Journal of Lie Theory
Volume **5** (1995) 1–14

$\mathbf{L^p}$–$\mathbf{L^q}$–Estimates for functions of the Laplace–Beltrami operator on noncompact symmetric spaces, II*

Michael Cowling, Saverio Giulini and Stefano Meda

Communicated by J. Faraut

In this paper we continue the study of functional calculus for the Laplace–Beltrami operator on symmetric spaces of the noncompact type begun in [3]; this paper is dedicated to a study of the Poisson semigroup, which we define shortly.

Let G and K be a connected noncompact semisimple Lie group with finite center and a maximal compact subgroup thereof, and consider the symmetric space G/K, also denoted by X. We denote by n the dimension of X, by ℓ its real rank, and by ν the "pseudo–dimension" $2\left|\Sigma_0^+\right| + \ell$, where $\left|\Sigma_0^+\right|$ is the cardinality of the set of the positive indivisible (restricted) roots.

There is a canonical invariant Riemannian metric on X; denote by $-\mathcal{L}_0$ the associated Laplace–Beltrami operator. By general nonsense, $\mathcal{L}_0$ is positive and essentially self-adjoint on $C_c^\infty(X)$; let $\mathcal{L}$ be the unique self-adjoint extension of $\mathcal{L}_0$ and $\{P_\eta\}$ the spectral resolution of the identity for which

$$\mathcal{L}f = \int_{R_0^2}^\infty \eta \, dP_\eta f \qquad \forall f \in \mathrm{Dom}(\mathcal{L}),$$

where $R_0 = \langle \rho, \rho \rangle^{1/2}$, ρ being the usual half-sum of the positive roots.

For θ in $[0,1]$ and σ in $(0,1)$, the θ–heat and the (σ,θ)–Poisson semigroups $(\mathcal{H}_{t,\theta})_{t>0}$ and $(\mathcal{P}_{t,\theta}^\sigma)_{t>0}$ are defined thus:

$$\mathcal{H}_{t,\theta} f = \int_{R_0^2}^\infty \exp\bigl(-t(\eta - \theta R_0^2)\bigr) \, dP_\eta f \qquad \forall t \in (0,\infty) \quad \forall f \in L^2(X)$$

$$\mathcal{P}_{t,\theta}^\sigma f = \int_{R_0^2}^\infty \exp\bigl(-t(\eta - \theta R_0^2)^\sigma\bigr) \, dP_\eta f \qquad \forall t \in (0,\infty) \quad \forall f \in L^2(X).$$

The (σ,θ)–Poisson semigroup may be obtained from the θ–heat semigroup by subordination. However, while estimates from above can be proved using this

* Work supported by the Australian Research Council and the Italian *M.U.R.S.T., fondi 60%.*

ISSN 0940–2268 / \$2.50

fact, estimates from below cannot. If $1 \leq p, q \leq \infty$ and the operator $\mathcal{P}_{t,\theta}^{\sigma}$ satisfies a norm inequality of the form

$$\left\|\mathcal{P}_{t,\theta}^{\sigma} f\right\|_{q} \leq C \left\|f\right\|_{p} \qquad \forall f \in L^{2}(X) \cap L^{p}(X),$$

$\mathcal{P}_{t,\theta}^{\sigma}$ is said to be L^p–L^q-bounded.

In this paper we examine for which p and q the operator $\mathcal{P}_{t,\theta}^{\sigma}$ is L^p–L^q-bounded, and we study the behaviour of the L^p–L^q-operator norms $|||\mathcal{P}_{t,\theta}^{\sigma}|||_{p;q}$ as t tends to 0 and to ∞ for all such p and q. As t tends to ∞, the expression describing the behaviour of $|||\mathcal{P}_{t,\theta}^{\sigma}|||_{p;q}$ involves powers of t, in which the indices ν and ℓ play an important rôle. Two features of our study are noteworthy. First, while $\mathcal{H}_{t,\theta}$ is L^p–L^q-bounded whenever $1 \leq p \leq q \leq \infty$, $\mathcal{P}_{t,\theta}^{\sigma}$ is not L^p–L^q-bounded for many such p and q. Second, when p or q reach the critical index for L^p–L^q-boundedness, the exponent $\ell + 1$ appears; in our previous work, we saw only ℓ. We refer to [3] for an account of related work.

In order to state our main theorem, we introduce a little notation: if $0 \leq \theta \leq 1$,

$$\begin{aligned} p_\theta &= 2\Big/\big(1 + (1-\theta)^{1/2}\big), \\ R_\theta &= [(1-\theta)\,\langle\rho,\rho\rangle]^{1/2}, \\ R_{\theta,p} &= \big[(\frac{4}{pp'} - \theta)\,\langle\rho,\rho\rangle\big]^{1/2}; \end{aligned}$$

in the definition of $R_{\theta,p}$, $p_\theta \leq p \leq p_\theta{}'$. Observe that $1 \leq p_\theta \leq 2$, that R_θ defined here agrees with R_0 as defined previously when $\theta = 0$, that $R_{\theta,p} = R_\theta$ when $p = 2$, and that $R_{\theta,p} = 0$ when $p = p_\theta$ or $p = p_\theta{}'$.

Theorem 1. *Suppose that $0 \leq \theta \leq 1$, $0 < \sigma < 1$, and $1 \leq p, q \leq \infty$. The following conditions hold:*

(i) *if $t > 0$, then $\mathcal{P}_{t,\theta}^{\sigma}$ is L^p–L^q-bounded only if $p \leq q$, $p \leq p_\theta{}'$, and $q \geq p_\theta$;*

(ii) *if $p_\theta \leq p \leq p_\theta{}'$, then*
$$|||\mathcal{P}_{t,\theta}^{\sigma}|||_{p;p} = \exp\big(-R_{\theta,p}^{2\sigma}\, t\big) \qquad \forall t \in (0,\infty);$$

(iii) *if $p \leq q$, $p \leq p_\theta{}'$ and $q \geq p_\theta$, then*
$$|||\mathcal{P}_{t,\theta}^{\sigma}|||_{p;q} \sim t^{-n(1/p-1/q)/2\sigma} \qquad \forall t \in (0,1];$$

(iv) *if $p < q = 2$ or $2 = p < q$, then*
$$|||\mathcal{P}_{t,\theta}^{\sigma}|||_{p;q} \sim t^{-\nu/4} \exp\big(-R_{\theta}^{2\sigma} t\big) \qquad \forall t \in [1,\infty);$$

