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# Homogeneous spaces admitting transitive semigroups

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**Abstract.**

Let $G$ be a semi-simple Lie group with finite center and $S \subset G$ a semigroup with $\operatorname{int} S \neq \emptyset$. A closed subgroup $L \subset G$ is said to be $S$-admissible if $S$ is transitive in $G/L$. In [10] it was proved that a necessary condition for $L$ to be $S$-admissible is that its action in $B(S)$ is minimal and contractive where $B(S)$ is the flag manifold associated with $S$, as in [9]. It is proved here, under an additional assumption, that this condition is also sufficient provided $S$ is a compression semigroup. A subgroup with a finite number of connected components is admissible if and only if its component of the identity is admissible, and if $L$ is a connected admissible group then $L$ is reductive and its semi-simple component $E$ is also admissible. Moreover, $E$ is transitive in $B(S)$ which turns out to be a flag manifold of $E$.

*Key words:* semigroups, semi-simple groups, flag manifolds, transitive groups, minimal actions.
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## 1. Introduction

Let $G$ be a connected and noncompact semi-simple Lie group. The problem we address here is that of finding the pairs $(S, L)$ formed by a semigroup $S \subset G$ with nonvoid interior and a closed subgroup $L$ of $G$ such that $S$ is transitive in the homogeneous space $G/L$. In such a pair we say that $L$ is $S$-admissible. We approach this problem by looking at the actions of $S$ and $L$ in the flag manifolds of $G$. In [9] one of these flag manifolds, say $B(S)$, was intrinsically attached to $S$ in such a way that the invariant control set for $S$ on $B(S)$ contracts to one point through iterations of regular elements inside the interior of $S$. Thus it becomes natural to search conditions on $L$ which involve its action on $B(S)$. There is indeed the necessary condition for the transitivity of $S$ on $G/L$, proved in [10], which ensures that the action of $L$ on $B(S)$ must be minimal and contractive. In this article we prove that these conditions are also sufficient as long as $S$ is a

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compression semigroup and the reflection $\theta(L)$ of $L$ under a Cartan involution of $G$ satisfies the same conditions as $L$. Once we have this necessary and sufficient conditions for transitivity we proceed to analyze those subgroups satisfying them. As we shall see they impose severe restrictions on $L$ so we get, as a rule, that transitivity of proper semigroups is a rare event. We consider mainly connected subgroups and make the basic assumption that $G$ is a simple group. In this case an application of a result by Vinberg [12], namely that a triangular linear group has a fixed point in any compact invariant subset of the projective space, allow us to show that any admissible connected subgroup $L$ is reductive and is such that its semi-simple part $E$ is noncompact and transitive in $B(S)$. Moreover, $E$ is also admissible and $B(S)$ is a flag manifold of $E$. So that the connected admissible subgroups are essentially those noncompact semi-simple subgroups of $G$ which have a common flag manifold with $G$. In particular, if $B(S)$ is the maximal flag manifold then there are no $S$-admissible proper connected subgroups.

As to the case where $L$ is not connected we observe first that if it has a finite number of connected components then $L$ is admissible for some $S$ if and only if its identity component $L_0$ is admissible so that this case is easily reduced to the connected case. On the other hand, we have little to say about the case where $L$ has an infinite number of components: We remark that the algebraic closure $\mathrm{zc}(L)$ of $L$ is also admissible and has a finite number of components so that the semi-simple part of the identity component of $\mathrm{zc}(L)$ is also admissible. A consequence of this fact is that the $S$-admissible subgroups are discrete if $G$ is simple and $B(S)$ is the maximal flag manifold of $G$. Also we reproduce here a proof for the well known fact in control theory which says that any semigroup with nonvoid interior in $G$ is transitive in $G/L$ if this homogeneous space admits a finite invariant measure. This shows in particular that any lattice in $G$ is admissible for any semigroup with interior points.

## 2. Preliminaries on semigroups and flags

In this section we recall some results about semigroups in semi-simple Lie groups which will be needed afterwards. Throughout the paper we let $G$ be a noncompact semi-simple Lie group with finite center and denote by $\mathfrak{g}$ its Lie algebra. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$ be a Cartan decomposition and denote by $\theta$ the corresponding Cartan involution either of $\mathfrak{g}$ or of $G$. Select a maximal abelian subalgebra $\mathfrak{a} \subset \mathfrak{s}$ and let $\Pi$ be the set of roots of the pair $(\mathfrak{g}, \mathfrak{a})$. We shall say that $H \in \mathfrak{a}$ is *regular real* provided $\alpha(H) \neq 0$ for every root $\alpha$. More generally, $H \in \mathfrak{g}$ is regular real if it is conjugate to a regular real element in $\mathfrak{a}$, or equivalently if it is regular real in some abelian subalgebra conjugate to $\mathfrak{a}$. In a similar way, we say that $h \in G$ is regular real if $h = \exp H$ for some regular real $H \in \mathfrak{g}$.

Choose a simple system of roots $\Sigma \subset \Pi$ and denote by $\Pi^+$ the corresponding set of positive roots. Let $\mathfrak{m}$ be the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$. The standard minimal parabolic subalgebra of $\mathfrak{g}$ is given by $\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$ where

$$\mathfrak{n} = \sum_{\alpha \in \Pi^+} \mathfrak{g}_\alpha$$

is the nilpotent constituent of the Iwasawa decomposition. As usual we denote by

$K$ and $N$ the connected subgroups whose Lie algebras are $\mathfrak{k}$ and $\mathfrak{n}$ respectively. The normalizer $P$ of $\mathfrak{p}$ in $G$ is a minimal parabolic subgroup and $G/P$ is the maximal flag manifold of $G$. It is well known that $\mathfrak{p}$ is the Lie algebra of $P$.

Given $\Theta \subset \Sigma$ let $\langle \Theta \rangle$ be the subset of positive roots generated by $\Theta$ and denote by $\mathfrak{n}^{-}(\Theta)$ the subalgebra spanned by the root spaces $\mathfrak{g}_{-\alpha}$, $\alpha \in \langle \Theta \rangle$. We denote by $\mathfrak{p}_\Theta$ the parabolic subalgebra

$$\mathfrak{p}_\Theta = \mathfrak{n}^{-}(\Theta) \oplus \mathfrak{p}\,.$$

Its normalizer $P_\Theta$ in $G$ is a parabolic subgroup whose Lie algebra is $\mathfrak{p}_\Theta$. We put $B_\Theta = G/P_\Theta$ for the corresponding flag manifold. If $M_\Theta = P_\Theta \cap K$ then $B_\Theta = K/M_\Theta$.

Recall that if $h \in G$ is regular real then it has a finite number of fixed points in $B_\Theta$. Each fixed point is hyperbolic and there is just one attractor, i.e., whose stable manifold is open and dense. The decomposition of $B_\Theta$ into stable manifolds of $h$ is the so called Bruhat decomposition. In fact, the stable manifolds are the orbits of the group $N^{-} = \exp \mathfrak{n}^{-}$ where $\mathfrak{n}^{-}$ is the subalgebra opposed to $\mathfrak{n}$. We note furthermore that the open stable manifold is also given by $N_\Theta^{-} b_0$ where $b_0 = P_\Theta$ is the origin in $B_\Theta = G/P_\Theta$ and $N_\Theta^{-} = \exp \mathfrak{n}_\Theta^{-}$. Here

$$\mathfrak{n}_\Theta^{-} = \sum_\alpha \mathfrak{g}_\alpha\,,$$

with the sum extended to the negative roots outside $-\langle \Theta \rangle$, is the subalgebra spanned by root spaces complementary to $\mathfrak{p}_\Theta$ in $\mathfrak{g}$.

Denote by $W$ the Weyl group of the pair $(\mathfrak{g}, \mathfrak{a})$, and by $W_\Theta$ the subgroup of $W$ generated by the reflections with respect to the roots in $\Theta \subset \Sigma$. In [9, Section 4] it was associated with a semigroup $S \subset G$ with $\operatorname{int} S \neq \emptyset$ a subgroup $W(S) \subset W$ which accounts for the number of $S$-control sets on the maximal boundary $B$. It was shown that $W(S) = W_{\Theta_S}$ for some subset $\Theta_S$ of the simple system of roots. We use the notation $B(S) = B_{\Theta_S}$. The main property of $B(S)$ which will be used here is that if $C \subset B(S)$ stands for the invariant control set for $S$ then $C$ is contained in the stable manifold of the attractor $b$ in $B(S)$ of any $h \in \operatorname{int} S$ which is regular real. This implies that $C$ is *contractive* with respect to $h$ in the sense that $h^k x \to b$ as $k \to \infty$ for all $x \in C$.

