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Automorphisms and quasi-conformal mappings of Heisenberg-type groups

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Abstract. The Lie algebras of trace-zero derivations of Heisenberg-type groups are explicitly characterized, along with the connected component of their groups of measure preserving automorphisms. We establish a general criterion on properties of the stabilizer of a lattice in a simply connected nilpotent Lie group and apply it to the full family of H-type Lie groups. A necessary condition for the existence of non-conformal quasi-conformal mappings on H-type groups is also given.

1. Introduction and Background

In this article we study some properties of Lie groups called of *Heisenberg type* which were first introduced by A. Kaplan ([10]) as a generalization of the Heisenberg group itself. We describe the structure of their automorphisms as well as some properties of their lattices.

A Heisenberg type Lie group N (or H-type group) is a connected and simply connected two-step nilpotent Lie group such that its commutator subgroup satisfies: $[N,N]=Z(N)$ and such that on its Lie algebra $\mathfrak{N}$ there is a positive definite real quadratic form $Q(\cdot)=\langle\cdot,\cdot\rangle$ which is *compatible* with the *natural decomposition*,

$$\mathfrak{N}=\mathfrak{Z}\oplus\mathfrak{V} \tag{1}$$

where $\mathfrak{Z}$ is the center of $\mathfrak{N}$ and $\mathfrak{V}$ is its orthocomplement with respect to $\langle\cdot,\cdot\rangle$. Here compatibility refers to Kaplan's basic assumption that the family of operators

$$\{\mathrm{ad}(X) : X\in\mathfrak{V} \mid \langle X,X\rangle=1\} \tag{2}$$

consists of partial isometries of $\ker(\mathrm{ad}(X))^{\perp}$ onto $\mathfrak{Z}$. Kaplan refers to this fact as *property H*. A. Korányi ([11]) has characterized the homogeneous left invariant Carnot-Carathéodori metrics on these manifolds. He has also shown that H-type groups can always be equipped with such a left invariant homogeneous metric. It is important to note that H-type groups, together with their solvable extensions

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$A \ltimes N$ by the one-dimensional groups of dilations have provided counterexamples to several conjectures in differential and spectral geometry (see, e.g. [3], [8], [19]).

The existence of the quadratic form gives rise to two other equivalent algebraic descriptions of H-type groups. Given an H-type Lie algebra $\mathfrak{N} = \mathfrak{Z} \oplus \mathfrak{V}$ with $m = \dim(\mathfrak{Z})$, consider the map $J : \mathfrak{Z} \to End(\mathfrak{V})$ defined, by:

$$\langle J_Z(V), V' \rangle \;=\; \langle Z, [V, V'] \rangle \qquad Z \in \mathfrak{Z},\; V, V' \in \mathfrak{V}. \tag{3}$$

By straightforward computation one can prove that, given an orthonormal basis $\{e_i\}$ of $\mathfrak{Z}$, the family $\{J_{e_i}\}$ satisfies the relations of the generators of the Clifford algebra $Cl(-Q, \mathfrak{Z}) = C_m$. In other words $J : \mathfrak{Z} \to End(\mathfrak{V})$ extends to a unitary representation of C_m on the orthocomplement of the center $\mathfrak{V}$. Thus we say that $\mathfrak{V}$ carries a structure of so called Clifford module over C_m (see [1], pgg. 22 ff.): $\mathfrak{Z}$ acts on $\mathfrak{V}$ as a set of linear transformations satisfying, for any two orthogonal $e_i, e_j \in \mathfrak{Z}$ of unit norm:

$$J_{e_i} J_{e_j} + J_{e_j} J_{e_i} = 0 \;\; and \;\; J_{e_i}^2 = -Id$$

The other description of H-type algebras can be given by using *compositions of quadratic forms*. Let (R, q_R) and (S, q_S) be two real quadratic spaces. Given a normalized composition of their quadratic forms $\mu : R \times S \to S$, we can define a Lie algebra as follows. Consider the dual map $\phi : S \times S \to R$ identified by the linear system

$$\langle r, \phi(s, s') \rangle \;=\; \langle \mu(r, s), s' \rangle, \qquad r \in R,\; s, s' \in S, \tag{4}$$

and an element r_0 satisfying $\mu(r_0, s) = s$ for all $s \in S$. Let π be the orthogonal projection onto $\mathbb{R}r_0^{\perp}$. If we choose $r \in \pi(R)$ and $r' = r_0$, the identity

$$\langle \mu(r, s), \mu(r', s) \rangle = \langle r, r' \rangle q_S(s).$$

implies

$$\langle \mu(r, s), s \rangle = \langle r, \phi(s, s) \rangle = 0.$$

Thus the new map $[\cdot, \cdot] = \pi \circ \phi(\cdot, \cdot)$ is skew symmetric: given (r_1, s_1) and (r_2, s_2) in $\pi(R) \oplus S$ we define write:

$$[(r_1, s_1), (r_2, s_2)] = (\pi \circ \phi(s_1, s_2), 0) \tag{5}$$

The last equations amounts to defining a two-step nilpotent Lie algebra structure on $\mathfrak{N} = \pi(R) \oplus S$. $\pi(R)$ becomes the center and S its orthocomplement. By computation one shows that such an $\mathfrak{N}$ satisfies property (H) and thus it is a Lie algebra of Heisenberg type with the map J in (3) given by: $J_r(s) = \mu(r, s)$ (see [10]).

An H-type algebra $\mathfrak{N}$ is said to be *irreducible* if the Clifford module identified by the map J is irreducible. Given any Clifford algebra with m generators there is up to equivalence one or possibly two (when $m = 3, 7 \; mod(8)$) irreducible

Clifford modules associated to it. The theory of quadratic forms provides a complete classification of all Clifford modules. A detailed account on the construction of these modules is given in [13] (Chapter I, sec. 5). The fact that $\dim(\mathfrak{Z})$ determines the dimension of $\mathfrak{V}$ and that two inequivalent Clifford modules of the same dimension are associated to isomorphic H-type algebras allows us to classify all of them (see [1], pg. 23). A new explicit realization of irreducible H-type Lie algebras for $0 \leq \dim(\mathfrak{Z}) \leq 7$ is given in the next paragraph. In the general case one observes that, since Clifford modules are completely reducible, any H-type Lie algebra $\mathfrak{N}$ is decomposable as: $\mathfrak{N} = \mathfrak{Z} \oplus \mathfrak{V}_1 \oplus \cdots \mathfrak{V}_k$ where all the H-type subalgebras $\mathfrak{Z} \oplus \mathfrak{V}_i$ are irreducible. H-type Lie algebras arise in a very natural way: consider a simple Lie group G of real-rank one. Classification tells us that, either it belongs to one of the families $SO_0(n,1)$, $SU_0(n,1)$ and $\mathrm{Sp}(n,1)$, or it is isomorphic to the exceptional group $F_{4,20}$. Such a group admits an Iwasawa decomposition: $G = K \cdot A \cdot N$. Korányi ([11]) has proved that $\mathfrak{N} = \mathrm{Lie}(N)$ is always of Heisenberg type: $\mathfrak{N}$ can be resp. $\mathbb{R}^{n-1}$, the classical $2(n-1)+1$ dimensional Heisenberg group $\mathfrak{N}_1^{n-1}$ or its quaternionic and octonionic analogues $\mathfrak{N}_3^{n-1}$ and $\mathfrak{N}_7$.

2. Real, Complex, Quaternionic, and Cayley algebras

By making use of the quadratic forms characterization of H-type algebras we will prove a new basic result that makes explicit computations considerably easier. Fix n and i, positive integers, $i \leq 7$ and let $\mathbf{K}^n$ be the quadratic space $\mathbb{C}^n$, $\mathbb{H}^n$ or $\mathbb{O}^n$ where $\mathbb{O}$ are the Cayley numbers. The real vector space $\mathbf{K}^n$ carries the scalar product: $\langle \underline{V}, \underline{V'} \rangle = \sum_k \mathrm{Re}(\underline{V_k} \cdot \overline{\underline{V'}_k}))$. Denote with $\mathbf{K}_i^*$ the quadratic space obtained by restricting the standard quadratic form to an arbitrarily fixed i-dimensional $\mathbb{R}$-subspace of the *imaginary* elements in $\mathbf{K}$. For each k between 1 and n we choose $\mu^k : \mathbf{K}_i^* \times \mathbf{K} \to \mathbf{K}$ to be the left or right composition. That is: $\mu^k(Z,X) = Z \cdot X$ or $\mu^k(Z,X) = X \cdot Z$. Given $\underline{Y} = (Y_1, \ldots, Y_n) \in \mathbf{K}^n$, we let, with slight abuse of notation, $\underline{Y_k} = (0, \ldots, 0, Y_k, 0, \ldots, 0)$, $Y_k \in \mathbf{K}$. Now define the composition of quadratic forms: $\mu : \mathbf{K}_i^* \times \mathbf{K}^n \to \mathbf{K}^n$ by:

$$\mu(X, \underline{Y}) \;=\; \sum_k \mu^k(X, \underline{Y_k}), \qquad \forall X \in \mathbf{K}_i^*,\ \underline{Y} \in \mathbf{K}^n.$$

Now set $R = \mathbb{R} \oplus \mathbf{K}_i^*$, $S = \mathbf{K}^n$ and perform the construction discussed in the introduction. The procedure we just described equips the vector space $\mathbf{K}_i^* \oplus \mathbf{K}^n$ with the structure of an H-type algebra.