(v) *if $p < 2 < q$, then*
$$|||\mathcal{P}_{t,\theta}^{\sigma}|||_{p;q} \sim t^{-\nu/2} \exp\big(-R_{\theta}^{2\sigma} t\big) \qquad \forall t \in [1,\infty);$$

(vi) *if $p < q < 2$ and $q > p_\theta$, then*

$$\|\mathcal{P}^{\sigma}_{t,\theta}\|_{p;q} \sim t^{-\ell/2q'} \exp\bigl(-R^{2\sigma}_{\theta,q}\, t\bigr) \qquad \forall t \in [1,\infty);$$

(vii) *if* $p < q = p_\theta$, *then*

$$\|\mathcal{P}^{\sigma}_{t,\theta}\|_{p;q} \sim t^{-(\ell+1)/2\sigma q'} \qquad \forall t \in [1,\infty);$$

(viii) *if* $2 < p < q$ *and* $p < p_\theta{}'$, *then*

$$\|\mathcal{P}^{\sigma}_{t,\theta}\|_{p;q} \sim t^{-\ell/2p} \exp\bigl(-R^{2\sigma}_{\theta,p}\, t\bigr) \qquad \forall t \in [1,\infty);$$

(ix) *if* $p_\theta{}' = p < q$, *then*

$$\|\mathcal{P}^{\sigma}_{t,\theta}\|_{p;q} \sim t^{-(\ell+1)/2\sigma p} \qquad \forall t \in [1,\infty).$$

Section 1 of this paper is devoted to notation, and a summary of relevant material, such as the the spherical Fourier transformation. In Section 2 we prove our theorem.

1. Notation and Background Material

We use the standard notation of the theory of Lie groups and symmetric spaces, as in, for instance, S. Helgason [6]. Our notation here is also consistent with our paper [3], to which we refer several times.

For any x in G, we denote by $A(x)$ the element of $\mathfrak{a}$ such that $x \in N \exp A(x)\, K$ (the Iwasawa decomposition). For any (complex-valued) linear form Λ on $\mathfrak{a}$, the elementary spherical function ϕ_Λ is defined by the rule

$$\phi_\Lambda(x) = \int_K \exp\bigl((i\Lambda + \rho)\, A(kx)\bigr)\, dk \qquad \forall x \in G.$$

The spherical Fourier transform $\widetilde{f}$ of an $L^1(G)$–function f is defined by the formula

$$\widetilde{f}(\Lambda) = \int_G f(x)\, \phi_{-\Lambda}(x)\, dx \qquad \forall \Lambda \in \mathfrak{a}^*.$$

Harish–Chandra proved an inversion formula and a Plancherel formula for the spherical Fourier transformation, namely

$$f(x) = \int_{\mathfrak{a}^*} \widetilde{f}(\Lambda)\, \phi_\Lambda(x)\, d\mu(\Lambda) \qquad \forall x \in G$$

for "nice" K–bi-invariant functions f on G, and

$$\|f\|_2 = \left[\int_{\mathfrak{a}^*} \bigl|\widetilde{f}(\Lambda)\bigr|^2\, d\mu(\Lambda)\right]^{1/2} \qquad \forall f \in L^2(K\backslash X),$$

where $d\mu(\Lambda) = c_G\, |\mathbf{c}(\Lambda)|^{-2}\, d\Lambda$, and $\mathbf{c}$ denotes the Harish–Chandra $\mathbf{c}$–function. For the details, see, for instance, Theorem IV.7.5 of Helgason [6]. We often deal with the inversion formula and the Plancherel formula with purely radial integrands. From the formula for the $\mathbf{c}$–function (Theorem IV.6.14 of [6]), it is clear that if $F : [0,\infty) \to \mathbf{C}$ and $f(\Lambda) = F(|\Lambda|)$ for all Λ in $\mathfrak{a}^*$, then

$$(1) \quad \int_{\mathfrak{a}^*} f(\Lambda)\, d\mu(\Lambda) = c_G \int_{\mathfrak{a}^*} F(|\Lambda|)\, |\mathbf{c}(\Lambda)|^{-2}\, d\Lambda \sim \int_0^\infty F(r)\, (1+r)^{n-\nu}\, r^{\nu-1}\, dr,$$

provided the integrals converge. We shall also use a modified version of the Plancherel measure, $\widetilde{\mu}$, defined by the rule

$$d\widetilde{\mu}(\Lambda)/d\Lambda = \prod_{\alpha \in \Sigma_0^+} (1 + |\langle \alpha, \Lambda \rangle|)^{d_\alpha}$$

(see [3] for the notation). Clearly

$$C \leq d\widetilde{\mu}(\Lambda)/d\Lambda \leq C'(1 + |\Lambda|^{n-\ell}),$$

where C and C' are constants. The modified Plancherel measure appears in several results of harmonic analysis on symmetric spaces; see [3].

Let $\mathbf{W}_1$ be the interior of the convex hull in $\mathfrak{a}^*$ of the images of ρ under the Weyl group of $(\mathfrak{g}, \mathfrak{a})$. For δ in $(0,1)$, we denote by $\mathbf{W}_\delta$ and $\mathbf{T}_\delta$ the dilate of $\mathbf{W}_1$ by δ and the tube over the polygon $\mathbf{W}_\delta$, i.e., $\mathbf{T}_\delta = \mathfrak{a}^* + i\delta\,\mathbf{W}_1$. If $1 \leq p < 2$, the spherical Fourier transform of an $L^p(G)$-function extends to a bounded holomorphic function in the tube $\mathbf{T}_{\delta(p)}$, where $\delta(p)$ is defined by the rule

$$\delta(p) = 2/p - 1.$$

We define the quadratic function Q_θ on $\mathfrak{a}^*_{\mathbf{C}}$:

$$Q_\theta(\Lambda) = \langle \Lambda, \Lambda \rangle + (1-\theta)\langle \rho, \rho \rangle \qquad \forall \Lambda \in \mathfrak{a}^*_{\mathbf{C}},$$

and denote by $p^\sigma_{t,\theta}$ the K–bi-invariant function on G such that

$$\widetilde{p}^\sigma_{t,\theta}(\Lambda) = \exp\bigl(-tQ_\theta(\Lambda)^\sigma\bigr) \qquad \forall \Lambda \in \mathbf{T}_{\delta(p_\theta)}.$$

Then

$$\mathcal{P}^\sigma_{t,\theta} f = f * p^\sigma_{t,\theta} \qquad \forall t \in (0,\infty) \quad \forall f \in L^2(X).$$

Note that Q^σ_θ and $\widetilde{p}^\sigma_{t,\theta}$ continue analytically to the tube $\mathbf{T}_{\delta(p_\theta)}$, but to no larger tube. We denote by h_t the kernel associated to the heat operator, i. e., $\widetilde{h}_t(\Lambda) = \exp\bigl(-tQ_0(\Lambda)\bigr)$.