In the sequel we shall use often the notion of *contractive sequences* in $G$ (see Guivarc'h and Raugi [3]): Let $g_k$ be a sequence in $G$ and write its polar decomposition as $g_k = v_k a_k u_k$ with $v_k, u_k \in K$ and $a_k \in \operatorname{cl} A^{+}$. Here $A^{+} = \exp \mathfrak{a}^{+}$, where $\mathfrak{a}^{+} \subset \mathfrak{a}$ is an open Weyl chamber. For a root $\alpha \in \Pi$ and $a \in \exp \mathfrak{a}$, put $\phi_\alpha(a) = \exp(\alpha(\log a))$. The sequence $g_k$ is said to be contractive if $\phi_\alpha(a_k) \to 0$ as $k \to +\infty$ for all negative roots $\alpha$. Moreover, the sequence is said to be contractive with respect to a flag manifold $B_\Theta$ if $\phi_\alpha(a_k) \to 0$ for every negative root $\alpha$ which is not in the subset $-\langle \Theta \rangle$ of roots spanned by $-\Theta$.

In general, given a sequence $g_k = v_k a_k u_k$ there exists a subsequence $g_{k_n} = v_{k_n} a_{k_n} u_{k_n}$ such that $v_{k_n} \to v$ and $u_{k_n} \to u$ in $K$, and there is a map $\tau : \mathfrak{n}_\Theta^{-} \to \mathfrak{n}_\Theta^{-}$ such that for every $Y \in \mathfrak{n}_\Theta^{-}$

$$g_{k_n} u^{-1} \exp(Y)\, b_0 \to v \exp(\tau Y)\, b_0. \tag{1}$$

(see [3] and [10, Prop. 2.5]). The linear map $\tau$ is diagonal and its eigenvalues are $\lim \phi_\alpha(a_k)$ with $\alpha$ a negative root outside $-\langle\Theta\rangle$. Because of this the limit in (1) is uniform for $Y$ in a compact subset of $\mathfrak{n}_\Theta^-$. The subsequence is contractive with respect to $B_\Theta$ if and only if $\tau = 0$. In this case (1) implies that $g_{k_n}x \to vb_0$ for all $x \in u^{-1}N_\Theta^- b_0$, that is, for all $x$ of the form $x = u^{-1}\exp(Y)\, b_0$ with $Y \in \mathfrak{n}_\Theta^-$.

From the limit in (1) we get the following criteria: If $g_k x \to x_0$ for all $x$ in an open subset of $B_\Theta$ then $g_k$ admits a contractive subsequence. In fact, it follows from (1) that $\tau Y = 0$ for $Y$ in an open subset of $\mathfrak{n}_\Theta^-$ so that $\tau = 0$ (c.f. Corollary 2.6 in [10]).

Another fact about contractive sequences we shall need below is that if $g_k$ is a contractive sequence with respect to $B_\Theta$ then for every $h, l \in G$, $hg_kl$ admits a contractive subsequence. This follows immediately from Corollary 2.6 in [10] (see also [3, Cor. 2.3]). In fact, if $g_k$ is contractive then $g_kl$ converges to a point for an open and dense subset. So that the same happens to $hg_kl$ and hence this sequence admits a contractive subsequence.

In the sequel we shall say that the action of a subgroup $L \subset G$ on a flag manifold $B$ is contractive, or simply that $L$ is contractive on $B$, if there is a contractive sequence $g_k \in L$ with respect to $B$.

Recall that the action of a group $L$ on the topological space $X$ is said to be *minimal* if every orbit $Lx$, $x \in X$ is dense in $X$. Equivalently, the action is minimal if there is no proper invariant closed subset. In the sequel we say simply that $L$ is minimal on $X$ if its action on $X$ is minimal.

## 3. Transitive semigroups

Let $S \subset G$ be a semigroup with $\operatorname{int} S \neq \emptyset$. A closed subgroup $L \subset G$ is said to be *$S$-admissible* if $S$ is transitive in $G/L$. From [10] we have the following necessary conditions for a subgroup to be $S$-admissible.

**Theorem 3.1.** *Let $S \subset G$ be a semigroup with $\operatorname{int} S \neq \emptyset$ and $L \subset G$ a closed subgroup. In order that $S$ is transitive on $G/L$ it is necessary that*

1. *the action of $L$ on $B(S)$ is minimal, and*
2. *$L$ admits a contractive sequence with respect to $B(S)$.*

The objective of this section is to prove a partial converse of this theorem, namely that the conditions on $L$ are sufficient as far as $S$ is a compression semigroup and the subgroup $\theta(L)$ satisfies the same conditions as $L$, where $\theta$ is a Cartan involution of $G$.

We consider here only those compression semigroups of their invariant control sets in a flag manifold. More precisely, let $C$ be the invariant control set for $S$ in $B(S)$. Then $S$ is the compression semigroup of $C$ if

$$S = \{g \in G : gC \subset C\}. \tag{2}$$

We note that such semigroups indeed exist. In fact, let $S_0$ be a semigroup with $\operatorname{int} S_0 \neq \emptyset$ and denote by $C$ its invariant control set on $B(S_0)$. If $S$ is defined by (2) then $C$ is the invariant control set for $S$ because $S_0 \subset S$. Any maximal semigroup with nonempty interior in $G$ is the compression semigroup of its invariant control set in some minimal flag manifold (see [9]).

We start by proving the following lemma. For its statement we fix an Iwasawa decomposition $G = KAN$ and the minimal parabolic subgroup $P = MAN$. If $Q \supset P$ is a parabolic subgroup we denote by $b_0$ the origin in the flag manifold $B = G/Q$.

**Lemma 3.2.** *Let $\sigma$ stand for the open Bruhat component containing $b_0$. Take $k \in K$ and let $C \subset k\sigma$ be a compact subset. Suppose that $g_i \in G$ is a sequence such that $g_i b_0 \to k b_0$. Then for some subsequence $g_{i_j}$ we have that $C \subset \theta\left(g_{i_j}\right)\sigma$, $j \geq j_0$.*

**Proof.** Let $g_i = u_i a_i n_i$ stand for the Iwasawa decomposition of $g_i$ with $u_i \in K$, $a_i \in A$ and $n_i \in N$. Then $g_i b_0 = u_i b_0$ because $b_0$ is invariant under $a_i n_i$. Also, $\theta(g_i) = u_i a_1^{-1}\theta(n_i)$, and since $\sigma$ is invariant under $AN^-$, $\theta(g_i)\sigma = u_i\sigma$. By taking a subsequence we can assume that $u_i \to u \in K$. Since $u_i b_0 \to k b_0$ and $u_i b_0 \to u b_0$ we have that $u b_0 = k b_0$. Therefore if we write $B = K/M_\Theta$ then $u = km$ with $m \in M_\Theta$.

Now, $M_\Theta$ normalizes $N_\Theta^-$ and fixes $b_0$ so that $m\sigma = \sigma$. Therefore if we put $v_i = u_i m^{-1}$ then $v_i \to k$, $v_i\sigma = u_i\sigma$ and $kv_i^{-1} \to 1$. On the other hand, the action of $K$ on $B$ is continuous with respect to the compact-open topology on the set of continuous maps of $B$. Therefore the fact that $C \subset k\sigma$ implies that $kv_i^{-1}C \subset k\sigma$ for $i$ big enough. This means that $v_i^{-1}C \subset \sigma$, that is, $C \subset v_i\sigma = u_i\sigma$. Since $\theta(g_i)\sigma = u_i\sigma$, the lemma follows. ■

The proof that a semigroup is transitive on a homogenous space is simplified by the following device.

**Lemma 3.3.** *Let $G$ be a topological group, $L \subset G$ a closed subgroup and $S \subset G$ a semigroup with* $\operatorname{int} S \neq \emptyset$. *We have,*

1. *If $S$ is transitive on $G/L$ then* $\operatorname{int} gSg^{-1} \cap L \neq \emptyset$ *for all $g \in G$.*
2. *Reciprocally, suppose that $G/L$ is connected. Then $S$ is transitive in $G/L$ if* $\operatorname{int} gSg^{-1} \cap L$ *is not empty for all $g \in G$.*

**Proof.** See Corollary 2.2 in [10]. ■

As mentioned above we shall prove the converse of Theorem 3.1 under the additional assumption that the action of $\theta(L)$ is contractive and minimal. At this regard we note that contractivity of the $L$-action ensures trivially that the action of $\theta(L)$ is contractive. In fact, let $g_n \in L$ be a contractive sequence and write $g_n = u_n a_n v_n$ for its polar decomposition. Then $\theta(g_n^{-1}) = v_n^{-1}a_n u_n^{-1}$ has the same radial component as $g_n$, so that $\theta(g_n^{-1})$ is also contractive.

We can prove now the main result of this section.