In order to determine ϕ we proceed as follows: for simplicity we assume $n = 1$ and $\mu(Z,X) = Z \cdot X$; by (4), with obvious notation,

$$\begin{aligned} \langle \mu_i(Z,X), X' \rangle &= \langle Z, \phi(X,X') \rangle \\ \mathrm{Re}(ZX \cdot \overline{X'}) &= \mathrm{Re}(Z \cdot \overline{\phi(X,X')}), \\ \mathrm{Re}(Z \cdot X\overline{X}') &= \mathrm{Re}(Z \cdot \overline{X'\overline{X}}); \end{aligned}$$

the last identity allows to conclude that

$$\phi(X,X') \;=\; X' \cdot \overline{X} \qquad \forall X, X' \in \mathbf{K}.$$

In case we choose the *right* action ($\mu(Z,X) = X \cdot Z$), the Lie bracket becomes: $[(Z,X),(Z',X')] = (\mathrm{Im}_i(\overline{X}X'),0)$. Note here that while our conclusion is trivial in the associative cases, for $\mathbf{K} = \mathbb{O}$ it can be deduced once the observation is made that the real part of the product of Cayley numbers is in fact associative (cf. e.g. [4], pg. 14).

Proposition 2.1. *Each irreducible H-type Lie algebra $\mathfrak{N} = \mathfrak{Z} \oplus \mathfrak{V}$ with $i = \dim(\mathfrak{Z}) \leq 7$ is isomorphic to an algebra $\mathfrak{N}_i \simeq \mathbf{K}_i^* \oplus \mathbf{K}$.*

Proof. In the case when $i = 0$, we define $\mathfrak{N} \simeq \mathbb{R}^n$ and there is nothing to prove. Otherwise, we note that the mapping $\pi \circ \phi(\cdot,\cdot)$ can be taken as the projection onto $\mathbf{K}_i^*$ of the standard Hermitian product defined on $\mathbf{K} \times \mathbf{K}$. If $i = 1$ we obtain $\mathfrak{N} = i\mathbb{R} \oplus \mathbb{C}$ and so $\mathfrak{N} \simeq \mathfrak{N}_1$, the classical Heisenberg Lie algebra.

In the remaining cases we will prove that we can construct an irreducible $\mathfrak{N}_i$ by direct computation.

If $i = 2,3$ we choose $\mathbf{K} = \mathbb{H}$; this way we can construct two quaternionic compositions of quadratic forms: $\mu_2 : \mathbb{H}_2^* \times \mathbb{H} \to \mathbb{H}$ and $\mu_3 : \mathbb{H}_3^* \times \mathbb{H} \to \mathbb{H}$. Both are defined by the equation:

$$\mu_i(X,Y) = X \cdot Y \qquad X \in \mathbb{H}_i^*, Y \in \mathbb{H}, \ \ i = 2,3$$

We also note that, in the case of $i = 3$, there is an inequivalent Clifford module corresponding to the composition defined by:

$$\mu_3'(X,Y) \ = \ Y \cdot X \qquad X \in \mathbb{H}^*, \, Y \in \mathbb{H}.$$

The Lie algebras with these two (left) compositions will be denoted by $\mathfrak{N}_2 \simeq \mathbb{H}_2^* \oplus \mathbb{H}$ and $\mathfrak{N}_3 \simeq \mathbb{H}_3^* \oplus \mathbb{H}$.

For $i = 4,5,6$ or 7 we choose $\mathbf{K} = \mathbb{O}$; the realizations are exactly the same as for the quaternionic cases. The algebras we get this way are $\mathfrak{N}_i \simeq \mathbb{O}_i^* \oplus \mathbb{O}$, $4 \leq i \leq 7$.

For all the Lie algebras described above the Lie bracket is defined as follows: given two elements (Z,X) and (Z',X') in $\mathfrak{N}_i$, we have:

$$\phi(X,X') = X' \cdot \overline{X}$$

and therefore:

$$[(Z,X),(Z',X')] \ = \ \left(\mathop{\mathrm{Im}}_i(X' \cdot \overline{X}), 0\right),$$

where Im_i is the projection onto the space $\mathbf{K}_i^*$. In doing so, we get the identity

$$\langle J_Z X, X' \rangle = \mathrm{Re}(ZX \cdot \overline{X'}) = \mathrm{Re}(Z \cdot \overline{\mathop{\mathrm{Im}}_i(X'\overline{X})}) = \langle Z, [X,X'] \rangle$$

as required by our definition (4). To prove irreducibility one has to observe that, since $J_Z(X) = Z \cdot X$, given an orthonormal basis $Z_1, \ldots, Z_n$ of $\mathbf{K}$ and any unit vector $X \in \mathbf{K}$ we have:

$$\begin{aligned} \langle Z_s \cdot X, Z_t \cdot X \rangle &= \mathrm{Re}(Z_s X \cdot \overline{Z_t X}) \\ &= \mathrm{Re}(Z_s \cdot X\overline{X} \cdot \overline{Z_t}) \ = \ \|X\|^2 \, \mathrm{Re}(Z_s \overline{Z_t}) \\ &= \delta_{s,t}. \end{aligned}$$

Therefore the family $\{Z_1X, \ldots, Z_nX\}$ is an orthonormal basis for $(\mathbf{K}, \langle\cdot,\cdot\rangle)$. In the case $i=2$ we can consider any two orthonormal vectors $Z_1,\ Z_2 \in \mathbb{H}_2^*$ and set $Z_3 = Z_1 \cdot Z_2$, $Z_4 = Z_1 \cdot \overline{Z_1} = 1$. Given any nonzero vector $X \in \mathbb{H}$, the action of $\mathbb{H}_2^*$ on X determines a submodule $M_X \subset \mathbb{H}$ which, by the previous observation, contains an orthonormal basis for $\mathbb{H}$ given by: $\{\frac{1}{\|X\|}Z_1X, \ldots, \frac{1}{\|X\|}Z_4X\}$, proving irreducibility.

For $i=4$ the same argument can be used by choosing any four orthonormal vectors $Z_1, \ldots, Z_4 \in \mathbb{O}_4^*$ and by observing that for any of them, say Z_4, it should hold that $Z_5 = Z_1 \cdot Z_4, Z_6 = Z_2 \cdot Z_4, Z_7 = Z_3 \cdot Z_4$, are actually orthogonal to $\{Z_1, \ldots, Z_4\}$. To see this just consider that, for imaginary orthonormal octonions Z_i, Z_j and Z_k, the identity

$$\langle Z_i \cdot Z_j, Z_k\rangle \ = \ \delta_{ijk}$$

holds, thanks to the associativity of the real part of the octonionic product.

By setting $Z_8 = Z_4 \cdot \overline{Z}_4 = 1$ we conclude that, given any nonzero $X \in \mathbb{O}$ $\{\frac{1}{\|X\|}Z_1X, \ldots, \frac{1}{\|X\|}Z_8X\}$ is an orthonormal basis for $\mathbb{O}$.

In the cases $i=3$ and $i=5,6,7$ irreducibility follows by choosing $\mathbf{K}_i^*$'s satisfying the inclusion relation $\mathbf{K}_{i-1}^* \subset \mathbf{K}_i^*$.

