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Good properties of algebras of invariants and defect of linear representations

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0. Introduction

If a reductive group G acts on an algebraic variety X, then the complexity of X, or of the action, is the minimal codimension of orbits of a Borel subgroup. The concept of complexity has the origin in theory of equivariant embeddings of homogeneous spaces. It has been shown in the fundamental paper by Luna and Vust [11] that an exhaustive theory can be developed for homogeneous spaces of complexity 0 and 1. Varieties of complexity 0 are usually called *spherical* and in this paper varieties of complexity one will be called *c1*-varieties. A full description of possible embeddings in the spirit of the Luna-Vust theory involves an explicit presentation of complicated combinatorial objects (e.g. the set of invariant valuations). Therefore up to now there are no applications of this theory to homogeneous spaces of complexity ≥ 2. In a sense, 'complexity equals 1' is a level separating good homogeneous spaces (from the point of view of Embedding Theory) and bad ones.

On the other hand, varieties of small complexity come into play in a number of other problems of Invariant Theory. As an example, one can mention a solution of Hilbert's 14^{th} problem for varieties of complexity ≤ 1 [7]. Another aspect is that appearance of a variety of complexity ≤ 1 frequently ensures that several related algebras of invariants, or in geometric terms, related quotient varieties have a simple enough structure. The main purpose of my paper is to present this relationship explicitly. But, if in the spherical case one gets a series of proved assertions, in *c1*-case majority of assertions exists as conjectures. Nevertheless, all results to be discussed in the paper aimed to the confirmation of the following

General principle: *spherical property implies that some related algebras of invariants are polynomial; c1-property implies that some related algebras of invari-*

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ants are complete intersections.

All algebraic varieties are defined over an algebraically closed field k of characteristic zero. Unless otherwise stated, all varieties are assumed to be irreducible, and a G-variety is a variety, endowed with a regular left action of an algebraic group G.

In sect. 1, we describe three types of algebras of invariants, which naturally attached to the given G-variety X. These are the algebra of covariants 1.1, invariants of the coisotropy representation (if X is a homogeneous space) 1.3, and invariants of the doubled action 1.2. If X is spherical, then under respective constraints these algebras appears to be polynomial. If X is a *c1*-variety, the desired result should be that these algebras are complete intersections. But at the moment, I can fulfill only one third of this program, i.e. there are a theorem on the algebra of covariants 1.6 and two conjectures 3.1,3.2. The proof of the theorem immediately amounts to the case of tori actions, and the latter is elaborated in sect. 2. In sect. 3, we confine ourselves with linear actions. We then prove that these two conjectures are consequences of a third one 3.5. A justification of such an abundance of conjectures is that all known to me examples confirm them and that our last conjecture resembles Popov's conjecture, which was stated almost 20 years ago and still is not disproved. Moreover, this one is checked for many classes of linear actions (see [24, 8.7] about history of the problem). To compare the conjectures, we introduce some notation.

Let $G \subset GL(V)$ be a finite dimensional linear representation of a reductive group and $\pi : V \to \operatorname{Spec} k[V]^G =: V/\!\!/G$ the quotient morphism. Define the *defect of equidimensionality* of G-module V to be the (non-negative) integer $\operatorname{def}_G(V) = \dim \pi^{-1}(\pi(0)) - \dim V + \dim V/\!\!/G$.

Popov's conjecture [18] *Suppose G is connected and $\operatorname{def}_G(V) = 0$. Then $V/\!\!/G$ is an affine space.*

Our conjecture looks as follows.

Conjecture 3.5. *Suppose G is connected, $\operatorname{def}_G(V) = 1$, and V is a self-dual G-module. Then $V/\!\!/G$ is a complete intersection (maybe even at worst a hypersurface).*

We shall prove this is true when G is a torus 3.10. The following is an illustration of these statements in the non-commutative case.

Example. Let $G = SL_2$ and V an irreducible G-module. As is well-known, any such V is self-dual. Then

- $\operatorname{def}_G(V) = 0 \Leftrightarrow \dim V \le 5$;
- $\operatorname{def}_G(V) \le 1 \Leftrightarrow \dim V \le 7$.

It is classical that $V/\!\!/G$ is an affine space, if $\dim V \le 5$, and $V/\!\!/G$ is a hypersurface, if $\dim V = 6$ or 7.

Our main reference in Invariant Theory is [24]. In general we will follow the notation and terminology therein.

1. Three types of algebras of invariants

First we fix notation and recall the relevant background about the complexity of actions of reductive groups. Throughout this paper, G denotes a connected reductive group with a selected Borel subgroup B and a selected maximal torus $T \subset B$. Denote by U the unipotent radical of B. The group operation in the character group $\mathcal{X}(T)$ of T will be written additively and $\mathcal{X}(T)$ is being considered as a lattice in $\mathbb{Q}$-vector space $E = \mathcal{X}(T) \otimes_{\mathbb{Z}} \mathbb{Q}$. We fix also respective subsets of E: the root system, the set of dominant weights $\mathcal{X}(T)_+$. Given a set M equipped with an action of a group $\tilde{G}$, then $M^{\tilde{G}}$ denotes the subset of $\tilde{G}$-fixed elements in it. We let $k[X]$ denote the algebra of regular functions and $k(X)$ denote the field of rational functions on an algebraic variety X. We say that affine X is factorial, if $k[X]$ is a unique factorization domain. Let G act on a variety X. The integer $c_G(X) = \min_{x\in X}\mathrm{codim} Bx$ is called the *complexity* of G-variety X. The integer $r_G(X) = \min_{x\in X}\mathrm{codim} Ux - \min_{x\in X}\mathrm{codim} Bx$ is called the *rank* of G-variety X. By the Rosenlicht theorem (see e.g. [24, 2.3]) $c_G(X) = \mathrm{trdeg}\, k(X)^B$ and $c_G(X) + r_G(X) = \mathrm{trdeg}\, k(X)^U$.

1.1 Algebra of covariants. Let X be an affine G-variety. The algebra of covariants $A_I := k[X]^U$ is finitely generated as well and we let $X/\!\!/U$ denote the respective affine variety. The natural T-action on $k[X]^U$ inverts $X/\!\!/U$ into a T-variety and determines the weight decomposition

$$k[X]^U = \bigoplus_{\lambda} k[X]^U_\lambda,$$

where $k[X]^U_\lambda = \{f \in k[X]^U \mid t\cdot f = \lambda(t)f, \text{ for any } t \in T\}$ and λ runs over all dominant weights. Let us denote

$$\Gamma = \Gamma_G(X) = \{\lambda \in \mathcal{X}(T)_+ \mid k[X]^U_\lambda \neq 0\}.$$

Since X is irreducible, it is clear that Γ is a (finitely generated) subsemigroup of $\mathcal{X}(T)_+$. We call it the *rank semigroup* of X, since $r_G(X)$ is dimension of the linear subspace of E generated by Γ [12]. The following properties of algebras of covariants are obvious:

1. If X is factorial, then so is $X/\!\!/U$.
2. If $k[X]$ does not have non-constant invertible functions, then so is $k[X]^U$.
3. $k(X)^U$ is the field of fractions of $k[X]^U$. Therefore $\dim X/\!\!/U = c_G(X)+r_G(X)$.

1.2 Doubled action. Let X be as in 1.1. Denote by X^* the dual G-variety. We refer to [12] or [14] for the definition of X^*. In this paper, we will be mostly interested in the case when X is a G-module (see sect. 3.) Then X^* is nothing but the dual G-module. Consider the diagonal ('doubled') action of G on $X\times X^*$ and set

$$A_{II} := k[X\times X^*]^G .$$

The doubled action is always stable [16]. This implies, in particular, that $k(X\times X^*)^G$ is the field of fractions of $k[X\times X^*]^G$. Therefore by [12, ch. 1] $\dim X\times X^*/\!\!/G = 2c_G(X) + r_G(X)$.

