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On a Theorem of S. Banach

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Abstract. In this note we discuss a Theorem of S. Banach which states that every Baire measurable homomorphism between polish groups is continuous. Furthermore we describe an application to the representation theory of the infinite dimensional unitary group.

I. Measurable and continuous group homomorphisms

In this note we discuss a Theorem of S. Banach which states that every Baire measurable homomorphism between polish groups is continuous. Since the assumption of Baire measurability might be difficult to verify, it is natural to ask to which extend Banach's Theorem holds under the assumption of Borel measurability. For G locally compact rather strong versions of Banach's Theorem can be derived using Haar measure on G (cf. [HR68] and Corollary I.9 below), so that the crucial case is when G is not locally compact.

The difference between Banach's original terminology and modern terminology concerning measurability apparently caused some confusion on the type of measurability assumption needed for Banach's Theorem. For instance, in [Mo76] Banach's Theorem is cited as if it would hold for Borel measurable homomorphisms, and there exist other places in the literature where Banach's Theorem is cited from Moore's paper in the same spirit. To clarify this point, we show in Lemma I.6 below that a Borel measurable function between metric spaces with a separable arcwise connected range is Baire measurable. Thus we end up with a version of Banach's Theorem which only needs Borel measurability for the homomorphism, but separability and arcwise connectedness for the range. We do not know whether the assumptions on the range are really necessary or not.

On the one hand, the version of Banach's Theorem mentioned above does not seem to be well accessible in the literature. Yet, on the other hand, it has quite remarkable applications to the representation theory of infinite dimensional groups which we explain in Section II. Therefore is appears to be worthwhile to put on record a suitable reference for an appropriate version of Banach's

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Theorem.

Definition I.1. (a) Let X be a topological space. A *measurable subset* of X means a Borel subset of X. The set $\mathfrak{B}(X)$ of Borel subsets of X is the σ-algebra generated by the open subsets of X.
(b) A subset F of a topological space X is called *nowhere dense* if its closure $\overline{F}$ does not have any interior point, i.e., $X \setminus \overline{F}$ is a dense open subset of X. We say that F is of *first category* if it is contained in a countable union of nowhere dense sets and of *second category* otherwise.
(c) A topological space X is called a *Baire space* if no subset of first category of X has interior points. ■

Since we will need it in the following we recall the following simple facts on Baire spaces.

Proposition I.2. (a) *If X is a Baire space and $Y \subseteq X$ of first category, then $X \setminus Y$ is a Baire space. Moreover each subset of first category in $X \setminus Y$ is also of first category in X.*
(b) *An open subset of a Baire space is a Baire space.*
(c) *A completely metrizable topological space is a Baire space.*

Proof. (a) Let $F \subseteq X \setminus Y$ be a subset of first category. Then we find nowhere dense subsets $F_n \subseteq X \setminus Y$ with $F = \bigcup_{n=1}^{\infty} F_n$. Let cl_X, resp. $\mathrm{cl}_{X\setminus Y}$, denote the closure of a set in X, resp. $X \setminus Y$. Then the fact that F_n is nowhere dense in $X \setminus Y$ implies that $\mathrm{cl}_{X\setminus Y} F_n = (X \setminus Y) \cap \mathrm{cl}_X F_n$ has empty interior. We conclude that for each open subset $U \subseteq X$ with $U \subseteq \mathrm{cl}_X F_n$ we have $U \cap (X \setminus Y) = \emptyset$, i.e., $U \subseteq Y$. Since Y is of first category in X, the assumption that X is Baire implies that U is empty. This shows that $\mathrm{cl}_X F_n$ has empty interior, i.e., F_n is also nowhere dense in X. So F is of first category in X.

If $V \subseteq F$ is an open subset of $X \setminus Y$, then there exists an open subset $\tilde{V} \subseteq X$ with $\tilde{V} \cap (X \setminus Y) = V$, i.e., $\tilde{V} \setminus Y = V$. Then $\tilde{V} \subseteq V \cup Y \subseteq F \cup Y$ is of first category in X and therefore empty because X is a Baire space. Thus F has empty interior in $X \setminus Y$, and we see that $X \setminus Y$ is a Baire space.
(b) Let $Y \subseteq X$ be open and $F \subseteq Y$ be a subset of first category. Then we find nowhere dense subsets $F_n \subseteq Y$ with $F = \bigcup_{n=1}^{\infty} F_n$. Let cl_X, resp. cl_Y, denote the closure of a set in X, resp. Y. Then the fact that F_n is nowhere dense in Y implies that $\mathrm{cl}_Y F_n = Y \cap \mathrm{cl}_X F_n$ has empty interior. We conclude that for each open subset $U \subseteq X$ with $U \subseteq \mathrm{cl}_X F_n$ we have $U \cap Y = \emptyset$. Now $X \setminus U$ is closed and contains Y and therefore F_n, hence also $\mathrm{cl}_X F_n$. Thus $U \subseteq X \setminus U$ entails that U is empty, and therefore that F_n is also nowhere dense in X. We conclude that F is of first category in X. If $V \subseteq F$ is an open subset of Y, then V is also open in X, and so $V = \emptyset$ follows from the fact that X is a Baire space, whence Y is also a Baire space.
(c) [Ru73, Th. 2.2] ■

