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Sums of Adjoint Orbits

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Abstract. We show that the sum of two adjoint orbits in the Lie algebra of an exponential Lie group coincides with the Campbell-Baker-Hausdorff product of these two orbits.

Introduction

N. Wildberger and others have recently investigated the structure of the hypergroup of the adjoint orbits in relation with the class hypergroup of compact Lie groups. A generalization of the notion of this type of hypergroup to non-compact groups, for instance to nilpotent or exponential Lie groups, leads to the problem of determining a precise relation between the sum of adjoint orbits in the Lie algebra and the product of the corresponding conjugacy classes in the group (see [1], and [4]). In ([3]) Wildberger has shown that for nilpotent Lie groups G the exponential of the sum of two adjoint orbits $\Omega_1 + \Omega_2$ is equal to the product $\exp\Omega_1 \cdot \exp\Omega_2$ in G. In this paper we consider the same problem for exponential groups.

Let us recall that by the definition of exponential Lie groups, the mappings

$$\exp\colon \mathfrak{g} \to G \quad \text{and} \quad \log\colon G \to \mathfrak{g}$$

are diffeomorphisms. We can transfer the group multiplication in G via exp to a group multiplication in the Lie algebra $\mathfrak{g}$ and we shall denote it by the symbol $*$. We obtain the so called Baker-Campbell-Hausdorff multiplication in $\mathfrak{g}$, which is given by

$$U * V = U + V + \frac{1}{2}[U,V] + \frac{1}{12}[U,[U,V]] + \frac{1}{12}[V,[V,U]] + \cdots$$

for small U and V in $\mathfrak{g}$.

Let X and Y be two elements of the Lie algebra $\mathfrak{g}$ of the exponential group G. We denote by ${}^{A}X = \mathrm{Ad}(A)X$ the adjoint action of the element A of G on X, and by

$${}^{G}X = \{{}^{A}X \mid A \in G\}$$

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the adjoint orbit of X. For $h \in G$, let

$$C(h) = \{g \cdot h \cdot g^{-1} \mid g \in G\}$$

be the conjugacy class of h.

We show in this note that $\exp({}^G X + {}^G Y)$ is equal to $C(\exp X) \cdot C(\exp Y)$.

Theorem A. *Let G be an exponential Lie group with Lie algebra $\mathfrak{g}$. For any elements X and Y of $\mathfrak{g}$ we have*

$${}^G X + {}^G Y = {}^G X * {}^G Y.$$

In fact in order to prove this identity, it suffices (see the end of the proof of Theorem A, after Lemma 11) to take two elements X and Y in $\mathfrak{g}$ and to show that there exist C, D, K, L in the subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ generated by X and Y, such that

$$X * Y = {}^C X + {}^D Y, \qquad X + Y = {}^K X * {}^L Y.$$

If we consider these identities on a purely formal level, they are almost trivial. Indeed, if $\mathfrak{h}_\infty$ is the free Lie algebra generated by X and Y, then we can form the formal CBH product $U * V$ as infinite power series in the brackets of X and Y and we obtain in this fashion a group structure on $\mathfrak{h}_\infty$. It is easy to see (for instance [3]) that

$$(0.1) \qquad {}^{\mathfrak{h}_\infty} X * {}^{\mathfrak{h}_\infty} Y = X + Y + [\mathfrak{h}_\infty, \mathfrak{h}_\infty] = {}^{\mathfrak{h}_\infty} X + {}^{\mathfrak{h}_\infty} Y.$$

If $\mathfrak{h}$ is nilpotent then we get from this formal identity that

$$(0.2) \qquad {}^{\mathfrak{h}} X * {}^{\mathfrak{h}} Y = X + Y + [\mathfrak{h}, \mathfrak{h}] = {}^{\mathfrak{h}} X + {}^{\mathfrak{h}} Y.$$

In the exponential non-nilpotent case, (0.2) is no longer true (see the first example in the last section of this paper) and we are forced to use closures.

Theorem B. *Let H be an exponential Lie group with Lie algebra $\mathfrak{h}$. If $\mathfrak{h}$ is generated by two elements X and Y, then*

$$({}^H X + {}^H Y)^- = X + Y + [\mathfrak{h}, \mathfrak{h}] = (C(X) * C(Y))^-.$$

(the symbol '$^-$' here means topological closure in $\mathfrak{h}$).

We see that in order to prove Theorem A we cannot use the result of Theorem B. In fact, Theorem A is much more delicate. Its proof requires a detailed analysis of the structure of a solvable Lie algebra generated by two elements.

The second example in the last section shows that in general solvable Lie groups the exponential of the sum of two adjoint orbits Ω_1 and Ω_2 may be much smaller than the product of $\exp \Omega_1$ with $\exp \Omega_2$.

This example allows us finally to present in Theorem C a new characterisation of solvable exponential groups.

Proof of Theorem A

The proof of Theorem A needs some preparations.

Definition . Let $\mathfrak{h}$ be an exponential Lie algebra generated by two elements X and Y. Let $\mathfrak{b}$ be an ideal of $\mathfrak{h}$. We denote by $\mathcal{S}_{\mathfrak{b}}^{*}$, resp. by $\mathcal{S}_{\mathfrak{b}}^{+}$, the set of all pairs (C, D), resp. (K, L) in $G \times G$, such that

$$(1) \qquad X * Y = {}^{C}X + {}^{D}Y \bmod \mathfrak{b}, \text{ resp. } X + Y = {}^{K}X * {}^{L}Y \bmod \mathfrak{b}.$$

If $\mathfrak{b} = \{0\}$, then we abbreviate $\mathcal{S}_{\beta}^{*}$, resp. $\mathcal{S}_{\beta}^{+}$ to $\mathcal{S}^{*}$, resp. $\mathcal{S}^{+}$.

Remark . If $\mathfrak{c}$ is another ideal of $\mathfrak{g}$ contained in $\mathfrak{b}$, then obviously

$$\mathcal{S}_{\mathfrak{c}}^{*} \subset \mathcal{S}_{\mathfrak{b}}^{*} \text{ resp. } \mathcal{S}_{\mathfrak{c}}^{+} \subset \mathcal{S}_{\mathfrak{b}}^{+}.$$

We need the following well known formula. For any $t \in \mathbb{R}$ define

$$f(t) = (-t)^{-1}(e^{-t} - 1).$$

Let us write for a linear operator Ψ on a finite dimensional vector space $\mathfrak{a}$

$$e(\Psi) = \exp(\Psi) = \sum_{k=0}^{\infty} \frac{1}{k!} \Psi^{k},$$

$$\begin{aligned} f(\Psi) &= \sum_{k=1}^{\infty} \frac{1}{k!} (-\Psi)^{k-1} \\ &= ((-\Psi)^{-1}(e(-\Psi) - 1) \text{ if } \Psi \text{ is invertible}). \end{aligned}$$

With these notations we have

Lemma 1. *Let G be an exponential Lie group with Lie algebra $\mathfrak{g}$, let $\mathfrak{a}$ and $\mathfrak{T}$ be two abelian subalgebras of $\mathfrak{g}$ such that $[\mathfrak{T}, \mathfrak{a}] \subset \mathfrak{a}$. Write $\Psi = \operatorname{ad} T_{|\mathfrak{a}}$ for $T \in \mathfrak{T}$. Then for any $T, T' \in \mathfrak{T}$, $A, A' \in \mathfrak{a}$ we have*

$$(4) \quad (T + A) * (T' + A') = T + T' + f(\Psi + \Psi')^{-1}(e(-\Psi') \cdot f(\Psi)A + f(\Psi')A').$$

In particular,

$$T * f(\Psi)A = T + A = (e(\Psi) \cdot f(\Psi)A) * T.$$

Proof. Let $\mathfrak{c} = \mathfrak{T} + \mathfrak{a}$ and let us realize the exponential group C of $\mathfrak{c}$ as a semi-direct product of $\mathfrak{T}$ with $\mathfrak{a}$, i. e.,

$$C = \mathfrak{T} \times \mathfrak{a} \text{ with multiplication } (T, A) \cdot (T', A') = (T + T', \exp(-\Psi')A + A')$$

for any $T, T' \in \mathfrak{T}$, $A, A' \in \mathfrak{a}$. It is easy to see that the exponential mapping $\exp\colon \mathfrak{c} \to C$ is given by:

$$\exp(T + A) = (T, f(\Psi)A).$$

Indeed, for any α and $\beta \in \mathbb{R}$ we have

$$\begin{aligned}(\alpha T, f(\alpha\Psi)\alpha A) \cdot (\beta T, f(\beta\Psi)\beta A) \\ &= ((\alpha+\beta)T, \exp(-\beta\Psi) \cdot f(\alpha\Psi)\alpha A + f(\beta\Psi)\beta A)) \\ &= ((\alpha+\beta)T, f((\alpha+\beta)\Psi)A).\end{aligned}$$

Hence our mapping exp satisfies the functional equation:

$$\exp(\alpha X) \cdot \exp(\beta X) = \exp(\alpha + \beta)X.$$

Also,

$$\frac{d}{dt}\exp(tX)_{|t=0} = X \quad \text{for any } X \in \mathfrak{c},\ \alpha, \beta \in \mathbb{R}.$$

Hence exp must be the exponential mapping. The inverse mapping log is thus given by:

$$\log(T, A) = T + f(\Psi)^{-1}A,\ T \in \mathfrak{T},\ A \in \mathfrak{a}.$$

We can now compute the CBH product $*$ on C. Indeed

$$\begin{aligned}(T + A) * (T' + A') &= \log(\exp(T + A) \cdot \exp(T' + A')) \\ &= \log((T, f(\Psi)A) \cdot (T', f(\Psi')A')) \\ &= \log((T + T', e(-\Psi') \cdot f(\Psi)A + f(\Psi')A')) \\ &= (T + T') + f(\Psi + \Psi')^{-1} \cdot (e(-\Psi') \cdot f(\Psi)A + f(\Psi')A').\end{aligned}$$

This finishes our proof. ∎

Remark . If $\mathfrak{a}_{\mathbb{C}}$ is the complexification of $\mathfrak{a}$ then we can extend Ψ, $f(\Psi)$ and $e(\Psi)$ $\mathbb{C}$-linearly to $\mathfrak{a}_{\mathbb{C}}$; we shall use the following relations for $Z = X + iY \in \mathfrak{a}_{\mathbb{C}}$:

(1.1) $\quad \Re(\Psi(Z)) = \Psi(X),\ \ \Re(f(\Psi)(Z)) = f(\Psi)(X),\ \ \Re(e(\Psi)(Z)) = e(\Psi)(X).$

The next lemma gives us special minimal ideals in $\mathfrak{h}$ which we shall use in the determination of the sets $\mathcal{S}^*$ and $\mathcal{S}^+$.

Lemma 2. *Let $\mathfrak{h}$ be an exponential Lie algebra which is generated by two elements X and Y. Let $\mathfrak{m}$ be a noncentral ideal in $\mathfrak{h}$. Then $\mathfrak{m}$ contains an ideal $\mathfrak{b}$ of $\mathfrak{h}$ which is one of the following five types.*

(i) $\mathfrak{b} = \mathbb{R}U$ *is one dimensional. There exists a nontrivial homomorphism* $\Psi\colon \mathfrak{h} \to \mathbb{R}$ *such that*

$$[A, U] = \Psi(A)U \text{ for any } A \in \mathfrak{h}$$

and $\Psi(X) \neq 0$ *or* $\Psi(Y) \neq 0$.

(ii) $\mathfrak{b} = \mathbb{R}U_1 + \mathbb{R}U_2$ *is two-dimensional. There exists a complex nontrivial homomorphism* Ψ *of* $\mathfrak{h}$ *such that*

$$[A, U_1 + iU_2] = \Psi(A)(U_1 + iU_2) \text{ for any } A \in \mathfrak{h}$$

and $\Psi(X) \neq 0$ *or* $\Psi(Y) \neq 0$.

(iii) $\mathfrak{b} = \mathbb{R}U + \mathbb{R}Z$ *is two-dimensional and* Z *is contained in the center* $\mathfrak{z}$ *of* $\mathfrak{h}$. *There exists a nontrivial linear functional* $\varphi : \mathfrak{h} \to \mathbb{R}$, *which is a homomorphism on* $[\mathfrak{h}, \mathfrak{h}]$, *and a nontrivial homomorphism* $\Psi : \mathfrak{h} \to \mathbb{R}$ *such that*

$$[A, U] = \Psi(A)U + \varphi(A)Z, \text{ for all } A \in \mathfrak{h}$$

and $\Psi(X) \cdot \Psi(Y) \neq 0$, $\varphi(X) \neq 0$ *or* $\varphi(Y) \neq 0$.

(iv) $\mathfrak{b} = \mathbb{R}U_1 + \mathbb{R}U_2 + \mathbb{R}Z_1 + \mathbb{R}Z_2$ *is three or four dimensional and* Z_1, Z_2 *are contained in the center of* $\mathfrak{h}$. *We have* $[U_1, U_2] = 0$ *and there exist a nontrivial linear functional* $\varphi: \mathfrak{h} \to \mathbb{C}$, *which is a homomorphism on* $[\mathfrak{h}, \mathfrak{h}]$, *and a complex valued homomorphism* $\Psi: \mathfrak{h} \to \mathbb{C}$ *such that*

$$[A, U_1 + iU_2] = \Psi(A)(U_1 + iU_2) + \varphi(A)(Z_1 + iZ_2)$$

and $\Psi(X) \cdot \Psi(Y) \neq 0$, $\varphi(X) \neq 0$ *or* $\varphi(Y) \neq 0$.

(v) *there exists an element* $U \neq 0$ *in* $\mathfrak{m}$ *such that* $(0) \neq [X, U]$, *resp.,* $0 \neq [Y, U]$ *is contained in the center of* $\mathfrak{h}$ *and* $\mathfrak{b} = \mathbb{R}[X, U]$, *resp.* $\mathfrak{b} = \mathbb{R}[Y, U]$.

Proof. Suppose first that there exists a minimal abelian ideal $\mathfrak{b}$ of $\mathfrak{h}$ contained in $\mathfrak{m}$ such that the intersection of $\mathfrak{b}$ with the center $\mathfrak{z}$ of $\mathfrak{h}$ is trivial. Since $\mathfrak{h}$ is solvable, $\mathfrak{b}$ must be of dimension 1 or 2. Furthermore since $\mathfrak{b}$ is not central we must have that $[\mathfrak{h}, \mathfrak{b}] \neq (0)$. This gives us the cases (i) and (ii).

If no such ideal exists then $\mathfrak{z}' = \mathfrak{m} \cap \mathfrak{z} \neq (0)$, since now any minimal ideal of $\mathfrak{h}$ contained in $\mathfrak{m}$ is central. Let us choose a proper minimal ideal $\widetilde{\mathfrak{b}}$ in $\widetilde{\mathfrak{m}} = \mathfrak{h}/\mathfrak{z}'$. If $\widetilde{\mathfrak{b}}$ is central in $\widetilde{\mathfrak{h}}$ then $\widetilde{\mathfrak{b}}$ is necessarily one dimensional since it is minimal, so we are in case (v). If $\widetilde{\mathfrak{b}}$ is one dimensional and not central then we choose U' in $\mathfrak{m}$ such that $\mathbb{R}(U' \bmod \mathfrak{z}') = \widetilde{\mathfrak{b}}$. We have for any $A \in \mathfrak{h}$

$$[A, U'] = \Psi(A)U' + Z_A$$

for some $\Psi(A)$ in $\mathbb{R}$ and some Z_A in $\mathfrak{z}'$. We can assume that the homomorphism Ψ of $\mathfrak{h}$ is not trivial, since otherwise $\widetilde{\mathfrak{b}}$ would be central in $\widetilde{\mathfrak{h}}$. Hence either $\Psi(X) \neq 0$ or $\Psi(Y) \neq 0$. If $\Psi(X) = 0$ and if $Z_X = 0$ then $\mathbb{R}(U' + \Psi(Y)^{-1} \cdot Z_Y)$ is $\mathrm{ad}(X)$ and $\mathrm{ad}(Y)$ invariant and hence is a noncentral ideal of $\mathfrak{h}$ contained in $\mathfrak{m}$, which is impossible. If $\Psi(X) = 0$ but $[X, U'] \neq 0$, then we are in case (v). If $\Psi(X) \neq 0$, we replace U' by $U = U' + \Psi(X)^{-1} Z_X$. Whence $[X, U] = \Psi(X)U$ and $[Y, U] = \Psi(Y)U + Z$ for some Z in $\mathfrak{z}'$. The vector Z is not 0 since then $\mathbb{R}U$ would be a minimal noncentral ideal of $\mathfrak{h}$ contained in $\mathfrak{m}$. Now since Y and X generate $\mathfrak{h}$, we must have that $[\mathfrak{h}, U] \subset \mathbb{R}U + \mathbb{R}Z$. An easy computation shows that φ is a homomorphism on $\ker \Psi \supset [\mathfrak{h}, \mathfrak{h}]$. This is case (iii).