1.3 Coisotropy representation. Let $X = G/H$ be an arbitrary homogeneous space. Denote by the respective small Gothik letters the Lie algebras of algebraic groups, e.g. $\mathfrak{h} = \text{Lie}H$. Denote by $\mathfrak{m}$ the orthogonal complement of $\mathfrak{h}$ with respect to a selected G-invariant inner product on $\mathfrak{g}$. Obviously, $\mathfrak{m}$ is a H-submodule of $\mathfrak{g}$. The representation $H \subset GL(\mathfrak{m})$ will be referred to as the *coisotropy representation* of G/H. Observe that $\mathfrak{h} \cap \mathfrak{m}$ is the nilpotent radical of $\mathfrak{h}$, and if H is reductive, then the coisotropy representation is orthogonal, i.e. $H \subset SO(\mathfrak{m})$. We set $A_{III} = k[\mathfrak{m}]^H$. The field of fractions of $k[\mathfrak{m}]^H$ coincides with $k(\mathfrak{m})^H$ at least, if H is reductive, or G/H is spherical. It allows us to conclude that in these cases also $\dim(\mathfrak{m}/\!\!/H) = 2c_G(X) + r_G(X)$ [6],[12].

We have thus defined three types of algebras of invariants. In the sequel, it will be referred to as algebras of type A_I, A_{II}, and A_{III} respectively.

1.4 Theorem. *Let us include the condition $c_G(X) = 0$ in constructions described in* 1.1, 1.2, *and* 1.3, *in other words, apply these constructions to spherical varieties. Suppose also that X is factorial and without non-constant regular invertible functions in* 1.1, 1.2 *and that H is connected in* 1.3. *Then algebras A_I, A_{II}, and A_{III} will be polynomial.*

Proof. 1. For A_I-case: see [8, III.3.1]. It essentially is equivalent to that in this case Γ is a free semigroup and $k[X]^U$ is the semigroup algebra.
2. For A_{II}-case: see [17, sect. 2].
3. For A_{III}-case: see [6, 7.2] or [12, ch. 3]. ■

It is worth to stress that spherical case is, as usual, a simplest one, because then several important conditions are automatically satisfied. By definition, spherical variety contains an open B-orbit. Therefore it is rational, contains also an open G-orbit, and $k[X]^G = k$. Second, any normal spherical X has only rational singularities [20] (see also [1] and [2] about rational singularities).

One of the main results in this paper concerns algebras of type A_I for *c1*-varieties, see 1.6. In order to state it, we recall some definitions. Given an affine variety Y, the minimal N such that there exists a closed embedding $Y \subset \mathbb{A}^N$ is called the *embedding dimension* of Y and is denoted by $\text{embdim}\, Y$. We set $\text{hd}\, k[Y] = \text{embdim}\, Y - \dim Y$. If $k[Y]$ is a Cohen-Macaulay graded domain, this integer is customary called the homological dimension of Y. Recall that Y (or its coordinate algebra $k[Y]$) is said to be a *complete intersection*, if there is a closed embedding $Y \subset \mathbb{A}^N$ such that the ideal of polynomials vanishing on Y is generated by $N - \dim Y$ elements. A *hypersurface* is a variety Y such that $\text{hd}\, Y = 1$.
Following [24, 8.3], we shall say that a variety Y is *strongly simply connected* (=s.s.c.), if for any closed subvariety $Z \subset Y$ of codimension at least 2 the variety $Y \setminus Z$ is simply connected. We mean here the algebraic fundamental group. In the case $k = \mathbb{C}$, one may also use the topological concept of the fundamental group. By the Zariski-Nagata theorem, a smooth simply connected variety is strongly simply connected. Therefore a normal variety X is s.s.c. if and only if this is true for its non-singular locus X^{reg}. The property of being s.s.c. is often inherited by the algebras of invariants. Let us describe a sufficiently general framework for this phenomenon.

1.5 Proposition. *Let X be an affine normal variety acted upon by an algebraic group $\hat{G}$. Suppose $k[X]^{\hat{G}}$ is finitely generated and the quotient morphism $\pi_{\hat{G},X} : X \to X/\!\!/\hat{G}$ has the following property: if $D \subset X$ is a $\hat{G}$-invariant divisor then $\pi_{\hat{G},X}(D)$ is a divisor as well. Then $X/\!\!/\hat{G}$ is s.s.c. whenever X is.*

Proof. In this situation, one can repeat, *mutatis mutandis*, the (topological) proof of Proposition 8.4 in [24]. It is also not difficult to give a purely algebraic proof (cf. e.g. [5, §1]). ∎

An essential feature of *c1*-varieties is that the algebra $k[X]^G$ is not necessarily trivial. It forces us to distinguish 2 cases, which may be considered as different generalizations of the spherical situation.

1.6 Theorem. *Let X be an affine unirational factorial variety acted upon by G, $c_G(X) = 1$, and $k[X]$ does not contain non-constant invertible functions.*
(1) *Assume $k[X]^G = k$. Then $X/\!\!/U$ is a complete intersection. Moreover,*
(a) *if X has only rational singularities, then $hd\,X/\!\!/U \le dim\,X/\!\!/U - 1$;*
(b) *if X is s.s.c. and has only rational singularities, then $hd\,X/\!\!/U \le dim\,X/\!\!/U - 2$.*
(2) *Assume $k[X]^G \ne k$ and $\Gamma \setminus \{0\}$ is contained in an open half-space of E. Then $X/\!\!/U$ is an affine space (of dimension $r_G(X) + 1$).*

Proof. It immediately amounts to the case of toric actions, because the T-variety $X/\!\!/U$ inherits all the required properties of the G-variety X. Indeed, $k(X/\!\!/U)^T = k(X)^B$, i.e. $c_T(X/\!\!/U) = c_G(X)$, and $k[X/\!\!/U]^T = k[X]^B = k[X]^G$. By the trivial reason, the rank semigroup of G-variety X coincides with that of T-variety $X/\!\!/U$. The assertion on descent of rationality of singularities is found in [20], and those on factoriality and invertible functions are standard. The part on the property of being s.s.c. follows from 1.5. The toric case will be elaborated in sect. 2. (see 2.1, and 2.3). ∎

1.7 Remark. As we will see in 2.2, the conditions of the part (1) in the theorem ensure that $\Gamma \setminus \{0\}$ is contained in an open half-space of E as well. The geometric counterpart of this property in our context is that the T-action on $X/\!\!/U$ is fixed-pointed, i.e. fixed points are the only closed T-orbits in $X/\!\!/U$. A sufficient condition for it is that G is semisimple, for then $\mathcal{X}(T)_+$ generates a strictly convex cone in E. I can say nothing about not fixed-pointed torus actions of complexity one. Fortunately, fixed-pointed actions are only important for applications. Instead of conditions of unirationality and $k[X]^G = k$ in the theorem, it suffices to assume that X contains a dense G-orbit (obviously, this is possible only if G has a non-trivial semisimple part).

The best class of varieties covered by theorem 1.6 deserves a special formulation.

1.8 Corollary. *Let H be a connected subgroup of a simply connected semisimple group G. Suppose $c_G(G/H) = 1$ and the character group of H is trivial. Then $k[G/H]^U$ is a complete intersection.*

Proof. By [7], the algebra $k[G/H]$ is finitely generated. Take X to be the respective affine variety. Then all the assumptions of 1.6 are fulfilled. (Factoriality of X follows from that of G and absence of characters of H.) ■

The key role in our considerations in sect. 2. plays the following result about the rank semigroup, cf. [13, §1]. It explains in part, why one might expect that some good properties would take place for *c1*-varieties.