**Theorem 3.4.** *Let $S \subset G$ be a semigroup with $\operatorname{int} S \neq \emptyset$ and denote by $C$ its invariant control set on $B(S)$. Suppose that $S$ is the compression semigroup of $C$, i.e.,*

$$S = \{g \in G : gC \subset C\}.$$

*Let $L$ be a closed subgroup and assume that its action on $B(S)$ is contractive. Assume also that both $L$ and $\theta(L)$ are minimal on $B(S)$. Then $S$ is transitive on $G/L$.*

**Proof.** We keep fixed an Iwasawa decomposition of $G$ which gives rise to a standard parabolic subgroup which is assumed to be the isotropy at the origin $b_0$ of $B(S)$. The corresponding open Bruhat component $N^- b_0$ will be denoted by $\sigma$. We can assume without loss of generality that $C \subset \sigma$.

Take $g \in G$. We wish to show that $L \cap \operatorname{int} gSg^{-1} \neq \emptyset$. The fact that $S$ is a compression semigroup implies that

$$gSg^{-1} = \{h \in G : hgC \subset gC\}.$$

Therefore in order to prove that the above intersection is not trivial it is enough to show that there exists $l \in L$ such that $lgC \subset \operatorname{int} gC$. In fact, the action of $G$ on $B(S)$ is continuous with respect to the compact-open topology so that $l$ belongs to $\operatorname{int}(gSg^{-1})$ if and only if it maps $gC$ into its interior. This $l \in L$ will be provided by a contracting sequence in $L$.

Let $g_i = x_i a_i y_i \in L$ with $a_i \in \operatorname{cl} A^+$ and $x_i, y_i \in K$ be a contracting sequence such that $x_i \to x$ and $y_i \to y^{-1}$ with $x, y \in K$. The limit in (1) implies that $g_i b \to x b_0$ for all $b \in y\sigma$. Since $L$ is minimal on $B(S)$, we can find $h \in L$ such that $b_1 = hxb_0$ belongs to $\operatorname{int} gC$. Hence the contracting sequence $l_i = hg_i \in L$ satisfies $l_i b \to b_1$ for all $b \in y\sigma$.

On the other hand, write $g = kan$, $k \in K$, $a \in A$, $n \in N^-$ for the Iwasawa decomposition of $g$. Then $g\sigma = k\sigma$ and $gC \subset k\sigma$ because $C \subset \sigma$. Since the action of $\theta(L)$ is minimal on $B(S)$, there exists a sequence $m_i \in \theta(L)$ such that $m_i y b_0 \to k b_0$. By Lemma 3.2 there exists $i$ big enough such that $gC \subset \theta(m_i y)\sigma$. Since $\theta(m_i y) = \theta(m_i) y$ we get $m \in L$ such that $gC \subset my\sigma$.

Now, take the contracting sequence $h_i = l_i m^{-1} \in L$. Then $h_i b \to b_1$ for all $b \in my\sigma$. In particular, this sequence contracts $gC$ to $b_1$, and the convergence is uniform in $gC$ because this set is compact. But $b_1 \in \operatorname{int} gC$, so that there exists $i_0$ such that $h_{i_0} gC \subset \operatorname{int} gC$. Since $h_{i_0} \in L$ this shows that $L$ intercepts the interior of the compression semigroup of $gC$, concluding the proof of the theorem. ■

This theorem raises the question of whether the minimal action of $L$ implies that the action of $\theta(L)$ is also minimal. For the time being we do not know the complete answer to this question. However there are some cases where the answer is in the affirmative. In fact, given a flag manifold $G/Q$ there is the dual flag manifold $G/\theta(Q)$ which is diffeomorphic to $G/Q$ through the commutative diagram

$$\begin{array}{ccc} G & \xrightarrow{\theta} & G \\ \downarrow & & \downarrow \\ G/Q & \xrightarrow{\overline{\theta}} & G/\theta(Q) \end{array}$$

where $\overline{\theta}(gQ) = \theta(g)\theta(Q)$. The actions of $G$ on $G/Q$ and $G/\theta(Q)$ are interchanged by $\overline{\theta}$ through the formula $\overline{\theta} \circ g = \theta(g) \circ \overline{\theta}$. This implies that $L$ is minimal on $G/Q$ if and only if $\theta(L)$ is minimal on $G/\theta(Q)$. In some cases $\theta(Q)$ is conjugate to $Q$ and hence $G/\theta(Q)$ coincides with $G/Q$. In these cases the minimal action of $L$ on $G/Q$ is equivalent to the minimal action of $\theta(L)$. Conjugacy of $Q$ and $\theta(Q)$ holds for instance if $Q$ is a minimal parabolic subgroup or for any parabolic subgroup if $\theta$ is an inner automorphism of $G$. We recall that if $\mathfrak{g}$ is a real form of a simple complex Lie algebra then $\theta$ is an inner automorphism if the diagram of the roots of the pair $(\mathfrak{g}, \mathfrak{a})$ is not of the type $A_l$, $D_l$, $l$ odd or $E_6$. Hence in a good deal of groups the minimal action of $L$ implies that of $\theta(L)$. In addition to these conditions on the flag manifolds, we mention that if $L$ is connected then it will be proved in the next section that $L$ is reductive. This implies that $L$ is invariant under some Cartan involution. Two Cartan involutions $\theta_1$ and $\theta_2$ are related by $\theta_1 = \theta_2 \circ C_g$ for some $g \in G$ where $C_g(h) = ghg^{-1}$. Hence if the action of $L$ is minimal on $G/Q$, the same happens to the action of $\theta(L)$.

Regarding still the minimal action of $\theta(L)$ we note that if $L$ is $S$-admissible then $\theta(L)$ is minimal in $B(S)$. In fact, $S$ is transitive in $G/L$ if and only if $S^{-1}$ is transitive. Now, it follows from the theory in [9] that $B(S^{-1}) = G/\theta(Q)$ if $B(S) = G/Q$. Therefore $L$ is minimal on $G/\theta(Q)$, which by the above duality implies that $\theta(L)$ is minimal on $B(S)$.

As a final comment about Theorem 3.4 we stress that it ensures that $S$ is transitive in $G/L$ only in case $S$ is the compression semigroup of the subset $C$. This condition is needed because our method of proof consists in showing that some element of $L$ maps $C$ into its interior. We do not know, however, if the conditions of Theorem 3.4 remain sufficient for more general semigroups with nonvoid interior.

## 4. Connected subgroups

The conditions of Theorem 3.1 open the way to a close analysis of the admissible subgroups. In this section we start this analysis by reducing the possibilities for the connected subgroups. We do not need here the contractive property of an admissible subgroup, but only its minimal action in some flag manifold. Therefore, along this section $B$ stands for a flag manifold of $G$ and $L$ for a connected Lie subgroup whose action on $B$ is minimal. We do not assume in advance that $L$ is closed in $G$.

Since we only consider semi-simple groups with finite center, we may assume without loss of generality that the center is trivial, so we are actually working in the identity component of the adjoint group $\mathrm{Ad}(G)$. Also, in order to simplify most of the arguments we assume from now on that $\mathfrak{g}$ is simple.

In this context we shall prove below that $L$ is reductive, i.e., its Lie algebra $\mathfrak{l}$ is the direct sum of its center $\mathfrak{z}$ and a semi-simple component, say $\mathfrak{e}$. With the further assumption that $L$ is closed it turns out that its action on $B$ is actually transitive.

The proof that $L$ is reductive uses successively the following lemma which is a consequence of a theorem of Vinberg [12]. In the sequel we say that a nilpotent subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is triangular in $\mathfrak{g}$ if the weights of the adjoint representation of

$\mathfrak{h}$ in $\mathfrak{g}$ are real.

**Lemma 4.1.** *Let $Q$ be a parabolic subgroup and put $B = G/Q$. Let also $L \subset G$ be a connected subgroup with Lie algebra $\mathfrak{l}$, and suppose that its action on $B$ is minimal. Then $\{0\}$ is the only nilpotent triangular ideal of $\mathfrak{l}$.*

**Proof.** Let $\mathfrak{i}$ be a nilpotent triangular ideal. The connected subgroup $I$ whose Lie algebra is $\mathfrak{i}$ is normal in $L$. If we check that $I$ has a fixed point in $B$ then we are done. In fact, if $x$ is fixed by $I$ then any $y \in Lx$ is also fixed because $I$ is normal. Since $Lx$ is dense in $B$ this implies that $Iz = z$ for all $z \in B$. But the action of $G$ on $B$ is effective because we are assuming $G$ to be simple and centerless. Therefore $I$ has a fixed point in $B$ if and only if $I = \{1\}$, or equivalently, $\mathfrak{i} = 0$.