Similarly, if $\widetilde{\mathfrak{b}}$ is two-dimensional then we can find U_1' and U_2' in $\mathfrak{m}$ such that $\widetilde{\mathfrak{b}} = \operatorname{span}(U_1', U_2') \bmod \mathfrak{z}'$ and such that for any A in $\mathfrak{h}$:

$$[A, U_1' + iU_2'] = \Psi(A)(U_1' + iU_2') + Z_A$$

for some $\Psi(A)$ in $\mathbb{C}$ and some Z_A in $(\mathfrak{z}')_{\mathbb{C}}$. The homomorphism Ψ is not trivial since $\widetilde{\mathfrak{b}}$ is two-dimensional. If $\Psi(X) = 0$ then we are either in the case ii) or in the case v). If $\Psi(X) \neq 0$, we replace $U_1' + iU_2'$ by $U_1 + iU_2 = U_1' + iU_2' + \Psi(X)^{-1} Z_X$ and we get for $\Xi = U_1 + iU_2$ the relations

$$[X, \Xi] = \Psi(X)(\Xi), \ [Y, \Xi] = \Psi(Y)(\Xi) + Z$$

for some Z in $(\mathfrak{z}')_{\mathbb{C}}$. The vector Z cannot be 0 since otherwise $\operatorname{span}(U_1, U_2)$ would be a minimal noncentral ideal in $\mathfrak{h}$. Since $[\overline{\Xi}, \Xi] \in \mathbb{C}Z$, necessarily, $0 = [X, [\overline{\Xi}, \Xi]]$. On the other hand, $[X, [\overline{\Xi}, \Xi]] = (\overline{\Psi(X)} + \Psi(X))[\overline{\Xi}, \Xi]$. Since G is exponential, we have $(\overline{\Psi(X)} + \Psi(X)) \neq 0$, hence $[\overline{\Xi}, \Xi] = 0$. This is case (iv).■

Definition . We say that an ideal $\mathfrak{b}$ of $\mathfrak{h}$ is *dangerous* if it has the form (iii) or (iv) in Lemma 2.

Lemma 3. (a) *Let $\mathfrak{h}$ be an exponential Lie algebra. Let U_1 and U_2 be two elements in $\mathfrak{h}$ such that $[U_1, U_2] = 0$ and such that there exists a nontrivial complex valued homomorphism Ψ of $\mathfrak{h}$ with*

$$[A, U_1 + iU_2] = \Psi(A)(U_1 + iU_2)$$

for any A in $\mathfrak{h}$.

Let X and Y be two elements in $\mathfrak{h}$ and suppose that $\Psi(Y) \neq 0$. Let B be an element in $\mathfrak{b} = \operatorname{span}(U_1, U_2)$. Then there exists for any (α_1, α_2) in $\mathbb{R}^2$ an element (β_1, β_2) such that

$$^{(\alpha_1 U_1 + \alpha_2 U_2)}X + {}^{(\beta_1 U_1 + \beta_2 U_2)}Y = X + Y + B.$$

Furthermore there exists for any (α_1, α_2) in $\mathbb{R}^2$ another element (β_1, β_2) such that

$$^{(\alpha_1 U_1 + \alpha_2 U_2)}X * {}^{(\beta_1 U_1 + \beta_2 U_2)}Y = X + Y + B.$$

(b) *Let $\mathfrak{h}$ be an exponential Lie algebra. Let U be an element in $\mathfrak{h}$ such that there exists a nontrivial real valued homomorphism Ψ of $\mathfrak{h}$ such that*

$$[A, U] = \Psi(A)U$$

for any A in $\mathfrak{h}$.

Let X and Y be two elements in $\mathfrak{h}$ and suppose that $\Psi(Y) \neq 0$. Let B be an element in $\mathfrak{b} = \operatorname{span}(U)$. There exists for any α in $\mathbb{R}$ an element β in $\mathbb{R}$ such that

$$^{\alpha U}X + {}^{\beta U}Y = X + Y + B.$$

Furthermore there exists for any α in $\mathbb{R}$ another element β such that

$$ {}^{\alpha U}X * {}^{\beta U}Y = X * Y + B. $$

Proof. (a) We must make some precise computations involving the complexification $\mathfrak{h}_{\mathbb{C}}$ of $\mathfrak{h}$. Any vector $C = \gamma_1 U_1 + \gamma_2 U_2$ of $\mathfrak{b}$ can be written as

$$ C = \Re(\gamma \cdot \Xi), $$

where $\gamma = \gamma_1 - i\gamma_2 \in \mathbb{C}$ and where $\Xi = U_1 + iU_2$. Let us write

$$ B = \Re(\omega \Xi). $$

Now if we set $\alpha = -\alpha_1 + i\alpha_2,\ \beta = -\beta_1 + i\beta_2$, we get

$$ \begin{aligned} &{}^{(\alpha_1 U_1 + \alpha_2 U_2)}X + {}^{(\beta_1 U_1 + \beta_2 U_2)}Y \\ &= X + [\alpha_1 U_1 + \alpha_2 U_2, X] + Y + [\beta_1 U_1 + \beta_2 U_2, Y] \\ &= X + Y + \Re((\Psi(X)\alpha + \Psi(Y)\beta)\Xi). \end{aligned} \tag{3.1} $$

We see now that for every α in $\mathbb{C}$ we find β in $\mathbb{C}$ such that $\Psi(X)\alpha + \Psi(Y)\beta = \omega$, i.e. such that

$$ {}^{(\alpha_1 U_1 + \alpha_2 U_2)}X + {}^{(\beta_1 U_1 + \beta_2 U_2)}Y = X + Y + B. $$

In the same way we treat:

$$ \begin{aligned} &{}^{(\alpha_1 U_1 + \alpha_2 U_2)}X * {}^{(\beta_1 U_1 + \beta_2 U_2)}Y \\ &= (X + [\alpha_1 U_1 + \alpha_2 U_2, X]) * (Y + [\beta_1 U_1 + \beta_2 U_2, Y]) \\ &= (X + \Re(\Psi(X)\alpha\Xi)) * (Y + \Re(\Psi(Y)\beta\Xi)) \\ &= X * (\Re(f(\Psi(X))\Psi(X)\alpha\Xi)) * (\Re(e(\Psi(Y))f(\Psi(Y))\Psi(Y)\beta\Xi) * Y \\ &= X * Y \\ &+ \Re(f(\Psi(X) + \Psi(Y))^{-1}\{(e(-\Psi(Y))f(\Psi(X))\Psi(X)\alpha + f(\Psi(Y))\Psi(Y)\beta)\Xi\}) \end{aligned} $$

(by Lemma 1). Whence if we set

$$ f(\Psi(X) + \Psi(Y))^{-1}\{e(-\Psi(Y))f(\Psi(X))\Psi(X)\alpha + f(\Psi(Y))\Psi(Y)\beta\} = \omega, \tag{3.2} $$

then we get for every (α_1, α_2) in $\mathbb{R}^2$ an element (β_1, β_2) in $\mathbb{R}^2$ such that

$$ {}^{(\alpha_1 U_1 + \alpha_2 U_2)}X * {}^{(\beta_1 U_1 + \beta_2 U_2)}Y = X * Y + B. $$

Part (b) is similar. ∎

Lemma 4. (a) *Let $\mathfrak{h}$ be an exponential Lie algebra. Let U_1 and U_2 be two elements of $\mathfrak{h}$ such that $[U_1, U_2] = 0$ and such that there exist a complex homomorphism Ψ, a complex linear functional φ of $\mathfrak{h}$, and a central vector $Z_1 + iZ_2 = Z \neq 0$ in $\mathfrak{h}_{\mathbb{C}}$, such that*

$$[A, \Xi] = \Psi(A)(\Xi) + \varphi(A)Z$$

for every A in $\mathfrak{h}$, where $\Xi = U_1 + iU_2$.

Let X and Y be two elements in $\mathfrak{h}$ so that $\Psi(X) \cdot \Psi(Y) \neq 0$. Suppose furthermore that

$$\det \begin{vmatrix} \Psi(X) & \Psi(Y) \\ \varphi(X) & \varphi(Y) \end{vmatrix} \neq 0.$$

Let $C = \Re(\gamma Z)$ and $B = \Re(\rho\Xi)$, for some $\gamma, \rho \in \mathbb{C}$. Then there exist (α_1, α_2) and (β_1, β_2) in $\mathbb{R}^2$ such that

$${}^{(\alpha_1 U_1 + \alpha_2 U_2)}X + {}^{(\beta_1 U_1 + \beta_2 U_2)}Y = X + Y + B + C.$$

Furthermore there exist (ω_1, ω_2) and (τ_1, τ_2) in $\mathbb{R}^2$ such that

$${}^{(\omega_1 U_1 + \omega_2 U_2)}X * {}^{(\tau_1 U_1 + \tau_2 U_2)}Y = X * Y + B + C.$$

(b) *Let $\mathfrak{h}$ be an exponential Lie algebra. Let U be an element of $\mathfrak{h}$ such that there exists a real homomorphism Ψ and a real linear functional φ of $\mathfrak{h}$ and a central vector $Z \neq 0$ in $\mathfrak{h}$, with*

$$[A, U] = \Psi(A)U + \varphi(A)Z \quad \textit{for any } A \in \mathfrak{h}.$$

Let X and Y be two elements in $\mathfrak{h}$ so that $\Psi(X) \cdot \Psi(Y) \neq 0$. Suppose furthermore that

$$\det \begin{vmatrix} \Psi(X) & \Psi(Y) \\ \varphi(X) & \varphi(Y) \end{vmatrix} \neq 0.$$

Let $C = cZ$ and $B = rU$ for some $c, r \in \mathbb{R}$.

Then there exist α and β in $\mathbb{R}$ such that

$${}^{\alpha U}X + {}^{\beta U}Y = X + Y + B + C.$$

Furthermore there exists for any ω in $\mathbb{R}$ another element τ such that

$${}^{\omega U}X * {}^{\tau U}Y = X * Y + B + C.$$

Proof. (a) We set

$$\alpha = -\alpha_1 + i\alpha_2, \text{ resp. } \beta = -\beta_1 + i\beta_2,$$

and have

$$\begin{aligned} &{}^{(\alpha_1 U_1 + \alpha_2 U_2)}X + {}^{(\beta_1 U_1 + \beta_2 U_2)}Y \\ &= X + [\alpha_1 U_1 + \alpha_2 U_2, X] + Y + [\beta_1 U_1 + \beta_2 U_2, Y] \\ &= X + \Re([X, \alpha \cdot \Xi]) + Y + \Re([Y, \beta \cdot \Xi]) \\ &= X + Y + \Re((\alpha\Psi(X) + \beta\Psi(Y)) \cdot \Xi) + \Re((\alpha\varphi(X) + \beta\varphi(Y))Z). \end{aligned}$$

Since $\det|\;|\neq 0$, there exists a unique pair (α,β) in $\mathbb{C}^2$ such that

$$\alpha\Psi(X)+\beta\Psi(Y)=\rho \text{ and } \alpha\varphi(X)+\beta\varphi(Y)=\gamma. \tag{4.1}$$

This means of course that

$$X+Y+B+C={}^{(\alpha_1U_1+\alpha_2U_2)}X+{}^{(\beta_1U_1+\beta_2U_2)}Y.$$

Finally, for $\omega=-\omega_1+i\omega_2$ and $\tau=-\tau_1+i\tau_2\in\mathbb{C}$ we have

$$\begin{aligned}E&={}^{(\omega_1U_1+\omega_2U_2)}X*{}^{(\tau_1U_1+\tau_2U_2)}Y\\&=(X+\Re([X,\omega\cdot\Xi])*(Y+\Re([Y,\tau\cdot\Xi])\\&=(X+\Re(\Psi(X)\omega(\Xi+\Psi(X)^{-1}\cdot\varphi(X)Z))*\\&\qquad*(Y+\Re(\Psi(Y)\tau(\Xi+\Psi(Y)^{-1}\cdot\varphi(Y)Z))\\&=X*\Re(\omega f(\Psi(X))\Psi(X)(\Xi+\Psi(X)^{-1}\cdot\varphi(X)Z))\\&\qquad*\Re(\tau\cdot e(\Psi(Y))f(\Psi(Y))\Psi(Y)(\Xi+\Psi(Y)^{-1}\cdot\varphi(Y)Z))*Y\\&=X*\Re((\omega f(\Psi(X))\Psi(X)+\tau e(\Psi(Y))f(\Psi(Y))\Psi(Y))\Xi)*Y\\&\qquad+\Re((\omega f(\Psi(X))\varphi(X)+\tau e(\Psi(Y))f(\Psi(Y))\varphi(Y))Z).\end{aligned}$$

by Lemma 1. Let us set

$$\omega f(\Psi(X))\Psi(X)+\tau e(\Psi Y))f(\Psi(Y))\Psi(Y)=a$$

and

$$\omega f(\Psi(X))\varphi(X)+\tau e(\Psi(Y))f(\Psi(Y))\varphi(Y)=b.$$

Then

$$\begin{aligned}E&=X*\Re(a(\Xi+\Psi(Y)^{-1}\varphi(Y)Z))*Y+\Re((-a\Psi(Y)^{-1}\varphi(Y)+b)Z)\\&=X*Y*\Re(e(-\Psi(Y))a(\Xi+\Psi(Y)^{-1}\varphi(Y)Z))+\Re((-a\Psi(Y)^{-1}\varphi(Y)+b)Z)\\&=X*Y*\Re(e(-\Psi(Y))a)\Xi+\Re((a(-\Psi(Y)^{-1}\varphi(Y)\\&\qquad+e(-\Psi(Y))\Psi(Y)^{-1}\varphi(Y))+b)Z)\\&=X*Y*\Re(e(-\Psi(Y))a(\Xi+\Psi(X*Y)^{-1}\varphi(X*Y)Z))\\&\qquad+\Re((a\{-\Psi(Y)^{-1}\varphi(Y)-e(-\Psi(Y))\Psi(X*Y)^{-1}\varphi(X*Y)\\&\qquad+e(-\Psi(Y))\Psi(Y)^{-1}\varphi(Y)\}+b)Z)\\&=X*Y+\Re((f(\Psi(X*Y))^{-1}e(-\Psi(Y))a)\Xi)\\&\qquad+\Re((a\{f(\Psi(X*Y))^{-1}e(-\Psi(Y))\Psi(X*Y)^{-1}\varphi(X*Y)-\Psi(Y)^{-1}\varphi(Y)\\&\qquad-e(-\Psi(Y))\Psi(X*Y)^{-1}\varphi(X*Y)+e(-\Psi(Y))\Psi(Y)^{-1}\varphi(Y)\}+b)Z).\end{aligned}$$

Since $\det|\;|\neq 0$ we can choose ω and τ such that

$$\omega f(\Psi(X))\Psi(X)+\tau\cdot e(\Psi(Y))f(\Psi(Y))\Psi(Y)=\rho e(\Psi(Y))f(\Psi(X*Y)) \tag{4.2}$$

and

$$\begin{aligned}&\omega f(\Psi(X))\varphi(X)+\tau e(\Psi(Y))f(\Psi(Y))\varphi(Y)\\&\quad=\gamma-\rho e(\Psi(Y))f(\Psi(X*Y))\{(f(\Psi(X*Y))^{-1}-1)\\&\qquad e(-\Psi(Y))\Psi(X*Y)^{-1}\varphi(X*Y)+(e(-\Psi(Y))-1)\Psi(Y)^{-1}\varphi(Y)\},\end{aligned}$$

and we get

$${}^{(\omega_1U_1+\omega_2U_2)}X*{}^{(\tau_1U_1+\tau_2U_2)}Y=X*Y+B+C.$$

The proof of (b) is similar to the proof (a). ■

Lemma 5. (a) *Let* $\mathfrak{h}$ *be an exponential Lie algebra. Let* U_1 *and* U_2 *be two elements in* $\mathfrak{h}$, *such that* $[U_1, U_2] = 0$, *let* Z_1, Z_2 *be two central elements of* $\mathfrak{h}$, *such that*

$$[A, U_1 + iU_2] = \Psi(A)(U_1 + iU_2) + \varphi(A)(Z_1 + iZ_2)$$

for any A *in* $\mathfrak{h}$. *Let* X *and* Y *be two elements in* $\mathfrak{h}$, *such that* $\Psi(X)\cdot\Psi(Y) \neq 0$, $\Psi(X) + \Psi(Y) = 0$, $\varphi(X) = 0$, $\varphi(Y) = 1$.