Throughout this paper, the following assumptions are made about the parameters:

$$\begin{gathered} 1 \leq p, q \leq \infty, \\ 1/p + 1/p' = 1, \\ 0 \leq \theta \leq 1, \\ 0 < \sigma < 1, \\ 0 < t < \infty. \end{gathered}$$

Recall that $p_\theta = 2\Big/\bigl[1+(1-\theta)^{1/2}\bigr]$, and note that $\delta(p_\theta) = (1-\theta)^{1/2}$. By C and C' we denote positive constants which may not be the same at different occurrences; C and C' may depend on anything quantified, implicitly or explicitly, before the formula in which they appear. The expression

$$A(t) \sim B(t) \qquad \forall t \in \mathbf{D},$$

where $\mathbf{D}$ is some subset of the domains of A and of B, means that there exist constants C and C' such that

$$C\,|A(t)| \leq |B(t)| \leq C'\,|A(t)| \qquad \forall t \in \mathbf{D}.$$

If the operator T on $L^2(X)$ satisfies a norm inequality of the form

$$\|Tf\|_q \leq C\,\|f\|_p \qquad \forall f \in L^2(X) \cap L^p(X),$$

then T extends uniquely to a bounded operator from $L^p(X)$ to $L^q(X)$ (where the weak-star topology should be used if $p = \infty$); conversely, if a continuous extension to a bounded operator from $L^p(X)$ to $L^q(X)$ exists, then such a norm inequality holds. We denote by $|||T|||_{p;q}$ the norm of the linear operator T from $L^p(X)$ to $L^q(X)$.

2. Estimates for the (σ, θ)–Poisson semigroup

First we prove a couple of technical results on integration, then we estimate the L^p–norms of $p^{\sigma}_{t,\theta}$ for various p. Finally we put the ingredients together to prove the main theorem.

Lemma 1. *Suppose that* $0 \leq \alpha < \beta \leq \infty$, $\omega > 0$, *and* $\eta > 0$. *Suppose also that* ψ *is a function on the interval* $[\alpha, \beta)$, *which is continuous and strictly positive throughout* $[\alpha, \beta)$, *such that, for constants* C *and* k,

$$|\psi(s)| \leq C\,(1+s)^k \qquad \forall s \in [\alpha, \beta).$$

Then

(i) *if* $\alpha = 0$,

$$\int_\alpha^\beta \exp(-ts^\omega)\,(s-\alpha)^{\eta-1}\,\psi(s)\,ds \sim t^{-\eta/\omega} \qquad \forall t \in [1, \infty);$$

(ii) *if* $\alpha > 0$,

$$\int_\alpha^\beta \exp(-ts^\omega)\,(s-\alpha)^{\eta-1}\,\psi(s)\,ds \sim t^{-\eta}\,\exp(-\alpha^\omega t) \qquad \forall t \in [1, \infty).$$

Proof. We assume initially that $\beta < \infty$.

We first prove (i). By changing variables ($v = ts^\omega$), we transform the integral to

$$\frac{t^{-\eta/\omega}}{\omega} \int_0^{t\beta^\omega} \exp(-v)\,v^{\eta/\omega-1}\,\psi\big((v/t)^{1/\omega}\big)\,dv;$$

the integrand is dominated by

$$\exp(-v)\,v^{\eta/\omega-1} \max\big\{|\psi(s)| : s \in [\alpha, \beta)\big\},$$

which is integrable on $\mathbf{R}^+$, and as t tends to ∞ the integral tends to

$$\int_0^\infty \exp(-v)\,v^{\eta/\omega-1}\,\psi(0)\,dv$$

by the Lebesgue dominated convergence theorem.

We now prove (ii). Let γ denote $\beta^\omega - \alpha^\omega$. Since ψ on $[\alpha,\beta)$ is continuous, bounded, and strictly positive, the function ϕ on $[0,\gamma)$, such that $\phi(0) = \omega^{1-\eta}\,\alpha^{\eta(1-\omega)}\,\psi(\alpha)$ and

$$\phi(v) = \Big[\frac{(v+\alpha^\omega)^{1/\omega}-\alpha}{v}\Big]^{\eta-1}\,\psi\big((v+\alpha^\omega)^{1/\omega}\big)\,(v+\alpha^\omega)^{1/\omega-1} \qquad \forall v\in(0,\gamma),$$

is too. By changing variables ($v = s^\omega - \alpha^\omega$), we transform the integral to

$$\frac{\exp(-t\alpha^\omega)}{\omega}\int_0^\gamma \exp(-tv)\,v^{\eta-1}\,\phi(v)\,dv.$$

We have therefore an integral of the form already treated in (i), and are done.

The case where $\beta=\infty$ now follows: we write the integral as the sum of an integral from α to $\alpha+1$ and one from $\alpha+1$ to ∞; the first integral is treated by the result already established, and the second is easily shown to be $O(t^{k+\eta+1}\exp\big(-(\alpha+1)^\omega t)\big)$. □

Lemma 2. *Suppose that* $0<\tau<\infty$ *and* $R>0$. *Then*

$$\int_0^1 \exp\big(-t[r^2+R^2]^\sigma\big)\,r^{\tau-1}\,dr \sim \begin{cases} 1 & \forall t\in(0,1] \\ t^{-\tau/2}\,\exp(-R^{2\sigma}t) & \forall t\in[1,\infty) \end{cases}$$

and

$$\int_1^\infty \exp\big(-t[r^2+R^2]^\sigma\big)\,r^{\tau-1}\,dr \sim \begin{cases} t^{-\tau/2\sigma} & \forall t\in(0,1] \\ t^{-1}\,\exp\big(-[1+R^2]^\sigma\,t\big) & \forall t\in[1,\infty). \end{cases}$$

Proof. It is trivial that the first integral behaves as claimed for t in $(0,1]$. To study its behaviour for t in $[1,\infty)$, we change variables $\big(s=[r^2+R^2]^{1/2}\big)$, and it becomes

$$\begin{aligned} &\int_R^{\sqrt{R^2+1}} \exp\big(-ts^{2\sigma}\big)\,(s^2-R^2)^{(\tau-2)/2}\,s\,ds \\ &= \int_R^{\sqrt{R^2+1}} \exp\big(-ts^{2\sigma}\big)\,(s-R)^{(\tau-2)/2}\,(s+R)^{(\tau-2)/2}\,s\,ds. \end{aligned}$$

The required behaviour for t in $[1,\infty)$ is a corollary of Lemma 1 (where $\omega=2\sigma$, $\eta=\tau/2$, $\alpha=R$, and $\psi(s)=(s+R)^{(\tau-2)/2}\,s$).