The above constructed irreducible H-type Lie algebras with centers $\mathfrak{Z}_i$ of real dimensions ranging from zero to seven exhaust all possibilities. ■

3. Automorphisms and Derivations

We start this section with some notation. If N is an H-type Lie group, $M(N)_0$ denotes the connected component of its group of (Haar) measure-preserving automorphisms. If we denote with $Aut(\mathfrak{N})_0$ the connected component of the (Lebesgue) measure-preserving automorphisms of $\mathfrak{N}$, we have that

$$\underset{0}{\mathrm{Der}}(\mathfrak{N}) \ = \ \mathrm{Lie}(\mathrm{Aut}(\mathfrak{N})_0) \ \simeq \ \mathrm{Lie}(M(N)_0)$$

Where $\mathrm{Der}_0(\mathfrak{N})$ is the Lie algebra of trace-zero derivations of $\mathfrak{N}$. Let $D_{\mathfrak{Z}}$ and $D_{\mathfrak{V}}$ be the restrictions of a derivation D resp. to the center and its orthocomplement:

$$D_{\mathfrak{Z}}([X,Y]) \ = \ \mathrm{ad}(D_{\mathfrak{V}}X)(Y) + \mathrm{ad}(X)(D_{\mathfrak{V}}Y) \qquad X, Y \in \mathfrak{V} \tag{6}$$

Therefore, any derivation $D \in \mathrm{Der}(\mathfrak{N})$ can be written as the sum $D_1 + D_2$ of a dilation D_1 and an block triangular matrix D_2:

$$D_1 = \begin{pmatrix} \lambda \cdot Id & 0 \\ 0 & \frac{\lambda}{2} \cdot Id \end{pmatrix} \qquad D_2 = \begin{pmatrix} C & B \\ 0 & A \end{pmatrix}.$$

where $C = D_{\mathfrak{Z}}$ is a square matrix acting on the center, A a square matrix acting on its orthocomplement and B a rectangular block. Note here that, due to 2-step nilpotency, the entries of B are not subject to any conditions. In this situation we see that the derivations of the type:

$$\begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} \tag{7}$$

form a Lie subalgebra of $\mathrm{Der}(\mathfrak{N})_0$ contained in its nilradical. Let us denote the Lie subalgebra of those derivations with $\mathcal{R}'$.

Definition: For the rest of this paper the notation $\mathrm{Aut}(\mathfrak{N})'_0$ indicates the connected component of the group of (Lebesgue) measure-preserving automorphisms of $\mathfrak{N}$, *mod* the non-grading preserving elements of its nilradical.

Let $T = \exp(D)$ be an element of $\mathrm{Aut}(\mathfrak{N})'_0$. Since it preserves the grading $\mathfrak{Z} \oplus \mathfrak{V}$ of $\mathfrak{N}$, the matrices $A = T_{\mathfrak{V}}$ and $C = T_{\mathfrak{Z}}$ satisfy:

$$\langle Z, [AX, AY] \rangle = \langle Z, C([X, Y]) \rangle \qquad X, Y \in \mathfrak{V},\ Z \in \mathfrak{Z}$$

which implies by (3):

$$\langle A^t J_Z(AX), Y \rangle = \langle J_{C^t(Z)}(X), Y \rangle,$$

and therefore

$$A^t \circ J_Z \circ A = J_{C^t(Z)} \qquad \forall Z \in \mathfrak{Z} \tag{8}$$

where J is the operator defining the Clifford module structure.

We immediately get the following

Lemma 3.1. *Given two irreducible H-type Lie algebras $\mathfrak{N}^1$ and $\mathfrak{N}^2$, with $dim(Z(\mathfrak{N}^1)) = dim(Z(\mathfrak{N}^2))\ mod(8)$, it holds:* $\mathrm{Aut}(\mathfrak{N}^1)'_0 \simeq \mathrm{Aut}(\mathfrak{N}^2)'_0$.

Proof. First we observe that, for the Clifford algebras in i and $i+8$ generators, the relation holds: $C_{i+8} = C_i \otimes C_8 = C_i \otimes \mathbb{R}(16)$, where $\mathbb{R}(16) = C_8$ is the full 16-dimensional matrix algebra over $\mathbb{R}$; hence C_{i+8} can be represented as the algebra of matrices $M(16, C_i)$ with coefficients in C_i (see [7], pg. 57). For $1 \leq n, m \leq 16$ we rewrite (m, n)-th entry of the matrix equation (8) for $\mathrm{Aut}(\mathfrak{N}_{i+8})'_0$ as:

$$\sum_{s,t=1}^{16} A^t_{s,m} \cdot X_{s,t} \cdot A_{t,n} = \sum_{k=1}^{16} C_{m,k} \cdot X_{k,n}, \tag{9}$$

where $X_{p,q} \in J(\mathfrak{Z}_i)$ and the blocks $A_{i,j}, C_{i,j}$ are real matrices.

If we choose $X = (X_{p,q})$ such that $X_{p,q} = \underline{0}$ unless $(p, q) = (m, n)$, (9) gives

$$A^t_{m,m} \cdot X \cdot A_{n,n} = C_{m,m} \cdot X, \qquad \forall X \in J(\mathfrak{Z}_i) \tag{10}$$

Since A in (8) is an automorphism and therefore invertible, and $J(\mathfrak{Z}_i)$ contains invertible elements, (10) forces the diagonal blocks of A and C to be invertible. Furthermore, if $n = m$ we get that $A_{m,m} \in Aut(\mathfrak{N}_i)'_0$ for all m's. The RHS in (10) does not depend on n, which in turn implies

$$C_{m,m} X = A^t_{m,m} \cdot X \cdot A_{n,n} = A_{m,m} \cdot X \cdot A_{m.m}, \qquad \forall m,\ \forall X \in J(\mathfrak{Z}_i).$$

Thus $A_{m,m} = A_{n,n}$ for all m and n. To show that $A_{m,n} = \underline{0}$ if $m \neq n$ we proceed in the same way: fix a pair (m, n) with $m \neq n$ and pick $X = (X_{p,q})$ so that $X_{p,q} = \underline{0}$ unless $(p, q) = (m, s)$ where $s \neq n$. This way we obtain:

$$A^t_{m,m} \cdot X_{m,s} \cdot A_{s,n} = \underline{0} \qquad \forall X \in J(\mathfrak{Z}_i)$$

By the same reasoning described above the last equation forces: $A_{m,n} = \underline{0}$ if $m \neq n$ and our proof is complete. ∎

The H-type Lie algebras with center on dimension zero and one are isomorphic to, resp. $\mathbb{R}^n$ and the real Heisenberg algebras $\mathfrak{N}_1^n$. The groups $\mathrm{Aut}(\mathfrak{N})'_0$ are isomorphic to $SL(n,\mathbb{R})$ and $\mathrm{Sp}(n,\mathbb{R})$ (see [15]), while $\mathrm{Aut}(\mathfrak{N}_3^n)'_0 \simeq \mathrm{Sp}(1)\times\mathrm{Sp}(n)$ (see [16], Prop. 10.1) We study the other cases.

To establish our notation we prove the following

Lemma 3.2. *$\mathfrak{N}_2$ is isomorphic to the complexification $\mathfrak{N}_1^{\mathbb{C}}$ of the real Heisenberg Lie algebra.*

Proof. Assume $\mathfrak{N} = \mathfrak{N}_2 \simeq \mathfrak{Z}\oplus\mathfrak{V}$; by the results proven in the previous section we can choose any two-dimensional subspace of $\mathbb{H}^*$ to be equal to $\mathfrak{Z}$. Given the standard basis $\{1,\underline{i},\underline{j},\underline{k}\}$ of $\mathbb{H}$, we set $\mathfrak{Z} = \mathbb{R}\underline{j}\oplus\mathbb{R}\underline{k}$, and define:

$$\underset{2}{\mathrm{Im}}(a+b\underline{i}+c\underline{j}+d\underline{k}) \;=\; c\underline{j}+d\underline{k}.$$

Given any $h\in\mathbb{H}$ can write: $h = z_1 + z_2\underline{j}$, where z_1 and z_2 are two complex numbers. For any $X, Y \in \mathfrak{V}\simeq\mathbb{H}$ we write: $X = x_1 + x_2\underline{j}$, $Y = y_1 + y_2\underline{j}$, so that

$$\begin{aligned}[(0,X),(0,Y)] &= (\underset{2}{\mathrm{Im}}(Y\cdot\overline{X}),0)\\ &= (\underset{2}{\mathrm{Im}}((y_1+y_2\underline{j})\cdot(\overline{x_1}+\overline{x_2\underline{j}})),0)\\ &= ((y_2x_1-y_1x_2)\underline{j},0)\\ &= ((x_1y_2-x_2y_1)\underline{j},0)\end{aligned}$$

With obvious notation we define $\phi:\mathfrak{N}_1^{\mathbb{C}}\to\mathfrak{N}_2$ by:

$$\phi(z_1,z_2,z_3) = (z_3\underline{j}, z_1+z_2\underline{j});$$

by the RHS of the previous equation ϕ is an isomorphism of Lie algebras and the lemma is proven. ∎

Proposition 3.3. $\mathrm{Aut}(\mathfrak{N}_2^n)'_0 \simeq U(1)\ltimes\mathrm{Sp}(2n,\mathbb{C})$

Proof. We consider here $n=1$, the general case being completely analogous. Let $\psi:\mathbb{C}^2\times\mathbb{C}^2\to\mathbb{C}$ be the complex symplectic form defined by: $\psi((x_1,x_2),(y_1,y_2)) = (x_1y_2-x_2y_1)$ so that, with the notation used for the Lemma:

$$[(0,\underline{X}),(0,\underline{Y})] \;=\; (\psi(\underline{X},\underline{Y}),\underline{0}) \;=\; (x_1y_2-x_2y_1,0,0).$$