1.9 Theorem. *Let X be an affine unirational factorial variety acted upon by G. Suppose $c_G(X) = 1$, $k[X]^G = k$ and $k[X]$ does not contain non-constant invertible functions. Then*
(1) $m_\lambda := \dim k[X]^U_\lambda < \infty$ *for any* $\lambda \in \Gamma$;
(2) *there is a unique* $\mu \in \Gamma$ *such that*
  (i) $m_\mu = 2$;
  (ii) *if* $\omega \in \Gamma$, $\omega - e\mu \in \Gamma$, *and* $\omega - (e+1)\mu \notin \Gamma$ $(e \in \mathbb{N})$, *then* $m_\omega = e+1$;
(3) *elements of any basis of* $k[X]^U_\mu$ *are relatively prime in* $k[X]^U$.

Proof. (1). It follows from the standard fact that $k[X]^U_\lambda$ is a finitely generated module over $k[X]^G$ [24, §3].
(2),(3). Since $\mathrm{trdeg} k(X)^B = 1$ and X is unirational, the field extension $k(X)^B/k$ is purely transcendental (the Lüroth theorem). Therefore $k(X)^B = k(t)$ for some rational function t. Write it as an irreducible fraction: $t = p/q$, where $p, q \in k[X]$. Then both p and q are semi-invariants of B of the same weight $\mu \in \Gamma$, i.e. $p, q \in k[X]^U_\mu$. This character μ is exactly what we need. By definition, p and q are relatively prime and, hence, the same holds for any basis of 2-dimensional subspace they are generate. Let us show μ satisfies the other conditions of the theorem. Take any $h \in k[X]^U_\mu$. Then $h/q \in k(t)$. After simple transformations, one obtains an equality in $k[X]^U$ of the form:

$$h(\sum_{i=0}^{l} \alpha_i p^i q^{l-i}) = q(\sum_{j=0}^{l} \beta_j p^j q^{l-j})$$

or, in more symmetric form,

$$h\prod_{i=1}^{l}(a_i p + b_i q) = \prod_{j=1}^{l+1}(c_j p + d_j q) \ .$$

Since different linear forms in p, q are relatively prime, h must be one of them. Hence, $m_\mu = 2$. A similar argument proves the remaining assertions. ■

Definition. The character μ that satisfies all the conditions of the theorem is said to be *remarkable*.

1.10 Examples. 1. Let $G = Sp_4$, $H = SL_2$, and the embedding $H \subset G$ is defined by the 4-dimensional irreducible representation of H. The homogeneous space G/H is of complexity 1 and rank 2. The algebra of covariants on it has been explicitly described in [4]. Its dimension is 3 and it has 4 generators, i.e. it is a hypersurface. Under the appropriate choice of generators the single relation is of the form: $a^3b^2 + c^3 + d^2 = 0$. The weights of generators are

$4\varphi_1, 3\varphi_2, 4\varphi_1 + 2\varphi_2, 6\varphi_1 + 3\varphi_2$ and the remarkable weight is $12\varphi_1 + 6\varphi_2$, where φ_i's are the fundamental weights of Sp_4.
2. Let $P \supset B$ be a minimal parabolic subgroup defined by a simple root α, and U_P its unipotent radical (G is simply connected and semisimple). It is easy to see that $c_G(G/U_P) = 1$ and $k[G/U_P]^U$ is a polynomial algebra of dimension $\dim T+1$. The fundamental weight corresponding α is remarkable here.

1.11 Remark. Actually, in all known examples related with 1.6 and actions of non-commutative groups on sufficiently good varieties, one obtains that $\mathrm{hd}\, X/\!\!/U \le 1$, i.e. $X/\!\!/U$ is at worst a hypersurface. For instance, I checked it in a number of linear actions and also in the following cases:

1. X is an affine homogeneous space and G is simple (see the table of such X in [13]).
2. X is an affine double cone and G is simple [16, §3].

2. Torus actions of complexity one

Let a torus T act on an affine variety Y. Then $k[Y] = \bigoplus_{\lambda\in\Gamma} k[Y]_\lambda$. Recall that $\Gamma = \Gamma_T(Y) \subset \mathcal{X}(T) \subset E$ is the rank semigroup. We keep in mind that Y could appear as a variety of the form $X/\!\!/U$. The purpose of this section is to prove the toric version of theorem 1.6. One part of proof is rather simple and another one is much involved.

2.1 Theorem. *Let Y be an affine unirational factorial variety acted upon by an algebraic torus T. Suppose $k[Y]$ does not contain non-constant invertible functions, $k[Y]^T \ne k$, $c_T(Y) = trdeg\, k(Y)^T = 1$, and $\Gamma \setminus \{0\}$ is contained in an open half-space of E. Then Y is an affine space.*

Proof. Consider the quotient morphism $\pi_T : Y \to Y/\!\!/T$. In our setting, $Y/\!\!/T$ is an affine smooth rational curve without non-constant invertible functions. Therefore it is an affine line and π_T is flat. Any isotypic component $k[Y]_\lambda$ is a flat(=free) $k[Y]^T$-module of rank 1 (the latter easily follows from the fact that the generic fibre of π_T is a toric variety). Let p_λ be its generator. Since Γ lies in an open half-space of E, the concept of the minimal generator system of Γ is well-defined. Let $\lambda_1, \dots, \lambda_s$ be that system. Then any $p_i := p_{\lambda_i}$ is prime in $k[Y]$. Assume Γ is not free and $\sum_i k_i\lambda_i = \sum_j l_j\lambda_j$ is a relation. Denote by ν the element of Γ represented by both parts of this equality. An easy consequence of factoriality is that (up to a scalar multiple) $p_\nu = \prod_i p_i^{k_i}$. But, the equality $\prod_i p_i^{k_i} = \prod_j p_j^{l_j}$ contradicts, in its turn, factoriality. Thus Γ is free and $k[Y]$ is freely generated by $p_1, \dots, p_s$ and a basic T-invariant. ■

In the remaining part of this section, we only consider torus actions without regular invariants.

2.2 Lemma. *Suppose $k[Y]^T = k$ and $k[Y]$ does not contain non-constant invertible functions. Then*
(1) $\Gamma \setminus \{0\}$ *is contained in an open half-space of E;*
(2) *the single closed T-orbit in Y is a (fixed) point;*
(3) *there is a natural embedding of Y into the tangent space of the T-fixed point.*

Proof. 1. Assume not. Then $\sum_i l_i\lambda_i = 0$ for some $\lambda_i \in \Gamma \setminus \{0\}$, where $l_i > 0$. Take a non-zero element $h_i \in k[Y]_{\lambda_i}$. Then $0 \neq \prod h_i^{l_i} \in k[Y]^T = k$, i.e. each h_i is invertible. A contradiction!
2. Since $k[Y]^T = k$, there is only one closed orbit. By the first part of proof, $\mathfrak{M} := \oplus_{\lambda\neq 0} k[Y]_\lambda$ is the T-invariant maximal ideal.
3. By the graded version of Nakayama's lemma, any basis of the ideal $\mathfrak{M}$ generates $k[Y]$. Therefore there is a surjective homomorphism of the symmetric algebra $S^\bullet(\mathfrak{M}/\mathfrak{M}^2)$ of $\mathfrak{M}/\mathfrak{M}^2$ onto $k[Y]$. ■