The existence of a point fixed by $I$ follows from [12]: Since $\mathfrak{i}$ is triangular the adjoint action of $\mathfrak{i}$ and $I$ on $\mathfrak{g}$ can be represented as a group of triangular matrices. This property is preserved when passing to exterior products of the adjoint representation. Since $Q$ is the normalizer of its Lie algebra, $B$ can be realized as a compact projective $G$-orbit in some exterior product. Now [12] ensures that any triangular group has a fixed point in a compact invariant subset of a projective space. It follows that $I$ has a fixed point in $B$, showing the lemma. ∎

From this lemma we get at once that the solvable radical $\mathfrak{r}$ of $\mathfrak{l}$, i.e., the maximal solvable ideal, is abelian. In fact, by the Theorem of Lie on representations of solvable Lie algebras, the derived algebra $\mathfrak{r}'$ is triangular in $\mathfrak{g}$. Hence the lemma implies that $\mathfrak{r}' = 0$, so that $\mathfrak{r}$ is abelian. Taking into account the Levi decomposition of $\mathfrak{l}$, we conclude:

**Lemma 4.2.** *Let the notations and assumptions be as in Lemma 4.1. Then*

$$\mathfrak{l} = \mathfrak{e} \oplus \mathfrak{v}$$

*with $\mathfrak{e}$ semi-simple and $\mathfrak{v}$ and abelian ideal of $\mathfrak{l}$.*

Once we have this lemma we proceed to show that $\mathfrak{l}$ is a reductive Lie algebra, or equivalently, that $X \in \mathfrak{v}$ commutes with $\mathfrak{l}$. In order to show this we must look in detail at the adjoint representation of $\mathfrak{v}$ in $\mathfrak{g}$, and afterwards at the adjoint representation of $\mathfrak{e}$ in $\mathfrak{v}$.

For $X \in \mathfrak{v}$ denote by $S_X$ and $N_X$ respectively the semi-simple and nilpotent parts of the Jordan decomposition of $\mathrm{ad}_{\mathfrak{g}}(X)$. Since $\mathfrak{g}$ is semi-simple, there are unique $X_S, X_N \in \mathfrak{g}$ such that $S_X = \mathrm{ad}(X_S)$ and $N_X = \mathrm{ad}(X_N)$. We have $X = X_S + X_N$, and of course these elements of $\mathfrak{g}$ commute between themselves.

Let $\mathsf{s}_{\mathfrak{v}}$ denote the subspace of $\mathfrak{g}$ spanned by $X_S$ with $X$ running through $\mathfrak{v}$. Similarly, denote by $\mathsf{n}_{\mathfrak{v}}$ the subspace spanned by $X_N$ with $X \in \mathfrak{v}$. We have,

**Lemma 4.3.** *The subspaces $\mathsf{s}_{\mathfrak{v}}$, $\mathsf{n}_{\mathfrak{v}}$ and $\mathsf{s}_{\mathfrak{v}} + \mathsf{n}_{\mathfrak{v}}$ are abelian subalgebras of $\mathfrak{g}$. Moreover, $\mathsf{n}_{\mathfrak{v}}$ is triangular whereas the elements of $\mathsf{s}_{\mathfrak{v}}$ are semi-simple.*

**Proof.** From the general theory of representations of nilpotent Lie algebras (see e.g. [11]), we can decompose the complexification $\mathfrak{g}_{\mathbb{C}}$ of $\mathfrak{g}$ into weight spaces for the adjoint representation of $\mathfrak{v}$. For each $X \in \mathfrak{v}$ the restriction of $\mathrm{ad}_{\mathfrak{g}_{\mathbb{C}}}(X)$ to a weight space has just one eigenvalue. Hence the restriction of $S_X$ to a weight space is a scalar matrix. This implies that

$$\mathrm{ad}_{\mathfrak{g}}[X_S, Y_S] = \mathrm{ad}_{\mathfrak{g}_{\mathbb{C}}}[X_S, Y_S] = [S_X, S_Y] = 0$$

for any pair $X, Y \in \mathfrak{v}$. Since $\mathfrak{g}$ is centerless, it follows that $[X_S, Y_S] = 0$ so that $\mathsf{s}_{\mathfrak{v}}$ is abelian. Using again the fact that $S_X$ is a scalar matrix inside a weight space and taking into account that each weight space is $N_X$-invariant, we get for all $X, Y \in \mathfrak{v}$, that $[X_S, Y_N] = [S_X, N_Y] = 0$. Since $\mathfrak{v}$ is abelian, it follows that

$$0 = [X, Y] = [X_N, Y_N].$$

Hence $\mathsf{n}_{\mathfrak{v}}$ is an abelian subalgebra commuting with $\mathsf{s}_{\mathfrak{v}}$, so that $\mathsf{s}_{\mathfrak{v}} + \mathsf{n}_{\mathfrak{v}}$ is also abelian.

The last statement follows from the fact that $N_X$, $X \in \mathfrak{v}$, are simultaneously triangular, and the complexifications of $S_X$, $X \in \mathfrak{v}$, are simultaneously diagonalizable. ■

From the construction of the subspaces, it is clear that

$$\mathfrak{v} \subset \mathsf{s}_{\mathfrak{v}} + \mathsf{n}_{\mathfrak{v}}, \mathsf{s}_{\mathfrak{v}} \subset \mathfrak{v} + \mathsf{n}_{\mathfrak{v}} \text{ and } \mathsf{n}_{\mathfrak{v}} \subset \mathfrak{v} + \mathsf{s}_{\mathfrak{v}}. \tag{3}$$

In order to continue we must look at the bracket relations between these subspaces and the semi-simple component $\mathfrak{e}$ of $\mathfrak{l}$. For this note first that since $\mathfrak{v}$ is an abelian ideal of $\mathfrak{l}$, $\mathrm{ad}_{\mathfrak{l}}(X)^2 = 0$ for all $X \in \mathfrak{v}$. On the other hand, the linear maps $S_X$ and $N_X$, $X \in \mathfrak{v}$ are polynomials in $\mathrm{ad}_{\mathfrak{g}}(X)$ so that they leave invariant both $\mathfrak{v}$ and $\mathfrak{l}$. Hence the restriction of $S_X$ to $\mathfrak{l}$ is zero. This implies that $[\mathfrak{e}, \mathsf{s}_{\mathfrak{v}}] = 0$. In particular, $\mathsf{s}_{\mathfrak{v}}$ is invariant under $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{e})$. Combining this with the inclusions (3) we conclude that $\mathsf{s}_{\mathfrak{v}} + \mathsf{n}_{\mathfrak{v}}$ is an invariant subspace under the adjoint representation of $\mathfrak{e}$ in $\mathfrak{g}$. We show below that $\mathsf{n}_{\mathfrak{v}}$ is also invariant. Before that we need the following lemma, which will be needed also afterwards in the proof that $L$ is transitive.

**Lemma 4.4.** *Let $X \in \mathsf{s}_{\mathfrak{v}}$ then the eigenvalues of $\mathrm{ad}_{\mathfrak{g}}(X)$ are purely imaginary.*

**Proof.** Since $\mathrm{ad}_{\mathfrak{g}}(X)$ is semi-simple, $X$ belongs to some Cartan subalgebra, say $\mathfrak{j}$ (see [13, Prop. 1.3.5.4]). For this subalgebra there exists a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$ such that

$$\mathfrak{j} = \mathfrak{j}_{\mathfrak{k}} \oplus \mathfrak{j}_{\mathfrak{s}}$$

where $\mathfrak{j}_{\mathfrak{k}} = \mathfrak{j} \cap \mathfrak{k}$ and $\mathfrak{j}_{\mathfrak{s}} = \mathfrak{j} \cap \mathfrak{s}$. The eigenvalues of the adjoint in $\mathfrak{g}$ of the elements in $\mathfrak{j}_{\mathfrak{k}}$ are purely imaginary and those of $\mathfrak{j}_{\mathfrak{s}}$ are real. Let $\theta$ be the Cartan involution associated with the Cartan decomposition and consider the inner product $B_\theta(Y, Z) = -\langle Y, \theta Z \rangle$ where $\langle \cdot, \cdot \rangle$ is the Cartan-Killing form of $\mathfrak{g}$. With respect to $B_\theta$, $\mathrm{ad}_{\mathfrak{g}}(Y)$, $Y \in \mathfrak{k}$ is skew-symmetric while $\mathrm{ad}_{\mathfrak{g}}(Z)$, $Z \in \mathfrak{s}$ is symmetric.