Let C *be any element in the span of* Z_1, Z_2. *Then there exist* α_1, α_2 *in* $\mathbb{R}$, *such that*

$${}^{\alpha_1 U_1 + \alpha_2 U_2}X + {}^{\alpha_1 U_1 + \alpha_2 U_2}Y = X + Y + C.$$

Furthermore, there exist ω_1, ω_2 *in* $\mathbb{R}$, *such that*

$${}^{\omega_1 U_1 + \omega_2 U_2}X * {}^{\omega_1 U_1 + \omega_2 U_2}Y = X * Y + C.$$

(b) *Let* $\mathfrak{h}$ *be an exponential Lie algebra. Let* U *be an element in* $\mathfrak{h}$, *let* Z *be a central element of* $\mathfrak{h}$, *such that*

$$[A, U_1] = \Psi(A)(U) + \varphi(A)(Z)$$

for any A *in* $\mathfrak{h}$. *Let* X *and* Y *be two elements in* $\mathfrak{h}$, *such that* $\Psi(X)\cdot\Psi(Y) \neq 0$, $\Psi(X) + \Psi(Y) = 0$, $\varphi(X) = 0$, *and* $\varphi(Y) = 1$.

Let C *be any element in the span of* Z. *Then there exists an* α *in* $\mathbb{R}$, *such that*

$${}^{\alpha U}X + {}^{\alpha U}Y = X + Y + C.$$

Furthermore, there exists ω *in* $\mathbb{R}$, *such that*

$${}^{\omega U}X * {}^{\omega U}Y = X * Y + C.$$

Proof. (a) Let us use the computations from the proof of Lemma 4 (a). For any (α_1, α_2) in $\mathbb{R}^2$ we have

$$\begin{aligned}
&{}^{(\alpha_1 U_1 + \alpha_2 U_2)}X + {}^{(\alpha_1 U_1 + \alpha_2 U_2)}Y \\
&= X + \Re([X, \alpha\cdot\Xi]) + Y + \Re([Y, \alpha\cdot\Xi]) \\
&= X + Y + \Re((\alpha\Psi(X) + \alpha\Psi(Y))\cdot\Xi) + \Re((\alpha\varphi(X) + \alpha\varphi(Y))Z) \\
&= X + Y + \Re(\alpha Z).
\end{aligned}$$

Hence it suffices to put $\alpha = \gamma$, where γ is such that $C = \Re(\gamma Z)$. Furthermore

$$\begin{aligned}
&{}^{(\omega_1 U_1 + \omega_2 U_2)}X * {}^{(\omega_1 U_1 + \omega_2 U_2)}Y \\
&= (X + \Re([X, \omega\cdot\Xi])) * (Y + \Re([Y, \omega\cdot\Xi])) \\
&= (X + \Re(\Psi(X)\omega(\Xi + \Psi(X)^{-1}\cdot\varphi(X)Z))) \\
&\qquad * (Y + \Re(\Psi(Y)\omega(\Xi + \Psi(Y)^{-1}\cdot\varphi(Y)Z))) \\
&= X * (\Re(\omega f(\Psi(X))\Psi(X)(\Xi + \Psi(X)^{-1}\cdot\varphi(X)Z))) \\
&\qquad * (\Re(\omega\cdot e(\Psi(Y))f(\Psi(Y))\Psi(Y)(\Xi + \Psi(Y)^{-1}\cdot\varphi(Y)Z))) * Y \\
&= X * (\Re((\omega f(\Psi(X))\Psi(X) + \omega\cdot e(\Psi(Y))f(\Psi(Y))\Psi(Y))\Xi)) * Y \\
&\qquad + \Re((\omega e(\Psi(Y))f(\Psi(Y))\varphi(Y))Z).
\end{aligned}$$

Since

$$\omega f(\Psi(X))\Psi(X) + \omega\cdot e(\Psi(Y))f(\Psi(Y))\Psi(Y) = \omega(e^{-\Psi(X)} - 1 + 1 - e^{\Psi(Y)}) = \omega\cdot 0 = 0,$$

it suffices to put $\omega = (e(\Psi(Y))f(\Psi(Y)))^{-1}\cdot\gamma$, where γ is such that $C = \Re(\gamma Z)$.

The proof of (b) is similar. ∎

The structure of $\mathfrak{h}$

We shall now construct inductively a sequence of ideals

$$\mathfrak{h} \supset [\mathfrak{h},\mathfrak{h}] = \mathfrak{n} \supset \mathfrak{n}_r \supset \mathfrak{n}_{r-1} \cdots \supset \mathfrak{n}_1 \supset \mathfrak{n}_0 = (0)$$

such that $\mathfrak{n}_i/\mathfrak{n}_{i-1} = \mathfrak{b}_i$ is an ideal in $\mathfrak{h}/\mathfrak{n}_i$ of the form i) to v) in Lemma 2 $(i = 1, 2, \cdots, r)$.

We shall use the root decomposition of $\mathfrak{h}$ relative to some regular element $T \in \mathfrak{h}$. Let us recall what a root Ψ of $\mathfrak{h}$ is. We choose any Jordan-Hölder sequence $\mathfrak{h} \supset \mathfrak{h}_1 \supset \cdots \supset \mathfrak{h}_p = \{0\}$ of $\mathfrak{h}$. The $\mathfrak{h}$-modules $\widetilde{\mathfrak{h}}_j = \mathfrak{h}_j/\mathfrak{h}_{j+1}$ are irreducible, for $j = 1$ to $p-1$, hence of dimension 1 or 2. We get in the dimension 1 case a real homomorphism Ψ_j of $\mathfrak{h}$. In the dimension 2 case, we have $(\widetilde{\mathfrak{h}}_j)_{\mathbb{C}} = \mathbb{C}\,\Xi + \mathbb{C}\,\overline{\Xi}$ and $[A,\Xi] = \Psi_j(A)\Xi$, $[A,\overline{\Xi}] = \overline{\Psi_j(A)}\,\overline{\Xi}$, for any $A \in \mathfrak{h}$ and $\Psi_j, \overline{\Psi}_j$ are complex homomorphisms. The homomorphisms Ψ_j's and $\overline{\Psi}_j$'s are called the roots of $\mathfrak{h}$. It is easy to see that the roots do not depend on a given Jordan-Hölder sequence and that for any T in $\mathfrak{h}$, the spectrum of $\operatorname{ad} T$ on $\mathfrak{h}_{\mathbb{C}}$ is given by the numbers $\Psi(T)$, $\Psi =$root of $\mathfrak{h}$.

Let first T be an element of $\mathfrak{h}$ which is in general position relatively to the roots of $\mathfrak{h}$, i. e. for any two distinct roots Ψ and Ψ' of $\mathfrak{h}$ we have $\Psi(T) \neq \Psi'(T)$. We denote by

$$\Psi^{\beta},\ \beta \in \sigma,$$

the corresponding root of $\mathfrak{h}$. Hence every root Ψ of $\mathfrak{h}$ is of the form Ψ^{β}, for some $\beta \in \sigma$. Furthermore, it is not difficult to see that a root Ψ, for which $\Psi(T) = -\beta$ for some $\beta \in \sigma$, is equal to $-\Psi^{\beta}$.

Let $\mathfrak{h}_{\mathbb{C}} = \sum_{\beta\in\sigma} (\mathfrak{h}_{\mathbb{C}})_{\beta}$ be the decomposition of $\mathfrak{h}_{\mathbb{C}}$ into the sum of the nilspaces of $\operatorname{ad} T$, the summation being made over the spectrum σ of $\operatorname{ad} T$. We have

$$[(\mathfrak{h}_{\mathbb{C}})_{\beta}, (\mathfrak{h}_{\mathbb{C}})_{\beta'}] \subset (\mathfrak{h}_{\mathbb{C}})_{\beta+\beta'},\ \text{for any } \beta,\ \beta' \text{ in } \sigma.$$

In particular $(\mathfrak{h}_{\mathbb{C}})_0 = (\mathfrak{h}_0)_{\mathbb{C}}$ is a subalgebra of $\mathfrak{h}_{\mathbb{C}}$ (which is in fact nilpotent) and

$$[(\mathfrak{h}_0)_{\mathbb{C}}, (\mathfrak{h}_{\mathbb{C}})_{\beta}] \subset (\mathfrak{h}_{\mathbb{C}})_{\beta}$$

for any β in σ. Furthermore for any S in $\mathfrak{h}_0$, $\operatorname{ad} S - \Psi^{\beta}(S)$ is nilpotent on $(\mathfrak{h}_{\mathbb{C}})_{\beta}$. Let

$$\mathfrak{h}_0 = (\mathfrak{h}_0)_{\mathbb{C}} \cap \mathfrak{h},\ \mathfrak{h}_{\beta} = ((\mathfrak{h}_{\mathbb{C}})_{\beta} + (\mathfrak{h}_{\mathbb{C}})_{\overline{\beta}})) \cap \mathfrak{h} = ((\mathfrak{h}_{\mathbb{C}})_{\beta} + \overline{(\mathfrak{h}_{\mathbb{C}})_{\beta}}) \cap \mathfrak{h}.$$

Then $\mathfrak{h} = \sum_{\beta\in\sigma} \mathfrak{h}_{\beta}$ and $\mathfrak{h}_{\beta} \subset \mathfrak{n}$ for any $\beta \neq 0$. If $\mathfrak{b}$ is any ideal of $\mathfrak{h}$ then

$$\mathfrak{b} = \mathfrak{b} \cap \mathfrak{h}_0 + \sum_{\beta\in\sigma\setminus 0} (\mathfrak{h}_{\beta}) \cap \mathfrak{b} \stackrel{\text{def}}{=} \mathfrak{b}_0 + \sum_{\beta\in\sigma\setminus 0} \mathfrak{b}_{\beta}$$

and

$$\mathfrak{h}/\mathfrak{b} = (\mathfrak{h}/\mathfrak{b})_0 + \sum_{\beta\in\sigma\backslash 0} (\mathfrak{h}/\mathfrak{b})_\beta = (\mathfrak{h}_0/\mathfrak{b}) + \sum_{\beta\in\sigma\backslash 0} (\mathfrak{h}_\beta/\mathfrak{b}).$$

Furthermore, let Ψ be a root of $\mathfrak{h}$ and let $\beta = \Psi(T)$. Suppose that $-\beta$ is also an eigenvalue of $\mathrm{ad}_{\mathfrak{h}_\mathbb{C}} T$. Let Ψ' be the root of $\mathfrak{h}$ corresponding to $-\beta$. If $(\mathfrak{h}_\mathbb{C})_0 \supset [(\mathfrak{h}_\mathbb{C})_\beta, (\mathfrak{h}_\mathbb{C})_{-\beta}] \neq \{0\}$, then for any S in $\mathfrak{h}_0$, since $\mathrm{ad}\, S - (\Psi(S) + \Psi'(S))$ is nilpotent on $[(\mathfrak{h}_\mathbb{C})_\beta, (\mathfrak{h}_\mathbb{C})_{-\beta}]$, we must have that $\Psi(S) + \Psi'(S) = 0$. On the other hand every root of $\mathfrak{h}$ is trivial on $\sum_{\beta\in\sigma\backslash 0} \mathfrak{h}_\beta$ and so $\Psi' = -\Psi$.

We begin by choosing in $[\mathfrak{n}, \mathfrak{n}]$ an ideal $\mathfrak{b}_1$ as in Lemma 2, if $[\mathfrak{n}, \mathfrak{n}]$ is not central. If $[\mathfrak{n}, \mathfrak{n}]$ is central, but $\mathfrak{n}$ is not, we choose the ideal $\mathfrak{b}_1$ in $\mathfrak{n}$. If $\mathfrak{n}$ is central, but $\mathfrak{h}$ is not abelian, we choose $\mathfrak{b}_1$ in $\mathfrak{h}$. If $\mathfrak{h}$ is abelian, we do nothing. Set $\mathfrak{n}_1 = \mathfrak{b}_1$.

If $\mathfrak{b}_1$ is dangerous, then we have Ξ_1 in $(\mathfrak{h}_\mathbb{C})_\beta$ for some $\beta \neq 0$ in σ and $Z_1 \neq 0$ in $(\mathfrak{h}_0)_\mathbb{C}$ and we obtain the linear functional $\varphi = \varphi_1$ of (iii) in Lemma 2 and the homomorphism Ψ_1, where

$$[A, \Xi_1] = \Psi_1(A)\Xi_1 + \varphi_1(A)Z_1,\ A \in \mathfrak{h},$$

and we let ${}_1\mathfrak{n} = \ker\varphi_1 \cap \mathfrak{n} = \{U \in \mathfrak{n} \mid [U, \Xi_1] = 0\}$. Thus ${}_1\mathfrak{n}$ is an ideal in $\mathfrak{h}$. If $\mathfrak{b}_1$ is of the form (i), (ii) or (v) then we set ${}_1\mathfrak{n} = \mathfrak{n}$. Continuing in this fashion, we find inductively the ideals $\mathfrak{n}_2 \subset \cdots \subset \mathfrak{n}_j$ of $\mathfrak{h}$ (contained in $[\mathfrak{n}, \mathfrak{n}]$ as long as $[\mathfrak{n}, \mathfrak{n}]/\mathfrak{n}_{j-1}$ is not central in $\mathfrak{h}/\mathfrak{n}_{j-1}$), the ideals ${}_1\mathfrak{n} \supset \cdots \supset {}_j\mathfrak{n} \supset [\mathfrak{n}, \mathfrak{n}]$, ${}_j\mathfrak{n} \supset \mathfrak{n}_j$. If ${}_j\mathfrak{n}/\mathfrak{n}_j$ is not central in $\mathfrak{h}/\mathfrak{n}_j$, then we again find an ideal $\mathfrak{b}_{j+1}$ in ${}_j\mathfrak{n}/\mathfrak{n}_j$. In the case where $\mathfrak{b}_{j+1}$is of the form (iii) or (iv) we have Ξ_j in $(\mathfrak{h}_\mathbb{C})_\beta$ for some $\beta_j = \beta \neq 0$ in σ and Z_j in $(\mathfrak{h}_0)_\mathbb{C}$ and we obtain the linear functional $\varphi = \varphi_j$ of iii) and the homomorphism Ψ_j in Lemma 2 where

$$[A, \Xi_j] = \Psi_j(A)\Xi_j + \varphi_j(A)Z_j \text{ mod } (\mathfrak{n}_j)_\mathbb{C},\ A \in \mathfrak{h}, \tag{5.1}$$

and we set ${}_{j+1}\mathfrak{n} = {}_j\mathfrak{n} \cap \ker\varphi_j = \{U \in {}_j\mathfrak{n} \mid [U, \Xi_j] = 0 \text{ mod } (\mathfrak{n}_j)_\mathbb{C}\}$. Hence ${}_{j+1}\mathfrak{n}$ is an ideal in $\mathfrak{h}$. If $\mathfrak{b}_j$ is of the form i), ii), or v) then we set ${}_{j+1}\mathfrak{n} = {}_j\mathfrak{n}$. Let $\mathfrak{n}_{j+1}$ be the set of all elements x in $\mathfrak{h}$, such that x mod $\mathfrak{n}_j \in \mathfrak{b}_{j+1}$. This finishes step j.

We continue this process until we find some r in $\mathbb{N}$, for which ${}_r\mathfrak{n}/\mathfrak{n}_r$ is central in $\mathfrak{h}/\mathfrak{n}_r$. We set

$$\mathfrak{m} = {}_r\mathfrak{n}.$$

We give now a precise description of $\mathfrak{h}/\mathfrak{m}$.

Definition 1. Let J be the set of all the indices j in $\{1, \cdots, r\}$ for which $\mathfrak{b}_j$ is dangerous.

Thus

$$[\mathfrak{n}, \mathfrak{n}] \subset \mathfrak{m} = \mathfrak{n} \cap \{\bigcap_{j\in J} \ker\varphi_j\} \subset \mathfrak{n}$$

and $\mathfrak{m}$ is an ideal of $\mathfrak{h}$. Indeed $\mathfrak{n} \cap \ker\varphi_j = \{u \in \mathfrak{n} \mid [u, \mathfrak{b}_j] = \{0\} \text{ mod } \mathfrak{n}_{j-1}\}$ is an ideal of $\mathfrak{h}$ for any j in J. Furthermore we see from (5.1) that for every j in J, since $[\mathfrak{h}_0, (\mathfrak{h}_\mathbb{C})_\beta] \subset (\mathfrak{h}_\mathbb{C})_\beta$,

$$\varphi_j(\mathfrak{h}_0 \cap \mathfrak{n}) = \{0\}. \tag{5.2}$$

Hence

$$\mathfrak{h}_0 \cap \mathfrak{n} \subset \mathfrak{m}.$$

Lemma 6. *Let* $\widetilde{\mathfrak{h}} = \mathfrak{h}/\mathfrak{m}$. *For any* $\beta \neq 0$ *in the spectrum* $\widetilde{\sigma}$ *of* $\mathrm{ad}\, T$ *on* $\widetilde{\mathfrak{h}}$ *there exists* j *in* J *such that* $\Psi^\beta = -\Psi_j$.