Let A and C be the restrictions of an automorphism $\phi\in\mathrm{Aut}(\mathfrak{N}_2)'_0$ to, resp., $\mathfrak{V}$ and $\mathfrak{Z}$. First suppose $C = Id$; we then have that $A\in\mathrm{GL}(4,\mathbb{R})$ satisfies:

$$\psi(A(\underline{X}),A(\underline{Y})) \;=\; (x_1y_2-x_2y_1) \;=\; \psi(\underline{X},\underline{Y})$$

Let $s\cdot\underline{X} = (sx_1,sx_2)$ for any $s\in\mathbb{C}$. It holds then:

$$\psi(A(s\cdot\underline{X}),A(\underline{Y})) = s\psi(\underline{X},\underline{Y})$$

which in turn, by the non-degeneracy of ψ, means that A is actually $\mathbb{C}$-linear and thus: $A \in \mathrm{Sp}(2,\mathbb{C})$, implying that all elements of $\mathrm{Aut}(\mathfrak{N}_2)'_0$ acting trivially on the center form a *complex* Lie group isomorphic to $\mathrm{Sp}(2,\mathbb{C})$.

Consider now the general case of an automorphism $\phi \in \mathrm{Aut}(\mathfrak{N}_2)'_0$:

$$[(0, A(\underline{X})), (0, A(\underline{Y}))] \;=\; C(x_1y_2 - x_2y_1) \tag{11}$$

Where $\phi_{\mathfrak{V}} = A \in \mathrm{GL}(4,\mathbb{R})$ and $\phi_{\mathfrak{Z}} = C \in \mathrm{GL}(2,\mathbb{R})$. If we denote with D the derivation in $\mathrm{Der}_0(\mathfrak{N}_2)$ such that $\exp(D) = \phi$ and write $D = \begin{pmatrix} S & 0 \\ 0 & T \end{pmatrix}$, equation (11) can be restated on the Lie algebra as:

$$[T(\underline{X}), \underline{Y}] + [\underline{X}, T(\underline{Y})] \;=\; S([\underline{X}, \underline{Y}]). \tag{12}$$

If we also write T in terms of 2×2 blocks:

$$T \;=\; \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix},$$

and choose complex vectors $X = (x, 0)$, $Y = (0, y)$, we obtain:

$$S(x\cdot y) \;=\; A'(x)\cdot y + x\cdot D'(y).$$

We now let $x = x_1 + ix_2$, $y = y_1 + iy_2$, $S = \begin{pmatrix} s_1 & s_2 \\ s_3 & s_4 \end{pmatrix}$ and so on for the blocks of T; at this point we rewrite the previous identity as:

$$\begin{pmatrix} s_1(x_1y_1 - x_2y_2) + s_2(x_1y_2 + x_2y_1) \\ s_3(x_1y_1 - x_2y_2) + s_4(x_1y_2 + x_2y_1) \end{pmatrix} \;=$$
$$\begin{pmatrix} (a_1+d_1)x_1y_1 - (a_4+d_4)x_2y_2 + (d_2-a_3)x_1y_2 + (a_2-d_3)x_2y_1 \\ (a_3+d_3)x_1y_1 - (a_2+d_2)x_2y_2 + (d_1+a_4)x_1y_2 + (a_4+d_1)x_2y_1 \end{pmatrix}$$

An analogous formula will be obtained by choosing $X = (0, x)$ and $Y = (y, 0)$ in terms of B' and C'. At this point a long and elementary computation shows that the matrices A', B', C' and D' have the form: $\begin{pmatrix} s & t \\ -t & s \end{pmatrix}$ and therefore T is a complex 2×2 matrix. The same immediately follows for S. Since the Lie algebra of a Lie group is complex if and only if the connected component of the underlying Lie group is a complex Lie group we get that C is contained in $\mathrm{GL}(1,\mathbb{C})$ and therefore: $C(z) = z_C \cdot z$ where $z_C \in \mathbb{C}$.

Equation (11) now gives:

$$[A(\underline{X}), A(\underline{Y})] \;=\; z_C \cdot (x_1y_2 - x_2y_1)$$

We observe that for complex numbers s, t such that $s\cdot t = z_C^{-1}$ we can define the complex linear automorphism $T_{s,t} : \mathfrak{N}_2 \to \mathfrak{N}_2$ as:

$$T_{s,t}(z_1, z_2, z_3) = (s\cdot z_1, t\cdot z_2, st\cdot z_3)$$

By doing so it holds: $T_{s,t}\circ\phi \in \mathrm{Sp}(2,\mathbb{C})$. Thus: $\phi = T_{s,t}^{-1}\circ\phi'$ with $\phi' \in \mathrm{Sp}(2,\mathbb{C})$. It is immediate to check that for any $\phi \in \mathrm{Sp}(2,\mathbb{C})$: $T_{s,t}\circ\phi\circ T_{s,t}^{-1} \in \mathrm{Sp}(2,\mathbb{C})$. We note that $T_{s,t}$ is measure preserving if and only if $|s\cdot t| = 1$. We conclude:

$$\mathrm{Aut}(\mathfrak{N}_2)'_0 \simeq U(1) \ltimes \mathrm{Sp}(2,\mathbb{C})$$

and the proposition is proven. ■

We present here a different proof of a known result for $\mathrm{Aut}(\mathfrak{N}_3)'_0$

Proposition 3.4. $\mathrm{Aut}(\mathfrak{N}_3)'_0 \simeq \mathrm{Sp}(1)\times\mathrm{Sp}(1)$

Proof. By using the notation as before, we get that for any $\underline{X}$ and $\underline{Y}$ in the orthocomplement of $\mathfrak{Z}$, the Lie bracket is given by:

$$[\underline{X},\underline{Y}] \;=\; \operatorname*{Im}_3(\underline{Y}\cdot\overline{\underline{X}}) \;=\; \operatorname*{Im}_3(y_1\overline{x_1}+\overline{x_2}y_2+(y_2x_1-x_2y_1)\underline{j}).$$

In other terms, given a $\phi\in Aut(\mathfrak{N}_3)'_0$ that fixes the center, its restriction A to $\mathfrak{V}$ should preserve the complex symplectic form ψ discussed in Proposition 3.3 (hence $A\in\mathrm{Sp}(2,\mathbb{C})$ is a complex transformation) *and* preserve the imaginary part of the form

$$\nu((x_1,x_2),(y_1,y_2)) \;=\; y_1\overline{x_1}+\overline{x_2}y_2.$$

Now, since A is a complex transformation, the latter is equivalent to saying that A actually preserves ν. To see this, consider the identity:

$$\mathrm{Im}(\nu(\underline{X},i\cdot\underline{Y}))=\mathrm{Re}(\nu(\underline{X},\underline{Y}));$$

in other words, for any A preserving the imaginary part of ν:

$$\mathrm{Im}(\nu(A(\underline{X}),A(i\cdot\underline{Y})) \;=\; \mathrm{Re}(\nu(\underline{X},\underline{Y})),$$

but since A is a complex linear transformation:

$$\mathrm{Im}\left(\nu(A(\underline{X}),A(i\cdot\underline{Y}))\right) \;=\; \mathrm{Re}\left(\nu(A(\underline{X}),A(\underline{Y}))\right),$$

this forces:

$$\mathrm{Re}\left(\nu(A(\underline{X}),A(\underline{Y}))\right) \;=\; \mathrm{Re}\left(\nu(\underline{X},\underline{Y})\right),$$

and thus A is a complex norm-preserving transformation: $A\in U(2)\cap\mathrm{Sp}(2,\mathbb{C})\simeq\mathrm{Sp}(1)$, (cf. e.g [6], pg. 80). The action of these automorphisms can be realized as right multiplication by a unit quaternion: $A(X)=X\cdot h$, $\|h\|=1$.