Now, put $\mathfrak{l}_1 = \mathfrak{l} + (\mathsf{s}_\mathfrak{v} + \mathsf{n}_\mathfrak{v})$ and take a basis of $\mathfrak{g}$ which is orthonormal with respect to $B_\theta$ and such that its first elements are in $\mathfrak{l}_1$. Since $S_X$ annihilates $\mathfrak{l}$ and $\mathsf{s}_\mathfrak{v} + \mathsf{n}_\mathfrak{v}$ is abelian it follows that the matrix of $\mathrm{ad}_\mathfrak{g}(X)$ in this basis is of the form

$$\mathrm{ad}_\mathfrak{g}(X) = \begin{pmatrix} 0 & * \\ 0 & * \end{pmatrix}.$$

On the other hand if we write $X = X_\mathfrak{k} + X_\mathfrak{s}$ with $X_\mathfrak{k} \in \mathfrak{j}_\mathfrak{k}$ and $X_\mathfrak{s} \in \mathfrak{j}_\mathfrak{s}$ then

$$\mathrm{ad}_\mathfrak{g}(X_\mathfrak{k}) = \begin{pmatrix} a & b \\ -b^t & c \end{pmatrix}$$

with $a$ and $c$ skew-symmetric because $X_\mathfrak{k} \in \mathfrak{k}$. Also, $X_\mathfrak{s} \in \mathfrak{s}$ hence

$$\mathrm{ad}_\mathfrak{g}(X_\mathfrak{s}) = \begin{pmatrix} \alpha & \beta \\ \beta^t & \gamma \end{pmatrix}$$

with $\alpha$ and $\gamma$ symmetric. Comparing these matrices we get $a = \alpha = 0$ and $\beta = b$. Moreover, $X_\mathfrak{k}, X_\mathfrak{s} \in \mathfrak{j}$ and $\mathfrak{j}$ is abelian. Hence

$$\mathrm{ad}_\mathfrak{g}[X_\mathfrak{k}, X_\mathfrak{s}] = \begin{pmatrix} 2bb^t & * \\ * & * \end{pmatrix} = 0,$$

which shows that $b = \beta = 0$. Therefore $X_\mathfrak{s}$ centralizes $\mathfrak{l}_1$. We can now apply Lemma 4.1 to show that $X_\mathfrak{s} = 0$: The subspace spanned by $X_\mathfrak{s}$ is an ideal of $\mathfrak{l}_1$. Let $L_1$ be the connected subgroup whose Lie algebra is $\mathfrak{l}_1$. Since $L$ is assumed to be connected, $L \subset L_1$, hence $L_1$ is minimal on $B$. Since $\mathrm{ad}_\mathfrak{g}(X_\mathfrak{s})$ has real eigenvalues, the ideal spanned by $X_\mathfrak{s}$ is triangular so that $X_\mathfrak{s} = 0$ concluding the proof of the lemma. ■

Now we can prove that $\mathsf{n}_\mathfrak{v}$ is invariant under the adjoint representation of $\mathfrak{e}$.

**Lemma 4.5.** *Let $\mathsf{s}_\mathfrak{v}^\perp$ denote the orthogonal complement of $\mathsf{s}_\mathfrak{v}$ with respect to the Cartan-Killing form $\langle \cdot, \cdot \rangle$ of $\mathfrak{g}$. Then $\mathsf{n}_\mathfrak{v} = \mathsf{s}_\mathfrak{v}^\perp \cap (\mathsf{s}_\mathfrak{v} + \mathsf{n}_\mathfrak{v})$, and $\mathsf{n}_\mathfrak{v}$ is $\mathrm{ad}_\mathfrak{g}(\mathfrak{e})$-invariant.*

**Proof.** If $X \in \mathsf{s}_\mathfrak{v}$ and $Y \in \mathsf{n}_\mathfrak{v}$ then $\mathrm{ad}_\mathfrak{g}(X)\,\mathrm{ad}_\mathfrak{g}(Y)$ is upper triangular. Hence

$$\mathrm{tr}\,(\mathrm{ad}_\mathfrak{g}(X)\,\mathrm{ad}_\mathfrak{g}(Y)) = 0.$$

This means that $\mathsf{n}_\mathfrak{v} \subset \mathsf{s}_\mathfrak{v}^\perp \cap (\mathsf{s}_\mathfrak{v} + \mathsf{n}_\mathfrak{v})$. On the other hand, Lemma 4.4 implies that $\langle X, X \rangle < 0$ if $0 \neq X \in \mathsf{s}_\mathfrak{v}$. It follows that the restriction to $\mathsf{s}_\mathfrak{v}$ of $\langle \cdot, \cdot \rangle$ is nondegenerate so that $\mathsf{s}_\mathfrak{v}^\perp \cap \mathsf{s}_\mathfrak{v} = 0$ and $\mathfrak{g} = \mathsf{s}_\mathfrak{v}^\perp \oplus \mathsf{s}_\mathfrak{v}$. Therefore

$$\mathsf{s}_\mathfrak{v} + \mathsf{n}_\mathfrak{v} = \mathsf{s}_\mathfrak{v} \oplus \left(\mathsf{s}_\mathfrak{v}^\perp \cap (\mathsf{s}_\mathfrak{v} + \mathsf{n}_\mathfrak{v})\right).$$

Since $\mathsf{n}_\mathfrak{v} \subset \mathsf{s}_\mathfrak{v}^\perp \cap (\mathsf{s}_\mathfrak{v} + \mathsf{n}_\mathfrak{v})$ these subspaces are equal.

Finally, note that the $\mathrm{ad}_\mathfrak{g}(\mathfrak{e})$-invariance of $\mathsf{s}_\mathfrak{v}$ implies that $\mathsf{s}_\mathfrak{v}^\perp$ is also invariant. Since $\mathsf{s}_\mathfrak{v} + \mathsf{n}_\mathfrak{v}$ is invariant we conclude that $\mathsf{n}_\mathfrak{v}$ is an invariant subspace under the adjoint action of $\mathfrak{e}$. ■

From this lemma it is easy to apply Lemma 4.1 to show that the subspace $\mathfrak{n}_{\mathfrak{v}}$ reduces to zero: Put $\mathfrak{l}_1 = \mathfrak{l} + (\mathfrak{s}_{\mathfrak{v}} + \mathfrak{n}_{\mathfrak{v}})$. Then $\mathfrak{l}_1$ is a subalgebra because $\mathfrak{s}_{\mathfrak{v}} + \mathfrak{n}_{\mathfrak{v}}$ is $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{e})$-invariant. Since $\mathfrak{n}_{\mathfrak{v}}$ is invariant under $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{e})$, it is an ideal in $\mathfrak{l}_1$. By construction $\mathfrak{n}_{\mathfrak{v}}$ is triangular. Let $L_1$ be the connected subgroup whose Lie algebra is $\mathfrak{l}_1$. The assumption that $L$ is connected implies that $L \subset L_1$ so that $L_1$ is minimal on $B$. Therefore by Lemma 4.1, $\mathfrak{n}_{\mathfrak{v}} = 0$. This implies that $\mathfrak{v} = \mathfrak{s}_{\mathfrak{v}}$ and since $\mathfrak{s}_{\mathfrak{v}}$ commutes with $\mathfrak{l}$, $\mathfrak{v}$ is the center of $\mathfrak{l}$.

Summarizing all the previous discussion in this section we have:

**Theorem 4.6.** *Suppose that a connected subgroup $L \subset G$ acts minimally on a flag manifold $B$ of $G$. Let $\mathfrak{l}$ be the Lie algebra of $L$. Then $\mathfrak{l}$ is reductive, that is, $\mathfrak{l} = \mathfrak{e} \oplus \mathfrak{z}$ with $\mathfrak{e}$ semi-simple and $\mathfrak{z}$ the center of $\mathfrak{l}$. Moreover, for all $X \in \mathfrak{z}$, $\mathrm{ad}_{\mathfrak{g}}(X)$ is semi-simple and has purely imaginary eigenvalues.*

Up to this point the only requirement about $L$, in addition to its minimal action, was that it is a connected Lie subgroup. We impose now the condition that $L$ is closed and prove that its action on $B$ is actually transitive. Note first that since we are assuming that $G$ is centerless, the fact that the eigenvalues of $\mathrm{ad}_{\mathfrak{g}}(X)$ are imaginary implies that the connected subgroup $Z$ whose Lie algebra is $\mathfrak{z}$ is relatively compact in $G$. Hence $Z$ is relatively compact in $L$ if $L$ is closed. Its closure $\overline{Z}$ is contained in the center of $L$. Hence the Lie algebra of $\overline{Z}$ is $\mathfrak{z}$. Since $\overline{Z}$ is connected it follows that $\overline{Z} = Z$ and $Z$ is compact.

**Theorem 4.7.** *Suppose that a closed and connected subgroup $L \subset G$ acts minimally on a flag manifold $B$ of $G$. Then $L$ is transitive on $B$.*

**Proof.** Write as above $\mathfrak{l} = \mathfrak{e} + \mathfrak{z}$ with $\mathfrak{e}$ semi-simple and $\mathfrak{z}$ the center of $\mathfrak{l}$. Denote by $E$ and $Z$ the connected subgroups whose Lie algebras are $\mathfrak{e}$ and $\mathfrak{z}$ respectively. Every $g \in L$ is a product of exponentials of elements of $\mathfrak{l}$. Since $Z$ is a central subgroup, $L = ZE$ with $Z$ is compact. However $E$ has a compact orbit in $B$ because it is a semi-simple subgroup. Let $Ex_0$ be this orbit. Then $Lx_0 = ZEx_0$ is compact. But the action of $L$ is minimal. Hence $Lx_0 = B$, and $L$ is transitive. ∎

## 5. Semi-simple subgroups

In the previous section we restricted attention to subgroups acting minimally on some flag manifold. We return here to the admissible groups, which satisfy also a contractive property.