2.3 Theorem. *Let Y be an affine unirational factorial variety acted upon by an algebraic torus T. Suppose $k[Y]$ does not contain non-constant invertible functions, $k[Y]^T = k$, and $c_T(Y) = trdeg\, k(Y)^T = 1$. Then Y is a complete intersection. Moreover,*

(a) *if Y has rational singularities, then $hdY \leq dimY - 1$;*

(b) *if Y is s.s.c. and has only rational singularities, then $hdY \leq dimY - 2$.*

The proof of the theorem occupies the rest of this section. Our plan looks as follows. First, we introduce a (possibly non connected) subgroup $\hat{T} \subset T$ of codimension 1 and show that $Y/\!\!/\hat{T} = \mathbb{A}^2$. Then we consider a minimal closed embedding $Y \subset V$, where V is a T-module and prove that $V/\!\!/\hat{T}^0$ is an affine space (Popov's conjecture for tori is applied at this point). Next step is to show that $Y/\!\!/\hat{T}^0$ is a complete intersection in $V/\!\!/\hat{T}^0$. This implies that Y is a complete intersection in V as well. At last, we show that $\dim V \leq 2\dim Y - 1$ whenever Y has only rational singularities, and moreover $\dim V \leq 2\dim Y - 2$, if Y is also s.s.c.

Without loss of generality one may assume T acts effectively on Y and then $\dim Y = \dim T + 1$. Further we write A for $k[Y]$ and $A(\lambda)$ for the weight subspace $k[Y]_\lambda$. Let $\mu \in \Gamma$ be the remarkable character of T 1.9 and define $\hat{T}$ to be the kernel of μ. Let p, q be a basis of the space $A(\mu)$. By 1.9, $\dim A(n\mu) = n + 1$, therefore $A(n\mu)$ is the nth symmetric power of $A(\mu)$. Since $A^{\hat{T}} = \bigoplus_{n=0}^{\infty} A(n\mu)$, one sees $A^{\hat{T}}$ is the polynomial algebra in variables p, q. Consider the quotient morphism $\pi_{\hat{T},Y} : Y \to Y/\!\!/\hat{T} \cong \mathbb{A}^2$. According to 2.2, there is a unique closed T-orbit in Y. For a moment, denote it by y_0. The fibre of $\pi_{\hat{T},Y}$ containing y_0 is the subset $\{y \in Y \mid p(y) = q(y) = 0\}$. Since p, q are relatively prime 1.9, it is of codimension 2 in Y. The mapping $\pi_{\hat{T},Y}$ is determined by homogeneous elements of A, therefore the fibre over $0 \in \mathbb{A}^2$, which we denote by $\mathfrak{N}_{\hat{T}}(Y)$, is of maximal dimension. This implies $\pi_{\hat{T},Y}$ is equidimensional, i.e. all its fibers are of dimension $\dim Y - 2 = \dim \hat{T}$.

We let V denote the tangent space of y_0 in Y. It is a T-module and Y is naturally embedded in V 2.2. This embedding sends y_0 into the origin $0 \in V$. Therefore we shall identify them in the sequel. It follows from the constructing that $\Gamma_T(V) = \Gamma_T(Y)$ and $k[V]^T = k$ as well. This implies that $\{0\}$ is the only closed T-orbit in V.

2.4 Lemma. (1) *The defining ideal $\Im$ of Y in V is generated by $\hat{T}$-invariant functions.* (2) *Moreover, $\Im$ is generated by functions of the weight μ.*

Proof. 1. Fix a basis of V consisting of T-weight vectors. Evidently, $\Im$ is generated by T-semi-invariant functions. Take any semi-invariant $g \in \Im$. Then $g = \sum m_i$, where m_i are monomials in T-weight coordinates on V, having the same weight relative to T-action. Denote by $\overline{m}_i$ the restriction of m_i on Y. Then each $\overline{m}_i \in A$ is not zero (the restriction on Y of *any* monomial is not equal zero) and $\sum \overline{m}_i = 0$. There is a $\chi \in \Gamma$ such that all $\overline{m}_i$ belong to $A(\chi)$. Observe that $m_\chi > 1$. (Otherwise, one gets an equality of the form $\overline{m}_i = c\overline{m}_j$, $c \in k \setminus \{0\}$, which contradicts factoriality of Y and the fact that Y is not contained in a proper subspace of V.) Then by 1.9 we have $\chi - \mu \in \Gamma$. Take the maximal $e \in \mathbb{N}$ such that $\delta := \chi - e\mu \in \Gamma$. Since $m_\delta = 1$ and $A(\chi) = A(\delta)A(e\mu)$, all the monomials $\overline{m}_i$ have a common factor $\overline{d} \in A(\delta)$ and $\overline{m}_i/\overline{d} \in A^{\hat{T}}$. By lifting it in $k[V]$, one finds a monomial h such that $g/h \in k[V]^{\hat{T}}$ and it is clear that $g/h \in \Im$.
2. According to the first part of the proof, $\Im$ is generated by functions of the weights $k\mu$, $k \in \mathbb{N}$. We let $\Im_0$ denote the ideal in $k[V]$ generated by the functions of the weight μ in $\Im$. Let us prove $\Im = \Im_0$. Choose two monomials $\tilde{p}, \tilde{q} \in k[V]^{\hat{T}}$ such that their images in A constitute a basis of $A(\mu)$. Without loss of generality, one may think these images are p, q. We let $F_l(\tilde{p}, \tilde{q})$ denotes a form of degree l in $\tilde{p}, \tilde{q}$.

Claim. *If $F_l(\tilde{p}, \tilde{q}) \in \Im$, then it equals identically zero, i.e. all the coefficients of F_l equal* 0.
(It immediately follows from the primeness of $\Im$ and constructing $\tilde{p}, \tilde{q}$.)
Let $\sum m_i \in \Im$ be a function of the weight $k\mu$, $k > 1$. Then $\overline{m}_i \in A(k\mu)$ for each i, i.e. $\overline{m}_i = F_k^{(i)}(p, q)$. This means $\overline{m}_i$ is a product of k linear forms in p, q. That is, $\overline{m}_i = \prod_{j=1}^{k} \overline{m}_i^{(j)}$, where $\overline{m}_i^{(j)} = F_1^{(j)}(p, q)$. On the other hand, $\overline{m}_i$ is a monomial in prime elements of A (=the images of coordinates on V). Therefore each $\overline{m}_i^{(j)}$ is the image of a monomial $m_i^{(j)} \in k[V]$. Hence one gets the functions $m_i^{(j)} - F_1^{(j)}(\tilde{p}, \tilde{q}) \in \Im_0$. It is now easy to see that $\sum m_i$ is the sum of a function from $\Im_0$ and a form of degree k in $\tilde{p}, \tilde{q}$. It thus follows from the claim that $\sum m_i \in \Im_0$ and we are done. ∎

Consider the quotient mappings $\pi_{\hat{T},Y} : Y \to Y/\!\!/\hat{T}$ and $\pi_{\hat{T},V} : V \to V/\!\!/\hat{T}$. Denote by $\mathfrak{N}_{\hat{T}}(V)$ the fibre of $\pi_{\hat{T},V}$ that contains zero, and recall that it is of maximal dimension [18]. By 2.4, we have $\mathfrak{N}_{\hat{T}}(Y) = \mathfrak{N}_{\hat{T}}(V)$. Therefore

$$\dim \hat{T} \le \dim V - \dim V/\!\!/\hat{T} \le \dim \mathfrak{N}_{\hat{T}}(V) = \dim \mathfrak{N}_{\hat{T}}(Y) = \dim \hat{T}.$$

Thus $\pi_{\hat{T},V}$ is equidimensional as well. Since the identity component $\hat{T}^0$ is a torus and Popov's conjecture for tori is proved [23], [25], one may conclude $V/\!\!/\hat{T}^0$ is an affine space, say of dimension s.