Proof. Suppose that β is not real. Choose a vector θ in $(\mathfrak{h}_\mathbb{C})_\beta$, such that $\theta \notin \mathfrak{m}_\mathbb{C}$ and such that

$$[T, \theta] = \beta\theta \bmod \mathfrak{m}_\mathbb{C} = \Psi^\beta(T)\theta \bmod \mathfrak{m}_\mathbb{C}.$$

Since $\theta \notin \mathfrak{m}_\mathbb{C}$, there exists j in J such that $\varphi_j(\theta) \neq 0$. Choose Ξ_j and $Z_j \neq 0$ in $(\mathfrak{b}_j)_\mathbb{C}$ such that

$$[A, \Xi_j] = \Psi_j(A)\Xi_j + \varphi_j(A) Z_j \bmod (\mathfrak{n}_{j-1})_\mathbb{C}$$

for any A in $\mathfrak{h}$. Hence

$$[[A,\theta],\Xi_j] = [A,[\theta,\Xi_j]] - [\theta,[A,\Xi_j]] = 0 - [\theta, \Psi_j(A)\Xi_j] \bmod \mathfrak{n}_{j-1}.$$

Hence $[[A,\theta]+\Psi_j(A)\theta, \Xi_j] = 0$ in $(\mathfrak{h}/\mathfrak{n}_{j-1})_\mathbb{C}$ and so also $\varphi_j([A,\theta]+\Psi_j(A)\theta) = 0$, i.e.

$$[A,\theta] = -\Psi_j(A)\theta \bmod (\ker\varphi_j)_\mathbb{C} \text{ for all } A \in \mathfrak{h}.$$

Since also

$$[T,\theta] = \beta\theta \bmod \mathfrak{m}_\mathbb{C} = \beta\theta \bmod (\ker\varphi_j)_\mathbb{C}$$

we see that $\Psi_j(T) = -\beta$. Hence $\Psi^\beta = -\Psi_j$ by 5.0. ∎

In Lemma 7 and 8 we give a precise description of the elements X and Y mod $\mathfrak{m}$ and in Lemma 9 we determine the structure of $\mathfrak{h}/\mathfrak{n}_r$.

Lemma 7. *Let* $\widetilde{\mathfrak{h}} = \mathfrak{h}/\mathfrak{m}$. *For any* $\beta \neq 0$ *in the spectrum* $\widetilde{\sigma}$ *of* $\mathrm{ad}\, T$ *on* $\widetilde{\mathfrak{h}}$, $((\widetilde{\mathfrak{h}})_\mathbb{C})_\beta$ *is one dimensional and for* T' *in* $\mathfrak{h}_0$, θ *in* $(\mathfrak{h}_\mathbb{C})_\beta$, *we have* $[T',\theta] = \Psi^\beta(T')\theta \bmod \mathfrak{m}_\mathbb{C}$. *Furthermore* $\widetilde{\mathfrak{h}}_0$ *is one or two-dimensional.*

Proof. Since $[\mathfrak{h}_0, \mathfrak{h}_0] \subset \mathfrak{m}$ by (5.2) we have $[\widetilde{\mathfrak{h}}_0, \widetilde{\mathfrak{h}}_0] = \{0\}$. Furthermore

$$[(\mathfrak{h}_\mathbb{C})_\beta, (\mathfrak{h}_\mathbb{C})_\gamma] \subset [\mathfrak{n},\mathfrak{n}] \subset \mathfrak{m} \text{ for all } \beta,\gamma \neq 0 \text{ in } \sigma$$

implies that

$$[(\widetilde{\mathfrak{h}}_\mathbb{C})_\beta, (\widetilde{\mathfrak{h}}_\mathbb{C})_\gamma] = \{0\}, \text{ for all } \beta,\gamma \neq 0 \text{ in } \sigma.$$

Let us show that for any T' in $\mathfrak{h}_0$, $[T',\theta] - \Psi^\beta(T')\theta \in \mathfrak{m}_\mathbb{C}$. Indeed, we have

$$[T',\theta] = \Psi^\beta(T')\cdot\theta + \theta_1 \text{ for some } \theta_1 \text{ in } ((\mathfrak{h}_\mathbb{C})_\beta)$$

and θ is not a scalar multiple of θ_1, since $\mathrm{ad}(T') - \Psi^\beta(T')$ is nilpotent on $((\mathfrak{h}_\mathbb{C})_\beta)$. The vector θ_1 must be contained in $\mathfrak{m}$. Since otherwise we would have an index j in J, such that $\varphi_j(\theta_1) \neq 0$. Since θ_1 is in $((\mathfrak{h}_\mathbb{C})_\beta)$, $[\theta_1, \Xi_j] \notin (\mathfrak{n}_{j-1})_\mathbb{C}$ implies that $\beta + \Psi_j(T) = 0$ by 5.0 and so Ξ_j must be in $(\mathfrak{h}_\mathbb{C})_{-\beta}$. Hence

$$[[T',\theta],\Xi_j] = [T',[\theta,\Xi_j]] - [\theta,[T',\Xi_j]] = 0 - \Psi_j(T')\varphi_j(\theta)Z_j \bmod (\mathfrak{n}_{j-1})_\mathbb{C}.$$

On the other hand we also have that

$$[[T',\theta],\Xi_j]=[\Psi^\beta(T')\theta+\theta_1,\Xi_j]=(\Psi^\beta(T')\varphi_j(\theta)+\varphi_j(\theta_1))Z_j \bmod (\mathfrak{n}_{j-1})_{\mathbb{C}}.$$

Thus $\varphi_j(\theta_1)=0$, a contradiction and θ_1 must be an element of $\mathfrak{m}_{\mathbb{C}}$.

Suppose that $(\widetilde{\mathfrak{h}}_{\mathbb{C}})_\beta$ is of dimension ≥ 2. Let V_1 and V_2 be two linearly independent vectors in $((\tilde{\mathfrak{h}})_{\mathbb{C}})_\beta$ and let $\mathfrak{B}_\beta \subset ((\tilde{\mathfrak{h}})_{\mathbb{C}})_\beta$ be a supplementary subspace, i.e.:

$$((\tilde{\mathfrak{h}})_{\mathbb{C}})_\beta=\mathbb{C}V_1\oplus\mathbb{C}V_2\oplus\mathfrak{B}_\beta.$$

Let

$$\mathfrak{n}''=\mathfrak{B}_\beta+\sum_{\beta'\neq\beta,\beta'\neq 0}(\widetilde{\mathfrak{h}}_{\mathbb{C}})_{\beta'},\ \mathfrak{h}'=\widetilde{\mathfrak{h}}_{\mathbb{C}}/\mathfrak{n}'',\ \theta_i=V_i \bmod \mathfrak{n}'',\ i=1,2.$$

Since X and Y generate $\mathfrak{h}$, the vectors $X'=X \bmod \mathfrak{n}''$ and $Y'=Y \bmod \mathfrak{n}''$ generate $\mathfrak{h}'$. Furthermore the subspace $\mathfrak{h}'_\beta$ of $\mathfrak{h}'$ is spanned by $\theta_i,\ i=1,2$ and $\mathfrak{h}'={\mathfrak{h}'}_0+{\mathfrak{h}'}_\beta$. We also have that $[\mathfrak{h}'_\beta,\mathfrak{h}'_\beta]$ and $[\widetilde{\mathfrak{h}}_0,\widetilde{\mathfrak{h}}_0]=(0)$. Let us write

$$X'=T_X+\alpha_1\theta_1+\alpha_2\theta_2,\ Y'=T_Y+\beta_1\theta_1+\beta_2\theta_2,$$

where T_X and T_Y are the components of X', resp. of Y', in $(\mathfrak{h}')_0$. WHence

$$[X',Y']=\Psi^\beta(X)(\beta_1\theta_1+\beta_2\theta_2)-\Psi^\beta(Y)(\alpha_1\theta_1+\alpha_2\theta_2).$$

We see that $\mathfrak{h}'_1=\operatorname{span}(X',Y',\Psi^\beta(X)(\beta_1\theta_1+\beta_2\theta_2)-\Psi^\beta(Y)(\alpha_1\theta_1+\alpha_2\theta_2))$ is a subalgebra of $\mathfrak{h}'$ containing X' and Y' and so $\mathfrak{h}'=\mathfrak{h}'_1$. But then $(\mathfrak{h}')_\beta$ is of dimension 1, a contradiction. Obviously, since $\mathfrak{h}_0\cap\mathfrak{n}\subset\mathfrak{m}$, we have $\dim(\widetilde{\mathfrak{h}})_0\leq \dim(\mathfrak{h}/[\mathfrak{h},\mathfrak{h}])\leq 2$. ∎

Let now

$$\sigma'=\widetilde{\sigma}\setminus\{0\}.$$

We have seen in Lemma 7 that for any β in σ', $((\mathfrak{h})_{\mathbb{C}})_\beta \bmod \mathfrak{m}_{\mathbb{C}}$ is of dimension 1 over $\mathbb{C}$. We choose a vector $\theta'_\beta\neq 0$ in $(\mathfrak{h}_{\mathbb{C}})_\beta \bmod \mathfrak{m}_{\mathbb{C}}$ and we write:

$$X=T_X+\sum_{\beta\in\sigma'}X_\beta\theta'_\beta \bmod \mathfrak{m}_{\mathbb{C}},$$

$$Y=T_Y+\sum_{\beta\in\sigma'}Y_\beta\theta'_\beta \bmod \mathfrak{m}_{\mathbb{C}},$$

where T_X, resp. $T_Y\in\mathfrak{h}_0$. We want to determine the X_β's, resp. Y_β's.

Lemma 8. *We can assume that for any β in σ', there exists a unique θ_β in $(\mathfrak{h}_{\mathbb{C}})_\beta$ with the following property: for every j in J we have $\varphi_j(\theta_\beta)=1$ or 0. Furthermore*

$$X=T_X \bmod \mathfrak{m},\ Y=T_Y+\Re(\sum_{\beta\in\sigma'}\theta_\beta) \bmod \mathfrak{m}.$$

For every j in J, there exists a unique β in σ', such that $\varphi_j(\theta_\beta) = 1$, $\varphi_j(\theta_\gamma) = 0$ for all γ in $\sigma', \gamma \neq \beta$, and such that $\Psi_j = -\Psi^\beta$ where Ψ^β is as in Lemma 7.

Proof. Let $R = \Re(\sum_{\beta\in\sigma'} \Psi_\beta(X)^{-1} \cdot X_\beta \theta'_\beta)$ and let $X' = {}^R X$, $Y' = {}^R Y$. Then $X' = T_X \mod \mathfrak{m}$ and $Y' = T_Y + \sum_{\beta\in\sigma'} Y_\beta \theta'_\beta \mod \mathfrak{m}_\mathbb{C}$ for some new coefficients Y_β. We remark that $\mathfrak{h}$ is also generated by X' and Y', since $R \in \mathfrak{h}$ and so $\mathfrak{h} = {}^R\mathfrak{h}$. We shall work from now on with X' and Y' and we shall show that there exist C, D, K, L in $\mathfrak{h}$ such that ${}^C X' + {}^D Y' = X' * Y'$ and ${}^K X' * {}^L Y' = X' + Y'$. But this means that

$${}^{R^{-1}*C*R}X + {}^{R^{-1}*D*R}Y = X * Y \text{ and } {}^{R^{-1}*K*R}X * {}^{R^{-1}*L*R}Y = X + Y.$$

We can thus forget about X and Y and write $X = X'$, $Y = Y'$.

Let now $j \in J$. There exists some γ in σ and some Ξ_j in $(\mathfrak{h}_\mathbb{C})_\gamma \cap \mathfrak{n}_j$ and Z_j in $(\mathfrak{h}_\mathbb{C})_0 \setminus \mathfrak{n}_{j-1}$ such that $(\mathfrak{n}_j)_\mathbb{C} \mod (\mathfrak{n}_{j-1})_\mathbb{C} = \mathbb{C}\Xi_j + \mathbb{C}(\Xi_j)^- + \mathbb{C}Z_j + \mathbb{C}(Z_j)^- \mod (\mathfrak{n}_{j-1})_\mathbb{C}$ and such that

$$[A, \Xi_j] = \Psi_j(A) \cdot \Xi_j + \varphi_j(A) \cdot Z_j \mod (\mathfrak{n}_{j-1})_\mathbb{C}, \text{ for all } A \in \mathfrak{h}.$$

Since $\varphi_j(\mathfrak{n}) \neq 0$, there exists some β in σ' such that $\varphi_j(\theta'_\beta) \neq 0$ which implies that $\gamma = -\beta$, whence $\Psi^{-\beta} = \Psi_j$. Furthermore $[\theta'_\mu, \Xi_j] \subset ((\mathfrak{h}_\mathbb{C})_{\mu-\beta} \cap (\mathfrak{n}_j)_\mathbb{C}) \subset (\mathfrak{n}_{j-1})_\mathbb{C}$ for all $\mu \neq \beta$. Hence $\varphi_j(\theta'_\mu) = 0$ for such a μ. Since $X = T_X \mod \mathfrak{m}$ we have $\varphi_j(X) = 0$ and so $\varphi_j(Y)$ must be $\neq 0$ for every j. On the other hand, for any β in σ', since $\theta'_\beta \notin \mathfrak{m}_\mathbb{C}$, there exists an index j in J, such that $\varphi_j(\theta'_\beta) \neq 0$. By rescaling Z_j, we may even assume that $\varphi_j(Y) = 1$ and so $Y_\beta \cdot \varphi_j(\theta'_\beta) = 1$ for any j in J associated with β. Replacing now θ'_β by $Y_\beta \cdot \theta'_\beta = \theta_\beta$ for all β in σ', we finally get:

$$Y = T_Y + \sum_{\beta\in\sigma'} \theta_\beta \mod \mathfrak{m}_\mathbb{C}.$$

■

(8.1) **Remark.** We have just seen that for any j in J, the restriction of the linear functional φ_j to $\mathfrak{n}$ is of the form $\varphi^{-\beta}$ for some unique β in σ', where $\varphi^{-\beta}(\theta_\beta) = 1$, $\varphi^{-\beta}(\theta_{\beta'}) = 0$ for $\beta' \neq \beta$, $\varphi^{-\beta}([\mathfrak{n}, \mathfrak{n}]) = \varphi^{-\beta}(\mathfrak{m}) = \{0\}$. Furthermore $\Psi^{-\beta} = \Psi_j$.

Since ${}_r\mathfrak{n}/\mathfrak{n}_r$ is central in $\mathfrak{h}/\mathfrak{n}_r$ we see from Lemma 8 that $(\mathfrak{h})_\mathbb{C}/(\mathfrak{n}_r)_\mathbb{C} = (\mathfrak{h}_0)_\mathbb{C} = \sum_{\beta\in\sigma'} \mathbb{C}\theta_\beta \mod (\mathfrak{n}_r)_\mathbb{C}$ and thus we can write

$$X = T_X \mod \mathfrak{n}_r, \quad Y = T_Y + \sum_{\beta\in\sigma'} \theta_\beta \mod (\mathfrak{n}_r)_\mathbb{C} \tag{8.2}$$

for some new elements T_X, T_Y in $\mathfrak{h}_0$. Let us set

$$S = [T_X, T_Y] \in [\mathfrak{h}_0, \mathfrak{h}_0]. \tag{8.3}$$

Lemma 9.

$$\mathfrak{h}_\mathbb{C} = \mathbb{C}T_X + \mathbb{C}T_Y + \sum_{\beta\in\sigma'} \mathbb{C}\theta_\beta + \mathbb{C}S + \sum_{\beta\in\sigma'} \mathbb{C}[\theta_\beta, \theta_{-\beta}] + (\mathfrak{n}_r)_\mathbb{C}.$$

Proof. First we observe that $[\mathfrak{h},\mathfrak{m}]\subset\mathfrak{n}_r$, since by the definition of $\mathfrak{n}_r$, $\mathfrak{m}/\mathfrak{n}_r$ is central in $\mathfrak{h}/\mathfrak{n}_r$. This implies especially that for $\beta\in\sigma'$, $((\mathfrak{h}_{\mathbb{C}})_\beta)\cap\mathfrak{m}$ is already contained in $(\mathfrak{n}_r)_{\mathbb{C}}$, since $\operatorname{ad}T_X$ is injective on $((\mathfrak{h}_{\mathbb{C}})_\beta)$. Especially, $[\theta_\beta,\theta'_\beta]\subset\mathfrak{n}_r$, for $\beta'\neq-\beta$. Since for any β in σ', $[\theta_\beta,\theta_{-\beta}]\subset\mathfrak{n}_{\mathbb{C}}\cap(\mathfrak{h}_0)_{\mathbb{C}}\subset\mathfrak{m}_{\mathbb{C}}$, we thus have that $[\mathfrak{h},[\theta_\beta,\theta_{-\beta}]]\subset(\mathfrak{n}_r)_{\mathbb{C}}$. We have seen in Lemma 7 that for $\beta\in\sigma'$, $T'\in\mathfrak{h}_0$, $[T',\theta_\beta]=\Psi^\beta(T')\theta_\beta \bmod \mathfrak{m}_{\mathbb{C}}$. Since $[T',\theta_\beta]\in(\mathfrak{h}_\beta)_{\mathbb{C}}$, we know now that $[T',\theta_\beta]=\Psi^\beta(T')\theta_\beta \bmod (\mathfrak{n}_r)_{\mathbb{C}}$.