For the admissible groups we have a slightly stronger result than Theorem 4.7, namely that the semi-simple component of $L$ is also admissible. In fact, let $S \subset G$ be a semigroup with $\mathrm{int}\, S \neq \emptyset$ and suppose that $S$ is transitive in $G/L$. Then $L = ZE$ with $Z$ a central compact subgroup and $E$ semi-simple. Hence $L/E$ is a compact group so that the canonical fibration

$$G/E \longrightarrow G/L$$

is a connected principal bundle with compact structure group $L/E$. This implies that $S$ is transitive on $G/L$ if and only if it is transitive on $G/E$ (see [6] or [2]).

Hence $E$ is $S$-admissible provided $L$ is $S$-admissible and if this is the case then $E$ is also transitive on the flag $B(S)$, as follows from Theorem 4.7.

After these preliminaries we proceed to the analysis of the admissible semi-simple groups.

Throughout this section we let $E$ stand for an admissible, noncompact and connected semi-simple subgroup of $G$. As before we denote by $S$ a semigroup with nonempty interior which is transitive on $G/E$, so that the action of $E$ on $B(S)$ is transitive and contractive.

Let $\mathfrak{e}$ be the Lie algebra of $E$ and write $\mathfrak{e} = \mathfrak{k}_1 + \mathfrak{s}_1$ for a Cartan decomposition of $\mathfrak{e}$. It is well known that there exists a Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{s}$ such that $\mathfrak{k}_1 \subset \mathfrak{k}$ and $\mathfrak{s}_1 \subset \mathfrak{s}$. Also, we can choose compatible abelian subalgebras $\mathfrak{a}_1 \subset \mathfrak{a}$ with $\mathfrak{a}_1$ maximal in $\mathfrak{s}_1$ and $\mathfrak{a}$ maximal in $\mathfrak{s}$. With these choices we denote by $\Pi$ the set of roots of the pair $(\mathfrak{g}, \mathfrak{a})$ and by $\Pi_1$ the set of roots of $(\mathfrak{g}_1, \mathfrak{a}_1)$.

Given a root $\alpha \in \Pi_1$ and $H \in \mathfrak{a}_1$, $\alpha(H)$ is an eigenvalue of $\mathrm{ad}_{\mathfrak{e}}(H)$ and hence of $\mathrm{ad}_{\mathfrak{g}}(H)$. However the eigenvalues of $\mathrm{ad}_{\mathfrak{g}}(H)$ are given by the values in $H$ of the roots in $\Pi$ because $\mathfrak{a}_1 \subset \mathfrak{a}$. Therefore every root in $\Pi_1$ is the restriction to $\mathfrak{a}_1$ of a root in $\Pi$. In general a set of positive roots in $\Pi$ does not restrict to a set of positive roots of $\Pi_1$. But we can choose a simple system of roots for $\Pi$ *compatible* with a simple system for $\Pi_1$, that is, in such a way that the positive roots in $\Pi$ contain the positive roots in $\Pi_1$. In fact, let $H \in \mathfrak{a}_1$ be a regular real element of $\mathfrak{e}$. Then $H$ belongs to the closure of a Weyl chamber in $\mathfrak{a}$. Let $\mathfrak{a}^+$ be one of these chambers, and denote by $\Sigma$ the corresponding simple system of roots and by $\Pi^+$ the set of positive roots thus obtained. The roots in $\Pi^+$ assume nonnegative values in $H$. By the same token, the values assumed in $H$ by the roots in $\Pi^- = -\Pi^+$ are $\leq 0$.

On the other hand, $H$ is regular real in $\mathfrak{e}$ so that it belongs to a Weyl chamber $\mathfrak{a}_1^+ \subset \mathfrak{a}_1$. Let

$$\Pi_1^+ = \{\alpha \in \Pi_1 : \alpha(H) > 0\}$$

be the set of positive roots and denote by $\Sigma_1$ the simple system associated with $\mathfrak{a}_1^+$. For $\alpha \in \Pi_1^+$ we have $\alpha(H) > 0$, and since the values of the roots in $\Pi^-$ are nonpositive in $H$, $\alpha$ is the restriction to $\mathfrak{a}_1$ of some $\beta \in \Pi^+$. Therefore, we have

**Lemma 5.1.** *Let $\Pi_1^+$ be a positive system of roots for $(\mathfrak{e}, \mathfrak{a}_1)$ and denote by $\mathfrak{a}_1^+$ the corresponding Weyl chamber. Pick $H \in \mathfrak{a}_1^+$. Then there is a positive system $\Pi^+$ for $(\mathfrak{g}, \mathfrak{a})$ such that $\alpha(H) \geq 0$ for all $\alpha \in \Pi^+$. In this case $\Pi_1^+$ and $\Pi^+$ are compatible, i.e., every $\beta \in \Pi_1^+$ is the restriction to $\mathfrak{a}_1$ of some $\alpha \in \Pi^+$.*

Compatible positive systems induce compatible Iwasawa decompositions. In fact, with the same notations as in the lemma, put

$$\mathfrak{n}_1 = \sum_{\beta \in \Pi_1^+} \mathfrak{e}_\beta.$$

Then $\mathfrak{n}_1$ is the sum of eigenspaces of $\mathrm{ad}_{\mathfrak{e}}(H)$ associated to positive eigenvalues. Since these eigenspaces are contained in $\mathfrak{n} = \sum_{\alpha \in \Pi^+}$ it follows that $\mathfrak{n}_1 \subset \mathfrak{n}$. We remark that even if the simple systems are compatible it may happen in general that $\mathfrak{a}_1^+$ is not contained in $\mathrm{cl}\, \mathfrak{a}^+$. This is due to the fact that some root $\alpha \in \Pi^+$

may be negative in some $H' \in \mathfrak{a}_1^+$. Nevertheless we shall prove below that for admissible subgroups the inclusion between the Weyl chambers actually holds.

Let $B$ be a flag manifold of $G$. If we take a simple system $\Sigma$ for $\Pi$, then $B = B_\Theta$ for some subset $\Theta \subset \Sigma$. Let $\Psi \subset \Sigma$ be the subset of simple roots which vanish identically on $\mathfrak{a}_1$. The next statement relates $\Theta$ and $\Psi$ in case the action of $E$ on $B_\Theta$ is contractive.

**Proposition 5.2.** *Take compatible simple systems $\Sigma_1$ and $\Sigma$, and suppose that $\Psi \subset \Theta$. Then the action of $E$ on $B_\Theta$ is contractive.*

*Reciprocally, if the action of $E$ on the flag manifold $B$ is contractive then there are Weyl chambers $\mathfrak{a}_1^+ \subset \mathfrak{a}_1$ and $\mathfrak{a}^+ \subset \mathfrak{a}$, defining compatible simple systems $\Sigma_1$ and $\Sigma$ such that if $B = B_\Theta$ with $\Theta \subset \Sigma$, and $\Psi \subset \Sigma$ is as above the annihilator of $\mathfrak{a}_1$ in $\Sigma$ then $\Psi \subset \Theta$.*

**Proof.** Keeping the notations as above, $\mathfrak{a}_1$ intercepts the closure of $\mathfrak{a}^+$ because the simple systems were chosen to be compatible. Set

$$\mathfrak{b} = \{H \in \mathfrak{a} : \alpha(H) = 0 \text{ for all } \alpha \in \Psi\}$$

if $\Psi \neq \emptyset$ and $\mathfrak{b} = \mathfrak{a}$ if $\Psi = \emptyset$. Clearly, $\mathfrak{a}_1 \subset \mathfrak{b}$. The intersection of $\mathfrak{b}$ with the closure of $\mathfrak{a}^+$ is given by those elements $H \in \mathfrak{b}$ such that $\alpha(H) \geq 0$ for all $\alpha \in \Sigma$. This is a cone in $\mathfrak{b}$ whose interior is the "subchamber"

$$\mathfrak{b}^+ = \{H \in \mathfrak{b} : \alpha(H) > 0, \alpha \in \Sigma - \Psi\}.$$

By the choice of $\mathfrak{a}^+$, $\mathfrak{a}_1$ intercepts the closure of $\mathfrak{b}^+$. Actually, $\mathfrak{a}_1$ intercepts $\mathfrak{b}^+$ itself because $\Psi$ is exactly the subset of simple roots which vanish on $\mathfrak{a}_1$. Hence there exists $H \in \mathfrak{a}_1 \cap \mathfrak{b}^+$. This means that $\alpha(H) > 0$ for every root $\alpha \in \Sigma - \Psi$. But by assumption $\Psi \subset \Theta$, so that $\alpha(H) < 0$ for every negative root outside $-\langle\Theta\rangle$. Therefore the sequence $\exp(nH)$, $n \geq 0$ is contractive in $B_\Theta$, showing the first statement.