Choose a one-dimensional subtorus T_1 of T such that $T \cong \hat{T}^0 \times T_1$. Then there is a finite group $F \subset T_1$ such that $\hat{T} \cong \hat{T}^0 \times F$. Consider the following commutative diagram:

$$\begin{array}{ccccc} 0 \in & Y & \longrightarrow & V & \\ & \Big\downarrow \pi_{\hat{T}^0,Y} & & \Big\downarrow \pi_{\hat{T}^0,V} & \\ \overline{0} \in & Y/\!\!/\hat{T}^0 & \longrightarrow & V/\!\!/\hat{T}^0 & \cong \mathbb{A}^s \\ & \Big\downarrow \pi_F & & \Big\downarrow & \\ \mathbb{A}^2 \cong & Y/\!\!/\hat{T} & \longrightarrow & V/\!\!/\hat{T} & \end{array} \tag{2.5}$$

All the vertical arrows here are quotient morphisms and the horizontal ones are closed embeddings. The torus T_1 acts on all varieties here and the whole diagram is T_1-equivariant, i.e. all the mappings commute with T_1-action. Consider the action of T_1 on $V/\!\!/\hat{T}^0$. It defines, a priori, a $\mathbb{Z}$-grading on $k[V/\!\!/\hat{T}^0]$. Since $k[V/\!\!/\hat{T}^0]^{T_1} = k[V]^T = k$, we have $\overline{0} := \pi_{\hat{T}^0,V}(0)$ is the only closed T_1-orbit in $V/\!\!/\hat{T}^0$. Therefore one obtains, in fact, a $\mathbb{N}$-grading on $k[V/\!\!/\hat{T}^0]$ and $k[Y/\!\!/\hat{T}^0]$. We let W denote the tangent space to $Y/\!\!/\hat{T}^0$ at $\overline{0}$. Since $k[V/\!\!/\hat{T}^0]$ does not contain non-constant invertible functions and $k[V/\!\!/\hat{T}^0]^{T_1} = k$, there is natural embedding of $Y/\!\!/\hat{T}^0$ in W 2.2.

2.6 Lemma. *If $Y/\!\!/\hat{T}^0$ is a complete intersection in W, then Y is a complete intersection in V.*

Proof. We let I and J denote the maximal ideals of $\overline{0}$ in $Y/\!\!/\hat{T}^0$ and $V/\!\!/\hat{T}^0$ respectively. Then $W = (I/I^2)^*$ and $V/\!\!/\hat{T}^0 \cong (J/J^2)^*$. In other words, $k[W] = S^\bullet(I/I^2)$ and $k[V/\!\!/\hat{T}^0] \cong S^\bullet(J/J^2)$. The restriction homomorphism $k[V/\!\!/\hat{T}^0] \to k[Y/\!\!/\hat{T}^0]$ induces a surjective linear mapping $J/J^2 \to I/I^2$. Now the commutative diagram

$$\begin{array}{ccc} S^\bullet(J/J^2) & \xrightarrow{\sim} & k[V/\!\!/\hat{T}^0] \\ \Big\downarrow & & \Big\downarrow \\ S^\bullet(I/I^2) & \longrightarrow & k[Y/\!\!/\hat{T}^0] \end{array}$$

and the assumption show that $Y/\!\!/\hat{T}^0$ is a complete intersection in $V/\!\!/\hat{T}^0$. In order to lift this condition in V, one have to use the first assertion in 2.4. ∎

Thus our next goal is to prove that $Y/\!\!/\hat{T}^0$ is a complete intersection in W. If $\hat{T} = \hat{T}^0$, i.e. $F = \{e\}$, then there is nothing to prove. So further we assume $F \neq \{e\}$. We set $C := k[Y/\!\!/\hat{T}^0] = \bigoplus_{\lambda \in \mathbb{Q}_+\mu} A(\lambda)$. Then $C^F := k[Y/\!\!/\hat{T}] = \bigoplus_{\lambda \in \mathbb{N}\mu} A(\lambda)$. If C is factorial, then it is a complete intersection by [15, thm. 2]. But, since this is not assumed here, one needs another argument. Our tool is that C^F is a polynomial algebra in 2 variables and that the whole stuff lies inside of a large factorial algebra A. However, it will appear as a by-product of our analysis, that C is factorial, at least, if C has only rational singularities 2.10.

Recall that $p, q \in A(\mu)$ generate C^F. Let $a \in C \setminus C^F$ belong to a minimal homogeneous generator system. Then $a \in A(\frac{n}{d}\mu)$, where g.c.d.$(n,d) = 1$ and $d > 1$.

2.7 Lemma. $n = 1$.

Proof. One has $a^d \in A(n\mu)$, i.e.

$$a^d = \prod_{i=1}^{l} (\alpha_i p + \beta_i q)^{n_i},$$

where $(\alpha_i : \beta_i) \neq (\alpha_j : \beta_j)$ and $\sum n_i = n$. By 1.9, different factors on the right hand side are relatively prime in A. Therefore all n_i's must be equal, i.e.

$$a^d = (\prod_{i=1}^{l} (\alpha_i p + \beta_i q))^{n_1},$$

and $n = n_1 l$. Since A is factorial, there is a $\chi \in \Gamma$ and a $z \in A(\chi)$ such that $z^{n_1} = a$, $z^d = \prod_i (\alpha_i p + \beta_i q)$. Hence $\chi = \frac{l}{d}\mu$ and $z \in C$. Therefore $n_1 = 1$, otherwise a does not belong to a minimal generator system. If $l > 1$, then similar arguments give us $a = b_1 b_2$, where $b_i \in C$. Thus $l = 1$ and $n = 1$. ■

Choose a minimal generator system of C an let $a_1, \ldots, a_r$ be its part lying in $C \setminus C^F$ (to get the whole generator system, one has to take, perhaps, something from $A(\mu)$). By 2.7 one has $a^{d_i} = \alpha_i p + \beta_i q$ for some $d_i > 1$.

2.8 Lemma. (1) *g.c.d.*$(d_i, d_j) = 1$ *and* (2) $\alpha_i \beta_j - \alpha_j \beta_i \neq 0$, *if* $i \neq j$.

Proof. 1. Assume *g.c.d.*$(d_i, d_j) = d_{ij} > 1$. Put $b_i = d_i/d_{ij}$ and $b_j = d_j/d_{ij}$. Then $a_i^{b_i}, a_j^{b_j} \in A(\frac{1}{d_{ij}}\mu)$. By 1.9 $\dim A(\frac{1}{d_{ij}}\mu) = 1$, i.e. $a_i^{b_i} = a_j^{b_j}$. Hence, there is a $c \in A$ such that $c^{b_j} = a_i$ and $c^{b_i} = a_j$. Again, by weight arguments one gets $c \in C$ and a_i, a_j can not thus belong together to a minimal generator system.
2. It goes through similar to the previous part. ■

The following assertion is a straightforward consequence of 2.8.

2.9 Proposition. 1. *If* $r \geq 2$ *then* $a_1, \ldots, a_r$ *form a minimal generator system of* C, *and* $Y /\!\!/ \hat{T}^0$ *is a complete intersection with* $hd(Y /\!\!/ \hat{T}^0) = r - 2$.
2. *If* $r = 1$, *then* C *is a polynomial ring in two variables (*a_1 *and any* $t \in A(\mu)$ *such that* $a_1^{d_1}$ *and* t *are linearly independent).*
3. $|F| = \prod d_i$ *and even* $F = \mathbb{Z}_{d_1} \times \ldots \times \mathbb{Z}_{d_r}$.