Definition I.3. A mapping $f\colon X \to Y$ between metric spaces is called *Baire measurable* if it is contained in the smallest class of functions $X \to Y$ containing all the continuous functions which is closed under taking pointwise limits. Note that the closedness under pointwise limits is a condition which is stable under arbitrary intersections, so that there is in fact a smallest set of functions with this property containing the continuous functions.

In the literature one finds several different names for this class of Baire measurable functions: they are called “opération measurable (B)” in Banach’s book [Ba32] or “fonction représentable analytiquement” in Lebesgue’s papers. ■

Note that if Y is a discrete space and X is connected, each continuous function $X \to Y$ is constant. Hence all Baire measurable functions $X \to Y$ are constant. It follows in particular that not every Borel function is Baire measurable.

Theorem I.4. (Baire) *If X is a Baire space, Y is a metric space, and $f\colon X \to Y$ Baire measurable, then there exists a subset $I \subseteq X$ of first category such that $f|_{X\setminus I}$ is continuous.*

Proof. (cf. [Ba30]) Since the set of Baire measurable functions is the smallest class of all functions containing the continuous functions which is closed under taking pointwise limits, we have to show that the class of functions satisfying the condition of the theorem is closed under taking pointwise limits because it trivially contains the continuous functions.

Suppose that the restriction of $f_n\colon X \to Y$ to $X \setminus I_n$ is continuous, where I_n is of first category in X. Then $I := \bigcup_{n\in\mathbb{N}} I_n$ is of first category in X, and all functions f_n are continuous on $X_1 := X \setminus I$. Suppose that $f = \lim_{n\to\infty} f_n$ holds pointwise on X. We write $d\colon Y \times Y \to \mathbb{R}$ for the metric on Y. Since the functions f_n are continuous on X_1, the sets

$$A_{n,\varepsilon} := \{x \in X_1\colon (\forall m \geq n) d\big(f_n(x), f_m(x)\big) \leq \varepsilon\}$$

are closed in X_1 and $f = \lim_{n\to\infty} f_n$ implies that $X_1 = \bigcup_{n=1}^\infty A_{n,\varepsilon}$. We put $B_\varepsilon := \bigcup_{n\in\mathbb{N}} A^0_{n,\varepsilon}$, where A^0 denotes the interior of A in X_1. Then B_ε is open and we claim that B_ε is dense in X_1. In fact, let $U \subseteq X_1$ be open. We first use Proposition I.2(a) to see that X_1 is a Baire space and then Proposition I.2(b) to see that U is also a Baire space. Hence $U = \bigcup_{n=1}^\infty (U \cap A_{n,\varepsilon})$ implies that at least one of the sets $U \cap A_{n,\varepsilon}$ is not nowhere dense in U. But these sets are closed subsets of U, so that there exists an $n \in \mathbb{N}$ for which $U \cap A_{n,\varepsilon}$ has interior points in U and therefore also in X_1, i.e., $A^0_{n,\varepsilon} \cap U \neq \emptyset$. Now $B_\varepsilon \cap U \neq \emptyset$ entails that B_ε is dense in X_1. This means that $X_1 \setminus B_\varepsilon$ is closed and has no interior points, i.e., $X_1 \setminus B_\varepsilon$ is nowhere dense. This proves that

$$J := \bigcup_{\varepsilon>0} X_1 \setminus B_\varepsilon = \bigcup_{n=1}^\infty X_1 \setminus B_{\frac{1}{n}}$$

is of first category in X_1.

Let $x \in X_1 \setminus J$ and $\varepsilon > 0$. Then $x \in B_\varepsilon$ and we find an $m \in \mathbb{N}$ with $x \in A^0_{m,\varepsilon}$. Then

$$d\big(f(y), f_m(y)\big) = \lim_{n\to\infty} d\big(f_n(y), f_m(y)\big) \leq \varepsilon$$

for all $y \in A_{m,\varepsilon}$ implies

$$d\big(f(y), f(x)\big) \leq 2\varepsilon + d\big(f_m(y), f_m(x)\big),$$

hence that f is continuous in x because $x \in A^0_{m,\varepsilon}$. This means that f is continuous on $X_1 \setminus J = X \setminus (I \cup J)$. Now the proof is complete because J is of first category in X (Proposition I.2(a)).