For the converse take at first an arbitrary Weyl chamber $\mathfrak{a}^+ \subset \mathfrak{a}$, and use the notations $A = \exp \mathfrak{a}$, $A^+ = \exp \mathfrak{a}^+$ and $A_1 = \exp \mathfrak{a}_1$. Then $B = B_\Theta$ for some $\Theta \subset \Sigma$, where $\Sigma$ is the simple system associated with $\mathfrak{a}^+$. Let $g_n \in E$ be a sequence which is contractive w.r.t. $B$. Then there are the two polar decompositions

- $g_n = u_n h_n v_n$, with $u_n, v_n \in K$ and $h_n \in \operatorname{cl} A^+$, and
- $g_n = s_n a_n t_n$ with $s_n, t_n \in K_1$ and $a_n \in A_1$.

The second of these decompositions is intrinsic in $E$. Since $A_1 \subset A$ it follows that $a_n \in A$. Hence there exists $w_n$ in the Weyl group of $\mathfrak{a}$ such that $b_n = w_n a_n w_n^{-1}$ belongs to the closure of $A^+$. By the second decomposition we can write $g_n = s_n w_n^{-1} b_n w_n t_n$, so that the uniqueness of the radial part in a polar decomposition ensures that $h_n = b_n = w_n a_n w_n^{-1}$.

By assumption $g_n$ is contractive. Hence $\phi_\alpha(h_n) \to 0$ for $\alpha$ outside $-\langle\Theta\rangle$. This means that $\phi_\alpha(w_n a_n w_n^{-1}) \to 0$, or, equivalently, $\phi_{w_n^{-1}\alpha}(a_n) \to 0$ for all $\alpha < 0$,

$\alpha \notin -\langle\Theta\rangle$. Taking a subsequence we can assume that $w_n = w$ is independent of $n$. Then there exists $a \in A_1$ such that $\phi_{w^{-1}\alpha}(a) < 1$ for every negative root $\alpha \notin -\langle\Theta\rangle$. In other words there exists $H \in \mathfrak{a}_1$ such that $\alpha(H) < 0$ for every negative root $\alpha \notin -\langle w^{-1}\Theta\rangle$. By continuity we can take $H$ to be regular real in $\mathfrak{a}_1$. Now consider the simple system $\Sigma' = w^{-1}\Sigma$ and the subset $\Theta' = w^{-1}\Theta \subset \Sigma'$. Of course $B = B_{\Theta'}$ with the construction made from $\Sigma'$. Let $W_{\Theta'}$ be the subgroup generated by the reflections with respect to the roots in $\Theta'$. By Lemma 5.3 below there exists $u \in W_{\Theta'}$ such that $(u^{-1}\alpha)(H) = \alpha(uH) \geq 0$ for all $\alpha \in \Theta'$. We have that $u^{-1}$ leaves $\langle\Theta\rangle$ invariant. Hence $(u^{-1}\alpha)(H) > 0$ for every positive root $\alpha \notin \langle\Theta\rangle$. Therefore if we put $\Sigma'' = u^{-1}\Sigma'$, $\Theta'' = u^{-1}\Theta'$ and let $\Psi''$ be the subset of roots in $\Sigma''$ which are identically zero on $\mathfrak{a}_1$, then $\Psi'' \subset \Theta''$ and the data $H \in \mathfrak{a}_1$, $\Psi''$, $\Theta''$ and $\Sigma''$ accomplish the requirements. ■

**Lemma 5.3.** *For a subset $\Theta \subset \Sigma$ denote by $W_\Theta$ the subgroup generated by the reflections with respect to the roots in $\Theta$. Take $H \in \mathfrak{a}$. Then there exists $u \in W_\Theta$ such that $\alpha(uH) \geq 0$ for all $\alpha \in \Theta$.*

**Proof.** For $\alpha \in \mathfrak{a}^*$ let $H_\alpha \in \mathfrak{a}$ be defined by $\alpha(\cdot) = \langle H_\alpha, \cdot\rangle$. Denote by $\mathfrak{a}_\Theta$ the subspace generated by $H_\alpha$, $\alpha \in \Theta$. The subset $\{H_\alpha : \alpha \in \langle\Theta\rangle\}$ is a root system in $\mathfrak{a}_\Theta$ containing $\Sigma_\Theta = \{H_\alpha : \alpha \in \Theta\}$ as a simple system and whose Weyl group is $W_\Theta$. Write $H = H_1 + H_2$ with $H_1 \in \mathfrak{a}_\Theta$ and $H_2$ orthogonal to $\mathfrak{a}_\Theta$. Then there exists $u \in W_\Theta$ such that $wH_2$ belongs to the Weyl chamber defined by $\Sigma_\Theta$. Since $uH_2 = H_2$ for all $u \in W_\Theta$ the lemma follows. ■

We can prove now that $B(S)$ is a flag manifold for semi-simple admissible subgroups.

**Theorem 5.4.** *Let $E \subset G$ be a connected semi-simple subgroup and suppose that its action on the flag manifold $B$ of $G$ is contractive and transitive. Then $B$ is also a flag manifold of $E$.*

**Proof.** Let $P_1 = M_1A_1N_1$ be a minimal parabolic subgroup of $E$. We must show that $P_1$ has a fixed point in $B$. Take an abelian subgroup $A \subset G$ such that $A_1 \subset A$. For a Weyl chamber $A^+ \subset A$ denote by $\Sigma$ the corresponding simple system of roots. By the above proposition we can choose a chamber $A^+ \subset A$ and a regular real $h \in A_1$ such that $h$ belongs to the closure of $A^+$ and $\alpha(\log h) < 0$ for every negative root $\alpha \notin -\langle\Theta\rangle$, where $\Theta \subset \Sigma$ is the subset defining the standard parabolic subgroup $P_\Theta$ which gives $B = G/P_\Theta$. Let $x_0$ be the origin in $G/P_\Theta$. By the choice of $h \in A_1$, we have that $x_0$ is its attractor with stable manifold $N^-x_0$, that is, $h^n x \to x_0$ as $n \to \infty$ for all $x \in N^-x_0$ where $N^-$ is the nilpotent group given by the negative roots defined by $\Sigma$.

In the minimal parabolic subgroup $P_1 \subset E$ we can take $N_1$ to be $\exp \mathfrak{n}_1$ where $\mathfrak{n}_1$ is the sum of the root spaces associated with the roots of $\mathfrak{a}_1$ which are positive on $\log h$. With this choice we have compatible Iwasawa decompositions so that $N_1 \subset N$. This implies that $A_1N_1x_0 = x_0$.

On the other hand, take $m \in M_1$. Then $mx_0$ is fixed by $h$ because $hm = mh$. From this we get that $x_0$ is fixed under the identity component $(M_1)_0$

of $M_1$. In fact, since the stable manifold $N^-x_0$ is open, there is a neighborhood of the identity $U \subset M_1$ such that $Ux_0 \subset N^-x_0$. But the only $h$-fixed point in $N^-x_0$ is $x_0$ itself. Hence $Ux_0 = x_0$ which shows that $(M_1)_0 x_0 = x_0$.

Let $Q \subset E$ be the isotropy at $x_0$ for the $E$-action. So far we have shown that $(M_1)_0 A_1 N_1$ is contained in $Q$. This implies that the Lie algebra $\mathfrak{m}_1 + \mathfrak{a}_1 + \mathfrak{n}_1$ of $(M_1)_0 A_1 N_1$ is contained in the isotropy subalgebra $\mathfrak{q}$ at $x_0$. Therefore $\mathfrak{q}$ is a parabolic subalgebra of $\mathfrak{e}$. Let $Q_1$ be the corresponding parabolic subgroup. Then $Q$ is a subgroup of finite index in $Q_1$ so that the canonical equivariant fibration

$$\pi : B = E/Q \longrightarrow E/Q_1$$

is a covering. The equivariance of $\pi$ implies that $y = \pi(x_0)$ is an attractor for the action of $h$ in $E/Q$. Hence the elements in the fiber over $y$ are also attractors. However the stable manifold at $x_0$ is dense, so that there is just one attractor and $B = E/Q_1$, showing that $B$ is indeed a flag manifold of $E$. ■

**Corollary 5.5.** *Let $E$ be as in the theorem above and suppose that the Weyl chambers $\mathfrak{a}_1^+$ and $\mathfrak{a}^+$ in $\mathfrak{e}$ and $\mathfrak{g}$ are compatible in the sense of Lemma 5.1. Then $\mathfrak{a}_1^+ \subset \mathrm{cl}\, \mathfrak{a}^+$.*