By performing the same change of variables, we transform the second integral to

$$\int_{\sqrt{R^2+1}}^\infty \exp\big(-ts^{2\sigma}\big)\,(s^2-R^2)^{(\tau-2)/2}\,s\,ds.$$

The result stated for t in $[1,\infty)$ follows from Lemma 1 (where $\omega=2\sigma$, $\eta=1$, $\alpha=[R^2+1]^{1/2}$, and $\psi(s)=(s^2-R^2)^{(\tau-2)/2}\,s$). By changing variables again, we transform the last integral to

$$t^{-\tau/2\sigma}\int_{\sqrt{R^2+1}\,t^{1/2\sigma}}^\infty \exp\big(-v^{2\sigma}\big)\,\big(v^2-(t^{1/2\sigma}R)^2\big)^{(\tau-2)/2}\,v\,dv$$

and the required behaviour for t in $(0,1]$ is an immediate consequence. $\square$

Note that the preceding lemma holds for any positive σ.
We now begin the harmonic analysis.

Lemma 3. *The following norm estimates for $p^\sigma_{t,\theta}$ hold:*

(i) *if* $p=2$, *then*
$$\left\|p^\sigma_{t,\theta}\right\|_p \sim \begin{cases} t^{-n/4\sigma} & \forall t\in(0,1] \\ t^{-\nu/4}\exp(-R_\theta^{2\sigma}t) & \forall t\in[1,\infty);\end{cases}$$
(ii) *if* $p=\infty$, *then*
$$\left\|p^\sigma_{t,\theta}\right\|_p \sim \begin{cases} t^{-n/2\sigma} & \forall t\in(0,1] \\ t^{-\nu/2}\exp(-R_\theta^{2\sigma}t) & \forall t\in[1,\infty);\end{cases}$$
(iii) *if* $p_\theta<p<2$, *then*
$$\left\|p^\sigma_{t,\theta}\right\|_p \sim \begin{cases} t^{-n/2\sigma p'} & \forall t\in(0,1] \\ t^{-\ell/2p'}\exp\big(-R_{\theta,p}^{2\sigma}\,t\big) & \forall t\in[1,\infty);\end{cases}$$
(iv) *if* $p=p_\theta<2$, *then*
$$\left\|p^\sigma_{t,\theta}\right\|_p \sim \begin{cases} t^{-n/2\sigma p'} & \forall t\in(0,1] \\ t^{-(\ell+1)/2\sigma p'} & \forall t\in[1,\infty).\end{cases}$$

Proof. To prove (i) we use the Plancherel formula and pass to polar coordinates, using formula (1):

$$\begin{aligned}
\left\|p^\sigma_{t,\theta}\right\|_2 &= \left[\int_{\mathfrak{a}^*}\exp\big(-2tQ_\theta(\Lambda)^\sigma\big)\,d\mu(\Lambda)\right]^{1/2}\\
&\sim \left[\int_0^\infty \exp\big(-2t[r^2+R_\theta^2]^\sigma\big)\,(1+r)^{n-\nu}\,r^{\nu-1}\,dr\right]^{1/2}\\
&\sim \left[\int_0^1 \exp\big(-2t[r^2+R_\theta^2]^\sigma\big)\,r^{\nu-1}\,dr\right.\\
&\qquad\left.+\int_1^\infty \exp\big(-2t[r^2+R_\theta^2]^\sigma\big)\,r^{n-1}\,dr\right]^{1/2} \qquad \forall t\in(0,\infty).
\end{aligned}$$

Now, from Lemma 2,

$$\int_0^1 \exp\big(-2t[r^2+R_\theta^2]^\sigma\big)\,r^{\nu-1}\,dr \sim \begin{cases} 1 & \forall t\in(0,1] \\ t^{-\nu/2}\exp(-2R_\theta^{2\sigma}t) & \forall t\in[1,\infty)\end{cases}$$

and

$$\int_1^\infty \exp\big(-2t[r^2+R_\theta^2]^\sigma\big)\,r^{n-1}\,dr \sim \begin{cases} t^{-n/2\sigma} & \forall t\in(0,1] \\ t^{-1}\exp\big(-2[1+R_\theta^2]^\sigma\,t\big) & \forall t\in[1,\infty).\end{cases}$$

This proves (i).

To prove (ii), we proceed similarly, using the inversion formula. For any x in G,

$$\begin{aligned}
\left|p^\sigma_{t,\theta}(x)\right| &= \left|\int_{\mathfrak{a}^*} \exp\bigl(-tQ_\theta(\Lambda)^\sigma\bigr)\, \phi_\Lambda(x)\, d\mu(\Lambda)\right| \\
&\le \int_{\mathfrak{a}^*} \exp\bigl(-tQ_\theta(\Lambda)^\sigma\bigr)\, |\phi_\Lambda(x)|\, d\mu(\Lambda) \\
&\le \int_{\mathfrak{a}^*} \exp\bigl(-tQ_\theta(\Lambda)^\sigma\bigr)\, \phi_\Lambda(e)\, d\mu(\Lambda) \\
&= p^\sigma_{t,\theta}(e),
\end{aligned}$$

so that $\left\|p^\sigma_{t,\theta}\right\|_\infty = p^\sigma_{t,\theta}(e)$. Now it is easy to see that

$$\left\|p^\sigma_{t,\theta}\right\|_\infty = \int_{\mathfrak{a}^*} \exp\bigl(-tQ_\theta(\Lambda)^\sigma\bigr)\, d\mu(\Lambda) \sim \begin{cases} t^{-n/2\sigma} & \forall t \in (0,1] \\ t^{-\nu/2}\exp(-R_\theta^{2\sigma} t) & \forall t \in [1,\infty) \end{cases}$$

by a calculation like that of the $L^2(G)$–norm of $p^\sigma_{t,\theta}$.

The hardest parts of this lemma are (iii) and (iv). We prove both cases by obtaining first a lower bound, then an upper bound.