This proves that the class of all those functions satisfying the assumptions of the theorem is closed under taking pointwise limits and contains the continuous functions, hence also the Baire measurable functions. ■

Theorem I.5. (Banach) *If G is a metrizable topological group which is a Baire space and H is a metrizable topological group, then every Baire measurable homomorphism $f: G \to H$ is continuous.*

Proof. (cf. [Ba55, p.23, Th. 4]) First Theorem I.4 shows that there exists a subset $I \subseteq G$ of first category such that f is continuous on $G \setminus I$. Let $x_n \to \mathbf{1}$ in G. Then the set $x_n.I \subseteq G$ is of first category for each $n \in \mathbb{N}$. Hence the same holds for

$$I \cup \bigcup_{n \in \mathbb{N}} x_n.I$$

which, in view of the fact that G is of second category, implies that this set must be different from G. Let x be in the complement of this set. Then $x \notin I$ and $x_n^{-1}x \notin I$ for all $n \in \mathbb{N}$. Hence the continuity of f on the complement of I implies that $f(x_n)^{-1}f(x) = f(x_n^{-1}x) \to f(x)$ which in turn implies that $f(x_n) \to \mathbf{1}$. Since G was assumed to be metrizable, i.e., has a countable local base in $\mathbf{1}$, we see that f is continuous in $\mathbf{1}$, and therefore f is continuous because it is a group homomorphism. ■

Next we will weaken the assumption that f is Baire measurable.

Lemma I.6. *Let X, Y be metric spaces, assume that Y is arcwise connected and separable, and $f: X \to Y$ a Borel measurable function. Then f is Baire measurable.*

Proof. First we show that f is the limit of a sequence $(f_n)_{n \in \mathbb{N}}$ of measurable functions with at most countably many values. Let $\varepsilon > 0$ and $(Y_n)_{n \in \mathbb{N}}$ be a basis for the topology consisting of sets whose diameter does not exceed ε. We put $Z_1 := Y_1$ and $Z_n := Y_n \setminus (Y_1 \cup \ldots \cup Y_{n-1})$ for $n > 1$. Deleting those of the Z_n which are empty, we may w.l.o.g. assume that the Z_n are non-empty. Then the sets Z_n are Borel sets and therefore $X_n := f^{-1}(Z_n)$ are Borel subsets of X. Choosing $z_n \in Z_n$ we define a new function $f_\varepsilon: X \to Z$ by $f_\varepsilon(x) := z_n$ for $x \in X_n$. Then

$$d_Y\big(f(x), f_\varepsilon(x)\big) \leq \varepsilon$$

for all $x \in X_n$, and $f_\varepsilon(X)$ is countable. Hence f is a uniform limit of functions with at most countably many values.

So it suffices to assume that $f(X)$ is countable. We write $f(X) = \{y_n : n \in \mathbb{N}\}$ and, using the arc connectedness of Y, we find a continuous function $\gamma: \mathbb{R} \to Y$ with $\gamma(n) = y_n$. We define a real valued function $h: X \to \mathbb{R}$ by

$h(x) := n$ whenever $f(x) = y_n$ and n is minimal with respect to this property. Then h is Borel measurable and $\gamma \circ h = f$.

The set of all functions $u: X \to \mathbb{R}$ for which $\gamma \circ u: X \to Y$ is Baire measurable contains the continuous functions and is closed under taking pointwise limits. This shows that for each Baire function $u: X \to \mathbb{R}$ the function $\gamma \circ u: X \to Y$ is Baire. Since h is a limit of finite linear combinations of characteristic functions, to show that h is a Baire function, it suffices to see that characteristic functions χ_B of Borel subsets $B \subseteq X$ are Baire. In fact, the set of all subsets $B \subseteq X$ for which χ_B is Baire contains all open subsets because for an open subset B we have

$$\chi_B(x) = \lim_{n\to\infty} \min\big(1, n \operatorname{dist}(x, X \setminus B)\big),$$

where $\operatorname{dist}(x, C) := \inf\{d(x,y): y \in C\}$. Furthermore $\chi_{X\setminus B} = 1 - \chi_B$ and

$$\chi_{\bigcap_{n\in\mathbb{N}} B_n} = \lim_{n\to\infty} \prod_{k=1}^{n} \chi_{B_k}.$$