Proof. (1). Since $a_1^{d_1}, a_2^{d_2}$ compose a basis of $A(\mu)$, the a_i's already generate C. Let

$$\sum \alpha_{i_1 \ldots i_r} a_1^{i_1} \ldots a_r^{i_r} = 0$$

be a homogeneous relation. Since all the monomials have the same weight, one sees that the number $\frac{i_1}{d_1} + \ldots + \frac{i_r}{d_r}$ does not depend on a monomial. It then follows from the mutual coprimeness of d_i's that $d_1 \,|\, (i_1 - j_1), \ldots, d_r \,|\, (i_r - j_r)$ for any two monomials $a_1^{i_1} \ldots a_r^{i_r}$ and $a_1^{j_1} \ldots a_r^{j_r}$. This implies that any relation is a consequence of a one for $a_i^{d_i}$'s. Obviously, the basic relations between them are the ones of the form $a_i^{d_i} = \zeta_i a_1^{d_1} + \eta_i a_2^{d_2}$, for each $i \geq 3$ and some $\zeta_i, \eta_i \in k \setminus \{0\}$.
(2), (3): Obvious. ■

Describing a minimal generator system of C is nothing else but describing the tangent space W. Thus, taking into account 2.6, we have proved that Y is a complete intersection. It only remains to prove the assertions on homological dimension.

2.10 Proposition. *Suppose Y has only rational singularities. Then $Y/\!\!/\hat{T}^0$ is either $\mathbb{A}^2$, or the hypersurface $\{ (x,y,z) \mid x^2+y^3+z^5=0\} \subset \mathbb{A}^3$. Moreover, if Y is s.s.c., then $Y/\!\!/\hat{T}^0 \cong \mathbb{A}^2$.*

Proof. 1. Assume Y has only rational singularities. By Boutot's theorem [1], $Y/\!\!/\hat{T}^0$ has rational singularities as well. According to 2.9, it suffices to treat the case $r \ge 3$. Then $\dim W = r$ and the ideal of $Y/\!\!/\hat{T}^0$ is generated by the functions $w_i^{d_i} - \zeta_i w_1^{d_1} - \eta_i w_2^{d_2}$ $(i \ge 3)$, where $\zeta_i, \eta_i \in k \setminus \{0\}$ and w_i's are coordinates on W. The surface $Y/\!\!/\hat{T}^0$ is normal and $\overline{0}$ is the only singular point on it. Since C is positively graded, $\overline{0}$ is a quasihomogeneous isolated singularity. Let us apply Flenner's criterion of rationality of singularities to it. We set $\deg w_i = (\prod_{j=1}^{r} d_j)/d_i$. Then the weight of each generating relation is $\prod_j d_j$. By [2, 3.8], the singularity in $\overline{0}$ is rational if and only if

$$(r-2)\prod_{i=1}^{r} d_i < \sum_{i=1}^{r}(\frac{1}{d_i}\prod_{j=1}^{r} d_j)$$

or $r-2 < \sum_i \frac{1}{d_i}$. Since $d_i > 1$ and g.c.d.$(d_i,d_j)=1$, the only solution of this inequality is $r=3$ and $\{d_1,d_2,d_3\}=\{2,3,5\}$.
2. Assume now that Y is also s.s.c. Since the mapping $\pi_Y : Y \to Y/\!\!/\hat{T}^0$ is equidimensional, all the assumptions of 1.5 are satisfied. Therefore $Y/\!\!/\hat{T}^0$ is s.s.c. as well. Now by Mumford's smoothness criterion for surfaces (see e.g. [5, th. 2.3]), one obtains that $\overline{0}$ is a smooth point and we are done. ■

2.11 An upper bound for hd Y. Our goal is estimating of dimension V. Let $V = V_1 \oplus V_2$, where V_1 is the subspace of $\hat{T}^0$-invariant vectors and V_2 is its $\hat{T}^0$-invariant complement. Since $\pi_{\hat{T}^0,V}$ is equidimensional, $\pi_{\hat{T}^0,V_2}$ is equidimensional as well. Then an easy application of the Hilbert-Mumford criterion shows $\dim V_2 \le 2\dim \hat{T}^0 = 2\dim Y - 4$. In order to estimate $\dim V_1$, one has the following inequalities:

$$\begin{aligned}\dim V_1 &= \#\{\ \hat{T}^0\text{-invariant elements in a minimal generator system of } A\ \}\\ &\le \#\{\ \text{minimal generator system of } C\ \}.\end{aligned}$$

After 2.10, we know that if Y has only rational singularities then C is generated by at most 3 elements, and if Y is also s.s.c. then C is generated by 2 elements. Hence $\dim V_1 \le 3$ or 2, and we are done.

The proof of 2.3 is thus finished. But, I think the estimating of homological dimension is not sharp and could be improved.

2.12 Remark. The ideal $\Im$ of polynomials vanishing on Y has a pretty structure in a basis of V consisting of the weight vectors. It follows from 2.4 that it is generated by $\hat{T}$-invariant polynomials of the weight μ. Since $\dim A(\mu) = 2$, one obtains $\Im$ is generated by polynomials that are the sum of at most 3 monomials. Actually, exactly 3 monomials, because a relation which is the sum of 2 monomials contradicts factoriality of A. Let $\tilde{p}, \tilde{q}$ be as in the proof of2.4. Then $\Im$ is generated by the polynomials of the form $\alpha_i\tilde{p} + \beta_i\tilde{q} - m_i$, where m_i's are some monomials. By the reason of factoriality of A, different monomials in these functions depend on different coordinates.

3. The defect of G-modules

In the previous section the *c1*-analog of the assertion in 1.4 on algebras of type A_I was proved. The following conjectures are *c1*-analogs of the assertions in 1.4 on algebras of type A_{II} and A_{III}.

3.1 Conjecture. *Let G/H be an affine homogeneous space (i.e. H is reductive). Suppose $c_G(G/H) = 1$ and H is connected. Then $k[\mathfrak{m}]^H$ is a complete intersection (maybe even at worst a hypersurface).*

Bearing in mind the following results of this section, we state the conjecture in the affine case. But I believe that if the affine version is true, then this also holds with weaker assumptions (e.g. G/H is quasiaffine).

3.2 Conjecture. *Let X be an affine factorial unirational c1-variety. Suppose $k[X]$ does not contain non-constant invertible functions. Then $k[X \times X^*]^G$ is a complete intersection (maybe even at worst a hypersurface).*

Let us present several examples illustrating these conjectures.

3.3 Examples. 1. Let G be a simple algebraic group. One finds in [13] the list of all affine homogeneous G-spaces of complexity one with connected isotropy subgroup. The case-by-case considerations show that algebras of invariants of coisotropy representations (i.e. of type A_{III}) are at worst hypersurfaces. Example 1.10 is one of the items of this list. Taking the coisotropy representation for Sp_4/SL_2 results in 7-dimensional irreducible representation of SL_2 (cf. example in Introduction).
2. Suppose $G = (SL_2)^n$ and H is a $(n–1)$-dimensional subtorus having non-trivial projection on each simple factor of G. Then the coisotropy representation is of the form: $\mathfrak{m} = W + W^* + \mathfrak{m}_0$, where $\mathfrak{m}_0$ is trivial one-dimensional module, W and W^* are dual H-modules, and the weights of H on W are connected by a linear relation with positive coefficients. A simple computation then shows that $k[\mathfrak{m}]^H$ is a hypersurface (cf. 3.10).
3. Suppose $G = SL_2 \times T_1$ and $R(3)$ is the space of 4-dimensional irreducible representation of SL_2 (one-dimensional torus T_1 acts on $R(3)$ by homotheties). It easily follows by dimension reason that $c_G(R(3)) = 1$. Generators and relations for the algebra $k[R(3) + R(3)^*]^{SL_2}$ were found in XIX century [3]. (Observe that $R(3) \cong R(3)^*$ as SL_2-module, but not as G-module.) By using it, one can check that $k[R(3) + R(3)^*]^G$ is generated by invariants of degrees $2, 4, 6, 8, 12$ and the single relation is of degree 24.

