras, one of the major achievements of the recent theory. Again LOEWNER's work anticipates future developments, beckons for re-examination in light of these developments, and suggests interesting geometric directions worthy of further investigation.

A. D. Wallace and Topological Semigroups

Contemporary with the work of LOEWNER in the United States was the school pioneered by A. D. WALLACE in topological semigroups, which was particularly active in the 1950's and 1960's. Most of this theory stands in contrast to a Lie theory of semigroups, chiefly because the semigroups considered were nearly always compact, and a compact cancellative semigroup is already a group. However, there were important semigroups that arose as compactifications of what today would be called Lie semigroups, and so there are scattered results that are closely related to a Lie theory of semigroups. These developments and connections with Lie theory have been adequately traced in the survey of HOFMANN [11], and there seems no compelling reason to present that material again on this occasion.

This brings us down to more recent times when certain developments in the areas of topological semigroups, geometric control theory, causality and chronogeometry, geometry, Lie and representation theory, and harmonic analysis all came together to initiate a lively and vigorous program of research and stimulate a rapid development of a Lie theory of semigroups. But an adequate treatment of these topics would require another survey.

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