In the case where A acts nontrivially on the center we have:

$$\operatorname*{Im}_3(A(\underline{Y})\cdot\overline{A(\underline{X})}) \;=\; C(\operatorname*{Im}_3(\underline{Y}\cdot\overline{\underline{X}}));$$

which implies, by direct computation (see appendix), that $C\in\mathrm{SO}(3)$. So if we set $A'(\underline{X})=L_{h_C}\circ A=h_C\cdot A(\underline{X})$ for a suitable unit quaternion h_C we have:

$$\operatorname*{Im}_3(A'(\underline{Y})\cdot\overline{A'(\underline{X})}) \;=\; h_C\cdot C(\operatorname*{Im}_3(\underline{Y}\cdot\overline{\underline{X}}))\cdot\overline{h_C} \;=\; \operatorname*{Im}_3(\underline{Y}\cdot\overline{\underline{X}});.$$

Hence, by $\mathbb{H}$-linearity of $\mathrm{Sp}(1)$, $A(\underline{X})=L_{h_C^{-1}}\circ A'=A'\circ L_{h_C^{-1}}$. The same argument shows that $\mathrm{Aut}(\mathfrak{N}_3^n)'_0\simeq\mathrm{Sp}(1)\times\mathrm{Sp}(n)$. ∎

4. $4 \leq \dim(Z) \leq 7$

In analogy to the quaternionic algebras it turns out to be convenient to use the representation of an octonion $c \in \mathbb{O}$ as $c = h_1 + h_2\underline{l}$ where h_1 and h_2 are in $\mathbb{H}$ and $\underline{l}$ one of the unit generators as done in ([4], pg. 15) so that, for any $\underline{X} = x_1 + x_2\underline{l}$, $\underline{Y} = y_1 + y_2\underline{l}$:

$$\begin{aligned}\underline{Y} \cdot \overline{\underline{X}} &= y_1\overline{x_1} + \overline{x_2}y_2 + (y_2x_1 - x_2y_1)\underline{l} \\ &= \nu((x_1, x_2), (y_1, y_2)) + \psi((x_1, x_2), (y_1, y_2))\underline{l};\end{aligned}$$

where the bar is now the quaternionic conjugate and ψ is a so-called anti-Hermitian form (cf.[6],pg. 91). The connected component of the Lie group stabilizing a anti-hermitian form over the vector space $\mathbb{H}^n$ is given by:

$$\mathrm{SU}_\psi(n, \mathbb{H}) = \mathrm{U}(n, n) \cap \mathrm{O}(2n, \mathbb{C}) \cap \mathrm{SL}(n, \mathbb{H}).$$

We are now able to prove the following

Proposition 4.1. $\mathrm{Aut}(\mathfrak{N}_4)'_0$ *is non compact.*

Proof. The convenient choice for $\mathfrak{Z}$ is the vector space $\mathbb{H} \cdot \underline{l}$. After making this choice, the Lie bracket becomes equivalent to:

$$[\underline{X}, \underline{Y}] = \psi(\underline{X}, \underline{Y}).$$

By a modification of the same argument exposed for $\mathfrak{N}_2$ we obtain that the connected component of the group of automorphisms of $\mathfrak{N}_4$ acting trivially on the center is isomorphic to a linear quaternionic group stabilizing the anti-Hermitian form ψ. We denote this group by $SU_\psi(2, \mathbb{H})$, and observe that, since it contains the measure preserving maps of the kind

$$\phi_h(\underline{X}) = (h \cdot x_1, x_2 \cdot h^{-1}), \ h \in \mathrm{GL}(1, \mathbb{H}),$$

it is non-compact. A detailed discussion of these forms and their corresponding orthogonal groups can be found in ([5], chapter I). Finall we observe that the automorphisms acting non-trivially on the center, contain a group isomorphic to the multiplicative group $\mathrm{GL}(1, \mathbb{H})$ given by the maps

$$M_h(\underline{X}) = (x_1 \cdot h, h^{-1} \cdot x_2), \ h \in \mathrm{GL}(1, \mathbb{H}),$$

■

Proposition 4.2. *For* $i = 5, 6, 7$ *the groups* $\mathrm{Aut}(\mathfrak{N}_i)'_0$ *are compact.*

Proof. The proof is achieved by direct computation with the help of a computer running MAPLE on a MATLAB platform.

We first write the equation on the Lie algebra:

$$Im\left(A(Y)\cdot\overline{X} + Y\cdot\overline{A(X)}\right) = C\left(\mathrm{Im}(Y\cdot\overline{X})\right)$$

as a set of $7\times 4 = 28$ linear equations by choosing $X\cdot Y = e_i\cdot e_j$ with $i\neq j$ and $\{e_i\}$ the standard basis of $\mathbb{O}^*$.

The long series of conditions on the coefficients of A and C immediately forces them to be skew-symmetric (see appendix). The same result follows also for the non-irreducible case. ■

We now return to the general case. Let $\mathfrak{N}$ be a H-type Lie algebra with center $\mathfrak{Z}$ of real dimension m admitting a natural decomposition

$$\mathfrak{N} = \mathfrak{N}^a_m = \mathfrak{Z}\oplus\mathfrak{V}^a,$$

where the subalgebra $\mathfrak{N}^n = \mathfrak{Z}\oplus\mathfrak{V}^a$ is defined by $\mathfrak{N}^a_m = \mathfrak{Z}\oplus\mathfrak{V}\oplus\ldots\oplus\mathfrak{V}$. It is easy to adapt the proof of Lemma 3.1 to show that $Aut(\mathfrak{N}^a_m)'_0 \simeq Aut(\mathfrak{N}^a_{m+8})'_0$. If $m = 3,7\ mod(8)$, the subalgebras $\mathfrak{N}_m = \mathfrak{Z}\oplus\mathfrak{V}^a$ and $\mathfrak{N}_m = \mathfrak{Z}\oplus\mathfrak{V}^b$ carry *inequivalent* Lie algebra structures. Thus $Aut(\mathfrak{N})'_0$ act separately on each inequivalent component, so that if $Aut(\mathfrak{Z}\oplus\mathfrak{V}^a)'_0 = A\ltimes B_1$ and $Aut(\mathfrak{Z}\oplus\mathfrak{V}^b)'_0 = A\ltimes B_2$, we have: $Aut(\mathfrak{Z}\oplus\mathfrak{V}^a\oplus\mathfrak{V}^b)'_0 = A\ltimes(B_1\times B_2)$, so that $Aut(\mathfrak{N}^{a,b}_m)'_0 \simeq Aut(\mathfrak{N}^{a,b}_{m+8})'_0$.

5. Stabilizers of Lattices

In this section we establish some general results on lattices of $M(N)$, the group of (Haar) measure-preserving automorphisms of a simply connected nilpotent Lie group N and its Lie algebra, $\mathrm{Der}_0(\mathfrak{N})$. These were first studied by R. Mosak and M. Moskowitz in [15]. There they assumed the quite general connected simply connected nilpotent group had a *log-lattice* Γ —that is: the set $\Lambda = \log(\Gamma)$ is a group in $\mathfrak{N} = \mathrm{Lie}(N)$. By Malcev's results these lattices can always be found in H-type groups. The stabilizer of Γ in $M(N)_0$ defined by:

$$\underset{M(N)_0}{\mathrm{Stab}}(\Gamma) = \{\phi\in M(N)_0 \mid \phi(\Gamma) = \Gamma\}$$

In [15] (Theorem 2.2.), a criterion was developed which shows when $\mathrm{Stab}_{M(N)_0}(\Gamma)$ is a lattice or a uniform lattice in $M(N)_0$. This criterion, established on the Lie algebra $\mathrm{Der}_0(\mathfrak{N}) = \mathrm{Lie}(M(N)_0)$, deals with the radical $\mathcal{R} = \mathrm{Rad}(\mathrm{Der}_0(\mathfrak{N}))$ and its maximal nilpotent ideal $\mathcal{R}_n = \mathrm{Rad}(\mathrm{Der}_0(\mathfrak{N}))_n$:

1. If $\mathcal{R} = \mathcal{R}_n$, then $\mathrm{Stab}_{M(N)_0}(\Gamma)$ is a lattice in $M(N)_0$.
2. If $\mathrm{Der}_0(\mathfrak{N})/\mathcal{R}_n$ is in addition of compact type, then $\mathrm{Stab}_{M(N)_0}(\Gamma)$ is uniform.

The above result remains valid also in the case we replaced $M(N)_0$ by any of its closed subgroups. Furthermore, the following holds:

Theorem 5.1. *Let Γ be a non log-lattice in a connected simply connected nilpotent Lie group N then there is a log-lattice $\Gamma'\subset N$ such that* $\mathrm{Stab}_{M(N)_0}(\Gamma)$ *has finite index in* $\mathrm{Stab}_{M(N)_0}(\Gamma')$.