Since by (5.2) $[\mathfrak{h}_0,\mathfrak{h}_0]\subset\mathfrak{m}$, necessarily $[[\mathfrak{h}_0,\mathfrak{h}_0],\mathfrak{h}]\subset(\mathfrak{n}_r)_{\mathbb{C}}$.

Now $\mathfrak{h}'=\mathbb{C}T_X+\mathbb{C}T_Y+\sum_{\beta\in\sigma'}\mathbb{C}\theta_\beta+\mathbb{C}S+\sum_{\beta\in\sigma'}\mathbb{C}[\theta_\beta,\theta_{-\beta}]+(\mathfrak{n}_r)_{\mathbb{C}}$ is a subalgebra of $\mathfrak{h}_{\mathbb{C}}$. This subalgebra evidently contains X and Y. Hence $\mathfrak{h}'=\mathfrak{h}_{\mathbb{C}}$. ∎

(9.1) **Remark.** Lemma 9 tells us that

$$\begin{aligned}(\mathfrak{m})_{\mathbb{C}}&=[\mathfrak{h}_0,\mathfrak{h}_0]_{\mathbb{C}}+\sum_{\beta\in\sigma'}\mathbb{C}[\theta_\beta,\theta_{-\beta}]+(\mathfrak{n}_r)_{\mathbb{C}}\\&=\mathbb{C}S+\sum_{\beta\in\sigma'}\mathbb{C}[\theta_\beta,\theta_{-\beta}]+(\mathfrak{n}_r)_{\mathbb{C}}.\end{aligned}$$

Let now for $\beta\in\sigma'$,

$$Z_\beta=[\theta_\beta,\theta_{-\beta}]\in(\mathfrak{h}_0)_{\mathbb{C}}.$$

Inductively we choose $\beta_1,\beta_2,\cdots,\beta_s$ in σ', such that

$$Z_{\beta_j}\notin\operatorname{span}_{\mathbb{C}}\{Z_{\beta_i},Z_{\overline{\beta}_i}\mid i<j\}+(\mathfrak{n}_r)_{\mathbb{C}}\tag{9.2}$$

and such that

$$\mathfrak{m}_{\mathbb{C}}=\mathbb{C}S+\operatorname{span}_{\mathbb{C}}\{Z_{\beta_i},Z_{\overline{\beta}_i}\mid i=1,\cdots,s\}+(\mathfrak{n}_r)_{\mathbb{C}}.$$

Let

$$\sigma^+=\{\beta_1,\cdots,\beta_s\}\subset\sigma'\subset\sigma.\tag{9.3}$$

In particular, condition (9.2) implies that $\sigma^+\cap\{-(\sigma^+)\}=\emptyset$. Let

$$\sigma^-=-\sigma^+,\ \sigma1'=\sigma'\setminus\{\sigma^-\cup(\overline{\sigma^-})\},\ \sigma0'=\sigma1'\setminus\{\sigma^+\cup\overline{\sigma^+}\}.$$

Let us choose a subset $\sigma0$ in $\sigma0'$ such that every real β in $\sigma0'$ is contained in $\sigma0$ and such that for any nonreal β in $\sigma0'$, $\{\beta,\overline{\beta}\}\cap\sigma0$ contains one element and let

$$\sigma1=\sigma0\,\dot\cup\,\sigma^+.\tag{9.4}$$

Let

$$\begin{aligned}({}_1\mathfrak{m})_{\mathbb{C}}&=(\mathfrak{n}_r)_{\mathbb{C}}+\operatorname{span}_{\mathbb{C}}\{\theta_\beta,\overline{\theta}_\beta\mid\beta\in\sigma^-\}+\operatorname{span}_{\mathbb{C}}\{Z_\beta,\overline{Z}_\beta\mid\beta\in\sigma^+\},\\({}_2\mathfrak{m})_{\mathbb{C}}&=({}_1\mathfrak{m})_{\mathbb{C}}+\sum_{\beta\in\sigma1}\mathbb{C}\theta_\beta.\end{aligned}$$

Hence

$$\begin{aligned}\mathfrak{h}_{\mathbb{C}} &= \mathbb{C}T_X + \mathbb{C}T_Y + \sum_{\beta\in\sigma 1'} \mathbb{C}\theta_\beta + \mathbb{C}S + ({}_1\mathfrak{m})_{\mathbb{C}} \\ &= \mathbb{C}T_X + \mathbb{C}T_Y + \mathbb{C}S + ({}_2\mathfrak{m})_{\mathbb{C}}\end{aligned}$$

and

$$(9.5)\quad \mathfrak{h} = \mathbb{R}T_X + \mathbb{R}T_Y + \sum_{\beta\in\sigma_1} \Re(\mathbb{C}\theta_\beta) + \mathbb{R}S + ({}_1\mathfrak{m}) = \mathbb{R}T_X + \mathbb{R}T_Y + \mathbb{R}S + ({}_2\mathfrak{m}),$$

$$\mathfrak{m}_1 = ({}_1\mathfrak{m})_{\mathbb{C}} \cap \mathfrak{h} = (\mathfrak{n}_r) + \mathrm{span}\{\Re(\mathbb{C}\theta_\beta) \mid \beta \in \sigma^-\} + \mathrm{span}\{\Re(\mathbb{C}Z_\beta) \mid \beta\in\sigma^+\}.$$

We now begin the determination of some elements in $\mathcal{S}^*$ and in $\mathcal{S}^+$.

Lemma 10. *$\mathcal{S}^*{}_{\mathfrak{n}_r}$, resp. $\mathcal{S}^+{}_{\mathfrak{n}_r}$, contains an element (C,D) resp. (K,L) such that*

$$0 \neq \det\begin{vmatrix} \Psi_j(X) & \Psi_j(Y) \\ \varphi_j({}^CX) & \varphi_j({}^DY)\end{vmatrix}$$

resp. such that

$$0 \neq \det\begin{vmatrix} \Psi_j(X) & \Psi_j(Y) \\ \varphi_j({}^KX) & \varphi_j({}^LY)\end{vmatrix}$$

for all $j \in J$.

Proof. The first step is to find a huge subset of $\mathcal{S}^*{}_{{}_2\mathfrak{m}} \supset \mathcal{S}^*{}_{{}_1\mathfrak{m}}$ resp. of $\mathcal{S}^+{}_{{}_2\mathfrak{m}} \supset \mathcal{S}^+{}_{{}_1\mathfrak{m}}$. We want first to determine

$$(C,D) = (c_0T_Y, d_0T_X) \text{ in } \mathcal{S}^*{}_{{}_2\mathfrak{m}}$$

resp.

$$(K,L) = (k_0T_y, l_0T_X) \text{ in } \mathcal{S}^+{}_{{}_2\mathfrak{m}}.$$

If S is an element of ${}_1\mathfrak{m}$, then we can take any c_0, d_0, k_0 and l_0 in $\mathbb{R}$. If $S \notin {}_1\mathfrak{m}$, then given d_0, resp. k_0, in $\mathbb{R}$, we set $c_0 = d_0 - \frac{1}{2}$, resp. $k_0 = \frac{1}{2} + l_0$.

Since ${}_2\mathfrak{m}$ is an ideal, such that $\mathfrak{h}/{}_2\mathfrak{m}$ is nilpotent of step ≤ 2, we have then that

$${}^CX + {}^DY = X + Y + (d_0 - c_0)S = X + Y + \frac{1}{2}S = X * Y \operatorname{mod}_2\mathfrak{m}.$$

Furthermore:

$${}^KX * {}^LY = X + Y + (\frac{1}{2} + l_0 - k_0)S = X + Y \operatorname{mod}_2\mathfrak{m}.$$

We shall from now on fix c_0, d_0, k_0 and l_0 and we look for a large number of elements in $\mathcal{S}^*{}_{{}_1\mathfrak{m}}$ resp. $\mathcal{S}^+{}_{{}_1\mathfrak{m}}$ of the form

$$(C,D) = \big((\Re(\sum_{\beta\in\sigma 1} c_\beta\theta_\beta)) * c_0T_Y, (\Re(\sum_{\beta\in\sigma 1} d_\beta\theta_\beta)) * d_0T_X\big)$$

resp.

$$(K,L) = \big(\Re(\sum_{\beta\in\sigma 1} k_\beta\theta)) * k_0 T_Y, (\Re(\sum_{\beta\in\sigma 1} l_\beta\theta)) * l_0 T_X\big).$$

Since

$$X = T_X \bmod {}_1\mathfrak{m},\ Y = T_Y + \sum_{\beta\in\sigma 1}\theta_\beta \bmod ({}_1\mathfrak{m})_{\mathbb{C}},$$

and since

$$X * Y = T_X + T_Y + \rho_0 S + \sum_{\beta\in\sigma 1}\rho_\beta\theta_\beta \bmod ({}_1\mathfrak{m})_{\mathbb{C}},$$

for some $\rho_0, \rho_\beta \in \mathbb{C},\ \beta\in\sigma 1$, we obtain the equations

$$(10.*)\quad \begin{aligned} &{}^C X + {}^D Y \\ &= T_X - \Re(\sum_{\beta\in\sigma 1} c_\beta\Psi^\beta(X)\theta_\beta) + T_Y + \Re(\sum_{\beta\in\sigma 1}(e^{d_0\Psi^\beta(X)} - \Psi^\beta(Y)d_\beta)\theta_\beta) \\ &= T_X + T_Y + \Re(\sum_{\beta\in\sigma 1}\rho_\beta\theta_\beta) \\ &= X * Y \bmod (\mathbb{R}S + {}_1\mathfrak{m}), \end{aligned}$$

whence for any β in $\sigma 1$:

$$(*_\beta)\qquad c_\beta\Psi^\beta(X) + (-e^{d_0\Psi^\beta(X)} + \Psi^\beta(Y)d_\beta) = -(X*Y)_\beta = -\rho_\beta.$$

We see that for any c_β in $\mathbb{R}$ we can find a unique d_β such that $(*_\beta)$ is satisfied. In the same way

$$(10.+)\quad \begin{aligned} &{}^K X * {}^L Y \\ &= (T_X - \Re(\sum_{\beta\in\sigma 1} k_\beta\Psi^\beta(X)\theta_\beta)) \\ &* (T_Y + \Re(\sum_{\beta\in\sigma 1}(e^{l_0\Psi^\beta(X)} - \Psi^\beta(Y)l_\beta)\theta_\beta) \\ &= X + Y \bmod (\mathbb{R}S + {}_1\mathfrak{m}) \end{aligned}$$

gives us for β in $\sigma 1$ the equation

$$(+_\beta)\quad \begin{aligned} &e(-\Psi^\beta(Y))f(\Psi^\beta(X))\Psi^\beta(X)(-k_\beta) + f(\Psi^\beta(Y))(\Psi^\beta(Y)(-l_\beta) + e^{l_o\Psi^\beta(X)}) \\ &= f(\Psi^\beta(X) + \Psi^\beta(Y))(X+Y)_\beta. \end{aligned}$$

(see Lemma 1). Again for any k_β in $\mathbb{C}$ we find a unique l_β such that $(+_\beta)$ is fulfilled.

In other words for any $C = (\Re(\sum_{\beta\in\sigma 1} c_\beta\theta)) * c_0 T_Y$, we find a unique $D = (\Re(\sum_{\beta\in\sigma 1} d_\beta\theta_\beta)) * d_0 T_X)$ such that $(C,D)\in \mathcal{S}^*{}_{{}_1\mathfrak{m}}$, and the numbers d_β depend linearly on c_β, resp. for any

$$K = (\Re(\sum_{\beta\in\sigma 1} k_\beta\theta)) * k_0 T_Y$$

there exists a unique $L = (\Re(\sum_{\beta\in\sigma 1} l_\beta\theta_\beta)) * l_0 T_X$ so that $(K,L) \in \mathcal{S}^+{}_{1\mathfrak{m}}$, and the numbers l_β depend linearly on k_β.

We now proceed with to investigate $\mathcal{S}^*{}_{\mathfrak{n}_r}$. As before we write $\sigma^+ = \{\beta_1, \cdots, \beta_s\}$, and now

$$\theta_i = \theta_{\beta_i},\ \theta_{-i} = \theta_{-\beta_i},\ Z_i = Z_{\beta_i},\ c_i = c_{\beta_i},\ d_i = d_{\beta_i},$$
$$\Psi^{\beta_i} = \Psi^i,\ \varphi^{\beta_i} = \varphi^i,\ \Psi^{-\beta_i} = \Psi^{-i},\ \varphi^{-\beta_i} = \varphi^{-i},$$

and so on, $i = 1$ to s.

We recall that $\varphi^i(\theta_{-i}) = 1,\ \varphi^i(\theta_\gamma) = 0$ for $\gamma \neq -\beta_i,\ \varphi^{-i}(\theta_i) = 1,\ \varphi^{-i}(\theta_\gamma) = 0$ for $\gamma \neq \beta_i$. We shall use the following coordinates in the group $H = \exp\mathfrak{h}$. Every element g in $(\exp\mathfrak{n}) \cdot \exp\mathbb{R}T_Y$ can be written by (9.5) in a unique way as a product:

$$g = g_r * (\prod_{i=1}^s \Re(c_{-i}\theta_{-i}) * (\sum_{\beta\in\sigma 1} \mathrm{re}(c_\beta\theta_\beta)) * c_0 T_Y,$$

where $g_r \in N_r = \exp\mathfrak{n}_r$ and where the c_{-i} and c_β are complex numbers. We will sometimes write

$$g_{\theta_{-i}} = c_{-i},\ \text{resp. } g_{\theta_\beta} = c_\beta.$$

Our goal is now to construct rational functions $c_{-i}, d_{-i},\ i = 1$ to s and $d_\beta,\ \beta \in \sigma 1$, defined on Zariski open subsets in the variables $c = \{c_\beta, \beta \in \sigma 1\}$, such that the pairs (C, D) with

$$C = \prod_{i=1}^s \Re(c_{-i}(c)\theta_{-i}) * (\sum_{\beta\in\sigma 1} \Re(c_\beta\theta_\beta)) * c_0 T_Y,$$
$$D = \prod_{i=1}^s \Re(d_{-i}(c)\theta_{-i}) * (\sum_{\beta\in\sigma 1} \Re(d_\beta(c)\theta_\beta)) * d_0 T_X)$$

are in $\mathcal{S}^*{}_{\mathfrak{n}_r}$ for any $c = (c_\beta)_{\beta\in\sigma 1}$ in the common domain of the functions c_{-i}, d_β, d_{-i} and that furthermore for any j in J, whenever $\Psi_j(X) + \Psi_j(Y) \neq 0$, we have

$$0 \neq \det \begin{vmatrix} \Psi_j(X) & \Psi_j(Y) \\ \varphi_j({}^C X) & \varphi_j({}^D Y) \end{vmatrix}.$$

A similar result will of course be shown for $\mathcal{S}^+{}_{\mathfrak{n}_r}$.

Let for $i = 1$ to s,

$$(\mathfrak{m}_i)_\mathbb{C} = (\mathfrak{n}_r)_\mathbb{C} + \sum_{k=1}^i (\mathbb{C}\theta_{-k} + \mathbb{C}(\theta_{-k})^- + \mathbb{C}Z_k + \mathbb{C}(Z_k)^-),\quad (\mathfrak{m}_0)_\mathbb{C} = (\mathfrak{m})_\mathbb{C}.$$

In particular, $(\mathfrak{m}_s)_\mathbb{C} = ({}_1\mathfrak{m})_\mathbb{C}$.

The subspace $\mathfrak{m}_i$ is an ideal of $\mathfrak{h}$ and $\mathfrak{m}_i/\mathfrak{m}_{i-1} \cong \mathbb{C}\theta_{-i} + \mathbb{C}(\theta_{-i})^- + \mathbb{C}Z_i + \mathbb{C}(Z_i)^-$ is dangerous. We determine now by induction on i the elements in $\mathcal{S}^*{}_{\mathfrak{m}_i}$, resp. $\mathcal{S}^+{}_{\mathfrak{m}_i}$, for $i = s$ to 0.

We have for any i in $\{1,\cdots,s\}$

$$[A,\theta_{-i}] = \Psi^{-i}(A)\cdot\theta_{-i} + \varphi^{-i}(A)Z_i \bmod \mathfrak{m}_{i-1},\ A\in\mathfrak{h}.$$

Taking (C,D) in ${\mathcal{S}^*}_{\mathfrak{m}_i}$ we try to find

$$(C',D') = \big(\Re(c_{-i}\theta_{-i}) * C, \Re(d_{-i}\theta_{-i}) * D\big) \text{ in } {\mathcal{S}^*}_{\mathfrak{m}_{i-1}},$$

using the formulas of Lemma 3, 4 and 5.