Therefore the fact that the Baire measurable functions $X \to \mathbb{R}$ form an algebra implies that $\{B \subseteq X: \chi_B \text{ Baire measurable}\}$ is a σ-algebra containing all Borel sets. Thus characteristic functions of Borel sets are Baire measurable. This proves that h is Baire measurable, and hence that f is Baire measurable. ■

Theorem I.7. *Every Borel measurable group homomorphism $f: G \to H$ from a completely metrizable separable topological group into an arcwise connected separable metrizable group is continuous.*

Proof. In view of Proposition I.2(c), the group G is a Baire space. Then Lemma I.6 shows that f is a Baire function. Now Banach's Theorem (Theorem I.5) implies that f is continuous. ■

The preceding theorem is cited from [Ba32] in [Mo76] without the assumption that H is arcwise connected. We do not know whether it is true without this assumption or not, but Banach certainly proves it only for Baire measurable homomorphisms.

In the monograph of Hewitt and Ross [HR68, Th. 22.18] one finds another "measurable implies continuity" result for locally compact groups. For the sake of completeness we recall this result for locally compact groups whose proof relies heavily on the use of Haar measure.

Theorem I.8. *Let G be a locally compact group and H a topological group which is σ-compact or separable, λ a left Haar measure for G, and $f: G \to H$ a group homomorphism for which there exists a λ-measurable subset $A \subseteq G$ with $0 < \lambda(A) < \infty$ such that for each open subset $U \subseteq H$ the set $f^{-1}(U) \cap A$ is λ-measurable. Then f is continuous.* ■

Corollary I.9. *Let G be a locally compact group and H a topological group which is σ-compact or separable, and $f: G \to H$ a Borel measurable group homomorphism. Then f is continuous.* ■

II. Applications to representation theory

In this section we show how Banach's Theorem (Theorem I.7) can be applied to study representations of groups which are not necessarily locally compact. The idea for these applications is due to Doug Pickrell (cf. [Pi90]). So this section can be viewed as a working out of his argument.

Proposition II.1. *If $\mathcal{H}$ is a separable Hilbert space and $U(\mathcal{H})$ is the unitary group of $\mathcal{H}$ endowed with the strong operator topology, then $U(\mathcal{H})$ is a completely metrizable topological group.*

Proof. The fact that $U(\mathcal{H})$ is a topological group follows easily from the oberservation that on $U(\mathcal{H})$ the weak and the strong operator topology coincide (cf. [HN93, Cor. 9.4]).

Let $(e_n)_{n\in\mathbb{N}}$ be an orthonormal basis. Then we obtain a map

$$\eta: U(\mathcal{H}) \to \mathcal{H}^{\mathbb{N}}, \quad g \mapsto (g.e_n)_{n\in\mathbb{N}}$$

which is injective and continuous with respect to the product toplogy on $\mathcal{H}^{\mathbb{N}}$ which turns it into a separable metric space. If $g_j \to g$ holds on all basis vectors e_n, $n \in \mathbb{N}$, then $g_j \to g$ holds pointwise on a dense subspace of $\mathcal{H}$ so that $\|g_j\| = 1$ for all j implies that $g_j \to g$ holds pointwise, hence in the strong operator topology. This proves that η is an embedding and therefore that $U(\mathcal{H})$ is metrizable.

Unfortunately $\eta\big(U(\mathcal{H})\big)$ is not a closed subset of $\mathcal{H}^{\mathbb{N}}$, i.e., $U(\mathcal{H})$ is not complete with respect to any inherited metric. The same argument as above shows that the monoid $\operatorname{Iso}(\mathcal{H}) = \{g \in B(\mathcal{H}): g^*g = \mathbf{1}\}$ of all isometries of $\mathcal{H}$ is complete with respect to the metric inherited from $\mathcal{H}^{\mathbb{N}}$. We claim that $U(\mathcal{H})$ is a G_δ-set in the complete separable metric space $\operatorname{Iso}(\mathcal{H}) \hookrightarrow \mathcal{H}^{\mathbb{N}}$. First we note that

$$U(\mathcal{H}) = \{g \in \operatorname{Iso}(\mathcal{H}): g^* \in \operatorname{Iso}(\mathcal{H})\}.$$

Let $(v_n)_{n\in\mathbb{N}}$ be a dense sequence in the closed unit ball $\{v \in \mathcal{H}: \|v\| \leq 1\}$ of $\mathcal{H}$. Then $g^* \in \operatorname{Iso}(\mathcal{H})\}$ is equivalent to $\|g^*.v_n\| = \|v_n\|$ for all $n \in \mathbb{N}$ which in turn means that

$$\sup\{|\langle v_n, g.v_m\rangle|: m \in \mathbb{N}\} = \sup\{|\langle g^*.v_n, v_m\rangle|: m \in \mathbb{N}\} = \|v_n\|.$$