Proof. One can show ([14], Theorem 2) that there is always a log-lattice Γ_α containing Γ. Since the intersection of two such log-lattices, say Γ_α and Γ_β, is also a log-lattice containing Γ, we can define the minimal object in that class:

$$\Gamma_2 \;=\; \bigcap_\alpha \Gamma_\alpha.$$

Consider now $\phi \in \mathrm{Stab}_{M(N)}(\Gamma)$; we claim that $\phi \in \mathrm{Stab}_{M(N)}(\Gamma_2)$. Let Γ_ϕ be the image of Γ_2 under the automorphism ϕ: $\Gamma_\phi = \phi(\Gamma_2)$. It holds that

$$\Gamma \;\subset\; \Gamma_\phi \cap \Gamma_2;$$

that means, by the minimality assumption: $\Gamma_\phi \;=\; \phi(\Gamma_2) \;\supset\; \Gamma_2$ and therefore $\Gamma_2 = \phi(\Gamma_2)$, thus:

$$\underset{M(N)_0}{\mathrm{Stab}}(\Gamma) \;\subset\; \underset{M(N)_0}{\mathrm{Stab}}(\Gamma_2).$$

Now, it can also be shown that Γ actually contains a log-lattice; let Γ_1 be such a lattice: $\Gamma_1 \subset \Gamma$. If we denote $\Lambda_i = \log(\Gamma_i)$, it follows from the construction of the Γ_i's that for some $K \in \mathbb{N}$:

$$\Lambda_1 \;=\; K \cdot \Lambda_2.$$

Consider now a measure preserving automorphism $\phi \in \mathrm{Stab}_{M(N)_0}(\Gamma_2)$. Its differential ϕ_* will yield:

$$\phi_*(\Lambda_1) \;=\; \phi_*(K \cdot \Lambda_2) \;=\; K \cdot \phi_*(\Lambda_2) \;=\; \Lambda_1.$$

And therefore: $\mathrm{Stab}_{M(N)_0}(\Gamma_1) \subset \mathrm{Stab}_{M(N)_0}(\Gamma_2)$. The same argument shows that $\mathrm{Stab}_{M(N)_0}(\Gamma_2) \subset \mathrm{Stab}_{M(N)_0}(\Gamma_1)$. Thus:

$$\underset{M(N)}{\mathrm{Stab}}(\Gamma_1) \;=\; \underset{M(N)}{\mathrm{Stab}}(\Gamma_2).$$

Take now $\psi \in \mathrm{Stab}_{M(N)}(\Gamma_1)$. We can write, for any positive integer k: $\Gamma^k = \psi^k(\Gamma)$

$$\Gamma_1 \subset \Gamma^k \subset \Gamma_2.$$

The Γ^k are therefore a family of subgroups contained between a group (Γ_2) and one of its subgroups of finite index (Γ_1). This implies that the number of Γ^k's has to be finite. Therefore there is a positive integer K_ψ such that: $\psi^{K_\psi} \in \mathrm{Stab}_{M(N)}(\Gamma)$. The set of all K_ψ's is bounded by above so if we take $K_{max} = max\{K_\psi \;\forall \psi \in \mathrm{Stab}_{M(N)}(\Gamma_1)\}$ we get that $\mathrm{Stab}_{M(N)}(\Gamma)$ is a subgroup of index K_{max} in $\mathrm{Stab}_{M(N)}(\Gamma_1)$. And the theorem is proven. ■

We would like to thank Professor G. Prasad for suggesting the following result. We first note that since N contains a lattice and is simply connected $\mathrm{Aut}(\mathfrak{N})_0 \simeq M(N)_0$ is the group of real points of an algebraic group, say $\mathbf{A}(N)$, defined over $\mathbb{Q}$ (see [15]).

Corollary 5.2. *Let Γ be a lattice in a simply connected nilpotent Lie group N. Then $\mathrm{Stab}_{M(N)_0}(\Gamma)$ is an arithmetic subgroup of $\mathbf{A}(N)$.*

Proof. Let $\phi : M(N)_0 \to \mathrm{Aut}(\mathfrak{N})_0$ be an algebraic isomorphism. By the previous result there is a log-lattice Γ_2 containing Γ such that $\mathrm{Stab}_{M(N)_0}(\Gamma)$ has finite index in $\mathrm{Stab}_{M(N)_0}(\Gamma_2)$. Let $\Lambda_2 = \log \Gamma_2$ and observe that $\phi(\mathrm{Stab}_{M(N)_0}(\Gamma_2))$ stabilizes the $\mathbb{Q}$-lattice Λ_2 and hence it is an arithmetic subgroup of $\mathbf{A}(N)$. Since $\mathrm{Stab}_{M(N)_0}(\Gamma)$ has finite index in $\mathrm{Stab}_{M(N)_0}(\Gamma_2)$, we have proven our claim. ∎

Theorem 5.3. *Let Γ be a lattice in simply connected nilpotent Lie group N; if the quotient $M(N)_0/Rad_n(M(N)_0)$ is compact,* $\mathrm{Stab}_{M(N)_0}(\Gamma)$ *is a uniform lattice.*

Proof. If $Rad(M(N)_0) = Rad_n(M(N)_0)$ the result follows from the criterion and Theorem 5.1. So the only case to consider is that $K = M(N)_0/Rad_n(M(N)_0)$ contains an abelian factor, say A:

$$K \;=\; K_1 \times A.$$

We apply our criterion to the closed subgroup $M_1(N) = K_1 \ltimes Rad_n(M(N)_0)$ and get, for any lattice $\Gamma \subset N$, that

$$\underset{M_1(N)}{\mathrm{Stab}}(\Gamma) \;\subset\; M(N)$$

is a uniform lattice in $M_1(N)$. Since $\mathrm{Stab}_A(\Gamma)$ is a finite set and the elements of A and K_1 commute, that proves the Theorem. ∎

We summarize the preceding results in the main result of this section.

Theorem 5.4. *If Γ is a lattice in an irreducible group of Heisenberg-type N with $7 \geq dim_{\mathbb{R}}(Z(N)) \geq 5 \;\; mod(8)$ or $dim_{\mathbb{R}}(Z(\mathfrak{N})) = 3 \;\; mod(8)$,* $\mathrm{Stab}_{M(N)_0}(\Gamma)$ *is a uniform lattice.*

Proof. In the Lie algebras $\mathrm{Lie}(M(N_i)_0)$ with $i = 3, 7 \;\; mod(8)$ the nilradical coincides with the radical; the quotient $\mathrm{Der}_0(\mathfrak{N})/\mathcal{R}_n$ is always of compact type and the above mentioned criterion applies directly. For the remaining two cases we should apply our extension of the result of Mosak and Moskowitz (Theorem 5.1). ∎

6. Isometries and Quasi-conformal Mappings

In this section we study the isometries of H-type groups and show how their structure gives a necessary and sufficient condition for the existence of non-conformal quasi-conformal mappings.

As a consequence of the proven results we first notice that any trace zero derivation D of a Lie algebra of Heisenberg-type with center of dimension equal to $3, 5, 6, 7 \, mod(8)$ can be decomposed as

$$D = D_K + D_N,$$

where D_N is nilpotent and D_K is in a Lie algebra of compact type.

From this it will follow that quasi-conformal mappings of certain H-type groups must be conformal.

Consider an H-type group, N, equipped with the left-invariant metric as in [2]. Let $Iso(N)$ be its group of isometries. Then, by [9], section 3:

$$Iso(N) \;=\; A(N) \ltimes N \tag{13}$$

The group $A(N)$ consists of those automorphisms of N whose differentials are isometries of the Lie algebra $\mathfrak{N}$:

$$A(N) \;=\; \{\phi \in \mathrm{Aut}(N) \mid \phi_* \in Iso(\mathfrak{N})\}.$$

We conclude that in the case of a Heisenberg-type group the Lie algebra of $A(N)$ (or $A(N)_0$, the connected component of $A(N)$) satisfies:

$$\mathrm{Lie}(A(N)) \;\subseteq\; \frac{\mathrm{Der}_0(\mathfrak{N})}{\mathcal{R}'}. \tag{14}$$

Our computations will be based on this latter fact. As a result we are able to deal with groups of automorphisms *locally* and thus avoid covering space arguments which make their appearance in previous work on the subject (see for example Pansu [16] and Riehm [18]). Since we are interested in the compactness of $A(N)$, and since there are a finite number of connected components, we can restrict our attention to the identity component of the automorphism group.

In his paper ([16]) P. Pansu establishes a result on conformal mappings for the groups N_3^n and N_7^n. A homeomorphism $T : U \to U'$ between open subsets of an H-type group is called λ-*quasiconformal* if there exists a real number $\lambda \in [1, \infty)$ such that for all $x \in U$, $\epsilon > 0$ and all sufficiently small r there is an $R > 0$ such that:

$$B(Tx, R) \subseteq T\,(B(x, r)) \subseteq B(Tx, (\lambda + \epsilon)R).$$

A quasiconformal map ϕ is said to be *conformal* when $\lambda = 1$.
This is equivalent to saying that ϕ is quasiconformal and $D(\phi)_e$, the differential of ϕ at the identity, is an isometry of the Lie algebra of N times a dilation ([16], pg. 44).

Pansu proves the following result ([16], Corollary 11.2.):

Theorem - A quasiconformal homeomorphism of N_3^n (resp. of N_7^n), acting as maximal unipotent group of isometries on the hyperbolic quaternionic (resp. octonionic) symmetric space, is conformal.