If $\Psi^i(X)+\Psi^i(Y)=0$, we proceed in the following way. we have ${}^CX + {}^DY = X*Y + B + Z'' \bmod \mathfrak{m}_{i-1}$ for some $B=\Re(\rho\theta_{-i})$ and $Z''=\Re(\gamma Z_i)$. By Lemma 3 we can find a unique d'_{-i}, which depends linearly on ρ, such that

$${}^CX + {}^{d'_{-i}\theta_{-i}*D}Y = X*Y + Z' \bmod (\mathfrak{m}_{i-1})$$

for some Z' in $\mathrm{span}_{\mathbb{C}}(Z_i,\overline{Z}_i)\cap\mathfrak{h}$. Then we can apply Lemma 5 in order to determine c_{-i} and d_{-i}. By Lemma 5, we can find α in $\mathbb{C}$ so that

$${}^{\Re(\alpha\theta_{-i})}X * {}^{\Re(\alpha\theta_{-i})}Y = X*Y - Z' \bmod \mathfrak{m}_{i-1}.$$

Whence

$$\begin{aligned}
&{}^{\Re(\alpha\theta_{-i})C}X + {}^{\Re(\alpha\theta_{-i})d'_{-i}\theta_{-i}*D}Y\\
&\qquad = {}^{\Re(\alpha\theta_{-i})}({}^CX + {}^{d'_{-i}\theta_{-i}*D}Y)\\
&\qquad = {}^{\Re(\alpha\theta_{-i})}(X*Y+Z' \bmod (\mathfrak{m}_{i-1}))\\
&\qquad = {}^{\Re(\alpha\theta_{-i})}X * {}^{\Re(\alpha\theta_{-i})}Y + Z' \bmod (\mathfrak{m}_{i-1})\\
&\qquad = X*Y \bmod (\mathfrak{m}_{i-1}).
\end{aligned}$$

Let us set $c_{-i}=\alpha,\ d_{-i}=\alpha+d'_{-i}$; c_{-i} and d_{-i} are rational functions of c_i and $c_\beta,\ \beta\in\sigma 0$. If $\Psi^i(X)+\Psi^i(Y)\neq 0$ consider the equations (given in 4.1)

$$(*{-i})\qquad \begin{aligned} c_{-i}\Psi^{-i}(X) + d_{-i}\Psi^{-i}(Y) &= (X*Y)_{-i} = \rho_i\\ c_{-i}\varphi^{-i}({}^CX) + d_{-i}\varphi^{-i}({}^DY) &= ({}^CX + {}^DY)_{Z_i} = \gamma_i, \end{aligned}$$

where γ_i depends on C and D. Since

$$\begin{aligned}
C &= \Re(c_{-(i+1)}\theta_{-(i+1)}) * \cdots * \Re(c_{-s}\theta_{-s}) * \Re(\sum_{\beta\in\sigma 1} c_\beta\theta_\beta) * c_0T_Y \text{ resp.}\\
D &= \Re(d_{-(i+1)}\theta_{-(i+1)}) * \cdots * \Re(d_{-s}\theta_{-s}) * \Re(\sum_{\beta\in\sigma 1} d_\beta\theta_\beta) * d_0T_X,
\end{aligned}$$

we see that γ_i is rational in $c_\beta,\ \beta\in\sigma 0$. In this way we obtain that the numbers c_{-i} and d_{-i} are rational functions in the c_β's, $\beta\in\sigma 1$ for fixed d_0.

The condition

$$\det\begin{vmatrix} \Psi^{-i}(X) & \Psi^{-i}(Y)\\ \varphi^{-i}({}^CX) & \varphi^{-i}({}^DY)\end{vmatrix} \neq 0$$

forces us to reject all the pairs (C, D) in $\mathcal{S}^*{}_{\mathfrak{m}_i}$ for which the corresponding determinant $= 0$. Since by (10.$*$), resp. (10.+), we have for any β in σ' that

$$ {}^{C}X = X - \Re(c_\beta \Psi_\beta(X)\theta_\beta) \bmod \ker \varphi^{-\beta}, \tag{10.1} $$

resp.

$$ {}^{D}Y = Y + \Re(e^{d_0 \cdot \Psi_\beta(X)} - d_\beta \Psi_\beta(Y)\theta_\beta) \bmod \ker \varphi^{-\beta} $$

we get $\varphi^{-i}({}^{C}X) = -c_i \Psi^i(X)$, $\varphi^{-i}({}^{D}Y) = e^{d_0 \cdot \Psi^i(X)} - d_i \Psi^i(Y)$ and so

$$ 0 = \det \begin{vmatrix} \Psi^{-i}(X) & \Psi^{-i}(Y) \\ \varphi^{-i}({}^{C}X) & \varphi^{-i}({}^{D}Y) \end{vmatrix} $$

$$ = \Psi^{-i}(X)(e^{d_0 \cdot \Psi^i(X)} - d_i \Psi^i(Y)) - \Psi^{-i}(Y)(-c_i \Psi^i(X)), $$

whence the condition $\det = 0$ is equivalent to

$$ c_i - d_i = -\Psi^i(Y)^{-1} e^{d_0 \cdot \Psi^i(X)}, \tag{10.2} $$

since $\Psi^{-i} + \Psi^i = 0$.

Together with the condition $(*\beta)$ we obtain for fixed d_0 a linear system in the variables c_i and d_i:

$$ \begin{aligned} c_i - d_i &= -\Psi^i(Y)^{-1} e^{d_0 \cdot \Psi^i(X)} \\ c_i \Psi^i(X) + d_i \Psi^i(Y) &= e^{d_0 \Psi^i(X)} - (X * Y)_i \end{aligned} \tag{$*i$} $$

with matrix

$$ \begin{pmatrix} 1 & -1 \\ \Psi^i(X) & \Psi^i(Y) \end{pmatrix} $$

whose determinant is $\Psi^i(X) + \Psi^i(Y)$.

Since $\Psi^i(X) + \Psi^i(Y) \neq 0$, there is a unique pair (c_i, d_i) which must be excluded, i.e. we must take out of $\mathcal{S}^*{}_{\mathfrak{m}_{i+1}}$ every pair (C, D) for which the coordinates c_i, resp. d_i satisfy $(*i)$. Finally when we arrive at $i = 1$ we have found elements C, D with $(C, D) \in \mathcal{S}^*{}_{\mathfrak{n}_r}$, and we have just seen that we can write C, D in the form

$$ \begin{aligned} C &= \Re(c_{-1}\theta_{-1}) * \cdots * \Re(c_{-s}\theta_{-s}) * \Re(\sum_{\beta \in \sigma 1} c_\beta \theta_\beta) * c_0 T_Y, \\ D &= \Re(d_{-1}\theta_{-1}) * \cdots * \Re(d_{-s}\theta_{-s}) * \mathrm{re}(\sum_{\beta \in \sigma 1} d_\beta \theta_\beta) * d_0 T_Y), \end{aligned} $$

where d_β, c_{-i} and d_{-i} are rational functions in the c_β's, $\beta \in \sigma 1$ and where the $c = (c_\beta)_{\beta \in \sigma 1}$ varies in a Zariski open set.

When we try to go from $\mathcal{S}^*{}_{\mathfrak{n}_r}$ to $\mathcal{S}^*$ we shall encounter the dangerous ideals $\mathfrak{b}_j$ with $j \in J$. If the corresponding root Ψ_j belongs to a root $-\beta$ with β in $\sigma 1$, i.e. if $\Psi_j = \Psi^{-\beta} = -\Psi^\beta$, $\varphi_j = \varphi^{-\beta}$ for some β in $\sigma 1$, and if

$\Psi_j(X) + \Psi_J(Y) \neq 0$, then we must again exclude all the solutions (C, D) in $\mathcal{S}^*{}_{\mathfrak{n}_r}$, for which

$$0 = \det \begin{vmatrix} \Psi_j(X) & \Psi_j(Y) \\ \varphi_j({}^C X) & \varphi_j({}^D Y) \end{vmatrix}.$$

But then the coordinates c_β and d_β are the unique solution of the system

$$(*\beta) \qquad \begin{aligned} c_\beta - d_\beta &= -\Psi^\beta(Y)^{-1} e^{d_0 \cdot \Psi^\beta(X)} \\ c_\beta \Psi^\beta(X) + d_\beta \Psi^\beta(Y) &= e^{d_0 \Psi^\beta(X)} - (X * Y)_\beta \end{aligned}$$

which gives us a Zariski closed subset of the β's, $\beta \in \sigma 1$, which we must throw away. The case where Ψ_j belongs to a root β_i in σ^+ is much more delicate. We recall that this means that $\Psi_j = \Psi^i$, $\varphi_j = \varphi^i$ and so

$$[A, \Xi_j] = \Psi^i(A)\Xi_j + \varphi^i(A) Z_j \bmod (\mathfrak{n}_{j-1})_{\mathbb{C}}, \; A \in \mathfrak{h}.$$

Now the coordinate c_{-i} of C had been obtained as solution of the system of equations:

$$(* - i) \qquad \begin{aligned} c_{-i}\Psi^{-i}(X) + d_{-i}\Psi^{-i}(Y) &= (X * Y)_{-i} \\ c_{-i}\varphi^{-i}({}^C X) + d_{-i}\varphi^{-i}({}^D Y) &= ({}^C X + {}^D Y)_{Z_i} = \gamma_i, \end{aligned}$$

where γ_i depends on the c_β's, $\beta \in \sigma 0$. The condition

$$0 = \det \begin{vmatrix} \Psi_j(X) & \Psi_j(Y) \\ \varphi_j({}^C X) & \varphi_j({}^D Y) \end{vmatrix} = \Psi_j(X)\varphi_j({}^D Y) - \Psi_j(Y)\varphi_j({}^C X)$$
$$\Leftrightarrow c_{-i} - d_{-i} = \Psi^i(Y)^{-1} e^{-d_0 \cdot \Psi^i(X)},$$

(see 10.1) imposes another constraint on the solution, which, we recall, is a rational function of the c_β, $\beta \in \sigma 1$.

Hence, if we determine c_{-i} and d_{-i} by the two equations $(* - i)$ we get

$$c_{-i} = \{\Psi^{-i}(X)\varphi^{-i}({}^D Y) - \Psi^{-i}(Y)\varphi^{-i}({}^C X)\}^{-1}\{(X*Y)_{-i}\varphi^{-i}({}^D Y) - \gamma_i \Psi^{-i}(Y)\}$$

$$d_{-i} = \{\Psi^{-i}(X)\varphi^{-i}({}^D Y) - \Psi^{-i}(Y)\varphi^{-i}({}^C X)\}^{-1}\{-(X*Y)_{-i}\varphi^{-i}({}^C X) + \gamma_i \Psi^{-i}(X)\}.$$

The condition $c_{-i} - d_{-i} = \Psi^i(Y)^{-1} e^{-d_0 \cdot \Psi^{-i}(X)}$ imposes another relation on c_i, namely:

$$\begin{aligned} &\Psi^i(Y)^{-1} e^{-d_0 \cdot \Psi^i(X)} \{\Psi^{-i}(X)\varphi^{-i}({}^D Y) - \Psi^{-i}(Y)\varphi^{-i}({}^C X)\} \\ &= (X*Y)_{-i}\varphi^{-i}({}^D Y) - \gamma_i \Psi^{-i}(Y) + (X*Y)_{-i}(\varphi^{-i}({}^C X)) - \gamma_i \Psi^{-i}(X). \end{aligned}$$

Since $\varphi^{-i}({}^C X + {}^D Y) = \varphi^{-i}(X * Y) = \text{const}$, we obtain a nontrivial relation between c_i and the other variables c_β, $\beta \in \sigma 0$:

$$\begin{aligned} &\{\Psi^i(Y)^{-1} e^{-d_0 \cdot \Psi^i(X)} (\Psi^{-i}(X) + \Psi^{-i}(Y))\}\varphi^{-i}({}^C X) \\ &\quad + \text{ rational function in } \{c_\beta, \beta \in \sigma 0\} = 0. \end{aligned}$$

Hence, using an appropriate induction hypothesis for $j = r$ to 1, we can conclude that there exist rational functions $c_{-i},\ i = 1$ to $s,\ d_\beta,\ \beta \in \sigma 1$, defined on Zariski open subsets in the variables $c_\beta,\ \beta \in \sigma 1$, such that the pairs

$$\begin{aligned}&(C, D) =\\ &(\prod_{i=1}^{s} \Re(c_{-i}\theta_{-i}) * (\sum_{\beta\in\sigma 1} \Re(c_\beta\theta_\beta)) * c_0 T_Y,\\ &\prod_{i=1}^{s} \Re(d_{-i}\theta_{-i}) * (\sum_{\beta\in\sigma 1} \Re(d_\beta\theta_\beta)) * d_0 T_X)\end{aligned}$$

are in ${\mathcal{S}^*}_{\mathfrak{n}_r}$ for any $(c_\beta)_{\beta\in\sigma 1}$ in the common domain of the functions c_{-i}, d_β, d_{-i} and that furthermore for any j in J, whenever $\Psi_j(X) + \Psi_j(Y) \neq 0$, we have

$$0 \neq \det \begin{vmatrix} \Psi_j(X) & \Psi_j(Y) \\ \varphi_j({}^C X) & \varphi_j({}^D Y) \end{vmatrix}.$$

We consider now ${\mathcal{S}^+}_{\mathfrak{n}_r}$. First we determine by induction on i the elements in ${\mathcal{S}^+}_{\mathfrak{m}_i}$. We have written

$$[A, \theta_{-i}] = \Psi^{-i}(A) \cdot \theta_{-i} + \varphi^{-i}(A) Z_i \bmod (\mathfrak{m}_{i-1})_{\mathbb{C}},\ A \in \mathfrak{h}.$$

Taking now (K, L) in ${\mathcal{S}^+}_{\mathfrak{m}_i}$, we try to find

$$(K' = \Re(k_{-i}\theta_{-i}) * K, L' = \Re(l_{-i}\theta_{-i}) * L) \text{ in } {\mathcal{S}^+}_{\mathfrak{m}_{i-1}}$$

using the formulas of Lemma 3, 4 and 5.

If $\Psi^i(X) + \Psi^i(Y) = 0$, we proceed in the following way. we have ${}^K X * {}^L Y = X + Y + B + Z'' \bmod \mathfrak{m}_{i-1}$ for some $B = \Re(\rho\theta_{-i})$ and $Z'' = \Re(\gamma Z_i)$. By Lemma 3 we can find a unique l'_{-i}, which depends linearly on ρ, such that

$${}^K X * {}^{l'_{-i}\theta_{-i} * L} Y = (X + Y) + Z' \bmod (\mathfrak{m}_{i-1})$$

for some Z' in $\operatorname{span}_{\mathbb{C}}(Z_i, \overline{Z}_i) \cap \mathfrak{h}$. Then we can apply Lemma 5 in order to determine k_{-i} and l_{-i}. We can find α in $\mathbb{C}$ so that

$${}^{\Re(\alpha\theta_{-i})} X + {}^{\Re(\alpha\theta_{-i})} Y = X + Y - Z' \bmod \mathfrak{m}_{i-1}.$$

Whence

$$\begin{aligned}&{}^{\Re(\alpha\theta_{-i}) K} X * {}^{\Re(\alpha\theta_{-i}) l'_{-i}\theta_{-i} * L} Y\\ &= {}^{\Re(\alpha\theta_{-i})}({}^K X * {}^{l'_{-i}\theta_{-i} * L} Y)\\ &= {}^{\Re(\alpha\theta_{-i})}(X + Y + Z' \bmod (\mathfrak{m}_{i-1}))\\ &= {}^{\Re(\alpha\theta_{-i})} X + {}^{\Re(\alpha\theta_{-i})} Y + Z' \bmod (\mathfrak{m}_{i-1})\\ &= X + Y \bmod (\mathfrak{m}_{i-1}).\end{aligned}$$

Let us set $k_{-i} = \alpha$, $l_{-i} = \alpha + l'_{-i}$, k_{-i} and l_{-i} are rational functions in k_β, $\beta \in \sigma 0$, and k_i. If $\Psi^i(X) + \Psi^i(Y) \neq 0$, consider the equations (4.2) for $X = {}^K X$, $Y = {}^L Y$, $\Psi = \Psi^{-i}$, $\varphi = \varphi^{-i}$. We get

$$(+-i)\qquad \begin{aligned} &k_{-i} f(\Psi^{-i}(X))\Psi^{-i}(X) + l_{-i} e(\Psi^{-i}(Y)) \cdot f(\Psi^{-i}(Y))\Psi^{-i}(Y) = \rho'_i \\ &(X+Y)_{Z_i} = k_{-i} f(\Psi^{-i}(X))\varphi^{-i}({}^K X) \\ &+ l_{-i} e(\Psi^{-i}(Y)) f(\Psi^{-i}(Y))\varphi^{-i}({}^L Y) \\ &= \gamma_i - \cdots = \gamma'_i, \end{aligned}$$

where γ_i depends rationally on k_β, $\beta \in \sigma 0$ and where ρ'_i is a constant.

