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Remarks on our paper: "On the exponential function of an invariant Lie semigroup"

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It has come to our attention that in our article [3] we may not have adequately attributed all the ideas on which the main result of the paper is based nor sufficiently documented their history. We would therefore like to supply at some clarifying remarks and include some references which we regret to have omitted in [3].

The concept of the characteristic function of a cone is due to Köcher who used this tool as early as 1957 in [5]. Vinberg elaborated on this idea in [6] and described the formalism which we outlined in the Section on "the characteristic function of a cone and the length functional of a cone" in our article [3] in this journal.

Our attention was first drawn to this subject through a lecture which E. B. Vinberg delivered on November 14, 1991 to the Darmstadt branch of the "Seminar Sophus Lie". He showed in which way the characteristic function is an effective tool in order to construct a "geodesic local exponential function" for a neighborhood of 0 in a pointed closed cone C of a Lie algebra $\mathfrak{g}$ of a Lie group G *onto* a local semigroup $S \subseteq G$ "generated by C". This was quite novel in as much as it has been a well-observed phenomenon in Lie semigroup theory, that the standard exponential function, namely, the restriction $\exp|C: C \to G$, is locally surjective onto a local semigroup generated by C only in very rare instances. Vinberg's proof of the local surjectivity used variational methods (the Euler Lagrange equations). To the best of our knowledge, Vinberg's results presented at Technische Hochschule Darmstadt in November 14, 1991, let alone the details, have not become a matter of published record so far.

While we certainly record with gratitude the inspiration we drew from Vinberg's ideas, without which our papers [1] and [3] would not have been written, the main thrust of our articles [1] and [3] is somewhat different. We address the question when the *global* exponential function $\exp: L(S) \to S$ of a closed subsemigroup S of a connected Lie group G is surjective. We attack this question in the special case that $L(S)$ is an invariant cone in $\mathfrak{g}$ (and therefore S is invariant in G under inner automorphisms).

A classical theorem in Lie group theory states that the exponential function $\exp: \mathfrak{g} \to G$ of a compact connected Lie group is surjective. The surjectivity of the exponential function of a connected Lie group in general is not satisfactorily settled to this day. One of us (Mittenhuber) had observed even before Vinberg taught us about the characteristic function of a cone that a new proof of this classical fact was possible as an application of PONTRYAGIN's Maximum Principle [2]. The point of our papers is that a blending of the ideas of the characteristic function of a cone and the associated distance maximizing curves with PONTRYAGIN's Maximum Principle yields the result that for any invariant Lie subsemigroup S of G every element $s \in S$ for which $S \cap sS^{-1}$ is compact can be reached by a one-parameter semigroup of S. This implies the classical result. In our paper [3] we show that the result is in effect a special case of a more general geometric result on affine connections on manifolds. One spin-off for Lie semigroup theory is that the exponential function of a closed divisible invariant Lie subsemigroup of a Lie group whose Lie algebra is compact is surjective.

The recent monograph [1] by HILGERT and one of the authors discusses orders on manifolds and their axiomatic foundations at considerable depth in Section 4. The historical notes of that Section contain additional informatin on the background of *conal homogeneous spaces*. This background includes literature on hyperbolic differential equations and Lorentzian geometry. The concept of monotone curves in the context of ordered homogeneous spaces is very much a part of the Lie semigroup theory in the last 15 years. The results on extremal curves which emerge in this context have their predecessors in Lorentzian geometry. VINBERG's idea to replace the Lorentzian arc-length by length functionals has given the field new direction, and our papers contribute to showing its effectivity.

References

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