Combining our results with those of Pansu we can prove a more general statement.

Theorem 6.1. *A quasiconformal homeomorphism of an H-type group with center of dimension $3, 5, 6, 7$ $mod(8)$ must be conformal.*

Proof. Let ϕ be the homeomorphism, N our H-type group and $\mathfrak{N} \simeq \mathfrak{Z} \oplus \mathfrak{V}$ its Lie algebra satisfying $\dim(Z(\mathfrak{N})) = 3,5,6,7\ mod(8)$. By Pansu's differentiability theorem ([16], sec.VII) the differential exists almost everywhere. We first observe that ([2], pg.12) that its differential at the identity is a grading-preserving automorphism [1]

$$D(\phi)_e(\mathfrak{V}) \subset \mathfrak{V}$$

By equation (7) it is clear that the component with respect to $\mathcal{R}'$ of any grading-preserving automorphism is zero. This in turn implies, by the hypothesis and equation (14), that the corresponding derivation yields $D_\phi = D_{\phi'} + D_{\phi''}$, where $D_{\phi'} \in \mathrm{Der}_0(\mathfrak{N})/\mathcal{R}'$ and $D_{\phi''}$ is a matrix of the type $D_{\phi''} = \begin{pmatrix} \lambda \cdot Id & 0 \\ 0 & \lambda/2 \cdot Id \end{pmatrix}$ and therefore $\phi = exp(D(\phi)_e)$ is a dilation times an isometry, which equivalent is to saying that the map is conformal. ∎

7. Appendix

Given the matrices $A = (x_{i,j}) \in \mathbb{R}(8)$ and $B \in \mathbb{R}(8)$ we compute explicitly the set of linear equations:

$$A(Y) \cdot \overline{X} + Y \cdot \overline{A(X)} = C\left(Y \cdot \overline{X}\right)$$

Where X, Y are elements of the standard basis $\{e_i\}_{1 \le i \le 8} = \{1, \underline{i}, \underline{j}, \ldots\}$ of the Cayley numbers $\mathbb{O}$.

For $X = e_1$, $Y = e_2$

$$\begin{pmatrix} x_{1,2} + x_{2,1} \\ x_{2,2} + x_{1,1} \\ x_{3,2} + x_{4,1} \\ x_{4,2} - x_{3,1} \\ x_{5,2} + x_{6,1} \\ x_{6,2} - x_{5,1} \\ x_{7,2} - x_{8,1} \\ x_{8,2} + x_{7,1} \end{pmatrix} = C(e_2)$$

for $X = e_1$, $Y = e_3$

$$\begin{pmatrix} x_{1,3} + x_{3,1} \\ x_{2,3} - x_{4,1} \\ x_{3,3} + x_{1,1} \\ x_{4,3} + x_{2,1} \\ x_{5,3} + x_{7,1} \\ x_{6,3} + x_{8,1} \\ x_{7,3} - x_{5,1} \\ x_{8,3} - x_{6,1} \end{pmatrix} = C(e_3)$$

[1] For the Heisenberg group this fact was proven by Korányi and Reimann ([12])

for $X = e_1$, $Y = e_4$

$$\begin{pmatrix} x_{1,4} + x_{4,1} \\ x_{2,4} + x_{3,1} \\ x_{3,4} - x_{2,1} \\ x_{4,4} + x_{1,1} \\ x_{5,4} + x_{8,1} \\ x_{6,4} - x_{7,1} \\ x_{7,4} + x_{6,1} \\ x_{8,4} - x_{5,1} \end{pmatrix} = C(e_4)$$

for $X = e_1$, $Y = e_5$

$$\begin{pmatrix} x_{1,5} + x_{5,1} \\ x_{2,5} - x_{6,1} \\ x_{3,5} - x_{7,1} \\ x_{4,5} - x_{8,1} \\ x_{5,5} + x_{1,1} \\ x_{6,5} + x_{2,1} \\ x_{7,5} + x_{3,1} \\ x_{8,5} + x_{4,1} \end{pmatrix} = C(e_5)$$

for $X = e_1$, $Y = e_6$

$$\begin{pmatrix} x_{1,6} + x_{6,1} \\ x_{2,6} + x_{5,1} \\ x_{3,6} - x_{8,1} \\ x_{4,6} + x_{7,1} \\ x_{5,6} - x_{2,1} \\ x_{6,6} + x_{1,1} \\ x_{7,6} - x_{4,1} \\ x_{8,6} + x_{3,1} \end{pmatrix} = C(e_6)$$

for $X = e_1$, $Y = e_7$

$$\begin{pmatrix} x_{1,7} + x_{7,1} \\ x_{2,7} + x_{8,1} \\ x_{3,7} + x_{5,1} \\ x_{4,7} - x_{6,1} \\ x_{5,7} - x_{3,1} \\ x_{6,7} + x_{4,1} \\ x_{7,7} + x_{1,1} \\ x_{8,7} - x_{2,1} \end{pmatrix} = -C(e_7)$$

for $X = e_1$, $Y = e_8$

$$\begin{pmatrix} x_{1,8} + x_{8,1} \\ x_{2,8} - x_{7,1} \\ x_{3,8} + x_{6,1} \\ x_{4,8} + x_{5,1} \\ x_{5,8} - x_{4,1} \\ x_{6,8} - x_{3,1} \\ x_{7,8} + x_{2,1} \\ x_{8,8} + x_{1,1} \end{pmatrix} = C(e_8)$$

for $X = e_2$, $Y = e_3$

$$\begin{pmatrix} x_{2,3} + x_{3,2} \\ -x_{1,3} - x_{4,2} \\ -x_{4,3} + x_{1,2} \\ x_{3,3} + x_{2,2} \\ -x_{6,3} + x_{7,2} \\ x_{5,3} + x_{8,2} \\ x_{8,3} - x_{5,2} \\ -x_{7,3} - x_{6,2} \end{pmatrix} = C(e_4)$$

for $X = e_2$, $Y = e_4$

$$\begin{pmatrix} x_{2,4} + x_{4,2} \\ -x_{1,4} + x_{3,2} \\ -x_{4,4} - x_{2,2} \\ x_{3,4} + x_{1,2} \\ -x_{6,4} + x_{8,2} \\ x_{5,4} - x_{7,2} \\ x_{8,4} + x_{6,2} \\ -x_{7,4} - x_{5,2} \end{pmatrix} = -C(e_3)$$

for $X = e_2$, $Y = e_5$

$$\begin{pmatrix} x_{2,5} + x_{5,2} \\ -x_{1,5} - x_{6,2} \\ -x_{4,5} - x_{7,2} \\ x_{3,5} - x_{8,2} \\ -x_{6,5} + x_{1,2} \\ x_{5,5} + x_{2,2} \\ x_{8,5} + x_{3,2} \\ -x_{7,5} + x_{4,2} \end{pmatrix} = C(e_6)$$

for $X = e_2$, $Y = e_6$

$$\begin{pmatrix} x_{2,6} + x_{6,2} \\ -x_{1,6} + x_{5,2} \\ -x_{4,6} - x_{8,2} \\ x_{3,6} + x_{7,2} \\ -x_{6,6} - x_{2,2} \\ x_{5,6} + x_{1,2} \\ x_{8,6} - x_{4,2} \\ -x_{7,6} + x_{3,2} \end{pmatrix} = -C(e_5)$$

for $X = e_2$, $Y = e_7$

$$\begin{pmatrix} x_{2,7} + x_{7,2} \\ -x_{1,7} + x_{8,2} \\ -x_{4,7} + x_{5,2} \\ x_{3,7} - x_{6,2} \\ -x_{6,7} - x_{3,2} \\ x_{5,7} + x_{4,2} \\ x_{8,7} + x_{1,2} \\ -x_{7,7} - x_{2,2} \end{pmatrix} = -C(e_8)$$

for $X = e_2$, $Y = e_8$

$$\begin{pmatrix} x_{2,8} + x_{8,2} \\ -x_{1,8} - x_{7,2} \\ -x_{4,8} + x_{6,2} \\ x_{3,8} + x_{5,2} \\ -x_{6,8} - x_{4,2} \\ x_{5,8} - x_{3,2} \\ x_{8,8} + x_{2,2} \\ -x_{7,8} + x_{1,2} \end{pmatrix} = -C(e_7)$$

for $X = e_3$, $Y = e_4$

$$\begin{pmatrix} x_{3,4} + x_{4,3} \\ x_{4,4} + x_{3,3} \\ -x_{1,4} - x_{2,3} \\ -x_{2,4} + x_{1,3} \\ -x_{7,4} + x_{8,3} \\ -x_{8,4} - x_{7,3} \\ x_{5,4} + x_{6,3} \\ x_{6,4} - x_{5,3} \end{pmatrix} = C(e_2)$$