Further, we restrict ourselves in 3.2 to the case when $X = V$ is a G-module. Then V^* is the dual G-module and both conjectures concern invariants of linear representations; moreover, in both cases these representations are self-dual. The purpose of this section is to show that the value of complexity in the case of algebras of type A_{II} and A_{III} imposes constraints on dimension of the fibers of the respective quotient morphisms 3.6,3.8. This relationship leads rather naturally to a plausible conjecture. But, we first state the conjecture and mutual relations inside of the whole stuff will be clarified after that.
One may put an idea forth that the best quotient morphisms are equidimensional ones. Hence, it is reasonable to introduce a measure of the deviation from equidimensionality. Let $\hat{G} \subset GL(V)$ be a rational representation of an algebraic group $\hat{G}$. Suppose the algebra of invariants $k[V]^{\hat{G}}$ is finitely generated and let $\pi_{\hat{G},V} : V \to V/\!\!/\hat{G}$ be the quotient morphism. Denote by $\mathfrak{N}_{\hat{G}}(V)$ the null-cone $\pi^{-1}_{\hat{G},V}(\pi_{\hat{G},V}(0))$.

Definition. The integer $\mathrm{def}_{\hat{G}}(V) := \dim \mathfrak{N}_{\hat{G}}(V) - (\dim V - \dim V/\!\!/\hat{G})$ is said to be the *defect (of equidimensionality)* of $\hat{G}$-module V.

Clearly $\mathrm{def}_{\hat{G}}(V) \geq 0$ and it follows from [18] that $\pi_{\hat{G},V}$ is equidimensional if and only if $\mathrm{def}_{\hat{G}}(V) = 0$. Let us give a simple property of the defect.

3.4 Lemma. *Let $\hat{H} \subset \hat{G}$ be a subgroup, and $W \subset V$ be a H-invariant subspace. Suppose the restriction homomorphism $k[V] \to k[W]$ inverts $k[W]^{\hat{H}}$ into a finite $k[V]^{\hat{G}}$-module. Then $\mathrm{def}_{\hat{G}}(V) \leq \mathrm{def}_{\hat{H}}(W)$.*

Proof. It is immediate that $\dim V/\!\!/\hat{G} = \dim W/\!\!/\hat{H}$ and $\mathfrak{N}_{\hat{H}}(W) = \mathfrak{N}_{\hat{G}}(V) \cap W$. Then the assertion follows by the standard inequality for dimension of intersections in smooth varieties. ■

At the rest of this section we work only with reductive groups and affine varieties. Now we are in position to state our next conjecture.

3.5 Conjecture. *Suppose G is connected reductive, V is a self-dual G-module, and $\mathrm{def}_G(V) = 1$. Then $V/\!\!/G$ is a complete intersection (maybe even at worst a hypersurface).*

The case of finite groups shows that the connectivity condition is essential here. There is an example of 4-dimensional representation of one-dimensional torus ($t \mapsto \mathrm{diag}\,(t,\, t^4,\, t^{-2},\, t^{-3})$) showing that this does not hold for not self-dual representations. However, my trust in this conjecture is based on the facts that it holds for representations of tori 3.10, resembles Popov's conjecture (cf. Introduction), and implies the preceding ones. In order to show it, I shall prove two propositions that yield a connection between defect and algebras of types A_{II} and A_{III}.
In what follows we use the concept of the *stabilizer in general position* (=s.g.p.), which is discussed in [24, §7], and a theory developed in [12] and [16], which yields, in particular, relations between s.g.p.'s for different actions under consideration.

3.6 Proposition. *Let G be a connected reductive group and $c_G(G/H) = c$. Suppose H is reductive (i.e. G/H is affine). Then $\mathrm{def}_H(\mathfrak{m}) \leq c$.*

Proof. (a) First, assume that the s.g.p. S for H-action on $\mathfrak{m}$ is finite. Then the s.g.p. for the B-action on G/H is also finite [12], whence $\dim H = \dim U - c$. The H-module $\mathfrak{m}$ is orthogonal, therefore by [21, 2.10] we have

$$\begin{aligned}\dim \mathfrak{N}_H(\mathfrak{m}) &\le \frac{\dim \mathfrak{m} - \dim \mathfrak{m}_0}{2} + \frac{\dim H - \dim T(H)}{2} \\ &= \frac{\dim G - \dim \mathfrak{m}_0 - \dim T(H)}{2},\end{aligned}$$

where $T(H)$ is a maximal torus of H and $\mathfrak{m}_0 = \mathfrak{m}^{T(H)}$ is the zero weight subspace. Since $\mathfrak{m}_0 + \mathfrak{t}(H)$ contains a Cartan subalgebra of $\mathfrak{g}$, one obtains $\dim \mathfrak{N}_H(\mathfrak{m}) \le \dim U$. On the other hand, $\dim \mathfrak{m} - \dim \mathfrak{m}/\!\!/H = \dim H - \dim S = \dim U - c$.
(b) Assume $\dim S > 0$. By the Luna-Richardson theorem [10], the restriction mapping $k[\mathfrak{m}]^H \to k[\mathfrak{m}^S]^{N_H(S)}$ is an isomorphism and hence $k[\mathfrak{m}^S]^{N_H(S)^0}$ is a finite $k[\mathfrak{m}]^H$-module. Let us introduce a connected reductive group $K \subset G$ such that $N_G(S)^0 = S^0 \cdot K$ and $S^0 \cap K$ is finite. Since $S \subset H$, we have $N_H(S)^0 = S^0 \cdot (K \cap H)^0$. Therefore, the Luna-Richardson theorem and 3.4 together imply that $\mathrm{def}_H(\mathfrak{m}) \le \mathrm{def}_{K\cap H}(\mathfrak{m}^S)$. Observe that $(K \cap H) \to GL(\mathfrak{m}^S)$ is the coisotropy representation of the homogeneous space $K/K \cap H$, and it follows from the constructing that the s.g.p. for it is finite. Therefore, by (a), $\mathrm{def}_{K\cap H}(\mathfrak{m}^S) \le c_K(K/K \cap H)$. Let us consider the subvariety $(G/H)^S$. By Luna's slice theorem [9] (see also [24, §6] about this theorem), it is smooth (maybe reducible) and each irreducible component is an orbit of $N_G(S)^0$. This easily implies that the component containing the coset eH is isomorphic to $K/K \cap H$. It is a principal component of $(G/H)^S$ in terminology of [16]. Now by [16, 1.9], one has $c_K(K/K \cap H) = c_G(G/H)$. ∎

3.7 Corollary. *Conjecture* 3.5 *implies conjecture* 3.1. ∎

In the following proposition a fact is used that to any G-action on X one can attach a parabolic subgroup containing B [6],[12]. It goes as follows. Let Bx be a generic B-orbit in X. Then $B_x = B \cap S$, where S is a (maybe non-connected) reductive subgroup of G. Moreover, B_x^0 is a Borel subgroup of S^0, and there is a semisimple $t \in B$ such that S lies between the centralizer $Z_G(t)$ and its commutator subgroup. Thus $P = B \cdot S$ is a parabolic subgroup of G. This S appears also as the stabilizer in general position for the G-action on $X \times X^*$ [12].