For $n, k \in \mathbb{N}$ the set

$$\begin{aligned} U_{n,k} &:= \{g \in \operatorname{Iso}(\mathcal{H}): (\exists m \in \mathbb{N})|\langle v_n, g.v_m\rangle| > (1 - \frac{1}{k})\|v_n\|\} \\ &= \{g \in \operatorname{Iso}(\mathcal{H}): \|g^*.v_n\| > (1 - \frac{1}{k})\|v_n\|\} \end{aligned}$$

is open and

$$\bigcap_{k,n=1}^{\infty} U_{n,k} = \{g \in \operatorname{Iso}(\mathcal{H}): (\forall n \in \mathbb{N})\|g^*.v_n\| = \|v_n\|\} = U(\mathcal{H}).$$

Thus $U(\mathcal{H})$ is a G_δ-set in the completely metrizable separable topological space $\operatorname{Iso}(\mathcal{H})$ and therefore $U(\mathcal{H})$ itself is completely metrizable (cf. [Sch73, Th. II.1, p.93]). ∎

If $(e_n)_{n\in\mathbb{N}}$ is an orthonormal basis in a separable Hilbert space, then the permutation

$$e_1 \mapsto e_2 \mapsto e_3 \mapsto \ldots \mapsto e_n \mapsto e_1$$

which fixes e_m for $m > n$ defines a unitary map $g_n \in U(\mathcal{H})$ and this sequnence converges strongly to the isometry $g\colon \mathcal{H} \to \mathcal{H}$ mapping $e_n \mapsto e_{n+1}$ for all $n \in \mathbb{N}$. So g is not surjective and therefore not unitary.

We have just seen that for a separable Hilbert space $\mathcal{H}$ the group $U(\mathcal{H})_s$ endowed with the strong operator topology is a Baire space. Let $U(\mathcal{H})_n$ denote the same group endowed with the norm topology. Then the identity map

$$\phi\colon U(\mathcal{H})_n \to U(\mathcal{H})_s$$

is bijective and continuous. Moreover, the map $\phi^{-1}\colon U(\mathcal{H})_s \to U(\mathcal{H})_n$ is Borel measurable. In fact, for each $\varepsilon > 0$ we have

$$\{g \in U(\mathcal{H})\colon \|g - \mathbf{1}\| \leq \varepsilon\} = \bigcap_{n\in\mathbb{N}} \{g \in U(\mathcal{H})\colon \|g.v_n - v_n\| \leq \varepsilon\}$$

for any dense sequence $(v_n)_{n\in\mathbb{N}}$ in the unit ball of $\mathcal{H}$. Since the set on the right hand side is a Borel set, we see that ϕ^{-1} is Borel measurable, so that ϕ is a Borel isomorphism.

Furthermore the group $U(\mathcal{H})_n$ is arc connected because the exponential function

$$\exp\colon i\,\mathrm{Herm}(\mathcal{H}) \to U(\mathcal{H})$$

is surjective (cf. [Ru73, Th. 12.37]). Nevertheless Lemma I.6 does not apply because the group $U(\mathcal{H})_n$ is not separable. Otherwise Theorem I.7 would imply that ϕ is a homeomorphism which is false.

Theorem II.2. (Pickrell's Theorem) *If $\pi\colon U(\mathcal{H})_n \to U(V)_s$ is a continuous representation of $U(\mathcal{H})_n$ on the separable Hilbert space V, then π is also continuous as a homomorphism $U(\mathcal{H})_s \to U(V)_s$.*

Proof. We know that $\pi \circ \phi^{-1}\colon U(\mathcal{H})_s \to U(V)_s$ is a Borel measurable group homomorphism. Further the group $U(V)_s$ is separable (cf. [HN93, Prop. 9.5]) so that Theorem I.7 shows that $\pi\colon U(\mathcal{H})_s \to U(V)_s$ is continuous. ∎

The implications of Pickrell's Theorem are explained in detail in [Pi88] (cf. also [Pi90]). Basically it implies that the separable representation theory of the group $U(\mathcal{H})_n$ is as well behaved as the representation theory of the group $U(\mathcal{H})_s$ which in turn is the same as for the dense subgroup $U_\infty(\mathcal{H}) = U(\mathcal{H}) \cap \big(\mathbf{1} + K(\mathcal{H})\big)$, where $K(\mathcal{H})$ denotes the set of all compact operators on $\mathcal{H}$.

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