In this way we see that the numbers k_{-i} and l_{-i} are rational functions in the k_β, $\beta \in \sigma 0$ and in k_i. The condition

$$\det \begin{vmatrix} \Psi^{-i}(X) & \Psi^{-i}(Y) \\ \varphi^{-i}({}^K X) & \varphi^{-i}({}^L Y) \end{vmatrix} \neq 0$$

forces us to reject all the pairs (K, L) in $\mathcal{S}^+{}_{\mathfrak{m}_i}$ for which the corresponding determinant $= 0$. Since

$$(10.3)\qquad {}^K X = X - \Re(k_i \Psi^i(X)\theta_i) \bmod \ker \varphi^{-i},$$

resp.

$${}^L Y = T_Y + \Re(e^{l_o \cdot \Psi^i(X)} - l_i \Psi^i(Y)\theta_i) \bmod \ker \varphi^{-i},$$

(see 10.1) we get $\varphi^{-i}({}^K X) = -k_i \Psi^i(X)$, $\varphi^{-i}({}^L Y) = e^{l_0 \cdot \Psi^i(X)} - l_i \Psi^i(Y)$ and so

$$0 = \det \begin{vmatrix} \Psi^{-i}(X) & \Psi^{-i}(Y) \\ \varphi^{-i}({}^K X) & \varphi^{-i}({}^L Y) \end{vmatrix}$$

is equivalent to

$$k_i - l_i = -\Psi^i(Y)^{-1} e^{l_0 \cdot \Psi^i(X)},$$

since $\Psi^{-i} + \Psi^i = 0$.

Together with the condition $({}^K X * {}^L Y)_{\theta_i} = (X+Y)_{\theta_i}$ we obtain a linear system in the variables k_i and l_i:

$$\begin{aligned} k_i - l_i &= -\Psi^i(Y)^{-1} e^{l_0 \cdot \Psi^i(X)} f(\Psi^i(X) + \Psi^i(Y))^{-1} \\ &\quad \{e(-\Psi^i(Y)) f(\Psi^i(X))(-\Psi^i(X)k_i) + f(\Psi^i(Y))(e^{l_0 \Psi^i(X)} - \Psi^i(Y)l_i)\} \\ &= (X+Y)_i. \end{aligned}$$

Replacing now k_i by $l_i - \Psi^i(Y)^{-1} e^{l_0 \Psi^i(X)}$ in the equation above, we get

$$\begin{aligned} &f(\Psi^i(X) + \Psi^i(Y))(X+Y)_i \\ &= (-1 + e(-\Psi^i(X) - \Psi^i(Y)))l_i + \cdots \text{ independent of } l_i \text{ and } k_i. \end{aligned}$$

Hence we find a unique l_i and k_i satisfying these equations. We must throw away all the pairs (K, L) for which the coordinates k_i and k_i satisfy these equations.

In this way we determine the elements (K, L) in $\mathcal{S}^+{}_{\mathfrak{n}_r}$ and we have just seen that we can write them in the form:

$$(K,L) = \big(\Re(k_{-1}\theta_{-1}) * \cdots * \Re(k_{-s}\theta_{-s}) * \Re(\sum_{\beta\in\sigma 1} k_\beta\theta_\beta) * k_o T_Y,$$
$$\Re(l_{-1}\theta_{-1}) * \cdots * \Re(l_{-s}\theta_{-s}) * \Re(\sum_{\beta\in\sigma 1} l_\beta\theta_\beta) * l_0 T_Y\big)$$

where the l_β's, k_{-i}'s and l_{-i}'s are rational functions in the variables k_β's, $\beta \in \sigma 1$ defined on a Zariski open subset.

When we go from $\mathcal{S}^+{}_{\mathfrak{n}_r}$ to $\mathcal{S}^+$ we shall have to deal again with the dangerous ideals $\mathfrak{b}_j$ where $j \in J$. If the correspondig root Ψ_j belongs to a root $-\beta$ contained in $-\sigma 1$, i.e. $\Psi_j = \Psi^{-\beta}$ and $\varphi_j = \varphi^{-\beta}$ for some β in $\sigma 1$ and if $\Psi_j(X) + \Psi_j(Y) \neq 0$, we must again exclude all the solutions (K, L) in $\mathcal{S}^+{}_{\mathfrak{n}_r}$, for which

$$0 = \det \begin{vmatrix} \Psi_j(X) & \Psi_j(Y) \\ \varphi_i({}^K X) & \varphi_j({}^L Y) \end{vmatrix}.$$

But then the coordinates $k_\beta = (K)_{\theta_\beta}$ and $l_\beta = (L)_{\theta_\beta}$ are given as the unique solution of the linear system of rank 2

$$k_\beta - l_\beta = -\Psi_\beta(Y)^{-1} e^{l_0 \cdot \Psi^\beta(X)}$$

and

$$(+_\beta) \quad \begin{aligned} &e(-\Psi^\beta(Y)) f(\Psi^\beta(X))\Psi^\beta(X)(-k_\beta) + f(\Psi^\beta(Y))(\Psi^\beta(Y)(-l_\beta) + e^{l_0\Psi^\beta(X)}) \\ &= f(\Psi^\beta(X) + \Psi^\beta(Y))(X+Y)_\beta. \end{aligned}$$

Hence it suffices to take out all the (K, L) which satisfy these two equations.

The case where Ψ_j belongs to a root β_i in σ^+ is much more delicate. We recall that this means that

$$[A, \Xi_j] = \Psi^i(A)\Xi_j + \varphi^i(A) Z_j, \; A \in \mathfrak{h}.$$

Indeed the coordinate k_{-i} of K had been obtained as solution of the equation:

$$(+-_i) \quad \begin{aligned} &k_{-i} f(\Psi^{-i}(X))\Psi^{-i}(X) + l_{-i} e(\Psi^{-i}(Y)) \cdot f(\Psi^{-i}(Y))\Psi^{-i}(Y) \\ &= \rho_i \cdot \exp(\Psi^{-i}(Y)) f(\Psi^{-i}(X*Y)) \\ &k_{-i} f(\Psi^{-i}(X))\varphi^{-i}({}^K X) + l_{-i} e(\Psi^{-i}(Y)) f(\Psi^{-i}(Y))\varphi^{-i}({}^L Y) = \gamma_i'. \end{aligned}$$

The condition

$$0 = \det \begin{vmatrix} \Psi_j(X) & \Psi_j(Y) \\ \varphi_j({}^K X) & \varphi_j({}^L Y) \end{vmatrix} = \Psi_j(X)\varphi_j({}^K Y) - \Psi_j(Y)\varphi_j({}^L X)$$
$$\Leftrightarrow k_{-i} - l_{-i} = \Psi^i(Y)^{-1} e^{l_0 \cdot \Psi^{-i}(X)},$$

imposes another constraint on the solution, which, we recall, is a rational function of the $k_\beta,\ \beta \in \sigma 1$. This gives us three equations relating $k_i = (K)_{\theta_i},\ l_i = (L)_{\theta_i}$ with k_{-i} and l_{-i}. The first two equations tell us that

$$\begin{aligned} &f(\Psi^{-i}(X))\{\Psi^{-i}(X)\varphi_{-i}({}^L Y) - \Psi^{-i}(Y)\varphi^{-i}({}^K X)\}k_{-i} \\ &\qquad = \rho_i'\varphi^{-i}({}^L Y) - \gamma_i' \cdot \Psi^{-i}(Y) \\ &\qquad f(\Psi^{-i}(Y))e(\Psi^{-i}(Y))\{\Psi^{-i}(X)\varphi_{-i}({}^L Y) - \Psi^{-i}(Y)\varphi^{-i}({}^K X)\}l_{-i} \\ &\qquad = \Psi^{-i}(X)\gamma_i' - \rho_i'\varphi^{-i}({}^K X) \end{aligned}$$

where $\gamma_i' = \gamma_i - \rho_i' \cdot \{(\exp(-\Psi^{-i}(Y)) - 1)\Psi^{-i}(Y)^{-1} \cdot \varphi^{-i}({}^L Y) + \cdots\}$ and where

$$\rho_i' = \rho_i \cdot \exp(\Psi^{-i}(Y))f(\Psi^{-i}(X * Y)),$$

as in (4.2). If we introduce these values into the last of the three equations and if we use the identity

$$\varphi_{-i}({}^K X * {}^L Y) = e(-\Psi^i(Y))f(\Psi^i(X))\varphi_{-i}({}^K X) + f(\Psi^i(Y))\varphi_{-i}({}^L X) \tag{10.4}$$

we get, since $\gamma_i' = \gamma_i - \cdots = -\rho_i'(\exp(-\Psi^{-i}(Y)) - 1)\Psi^{-i}(Y)^{-1} \cdot \varphi^{-i}({}^L Y) +$ a function in the variables $k_\beta,\ \beta \in \sigma 0$:

$$\begin{aligned} &\Psi^i(Y)^{-1}e^{l_0 \cdot \Psi^{-i}(X)} \cdot \{\Psi^{-i}(X)\varphi^{-i}({}^L Y) - \Psi^{-i}(Y)\varphi^{-i}({}^K X)\} \\ &= f(\Psi^{-i}(X))^{-1} \cdot (\rho_i'\varphi^{-i}({}^L Y) - \gamma_i'\Psi^{-i}(Y)) \\ &- (f(\Psi^{-i}(Y))e(\Psi^{-i}(Y))^{-1}(\Psi^{-i}(X)\gamma_i' - \rho_i'\varphi^{-i}({}^K X)) \\ &= \rho_i' f(\Psi^{-i}(X))^{-1}\varphi^{-i}({}^L Y)(1 + (\exp(-\Psi^{-i}(Y)) - 1)\Psi^{-i}(Y)^{-1}\Psi^{-i}(Y)) \\ &- \rho_i' f(-\Psi^{-i}(Y))^{-1}\{(-\Psi^{-i}(X)(\exp(-\Psi^{-i}(Y)) - 1)\Psi^{-i}(Y)^{-1}\varphi^{-i}({}^L Y)) \\ &- \varphi^{-i}({}^K X)\} + \text{ a function in the other variables} \\ &= \rho_i' f(\Psi^{-i}(X))^{-1} f(\Psi^i(Y))^{-1}\{\varphi^{-i}({}^L Y)((f(\Psi^i(Y))(1 + (\exp(\Psi^i(Y)) - 1) \\ &+ f(\Psi^{-i}(X))(-\Psi^{-i}(X))(\exp(\Psi^i(Y)) - 1)\Psi^{-i}(Y)^{-1}) \\ &+ f(\Psi^{-i}(X)) \cdot \varphi^{-i}({}^K X)\} + \text{ a function in the other variables} \\ &= \rho_i' f(\Psi^{-i}(X))^{-1} f(\Psi^i(Y))^{-1}. \end{aligned}$$

$$\begin{aligned} &\{\varphi^{-i}({}^L Y)(f(\Psi^i(Y))\exp(\Psi^i(Y)) + (\exp(\Psi^i(X)) - 1)f(\Psi^{-i}(Y))) \\ &+ \varphi^{-i}({}^K X)f(\Psi^{-i}(X))\} \\ &+ \text{ a function in the other variables} \\ &= \rho_i' f(\Psi^{-i}(X))^{-1} f(\Psi^i(Y))^{-1}. \end{aligned}$$

$$\begin{aligned} &\{\varphi^{-i}({}^L Y)f(\Psi^{-i}(Y))e(\Psi^i(X)) + f(\Psi^{-i}(X))\varphi^{-i}({}^K X)\} \\ &+ \text{ a function in the other variables} \\ &= \rho_i'(f(\Psi^{-i}(X))f(\Psi^i(Y)))^{-1} \cdot (e^{\Psi^i(Y) + \Psi^i(X)}). \end{aligned}$$

$$\{e(-\Psi^i(Y))f(\Psi^i(X))\varphi_{-i}({}^KX)+f(\Psi^i(Y))\varphi_{-i}({}^LY)\}$$
$$+ \text{ a function in the other variables}$$
$$= \rho_i'(f(\Psi^{-i}(X))f(\Psi^i(Y)))^{-1}\cdot(e^{\Psi^i(Y)+\Psi^i(X)})\cdot\varphi^{-i}({}^KX*{}^LY)$$
$$+ \text{ a function in the other variables}$$
$$= \text{ a function in the other variables },$$

by (10.4) and since $\varphi^{-i}({}^KX*{}^LY)=\varphi^{-i}(X+Y)+\text{constant}$.

On the other hand , using again (10.4)

$$\Psi^i(Y)^{-1}e^{l_0\cdot\Psi^{-i}(X)}\cdot\{\Psi^{-i}(X)\varphi^{-i}({}^LY)-\Psi^{-i}(Y)\varphi^{-i}({}^KX)\}$$
$$=\cdots=\Psi^i(Y)^{-1}e^{l_0\cdot\Psi^{-i}(X)}f(\Psi^{-i}(Y))^{-1}\{e^{-\Psi^i(X)}-e^{\Psi^i(Y)}\}\varphi^{-i}({}^KX)$$
$$+ \text{ a function in the other variables}$$

Hence we get an identity of the form:

$$\varphi^{-i}({}^KX)+\text{ a rational function of the other variables}=0.$$

This gives us a nontrivial rational condition on the k_β's, $\beta\in\sigma0$ and k_i. Hence, using an appropriate induction hypothesis for $j=r$ to 1, we can conclude that for any $l_0\in\mathbb{R}$, there exist rational functions $l_i,\ i=1$ to s, $l_\beta,\ \beta\in\sigma1,\ k_-i,\ i=1$ to s in the variables $k_\beta\in\mathbb{R},\ \beta\in\sigma1$, defined on Zariski open subsets, such that the pairs

$$(K,L)=$$
$$(\prod_{i=1}^{s}\Re(k_{-i}\theta_{-i})*(\sum_{\beta\in\sigma1}\Re(k_\beta\theta_\beta))*k_0T_Y,\prod_{i=1}^{s}\Re(l_{-i}\theta_{-i})*(\sum_{\beta\in\sigma1}\Re(l_\beta\theta_\beta))*l_0T_X)$$

are in ${\mathcal{S}^+}_{\mathfrak{n}_r}$ for any $(k_\beta)_{\beta\in\sigma1}$ in the common domain of the functions k_{-i},l_β,l_{-i} and such the determinants are $\neq 0$. ∎

Lemma 11. *Let $\mathfrak{h}$ be a subalgebra generated by two elements X and Y of the exponential Lie algebra $\mathfrak{g}$. Then $\mathcal{S}^*$ and $\mathcal{S}^+$ are not empty.*

Proof. Let us take any element (C,D) in ${\mathcal{S}^*}_{\mathfrak{n}_r}$ with the property of Lemma 10. It is now easy to see that for some appropriate elements N,M in $\mathfrak{n}_r$, we have $(N*C,M*D)\in\mathcal{S}^*$. We proceed by a backwards induction on $j=r$ to 1. Having found (C_j,D_j) in $\mathcal{S}^*_{\mathfrak{n}_|}$, such that $(C_j \bmod \mathfrak{n}_r, D_j \bmod \mathfrak{n}_r)=(C \bmod \mathfrak{n}_r, D \bmod \mathfrak{n}_r)$ we look for elements N_{j-1},M_{j-1} in $\mathfrak{h}_{j-1}$, such that $(N_{j-1}*C_j,M_{j-1}*D_j)\in{\mathcal{S}^*}_{\mathfrak{n}_{j-1}}$. It suffices to consider the corresponding Lemma 3, 4 or 5. Since for any N in $\mathfrak{m}$ and U in $\mathfrak{h}$ we have $\varphi_j(N*U)=\varphi_j(U)$ we see that in the dangerous cases

$$\det\begin{vmatrix}\Psi_j(X) & \Psi_j(Y)\\ \varphi_j({}^{C_{j-1}}X) & \varphi_j({}^{D_{j-1}}Y)\end{vmatrix}=\det\begin{vmatrix}\Psi_j(X) & \Psi_j(Y)\\ \varphi_j({}^{C}X) & \varphi_j({}^{D}Y)\end{vmatrix}\neq 0$$

and so we can solve the given equations. This shows us that $\mathcal{S}^*\neq\emptyset$. We proceed in the same way to show that $\mathcal{S}^+$ is not empty. ∎

End of the proof of Theorem A: Let $A = {}^{C}X + {}^{D}Y$ be an element of ${}^{G}X + {}^{G}Y$. Let us write ${}^{C}X = X'$ and ${}^{D}Y = Y'$. Let $\mathfrak{h}'$ be the subalgebra generated by X' and Y'. By Lemma 11, there exist K' and L' in $\mathfrak{h}'$, such that

$$X' + Y' = ({}^{K'}X' * {}^{L'}Y').$$

Hence $A = ({}^{K'*C}X) * ({}^{L'*D}Y) \in ({}^{G}X) * ({}^{G}Y)$. If $B = {}^{K}X * {}^{L}Y \in {}^{G}X * {}^{G}Y$, then writing ${}^{K}X = X'$, ${}^{L}Y = Y'$, we can find by Lemma 11 elements C' and D' in the subalgebra generated by X' and Y', such that $X' * Y' = {}^{C'}X' + {}^{D'}Y'$. Hence

$$B = {}^{K}X * {}^{L}Y = X' * Y' = {}^{C'*K}X + {}^{D'*L}Y \in {}^{G}X + {}^{G}Y.$$

■

Proof of Theorem B

It is easy to see that ${}^{H}X + {}^{H}Y \subset X + Y + [\mathfrak{h}, \mathfrak{h}]$, hence also

$$({}^{H}X + {}^{H}Y)^{-} \subset X + Y + [\mathfrak{h}, \mathfrak{h}].$$

In order to prove that $X + Y + [\mathfrak{h}, \mathfrak{h}] \subset ({}^{H}X + {}^{H}Y)^{-}$ we proceed by induction on the dimension of $\mathfrak{h}$. If $\mathfrak{h}$ is one dimensional then there is nothing to prove.