3.8 Proposition. *Let V be a G-module and $c_G(V) = c$. Then $\mathrm{def}_G(V \oplus V^*) \le c - \dim V^L$, where L is a Levi factor of a parabolic, which is attached to the given G-action on V.*

Proof. The method of proof is close to that of 3.6. If $c = 0$, this was proved in [17].
(a) First, assume that the s.g.p. S for the G-action on $V \oplus V^*$ is finite. Then, as explained before, there is a point $v \in V$ such that B_v is finite and, hence, $U_v = \{e\}$. Therefore $\dim V/\!\!/U = \dim V - \dim U$ and, by 1.2, $\dim (V \oplus V^*)/\!\!/G = \dim V - \dim U + c$. On the other hand, making use of [21, 2.10] one gets a bound for dimension of the null-cone: $\dim \mathfrak{N}_G(V \oplus V^*) \le \dim V + \dim U - \dim V^T$. Thus $\mathrm{def}_G(V \oplus V^*) \le c - \dim V^T$. In this case a parabolic attached to the action is B.

(b)Now assume that $\dim S > 0$. Again, by the Luna-Richardson theorem, we have $(V\oplus V^*)/\!\!/G \cong (V\oplus V^*)^S/\!\!/N_G(S)$. Let us introduce $K \subset G$ as in 3.6. By 3.4, we then have $\mathrm{def}_G(V\oplus V^*) \le \mathrm{def}_K((V\oplus V^*)^S)$. By constructing, the s.g.p. for the K-action on $(V\oplus V^*)^S$ is equal $K\cap S$, i.e. is finite. Therefore, according to (a), $\mathrm{def}_K((V\oplus V^*)^S) \le c_K(V^S) - \dim\,(V^S)^{T(K)}$, where $T(K)$ is a maximal torus of K. Now two final remarks: $S{\cdot}T(K)$ is conjugated to a Levi factor of $S{\cdot}B$ and, by [16, 1.9], one has $c_G(V) = c_K(V^S)$. ∎

3.9 Corollary. *Conjecture* 3.5 *implies* 3.2 *for linear actions.* ∎

The following is the promised confirmation of 3.5 for tori actions.

3.10 Proposition. *Let V be a self-dual T-module, where T is a torus. Suppose $\mathrm{def}_T(V) = 1$. Then $V/\!\!/T$ is a hypersurface.*

Proof. Let us say a representation $T \subset GL(V)$ is decomposable, if there is a decomposition $V = V_1 \oplus V_2$ such that $T\mid_{V_1\oplus V_2} = T_1 \times T_2$, where $T_i \subset GL(V_i)$. Obviously, in this case $V/\!\!/T \cong V_1/\!\!/T_1{\times}V_2/\!\!/T_2$ and $\mathrm{def}_T(V) = \mathrm{def}_{T_1}(V_1) + \mathrm{def}_{T_2}(V_2)$. Hence, taking into account Popov's conjecture for tori, it suffices to treat the indecomposable case. (The indecomposability means, in particular, that $V^T = \{0\}$.) We set $m = \dim T$. Clearly, one can assume the kernel of the action is trivial. It then follows from the Hilbert-Mumford criterion that $\dim V \le 2(m + \mathrm{def}_T(V))$ (regardless of self-duality), i.e. $\dim V \le 2m + 2$. On the other hand, since V is self-dual, we have $\dim V$ is even and $\ge 2m$. If $\dim V = 2m$ then $\mathrm{def}_T(V) = 0$. Thus $\dim V = 2m + 2$. Now, it is easy to see that one can choose a 'Lagrangian' subspace $W \subset V$ such that the weights of T on W are connected by a linear relation with *positive* coefficients. Obviously, the same relation holds for the (opposite) weights of the complementary subspace W^*. That is, one essentially reduces to the situation of example 2 in 3.3. Let $\lambda_1, \ldots, \lambda_{m+1}$ be the weights of T in W and $\sum_i l_i\lambda_i$ the connecting relation with g.c.d.$(l_1, \ldots, l_{m+1}) = 1$. Then $k[V]^T$ is generated by $x_1y_1, \ldots, x_{m+1}y_{m+1}$, $\prod\limits_i x_i^{l_i}$, $\prod\limits_i y_i^{l_i}$, where $\{x_i\}$, $\{y_i\}$ are the coordinates on W and W^* respectively. ∎

3.11 Remark. It is worth to mention that the condition $\mathrm{def}_G(V) \le d$ is *inheritable* in the sense of [24]. This means that the same inequality holds

(1) for any G-invariant subspace $W \subset V$,

(2) For the slice-group of any semisimple element $v \in V$.

To prove the first property, one should slightly modify the argument in [24, 8.2], where the case $d = 0$ is treated. The second property is an easy consequence of Luna's slice theorem [9].

3.12 Examples. In conclusion, I present several examples in support of conjecture 3.5.

1. Making use of the Hilbert-Mumford criterion, one can easily determine all SL_2-modules with defect one. Denote by $R(n)$ an irreducible SL_2-module of dimension $n+1$. Then the list looks as follows: $R(6)$, $R(4) + R(2)$, $R(2) +$

$R(2)+R(2)$, $R(4)+R(1)$, $R(2)+R(2)+R(1)$, $R(2)+R(3)$, $R(2)+R(1)+R(1)$, $R(3)+R(1)$, $R(5)$, $R(1)+R(1)+R(1)$. It is classically known that the last module has a polynomial algebra of invariants and in all other cases this algebra is a hypersurface. (See e.g. the table and references in [19].)
2. Let $G = SL(W)$ and $V = W + W^* + \mathfrak{sl}(W)$. Then V is a self-dual G-module. The respective G-action is stable and has trivial s.g.p. Therefore $\dim V - \dim V/\!\!/G = \dim G = n^2-1$, where $n = \dim W$. On the other hand, by [21, 2.10], one obtains $\dim \mathfrak{N}_G(V) \le \frac{1}{2}(\dim V - \dim V^T) + \dim U = n^2$. That is, $\mathrm{def}_G(V) \le 1$ and, in fact, it equals 1. The algebra of invariants here is a hypersurface and one has a rare opportunity to write explicitly down the generators and the relation. In order to prove that the functions listed below generate the algebra of invariants, one can probably use Classical Invariant Theory. However, I can do it by using a method, which will be described elsewhere. Let $x = (x_1, \dots, x_n)^t$, A, $\xi = (\xi_1, \dots, \xi_n)$ be generic elements in $W, \mathfrak{sl}(W), W^*$ respectively. (A is a matrix and x is a vector-column.) Denote by $\langle\ ,\ \rangle$ the canonical pairing between W and W^*. The generators of the algebra of invariants are

$$\mathrm{tr}\,(A^i),\ i = 2, \dots, n; \quad \langle \xi, A^l x\rangle,\ l = 0, \dots, n-1\ ,$$

$$D_x = \det\,[x, Ax, \dots, A^{n-1}x], \quad D_\xi = \det\,[\xi, \xi A, \dots, \xi A^{n-1}]\,.$$

The single relation is

$$D_x D_\xi = \det\,(\langle \xi, A^{i+j}x\rangle)_{i,j=1}^n.$$

If $i+j \ge n$, the function $\langle \xi, A^{i+j}x\rangle$ is expressed via the generators by the Cayley-Hamilton theorem.
3. Let $G = SL_3$ and $V = \mathfrak{sl}_3 + \mathfrak{sl}_3$. Then $\mathrm{def}_G(V) = 1$ and it was shown in [22] that the corresponding quotient variety is a hypersurface.

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