We begin the proof of (iii). Suppose that $p_\theta < p < 2$. By Theorem 2.1 of [3], and the fact that $d\widetilde{\mu}(\Lambda)/d\Lambda \ge C$,

$$\begin{aligned}
\text{(2)}\quad \left\|p^\sigma_{t,\theta}\right\|_p &\ge C\left[\int_{\mathfrak{a}^*} \left|\widetilde{p}^\sigma_{t,\theta}(\Lambda + i\delta(p)\rho)\right|^{p'} d\widetilde{\mu}(\Lambda)\right]^{1/p'} \\
&\ge C\left[\int_{\mathfrak{b}} \exp\Bigl(-tp' \operatorname{Re}\bigl([Q_\theta(\Lambda + i\delta(p)\rho)]^\sigma\bigr)\Bigr)\, d\Lambda\right]^{1/p'} \qquad \forall t \in [1,\infty),
\end{aligned}$$

where $\mathfrak{b}$ denotes the unit ball in $\mathfrak{a}^*$. We denote the right hand side of the last inequality by $I(t)$. Now for all Λ in $\mathfrak{a}^*$,

$$\begin{aligned}
\operatorname{Re}\bigl([Q_\theta(\Lambda + i\delta(p)\rho)]^\sigma\bigr) &\le |Q_\theta(\Lambda + i\delta(p)\rho)|^\sigma \\
&= \bigl(\langle\Lambda,\Lambda\rangle + R^2_{\theta,p} + 2i\delta(p)\langle\rho,\Lambda\rangle\bigr)^{\sigma/2} \\
&\le \bigl([\langle\Lambda,\Lambda\rangle + R^2_{\theta,p}]^2 + 4\delta(p)^2 R_0^2 \langle\Lambda,\Lambda\rangle\bigr)^{\sigma/2} \\
&= \bigl(\langle\Lambda,\Lambda\rangle^2 + \gamma_1\langle\Lambda,\Lambda\rangle + \gamma_2\bigr)^{\sigma/2},
\end{aligned}$$

where $\gamma_1 = 2R^2_{\theta,p} + 4\delta(p)^2 R_0^2$ and $\gamma_2 = R^4_{\theta,p}$, so, by using polar co-ordinates, and then changing variables $\bigl(s = (r^4 + \gamma_1 r^2 + \gamma_2)^{1/4}\bigr)$, we see that

$$\begin{aligned}
I(t) &\ge C\left[\int_0^1 \exp\left(-tp'\bigl(r^4 + \gamma_1 r^2 + \gamma_2\bigr)^{\sigma/2}\right) r^{\ell-1}\, dr\right]^{1/p'} \\
&= C\left[\int_{R_{\theta,p}}^{\gamma} \exp\bigl(-tp' s^{2\sigma}\bigr)\, (s - R_{\theta,p})^{\ell/2-1}\, \psi(s)\, ds\right]^{1/p'} \qquad \forall t \in [1,\infty),
\end{aligned}$$

where $\gamma = (1+\gamma_1+\gamma_2)^{1/4}$ and ψ is a continuous strictly positive function on the interval $[R_{\theta,p}, \gamma]$. Now, Lemma 1 applies (where $\omega = 2\sigma$, $\eta = \ell/2$ and $\alpha = R_{\theta,p}$) and we conclude that

$$I(t) \geq C\, t^{-\ell/2p'} \exp(-R_{\theta,p}^{2\sigma}\, t) \qquad \forall t \in [1,\infty); \tag{3}$$

combined with (2), this proves the lower bound of (iii).

We now prove the upper bound of (iii). From [3], Theorem 2.4, if $p_\theta \leq p < 2$, then

$$\left\| p_{t,\theta}^{\sigma} \right\|_p \leq \left\| p_{t,\theta}^{\sigma}\, \phi_{i\delta(p)\rho} \right\|_1^{\delta(p)} N^{1-\delta(p)},$$

where

$$\begin{aligned} N &= \left[\int_{\mathfrak{a}^*} \left| \widetilde{p}_{t,\theta}^{\sigma}(\Lambda + i\delta(p)\rho) \right|^2 d\widetilde{\mu}(\Lambda) \right]^{1/2} \\ &= \left[\int_{\mathfrak{a}^*} \left| \exp\bigl(-t[Q_\theta(\Lambda + i\delta(p)\rho)]^{\sigma}\bigr) \right|^2 d\widetilde{\mu}(\Lambda) \right]^{1/2} \\ &= \left[\int_{\mathfrak{a}^*} \exp\Bigl(-2t\, \mathrm{Re}\bigl([Q_\theta(\Lambda + i\delta(p)\rho)]^{\sigma}\bigr)\Bigr) d\widetilde{\mu}(\Lambda) \right]^{1/2}. \end{aligned}$$

Since $p_{t,\theta}^{\sigma}$ is a positive function,

$$\left\| p_{t,\theta}^{\sigma}\, \phi_{i\delta(p)\rho} \right\|_1 = \widetilde{p}_{t,\theta}^{\sigma}(-i\delta(p)\rho) = \exp\bigl(-R_{\theta,p}^{2\sigma}\, t\bigr).$$

Thus

$$\left\| p_{t,\theta}^{\sigma} \right\|_p \leq \exp\bigl(-\delta(p) R_{\theta,p}^{2\sigma}\, t\bigr)\, N^{1-\delta(p)}. \tag{4}$$

To estimate N, we observe that

$$\mathrm{Re}\bigl([Q_\theta(\Lambda + i\delta(p)\rho)]^{\sigma}\bigr) \geq \bigl(\mathrm{Re}[Q_\theta(\Lambda + i\delta(p)\rho)]\bigr)^{\sigma} = [\langle \Lambda, \Lambda \rangle + R_{\theta,p}^2]^{\sigma} \qquad \forall \Lambda \in \mathfrak{a}^*,$$

because, for any complex number z with nonnegative real part and any σ in $(0,1)$, $\mathrm{Re}\bigl(z^{\sigma}\bigr) \geq \bigl(\mathrm{Re}(z)\bigr)^{\sigma}$. We recall that $d\widetilde{\mu}(\Lambda)/d\Lambda \leq C(1+|\Lambda|^{n-\ell})$, and pass to polar co-ordinates, to deduce that

$$\begin{aligned} N &\leq C\left[\int_0^{\infty} \exp\bigl(-2t[r^2 + R_{\theta,p}^2]^{\sigma}\bigr) \bigl(1 + r^{n-\ell}\bigr)\, r^{\ell-1}\, dr \right]^{1/2} \\ &\sim \Biggl[\int_0^{1} \exp\bigl(-2t[r^2 + R_{\theta,p}^2]^{\sigma}\bigr)\, r^{\ell-1}\, dr \\ &\qquad\qquad + \int_1^{\infty} \exp\bigl(-2t[r^2 + R_{\theta,p}^2]^{\sigma}\bigr)\, r^{n-1}\, dr \Biggr]^{1/2} \\ &\leq C\, t^{-\ell/4} \exp\bigl(-R_{\theta,p}^{2\sigma}\, t\bigr) \qquad \forall t \in [1,\infty), \end{aligned}$$

by Lemma 2, so that

$$\left\| p_{t,\theta}^{\sigma} \right\|_p \leq \exp\bigl(-\delta(p) R_{\theta,p}^{2\sigma}\, t\bigr)\, N^{1-\delta(p)} \leq C\, t^{-\ell/2p'} \exp\bigl(-R_{\theta,p}^{2\sigma}\, t\bigr) \qquad \forall t \in [1,\infty),$$

as required to prove (iii).