We may suppose that $[\mathfrak{h}, \mathfrak{h}] = \mathfrak{n}$ is not central, since otherwise $\mathfrak{h}$ is nilpotent and we know then by Wildberger's result that $X + Y + [\mathfrak{h}, \mathfrak{h}] = {}^{H}X + {}^{H}Y$. We look at minimal noncentral ideals $\mathfrak{b}$ contained in $\mathfrak{n}$. If $\mathfrak{b} \cap \mathfrak{z} = \{0\}$, then we consider $\mathfrak{p} : \mathfrak{h} \to \mathfrak{h}/\mathfrak{b} = \widetilde{\mathfrak{h}}$. We have cases (i) and (ii) of Lemma 2. We shall treat only the case (ii) and leave the other case to the reader.

There exists a basis U_1, U_2 of $\mathfrak{b}$ such that for any $A \in \mathfrak{h}$,

$$[A, U_1 + iU_2] = \Psi(A)(U_1 + iU_2),$$

where $A \to \Psi(A)$ is a nontrivial linear functional which satisfies $\Psi([\mathfrak{h}, \mathfrak{h}]) = \{0\}$. Let us suppose that $\Psi(Y) \neq 0$ (otherwise we replace Y by X). We shall use now the induction hypothesis for $\widetilde{\mathfrak{h}} = \mathfrak{h}/\mathfrak{b}$. Let $\widetilde{X} = X \bmod \mathfrak{b}$, $\widetilde{Y} = Y \bmod \mathfrak{b}$ etc. We get:

$$({}^{\widetilde{H}}\widetilde{X} + {}^{\widetilde{H}}\widetilde{Y})^{-} = \widetilde{X} + \widetilde{Y} + [\widetilde{\mathfrak{h}}, \widetilde{\mathfrak{h}}].$$

Let now $p \in [\mathfrak{h}, \mathfrak{h}]$. By the induction hypothesis, there exists for $\epsilon > 0$ an element $O(\epsilon)$ in $\mathfrak{h}$ of length $< \epsilon$, an element B in $\mathfrak{b}$, C, D in $\mathfrak{h}$ such that

$$X + Y + P = {}^{C}X + {}^{D}Y + B + O(\epsilon).$$

We have seen in Lemma 3 that there exists β_1, β_2 in $\mathbb{R}$ such that

$${}^{C}X + {}^{(\beta_1 U_1 + \beta_2 U_2)*D}Y = X + Y + P - O(\epsilon).$$

We continue now with case (iv) of Lemma 2. We write as before:

$$[X, \theta] = \Psi(X)\theta, \ [Y, \theta] = \Psi(Y)\theta + Z,$$

where $Z \in \mathfrak{z} + i\mathfrak{z}$ and where $\Psi(X)$ and $\Psi(Y)$ are the two nonreal complex numbers, which are not purely imaginary (since $\mathfrak{h}$ is exponential). We divide through $\mathfrak{z}' = \mathrm{span}_{\mathbb{C}}(Z, Z^-) \cap \mathfrak{h}$ and we apply the induction hypothesis.

For any $P \in [\mathfrak{h}, \mathfrak{h}]$ and $\epsilon > 0$ there exist $C, D, O(\frac{\epsilon}{2})$ in $\mathfrak{h}$, $\|O(\frac{\epsilon}{2})\| < \frac{\epsilon}{2}$, W in $\mathfrak{z}'$ such that

$$X + Y + P = {}^{C}X + {}^{D}Y + W + O(\frac{\epsilon}{2}).$$

We look at the complex matrices

$$M = \begin{pmatrix} \Psi(X) & \Psi(Y) \\ \varphi({}^{C}X) & \varphi({}^{D}Y) \end{pmatrix}.$$

If the rank of M is not 2, we take

$$R = (-\Psi(X))^{-1}[X, Y]$$

and we set for any $\delta \neq 0$ in $\mathbb{R}$:

$$C' = (\delta R) * C.$$

The corresponding matrix

$$M' = \begin{pmatrix} \Psi(X) & \Psi(Y) \\ \varphi({}^{C'}X) & \varphi({}^{D}Y) \end{pmatrix}$$

is then of rank 2 for all $\delta \neq 0$. Indeed we have

$$\begin{aligned}[R, \theta] &= (-\Psi(X))^{-1}[[X, Y], \theta] = (-\Psi(X))^{-1}([X, [Y, \theta]] - [Y, [X, \theta]]) \\ &= (-\Psi(X))^{-1}(\Psi(X)\Psi(Y)\theta - \Psi(X)\Psi(Y)\theta - \Psi(X)Z) = Z.\end{aligned}$$

Hence

$$\begin{aligned}[{}^{C'}X, \theta] &= {}^{\delta R}[{}^{C}X, {}^{-\delta R}\theta] = {}^{\delta R}[{}^{C}X, \theta] = {}^{\delta R}(\Psi(X)\theta + \varphi({}^{C}X)Z) \\ &= \Psi(X)\theta + (\varphi({}^{C}X) + \delta\Psi(X))Z.\end{aligned}$$

Thus

$$\varphi({}^{C'}X) = \varphi({}^{C}X) + \Psi(X)\delta$$

and

$$\det M' = -\Psi(X)\Psi(Y)\delta \neq 0.$$

For δ very small the element

$$O(\delta) = {}^{C}X - {}^{C'}X$$

of $[\mathfrak{h}, \mathfrak{h}]$ is of length $< \frac{\epsilon}{2}$ and so we can write $O(\delta) + O(\frac{\epsilon}{2}) = O(\epsilon)$ and also

$$X + Y + P = {}^{C'}X + {}^{D}Y + W + O(\epsilon)$$

and if we now write C instead of C' we can assume that rank $M = 2$.

By Lemma 4 we can choose $\alpha_1, \alpha_2, \beta_1, \beta_2$ in $\mathbb{R}$ such that

$${}^{(\alpha_1 U_1 + \alpha_2 U_2) * C}X + {}^{(\beta_1 U_1 + \beta_2 U_2) * D}Y = {}^{C}X + {}^{D}Y + W.$$

This means of course that

$$X + Y + P = {}^{(\alpha_1 U_1 + \alpha_2 U_2) * C}X + {}^{(\beta_1 U_1 + \beta_2 U_2) * D}Y + O(\epsilon).$$

The case (v) of Lemma 2 is very easy and is left to the reader. ■

It has been shown by Wildberger in [3] that if H is nilpotent then we do not need closures. In fact the following slightly stronger result holds.

Proposition 1. *Let $\mathfrak{h}$ be a nilpotent Lie algebra generated by two elements X and Y as an ideal. Then:*

$$^{H}X + {}^{H}Y = X + Y + [\mathfrak{h}, \mathfrak{h}] = {}^{H}X * {}^{H}Y.$$

Proof. Indeed, if $\mathfrak{h}$ is abelian, then the result is clear. If not, let us proceed by induction on the dimension of $\mathfrak{h}$. There exists a noncentral element U in $\mathfrak{h}$, such that $[U, \mathfrak{h}] \subset \mathfrak{z}$. If now

$$[X, U] = 0 \text{ and } [Y, U] = 0,$$

then X and Y are contained in the centralizer $\mathfrak{z}(U)$ of U. But $\mathfrak{z}(U)$ is an ideal in $\mathfrak{h}$. This implies that $\mathfrak{z}(U) = \mathfrak{h}$ and so U is central in $\mathfrak{h}$. Hence we may suppose that $[X, U] = Z \neq 0$. We divide through $\mathbb{R}Z$ and we use the induction hypothesis for $\mathfrak{h}/\mathbb{R}Z$ and so on. ∎

Two examples

First Example: $^{H}X + {}^{H}Y \neq X + Y + [\mathfrak{h}, \mathfrak{h}]$.

We give now an example of an exponential Lie algebra $\mathfrak{g}$ generated by two elements X, Y in $\mathfrak{g}$ such that

$$^{G}X + {}^{G}Y = {}^{G}X * {}^{G}Y \neq X + Y + [\mathfrak{g}, \mathfrak{g}].$$

Let $\mathfrak{g}$ be a Lie algebra spanned by the vectors T, U, V, Z and equipped with the following nontrivial brackets:

$$[T, U] = -U, \ [T, V] = V, \ [U, V] = Z.$$

The Lie algebra $\mathfrak{g}$ is an extension by T of the two step nilpotent algebra $\mathfrak{n} = [\mathfrak{g}, \mathfrak{g}] = \mathrm{span}(U, V, Z)$ and the center of $\mathfrak{g}$ is given by the span of Z. Let now

$$X = T + U, \ Y = -T + V.$$

X and Y generate $\mathfrak{g}$: indeed let $\mathfrak{g}_0$ be the subalgebra of $\mathfrak{g}$ generated by X and Y. Then $X + Y = U + V \in \mathfrak{g}_0$, $[X, U + V] = -U + V \in \mathfrak{g}_0 \mod \mathbb{R}Z$ and thus $[U, V] = Z \in \mathfrak{g}_0$. Finally $\mathfrak{g}_0$ contains T, U, V and Z and so $\mathfrak{g} = \mathfrak{g}_0$.

We shall realize the group G associated with $\mathfrak{g}$ as a semidirect product of $\mathbb{R}$ with $\mathfrak{n} = [\mathfrak{g}, \mathfrak{g}]$, i.e. $G = \mathbb{R} \times \mathfrak{n}$ and the multiplication in G is given by:

$$\begin{aligned}(t, &uU + vV + zZ) \cdot (t', u'U + v'V + z'Z) \\ &= (t + t', (e^{t'}u + u')U + (e^{-t'}v + v')V + (z + z' + \frac{1}{2}(e^{t'}uv' - e^{-t'}v'u)Z).\end{aligned}$$

Let us show now that $^{G}X + {}^{G}Y \neq X + Y + [\mathfrak{g}, \mathfrak{g}]$. We remark first that $G = [G, G] \cdot \exp(\mathbb{R}X) = [G, G] \cdot \exp(\mathbb{R}Y)$, hence

$$^{G}X = {}^{[G,G]}X, \text{ resp. } {}^{G}Y = {}^{[G,G]}Y$$

and so

$$\begin{aligned} {}^G X &= \{^{\exp uU * \exp vV}(T+U) \mid u,v \in \mathbb{R}\} \\ &= \{T + (1+u)U + (-v)V + (-v-uv)Z \mid u,v \in \mathbb{R}\}. \end{aligned}$$

And similarly

$$\begin{aligned} {}^G Y &= \{^{\exp u'U * \exp v'V}(-T+V) \mid u',v' \in \mathbb{R}\} \\ &= \{-T + (-u')U + (1+v')V + (u'(1+v')Z) \mid u',v' \in \mathbb{R}\}. \end{aligned}$$

Hence

$$\begin{aligned} &{}^G X + {}^G Y = \\ &= \{(1+u-u')U + (1-v+v')V + (v(-1-u) + u'(1+v')Z \mid u,v,u',v' \in \mathbb{R}\}. \end{aligned}$$

Let $A = \alpha U + \beta V + \delta Z$ be any element in $[\mathfrak{g},\mathfrak{g}]$. We try to solve the equation:

$$A \in {}^G X + {}^G Y,$$

i.e.

$$\alpha = 1 + u - u',\ \beta = -v + (1+v'),\ \delta = v(-1-u) + u'(1+v'),$$

for some $u, u', v, v' \in \mathbb{R}$. If now $1+u-u' = \alpha = 0$, then $\delta = -u'v + u'(1+v') = u'(-v + (1+v')) = u'\beta$. Thus if $\beta = 0$, δ must also be 0 and no element

$$A = \delta Z \text{ of } [\mathfrak{g},\mathfrak{g}],\ \delta \neq 0,$$

is contained in ${}^G X + {}^G Y$. We also see that ${}^G X + {}^G Y$ contains every element $B = \alpha U + \beta V + \delta Z$, with $\alpha^2 + \beta^2 \neq 0$ and finally

$${}^G X + {}^G Y = [\mathfrak{g},\mathfrak{g}] \setminus \mathbb{R}^* Z.$$

Second Example: $\exp({}^H X + {}^H Y) \neq C(\exp X) \cdot C(\exp Y)$.

Let us show by a last example that for solvable nonexponential groups we do no longer have that

$$\exp({}^G X + {}^G Y) = C(\exp X) \cdot C(\exp Y).$$

Let $\mathfrak{g} = \mathfrak{e}(2) = \mathbb{R}T + \mathbb{C}$ be the three dimensional Lie algebra with the brackets:

$$[T, \Xi] = i\ \Xi,\ \ \Xi \in \mathbb{C}.$$

The Lie group associated with $\mathfrak{g}$ can be described as $G = E(2) = \mathbb{R} \times \mathbb{C}$ with group law:

$$(t, \Xi) \cdot (t', \Xi') = (t + t', e^{-it'} \cdot \Xi + \Xi').$$

In Lemma 1 we have seen that

$$\exp(tT + \Xi) = (t, \frac{(e^{-it} - 1)}{-it} \cdot \Xi).$$

Let now $X = sT$, $Y = tT + 1$, with $s \cdot t \neq 0$. Then X and Y generate $\mathfrak{g}$ and we have

$$ {}^G X = {}^{\mathbb{C}} X = sT + \mathbb{C}, {}^G Y = tT + \mathbb{C} \text{ and } {}^G X + {}^G Y = (s+t)T + \mathbb{C} $$

and also

$$ C(\exp X) = (s, \mathbb{C}) \text{ and } C(\exp Y) = (t, \mathbb{C}). $$

Thus

$$ C(\exp X) \cdot C(\exp Y) = ((s+t), \mathbb{C}). $$

But

$$ \exp({}^G X + {}^G Y) = \exp((s+t)T + \mathbb{C}) = ((s+t), f(s+t)\mathbb{C}). $$

Hence, if $s + t = 2k\pi \neq 0$, then

$$ \begin{aligned} &\exp({}^G X + {}^G Y) = \exp(2k\pi + \mathbb{C}) = (2k\pi, \{0 \cdot \mathbb{C}\}) \\ &= (2k\pi, \{0\}) \neq \exp({}^G X) \cdot \exp({}^G Y) = (2k\pi, \mathbb{C}). \end{aligned} $$

Theorem C. *Let $G = \exp \mathfrak{g}$ be a simply connected, connected solvable Lie group. Then G is exponential if and only if for every X and Y in $\mathfrak{g}$, $\exp X \cdot \exp Y \in \exp({}^G X + {}^G Y)$.*

Proof. If G is exponential, then the condition is satisfied by Theorem A. If G is not exponential, then the exponential mapping is not surjective. However, for any solvable Lie group S with Lie algebra $\mathfrak{s}$, for any subspace $\mathfrak{w}$ of $\mathfrak{s}$ such that $\mathfrak{s} = \mathfrak{w} + [\mathfrak{s}, \mathfrak{s}]$, we have

$$ S = \exp \mathfrak{w} \cdot \exp[\mathfrak{s}, \mathfrak{s}] = \exp \mathfrak{w} \cdot \exp \mathfrak{s}. $$

Hence if we choose g in G, such that $g \notin \exp \mathfrak{g}$, then we can take X, Y in $\mathfrak{g}$, such that

$$ g = \exp X \cdot \exp Y. $$

Hence $g \in \exp {}^G X \cdot \exp {}^G Y$, but $g \notin \exp({}^G X + {}^G Y)$. ■

Final Question: Would it be possible to obtain our result directly by using a special expression for the Baker-Campbell-Hausdorff-formula (see for instance [2] in a different context)?

References

[1] Dooley, A. H., Repka. J, and N. J. Wildberger, *Sums of adjoint orbits*, Lin. Multilin. Alg. **36** (1993), 79–101.

[2] Rouvière, F., *Espaces symétriques et méthode de Kashiwara-Vergne*, Ann. scient. Ec. Norm. Sup. 4^e série, t.**19**, 1986, 553–581.

[3] Wildberger, N. J., *On a relationship between adjoint orbits and conjugacy classes of a Lie group*, Canad. Math. Bull. Vol. **33**, (1990), 297–304.

[4] —, *Hypergroups and harmonic analysis*, Proc. Center Mathem. Anal. (ANU), **29** (1992), 238–253.

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