for $X = e_3$, $Y = e_5$

$$\begin{pmatrix} x_{3,5} + x_{5,3} \\ x_{4,5} - x_{6,3} \\ -x_{1,5} - x_{7,3} \\ -x_{2,5} - x_{8,3} \\ -x_{7,5} + x_{1,3} \\ -x_{8,5} + x_{2,3} \\ x_{5,5} + x_{3,3} \\ x_{6,5} + x_{4,3} \end{pmatrix} = -C(e_7)$$

for $X = e_3$, $Y = e_6$

$$\begin{pmatrix} x_{3,6} + x_{6,3} \\ x_{4,6} + x_{5,3} \\ -x_{1,6} - x_{8,3} \\ -x_{2,6} + x_{7,3} \\ -x_{7,6} - x_{2,3} \\ -x_{8,6} + x_{1,3} \\ x_{5,6} - x_{4,3} \\ x_{6,6} + x_{3,3} \end{pmatrix} = C(e_8)$$

for $X = e_3$, $Y = e_7$

$$\begin{pmatrix} x_{3,7} + x_{7,3} \\ x_{4,7} + x_{8,3} \\ -x_{1,7} + x_{5,3} \\ -x_{2,7} - x_{6,3} \\ -x_{7,7} - x_{3,3} \\ -x_{8,7} + x_{4,3} \\ x_{5,7} + x_{1,3} \\ x_{6,7} - x_{2,3} \end{pmatrix} = -C(e_5)$$

for $X = e_3$, $Y = e_8$

$$\begin{pmatrix} x_{3,8} + x_{8,3} \\ x_{4,8} - x_{7,3} \\ -x_{1,8} + x_{6,3} \\ -x_{2,8} + x_{5,3} \\ -x_{7,8} - x_{4,3} \\ -x_{8,8} - x_{3,3} \\ x_{5,8} + x_{2,3} \\ x_{6,8} + x_{1,3} \end{pmatrix} = -C(e_6)$$

for $X = e_4$, $Y = e_5$

$$\begin{pmatrix} x_{4,5} + x_{5,4} \\ -x_{3,5} - x_{6,4} \\ x_{2,5} - x_{7,4} \\ -x_{1,5} - x_{8,4} \\ -x_{8,5} + x_{1,4} \\ x_{7,5} + x_{2,4} \\ -x_{6,5} + x_{3,4} \\ x_{5,5} + x_{4,4} \end{pmatrix} = C(e_8)$$

for $X = e_4$, $Y = e_6$

$$\begin{pmatrix} x_{4,6} + x_{6,4} \\ -x_{3,6} + x_{5,4} \\ x_{2,6} - x_{8,4} \\ -x_{1,6} + x_{7,4} \\ -x_{8,6} - x_{2,4} \\ x_{7,6} + x_{1,4} \\ -x_{6,6} - x_{4,4} \\ x_{5,6} + x_{3,4} \end{pmatrix} = C(e_7)$$

for $X = e_4$, $Y = e_7$

$$\begin{pmatrix} x_{4,7} + x_{7,4} \\ -x_{3,7} + x_{8,4} \\ x_{2,7} + x_{5,4} \\ -x_{1,7} - x_{6,4} \\ -x_{8,7} - x_{3,4} \\ x_{7,7} + x_{4,4} \\ -x_{6,7} + x_{1,4} \\ x_{5,7} - x_{2,4} \end{pmatrix} = C(e_6)$$

for $X = e_4$, $Y = e_8$

$$\begin{pmatrix} x_{4,8} + x_{8,4} \\ -x_{3,8} - x_{7,4} \\ x_{2,8} + x_{6,4} \\ -x_{1,8} + x_{5,4} \\ -x_{8,8} - x_{4,4} \\ x_{7,8} - x_{3,4} \\ -x_{6,8} + x_{2,4} \\ x_{5,8} + x_{1,4} \end{pmatrix} = -C(e_5)$$

for $X = e_5$, $Y = e_6$

$$\begin{pmatrix} x_{5,6} + x_{6,5} \\ x_{6,6} + x_{5,5} \\ x_{7,6} - x_{8,5} \\ x_{8,6} + x_{7,5} \\ -x_{1,6} - x_{2,5} \\ -x_{2,6} + x_{1,5} \\ -x_{3,6} - x_{4,5} \\ -x_{4,6} + x_{3,5} \end{pmatrix} = C(e_2)$$

for $X = e_5$, $Y = e_7$

$$\begin{pmatrix} x_{5,7} + x_{7,5} \\ x_{6,7} + x_{8,5} \\ x_{7,7} + x_{5,5} \\ x_{8,7} - x_{6,5} \\ -x_{1,7} - x_{3,5} \\ -x_{2,7} + x_{4,5} \\ -x_{3,7} + x_{1,5} \\ -x_{4,7} - x_{2,5} \end{pmatrix} = C(e_3)$$

for $X = e_5$, $Y = e_8$

$$\begin{pmatrix} x_{5,8} + x_{8,5} \\ x_{6,8} - x_{7,5} \\ x_{7,8} + x_{6,5} \\ x_{8,8} + x_{5,5} \\ -x_{1,8} - x_{4,5} \\ -x_{2,8} - x_{3,5} \\ -x_{3,8} + x_{2,5} \\ -x_{4,8} + x_{1,5} \end{pmatrix} = C(e_4)$$

for $X = e_6$, $Y = e_7$

$$\begin{pmatrix} x_{6,7} + x_{7,6} \\ -x_{5,7} + x_{8,6} \\ x_{8,7} + x_{5,6} \\ -x_{7,7} - x_{6,6} \\ x_{2,7} - x_{3,6} \\ -x_{1,7} + x_{4,6} \\ x_{4,7} + x_{1,6} \\ -x_{3,7} - x_{2,6} \end{pmatrix} = -C(e_4)$$

for $X = e_6$, $Y = e_8$

$$\begin{pmatrix} x_{6,8} + x_{8,6} \\ -x_{5,8} - x_{7,6} \\ x_{8,8} + x_{6,6} \\ -x_{7,8} + x_{5,6} \\ x_{2,8} - x_{4,6} \\ -x_{1,8} - x_{3,6} \\ x_{4,8} + x_{2,6} \\ -x_{3,8} + x_{1,6} \end{pmatrix} = C(e_3)$$

for $X = e_7$, $Y = e_8$

$$\begin{pmatrix} x_{7,8} + x_{8,7} \\ -x_{8,8} - x_{7,7} \\ -x_{5,8} + x_{6,7} \\ x_{6,8} + x_{5,7} \\ x_{3,8} - x_{4,7} \\ -x_{4,8} - x_{3,7} \\ -x_{1,8} + x_{2,7} \\ x_{2,8} + x_{1,7} \end{pmatrix} = -C(e_2)$$

We then use the results of the computation to solve the equation:

$$P_5\left(A(e_j)\cdot\overline{e_i} + e_j\cdot\overline{A(e_i)}\right) = C\left(P_5(e_j\cdot\overline{e_i})\right) = 0 \tag{15}$$

Where P_5 is the projection onto the five-dimensional subspace $\mathbb{O}_5^*$ of the imaginary Cayley numbers. In doing so we get that

$$x_{1,1} = x_{2,2} = \ldots = x_{8,8} = \frac{tr(A)}{8}.$$

As well as a set of seven linear systems of the kind (for brevity we write only one of them, the others being derived in the exact same way):

$$\begin{cases} x_{7,2} - x_{8,1} = x_{5,4} + x_{6,3} = -x_{3,6} - x_{4,5} = x_{1,8} - x_{2,7} \\ x_{6,3} + x_{8,1} = -x_{5,4} + x_{7,2} = -x_{2,7} + x_{4,5} = -x_{1,8} - x_{3,6} \\ x_{5,4} + x_{8,1} = -x_{6,3} + x_{7,2} = -x_{8,1} - x_{4,5} = -x_{2,7} + x_{3,6} \\ -x_{4,5} + x_{8,1} = x_{3,6} + x_{7,2} = x_{2,7} - x_{6,3} = -x_{1,8} + x_{5,4} \end{cases}$$

those can be solved directly and give seven equations of the type:

$$x_{7,2} + x_{2,7} = x_{8,1} + x_{1,8} = x_{5,4} + x_{4,5} = x_{6,3} + x_{3,6} = -(x_{6,3} + x_{3,6})$$

so that, in general:

$$x_{i,j} = -x_{j,i} \quad 1 \le i,j \le 8,$$

thus A is actually the sum of a skew-symmetric matrix and scalar multiple of $Id_{\mathbb{R}(8)}$. The same conclusion can reached by taking P_6 or P_7 instead of P_5 since the conditions are actually redundant in (15).

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