We now consider (iv). If $p > 1$ and $\ell > 1$, then, much as argued to prove (2),

$$\begin{aligned}\left\|p^\sigma_{t,\theta}\right\|_p &\geq C\left[\int_{\mathfrak{a}^*} \left|\widetilde{p}^\sigma_{t,\theta}(\Lambda + i\delta(p)\rho)\right|^{p'} d\widetilde{\mu}(\Lambda)\right]^{1/p'} \\ &\geq C\left[\int_{\mathfrak{c}} \exp\bigl(-tp'\left|Q_\theta(\Lambda + i\delta(p)\rho)\right|^\sigma\bigr)\, d\Lambda\right]^{1/p'} \qquad \forall t \in [1,\infty),\end{aligned}$$

where $\mathfrak{c}$ denotes the subset of $\mathfrak{a}^*$ of all elements of the form $\Lambda_0 + \lambda_1\rho$, such that $\langle\Lambda_0,\rho\rangle = 0$, $\langle\Lambda_0,\Lambda_0\rangle \leq 1$, and $0 \leq \lambda_1 \leq 1$. We denote the right hand side of the last inequality by $J(t)$. Now if Λ may be written in this way, then

$$\begin{aligned}|Q_\theta(\Lambda + i\delta(p)\rho)| &= \left|\langle\Lambda_0,\Lambda_0\rangle + \lambda_1^2 R_0^2 + 2i\delta(p)\lambda_1 R_0^2\right| \\ &\leq \langle\Lambda_0,\Lambda_0\rangle + \lambda_1^2 R_0^2 + 2\lambda_1 R_0^2 \\ &\leq \langle\Lambda_0,\Lambda_0\rangle + 3\lambda_1 R_0^2,\end{aligned}$$

so, passing to polar co-ordinates in $\rho^\perp$, and then changing variables, we deduce that

$$\begin{aligned}J(t) &\geq C\left[\int_0^1\int_0^1 \exp\bigl(-tp'\left|\lambda_0^2 + 3\lambda_1 R_0^2\right|^\sigma\bigr)\,\lambda_0^{\ell-2}\,d\lambda_0\,d\lambda_1\right]^{1/p'} \\ &\geq C\,t^{-(\ell+1)/2\sigma p'}\left[\int_0^{t^{1/2\sigma}}\int_0^{t^{1/\sigma}} \exp\bigl(-p'\left|\lambda_0^2 + 3\lambda_1 R_0^2\right|^\sigma\bigr)\,\lambda_0^{\ell-2}\,d\lambda_0\,d\lambda_1\right]^{1/p'} \\ &\geq C\,t^{-(\ell+1)/2\sigma p'} \qquad \forall t \in [1,\infty),\end{aligned}$$

as required to prove the lower bound. If $p = 1$ or $\ell = 1$, the argument simplifies but the conclusion is the same.

To prove the upper bound, we note first that (4) continues to hold, and that $R^{2\sigma}_{\theta,p} = 0$, so that

$$\left\|p^\sigma_{t,\theta}\right\|_p \leq N^{1-\delta(p)}.$$

To estimate N, we write $\Lambda = \Lambda_0 + \lambda_1\rho$, where $\langle\Lambda_0,\rho\rangle = 0$, as above, and note that

$$\begin{aligned}\operatorname{Re}\bigl([Q_\theta(\Lambda + i\delta(p)\rho)]^\sigma\bigr) &= \operatorname{Re}\bigl([\langle\Lambda_0,\Lambda_0\rangle + \lambda_1^2 R_0^2 + 2i\delta(p)\lambda_1 R_0^2]^\sigma\bigr) \\ &\geq \operatorname{Re}\bigl([\langle\Lambda_0,\Lambda_0\rangle + 2i\delta(p)\lambda_1 R_0^2]^\sigma\bigr).\end{aligned}$$

We let Φ and $\Psi : \mathbf{R}^+ \times \mathbf{R} \to \mathbf{R}$ be the functions given by the formulae

$$\begin{aligned}\Phi(\lambda_0,\lambda_1) &= \operatorname{Re}\bigl([\lambda_0^2 + 2i\delta(p)R_0^2\lambda_1]^\sigma\bigr) \qquad \forall\lambda_0 \in (0,\infty), \quad \forall\lambda_1 \in (-\infty,\infty) \\ \Psi(\lambda_0,\lambda_1) &= \bigl(1 + \lambda_0 + |\lambda_1|\bigr)^{n-\ell} \qquad \forall\lambda_0 \in (0,\infty), \quad \forall\lambda_1 \in (-\infty,\infty),\end{aligned}$$

and recall that $d\widetilde{\mu}(\Lambda)/d\Lambda \leq C(1+|\Lambda|^{n-\ell})$. By passing to polar co-ordinates in $\rho^\perp$ and changing variables, we deduce that

$$\begin{aligned} N &= \Big[\int_{\mathfrak{a}^*} \exp\Big(-2t\,\mathrm{Re}\big([Q_\theta(\Lambda+i\delta(p)\rho)]^\sigma\big)\Big)\,d\widetilde{\mu}(\Lambda)\Big]^{1/2} \\ &\leq C\Big[\int_{-\infty}^{\infty}\int_0^{\infty} \exp\big(-2t\,\Phi(\lambda_0,\lambda_1)\big)\,\Psi(\lambda_0,\lambda_1)\,\lambda_0^{\ell-2}\,d\lambda_0\,d\lambda_1\Big]^{1/2} \\ &= C\,t^{-(\ell+1)/4\sigma}\Big[\int_{-\infty}^{\infty}\int_0^{\infty} \exp\big(-2\,\Phi(\lambda_0,\lambda_1)\big)\,\Psi\big(\frac{\lambda_0}{t^{1/2\sigma}},\frac{\lambda_1}{t^{1/\sigma}}\big)\,\lambda_0^{\ell-2}\,d\lambda_0\,d\lambda_1\Big]^{1/2} \\ &\leq C\,t^{-(\ell+1)/4\sigma} \qquad \forall t\in[1,\infty). \end{aligned}$$

Now we can conclude that

$$\big\|p_{t,\theta}^\sigma\big\|_p \leq C\,t^{-(1-\delta(p))(\ell+1)/4\sigma} = C\,t^{-(\ell+1)/2\sigma p'} \qquad \forall t\in[1,\infty),$$

as required. This finishes the proof of (iv), and of Lemma 3. □

For convenience, we list the results of our main theorem.

(i) if $t>0$, then $\mathcal{P}_{t,\theta}^\sigma$ is L^p-L^q-bounded only if $p\leq q$, $p\leq p_\theta{}'$, and $q\geq p_\theta$;

(ii) if $p_\theta\leq p\leq p_\theta{}'$, $|||\mathcal{P}_{t,\theta}^\sigma|||_{p;p} = \exp\big(-R_{\theta,p}^{2\sigma}\,t\big)$ for all t in $(0,\infty)$;

(iii) if $p\leq p_\theta{}'$ and $q\geq p_\theta$, $|||\mathcal{P}_{t,\theta}^\sigma|||_{p;q} \sim t^{-n(1/p-1/q)/2\sigma}$ for all t in $(0,1]$;

(iv) if $p<q=2$ or $2=p<q$, $|||\mathcal{P}_{t,\theta}^\sigma|||_{p;q} \sim t^{-\nu/4}\exp(-R_\theta^{2\sigma}t)$ for all t in $[1,\infty)$;

(v) if $p<2<q$, $|||\mathcal{P}_{t,\theta}^\sigma|||_{p;q} \sim t^{-\nu/2}\exp(-R_\theta^{2\sigma}t)$ for all t in $[1,\infty)$;

(vi) if $p<q<2$ and $q>p_\theta$, $|||\mathcal{P}_{t,\theta}^\sigma|||_{p;q} \sim t^{-\ell/2q'}\exp\big(-R_{\theta,q}^{2\sigma}\,t\big)$ for all t in $[1,\infty)$;

(vii) if $p<q=p_\theta$, $|||\mathcal{P}_{t,\theta}^\sigma|||_{p;q} \sim t^{-(\ell+1)/2\sigma q'}$ for all t in $[1,\infty)$;

(viii) if $2<p<q$, $|||\mathcal{P}_{t,\theta}^\sigma|||_{p;q} \sim t^{-\ell/2p}\exp\big(-R_{\theta,p}^{2\sigma}\,t\big)$ for all t in $[1,\infty)$;

(ix) if $p_\theta{}'=p<q$, $|||\mathcal{P}_{t,\theta}^\sigma|||_{p;q} \sim t^{-(\ell+1)/2\sigma p}$ for all t in $[1,\infty)$.

Proof of the theorem. First, a result of L. Hörmander [8] shows that $\mathcal{P}_{t,\theta}^\sigma$ cannot be L^p-L^q-bounded unless $p\leq q$. Next, $\mathcal{P}_{t,\theta}^\sigma$ cannot be L^p-L^q-bounded when $q<p_\theta$, by the sufficient condition of J. L. Clerc and E. M. Stein [1] and the fact that $\widetilde{p}_{t,\theta}^\sigma$ continues analytically to $\mathbf{T}_{\delta(p_\theta)}$ but to no larger tube. By duality, $\mathcal{P}_{t,\theta}^\sigma$ cannot be L^p-L^q-bounded if $p>p_\theta{}'$. This proves (i). Observe that parts (ii) to (ix) imply that $\mathcal{P}_{t,\theta}^\sigma$ is L^p-L^q-bounded if $p\leq q$, $p\leq p_\theta{}'$, and $q\geq p_\theta$.

We now prove (ii), for p in $[p_\theta,2]$. Define Λ_p to be $i\delta(p)\rho$. By C. S. Herz' *principe de majoration* [7] and spherical Fourier analysis, we have that

$$|||\mathcal{P}_{t,\theta}^\sigma|||_{p;p} = \int_G p_{t,\theta}^\sigma(x)\,\phi_{\Lambda_p}(x)\,dx = \widetilde{p}_{t,\theta}^\sigma(-\Lambda_p) = \exp\big(-R_{\theta,p}^{2\sigma}\,t\big) \qquad \forall t\in(0,\infty).$$

By duality, this result also holds for p in $[2, p_\theta{}']$.

Estimate (iii) is a consequence of the theory of ultracontractive semigroups, combined with the fact that $\|\mathcal{P}^\sigma_{t,\theta}\|_{1;\infty} = \left\|p^\sigma_{t,\theta}\right\|_\infty$ and the estimate for $\left\|p^\sigma_{t,\theta}\right\|_\infty$ in Lemma 3 above. See, e.g., Cowling and Meda [4], E. B. Davies [5], or N. Th. Varopoulos *et al.* [9].

Next we prove (iv). By duality, it suffices to treat the case where $p < 2$ and $q = 2$. On the one hand, by the Kunze–Stein phenomenon [2], $\|\mathcal{P}^\sigma_{t,\theta}\|_{p;2} \leq C \left\|p^\sigma_{t,\theta}\right\|_2$, so that

$$\|\mathcal{P}^\sigma_{t,\theta}\|_{p;2} \leq C\, t^{-\nu/4} \exp(-R_\theta^{2\sigma} t) \qquad \forall t \in [1,\infty),$$

from Lemma 3. On the other hand,

$$\left\|h_1 * p^\sigma_{t,\theta}\right\|_2 \leq \|\mathcal{P}^\sigma_{t,\theta}\|_{p;2} \left\|h_1\right\|_p = C \, \|\mathcal{P}^\sigma_{t,\theta}\|_{p;2} \qquad \forall t \in [1,\infty);$$

by the Plancherel formula, formula (1), and Lemma 2,

$$\begin{aligned}
\left\|h_1 * p^\sigma_{t,\theta}\right\|_2 &= \left[\int_{\mathfrak{a}^*} \exp\bigl(-2tQ_\theta(\Lambda)^\sigma - 2Q_0(\Lambda)\bigr)\, d\mu(\Lambda)\right]^{1/2} \\
&\sim \left[\int_0^\infty \exp\bigl(-2t[r^2 + R_\theta^2]^\sigma - 2[r^2 + R_0^2]\bigr)\, (1+r)^{n-\nu}\, r^{\nu-1}\, dr\right]^{1/2} \\
&\geq C\left[\int_0^1 \exp\bigl(-2t[r^2 + R_\theta^2]^\sigma\bigr)\, r^{\nu-1}\, dr\right]^{1/2} \\
&\sim t^{-\nu/4} \exp(-R_\theta^{2\sigma} t) \qquad \forall t \in [1,\infty),
\end{aligned}$$

so that

$$\|\mathcal{P}^\sigma_{t,\theta}\|_{p;2} \geq C\, t^{-\nu/4} \exp(-R_\theta^{2\sigma} t) \qquad \forall t \in [1,\infty).$$

It follows that $\|\mathcal{P}^\sigma_{t,\theta}\|_{p;2} \sim t^{-\nu/4} \exp(-R_\theta^{2\sigma} t)$ for all t in $[1,\infty)$, and (iv) is proved.

To prove (v), we proceed similarly. On the one hand, from (iv),

$$\|\mathcal{P}^\sigma_{t,\theta}\|_{p;q} \leq \|\mathcal{P}^\sigma_{t/2,\theta}\|_{p;2} \, \|\mathcal{P}^\sigma_{t/2,\theta}\|_{2;q} \sim t^{-\nu/2} \exp(-R_\theta^{2\sigma} t) \qquad \forall t \in [1,\infty).$$

On the other hand,

$$\|\mathcal{H}_{1,0}\|_{1;p} \, \|\mathcal{P}^\sigma_{t,\theta}\|_{p;q} \, \|\mathcal{H}_{1,0}\|_{q;\infty} \geq \left\|h_2 * p^\sigma_{t,\theta}\right\|_\infty \sim t^{-\nu/2} \exp(-R_\theta^{2\sigma} t) \qquad \forall t \in [1,\infty),$$

by an argument similar to that used in the proof of (iv) above. This completes the proof of (v).

We now prove (vi). On the one hand, Theorem 2.2 in [3] and Lemma 3 may be invoked to show that

$$\|\mathcal{P}^\sigma_{t,\theta}\|_{p;q} \leq C \left\|p^\sigma_{t,\theta}\right\|_q \leq C\, t^{-\ell/2q'} \exp\bigl(-R_{\theta,q}^{2\sigma}\, t\bigr) \qquad \forall t \in [1,\infty).$$

On the other hand,

$$\left\|h_1 * p^\sigma_{t,\theta}\right\|_q \leq \left\|\left|\mathcal{P}^\sigma_{t,\theta}\right|\right\|_{p;q} \left\|h_1\right\|_p = C \left\|\left|\mathcal{P}^\sigma_{t,\theta}\right|\right\|_{p;q} \qquad \forall t \in [1,\infty).$$

As argued to prove (2), if $\mathfrak{b}$ again denotes the unit ball in $\mathfrak{a}^*$,

$$\begin{aligned}\left\|h_1 * p^\sigma_{t,\theta}\right\|_q &\geq C \left[\int_{\mathfrak{a}^*} \left|\widetilde{h}_1(\Lambda + i\delta(q)\rho)\, \widetilde{p}^\sigma_{t,\theta}(\Lambda + i\delta(q)\rho)\right|^{q'} d\widetilde{\mu}(\Lambda)\right]^{1/q'} \\ &\geq C \left[\int_{\mathfrak{b}} \left|\widetilde{h}_1(\Lambda + i\delta(q)\rho)\, \widetilde{p}^\sigma_{t,\theta}(\Lambda + i\delta(q)\rho)\right|^{q'} d\Lambda\right]^{1/q'} \\ &\geq C \left[\int_{\mathfrak{b}} \left|\widetilde{p}^\sigma_{t,\theta}(\Lambda + i\delta(q)\rho)\right|^{q'} d\Lambda\right]^{1/q'} \qquad \forall t \in [1,\infty),\end{aligned}$$

since for Λ in $\mathfrak{b}$, $\left|\widetilde{h}_1(\Lambda + i\delta(q)\rho)\right|$ is bounded away from 0. This integral was treated in the proof of Lemma 3 (see (3)), and we conclude that

$$\left\|\left|\mathcal{P}^\sigma_{t,\theta}\right|\right\|_{p;q} \geq C\, t^{-\ell/2q'} \exp\left(-R^{2\sigma}_{\theta,q}\, t\right) \qquad \forall t \in [1,\infty),$$

completing the proof of (vi).

Finally, (vii) is proved in the same way as (vi), but part (iv) of Lemma 3 is used instead of part (iii), and (viii) and (ix) follow from (vi) and (vii) by duality. $\square$

References

[1] Clerc, J. L., and E. M. Stein, *L^p multipliers for noncompact symmetric spaces*, Proc. Nat. Acad. Sci. U.S.A. **71** (1974), 3911–3912.

[2] Cowling, M. G., *The Kunze–Stein phenomenon*, Ann. of Math. **107** (1978), 209–234.

[3] Cowling, M. G., S. Giulini and S. Meda, *L^p–L^q estimates for functions of the Laplace–Beltrami operator on noncompact symmetric spaces. I*, Duke Math. J. **72** (1993), 109–150.

[4] Cowling, M. G., and S. Meda, *Harmonic analysis and ultracontractivity*, Trans. Amer. Math. Soc. **340** (1993), 733–752..

[5] Davies, E. B., "Heat Kernels and Spectral Theory," Cambridge University Press, Cambridge, 1989.

[6] Helgason, S., "Groups and Geometric Analysis," Academic Press, 1984.

[7] Herz, C. S., *Sur le phénomène de Kunze–Stein*, C. R. Acad. Sci. Paris (Série A) **271** (1970), 491–493.

[8] Hörmander, L., *Estimates for translation invariant operators in L^p spaces*, Acta Math. **104** (1960), 93–140.

[9] Varopoulos, N. Th., L. Saloff–Coste, and T. Coulhon, "Analysis and Geometry on Groups," Cambridge University Press, Cambridge, 1992.

M. G. Cowling
School of Mathematics
University of New South Wales,
Sydney N.S.W. 2052
Australia
e-mail: m.cowling@unsw.edu.au

S. Giulini
Dipartimento di Matematica
Università di Genova
via L. B. Alberti 4
I–16132 Genova, Italia
e-mail: giulini@dima.unige.it

S. Meda
Dipartimento di Matematica
Politecnico di Milano
via Bonardi 9
I-20133 Milano, Italia
e-mail: stemed@ipmma1.mate.polimi.it

Received November 28, 1994