**Proof.** Since $B$ is a flag manifold of $E$ there is a point $x_0 \in B$ which is a common attractor for $h \in A_1^+$. The stable manifold of this attractor is $N_\Theta^+ x_0 = \left(\exp \mathfrak{n}_\Theta^+\right) x_0$ where

$$\mathfrak{n}_\Theta^+ = \sum \mathfrak{g}_\alpha$$

with the sum running through the negative roots $\alpha \in -\langle\Theta\rangle$. On the other hand, let $\mathfrak{b}^+$ be as in the proof of Proposition 5.2 and suppose that $H \in \mathfrak{a}_1^+$ is not in the closure of $\mathfrak{b}^+$. Then there exists a negative root $\alpha \notin -\langle\Theta\rangle$ such that $\alpha(H) > 0$ because $\Psi \subset \Theta$ and hence $x_0$ is not an attractor for $\exp H$. Therefore every $H \in \mathfrak{a}_1^+$ is in the closure of $\mathfrak{a}^+$. ■

With the above theorem it is virtually possible to get all the admissible connected subgroups by checking the list of flags for the different noncompact semi-simple real Lie groups. As an example we mention that the projective space $\mathbb{P}^{n-1}$ is a flag manifold for $\mathrm{Sl}(n,\mathbb{R})$. The list of semi-simple groups transitive on $\mathbb{P}^{n-1}$ was provided by Boothby and Wilson [1]. The connected subgroups which are admissible for a semigroup $S$ with $B(S) = \mathbb{P}^{n-1}$ are in that list. By checking it we can see that the only one which is contractive and hence admissible is $\mathrm{Sp}(m,\mathbb{R})$ with $n = 2m$ (c.f. [10]).

In addition to this specific example we also obtain that $G$ is the only connected subgroup admissible for semigroups associated to the maximal flag manifold:

**Proposition 5.6.** *Suppose that in Theorem 5.4 the group $G$ is simple and $B$ is the maximal flag manifold of $G$. Then $E = G$.*

**Proof.** Take compatible Cartan decompositions $\mathfrak{e} = \mathfrak{k}_1 \oplus \mathfrak{s}_1$ and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$, and let $\mathfrak{a}_1 \subset \mathfrak{s}_1$ be a maximal abelian subalgebra. By Proposition 5.2 there exists $H \in \mathfrak{a}_1$ which is regular real in $\mathfrak{g}$. Since $B$ is a flag manifold of $E$, the compact group $K_1 = \exp \mathfrak{k}_1$ is transitive on it. On the other hand since $B$ is the maximal flag manifold of $G$ it is the $\mathrm{Ad}(K)$-orbit of $H$ and therefore $B = \mathrm{Ad}(K_1) H$. This orbit is contained in $\mathfrak{s}_1$ and generates $\mathfrak{s}$. Hence $\mathfrak{s}_1 = \mathfrak{s}$. Since $\mathfrak{g}$ is simple $\mathfrak{s}$ generates $\mathfrak{g}$ so that $\mathfrak{e} = \mathfrak{g}$. ■

## 6. Nonconnected subgroups

In the preceding sections we made the basic assumption that the admissible subgroup $L$ is connected. Here we make some comments about the nonconnected admissible subgroups. First of all, the case where there are a finite number of connected components is easily reduced to the connected case. In fact, if $L_0$ stands for the identity component in $L$ then there is the canonical fibration

$$G/L_0 \longrightarrow G/L$$

which makes $G/L_0$ a principal bundle with $L/L_0$ as structure group. If the number of connected components of $L$ is finite then $L/L_0$ is finite and hence a semigroup $S \subset G$ with $\mathrm{int}\, S \neq \emptyset$ is transitive in $G/L_0$ if and only if it is transitive in $G/L$ (see [6], [2]). Therefore $L$ is admissible if and only if $L_0$ is admissible and we are back to the situation of the previous sections.

We say now a few words about subgroups with infinite connected components. Suppose that $G$ is centerless. Then $G$ is the identity component of an algebraic group and we can consider it in the Zariski topology. For a subset $C \subset G$ we let $\mathrm{zc}(C)$ stand for its Zariski closure. If $L \subset G$ is a subgroup then $\mathrm{zc}(L)$ is a closed subgroup with a finite number of connected components. Moreover, since $L \subset \mathrm{zc}(L)$ we have that $\mathrm{zc}(L)$ is admissible for the semigroup $S$ if $L$ is $S$-admissible. Let $(\mathrm{zc}(L))_0$ be the identity component of $\mathrm{zc}(L)$ and put $L_1 = L \cap (\mathrm{zc}(L))_0$. Then $L/L_1$ is isomorphic to a subgroup of $(\mathrm{zc}(L)) / (\mathrm{zc}(L))_0$ so $L_1$ is a subgroup of finite index in $L$ which is also admissible. We have thus proved that any admissible subgroup has a subgroup of finite index which is Zariski dense in a connected admissible group.

Recall that if a subspace of $\mathfrak{g}$ is invariant under $L$ then it is also invariant under the algebraic closure $(\mathrm{zc}(L))_0$. Therefore the Lie algebra $\mathfrak{l}$ of $L$ is an ideal of the Lie algebra $\widetilde{\mathfrak{l}}$ of $\mathrm{zc}(L)$. Hence if $(\mathrm{zc}(L))_0$ is simple either $\mathfrak{l} = 0$ or $\mathfrak{l} = \widetilde{\mathfrak{l}}$. In the latter case $L$ contains $(\mathrm{zc}(L))_0$ so it has a finite number of connected components and we are back to the previous situation. On the other hand if $\mathfrak{l} = 0$ then $L$ is discrete. Combining this fact with Proposition 5.6 we obtain:

**Proposition 6.1.** *Suppose that $G$ is simple with finite center and $B(S)$ is the maximal flag manifold. Suppose also that $L \neq G$ is a subgroup such that $S$ is transitive on $G/L$. Then $L$ is discrete.*

Concerning the discrete infinite subgroups which are admissible we mention that there is an easy case, namely when $L$ is a lattice in $G$. Recall that a discrete subgroup $H$ of a topological group is said to be a lattice if there exists a $G$-invariant probability measure $\mu$ on $G/H$, that is, $\mu(g^{-1}A) = \mu(A)$ for all $g \in G$ and measurable sets $A \in G/H$. In case $L$ is a lattice, the Recurrence Theorem implies that any semigroup $S \subset G$ with $\text{int}\, S \neq \emptyset$ is transitive in $G/L$. This is a well known fact in the context of control systems (see [5], [8]), whose proof extends to more general semigroup actions. For the sake of completeness we provide here a proof which works for the action of a semigroup of homeomorphisms in a topological space. We need first the

**Lemma 6.2.** *Let $M$ be a topological space and $T$ a semigroup of homeomorphisms of $M$. Denote by $H$ the group of homeomorphisms of $M$ generated by $T \cup T^{-1}$. Assume that*
(1) *There exists a finite measure $\mu$ on the Borel subsets of $M$ which is invariant under $H$ and such that $\mu(A) > 0$ for any open set $A \subset M$.*
(2) *The action of $H$ on $M$ is transitive. Let $U \subset M$ be a non empty open subset which is invariant under $T$. Then $U$ is dense in $M$.*

**Proof.** Since $H$ is transitive in $M$, we must show that $\text{cl}\, U$ is $H$-invariant. This invariance follows at once if we show that $\text{cl}\, U$ is invariant under $T^{-1}$. In order to check this, suppose to the contrary that there exists $x \in \text{cl}\, U$, an open subset $V$ with $V \cap \text{cl}\, U = \emptyset$ and $g \in T$ such that $g^{-1}x \in V$. Apply the Recurrence Theorem (see e.g. [4]) to $g$ to get a subset $F \subset gV$ with $\mu(F) = \mu(gV)$ such that for any $y \in F$ there exists an integer $k > 1$ such that $g^k y \in gV$. Since $\mu$ is positive on open sets and $\mu(F) = \mu(gV)$ it follows that $F$ is dense in $gV$. Hence $U \cap F \neq \emptyset$ because $U \cap gV$ is open in $gV$. If $y \in U \cap F$ and $k > 1$ is such that $g^k y \in gV$ then $g^{k-1} y \in V$ which contradicts the fact that $U$ is $T$-invariant. ■

From this Lemma we obtain sufficient conditions for the transitivity of semigroups.

**Proposition 6.3.** *Suppose that $T$ and $M$ are as in the previous lemma and assume further that for all $x \in M$, $\text{int}\,(Tx)$ and $\text{int}\,(T^{-1}x)$ are not empty. Then $T$ is transitive on $M$.*

**Proof.** Pick $x, y \in M$. By the lemma $\text{int}\,(Tx)$ is dense in $M$. Hence $\text{int}\,(T^{-1}y) \cap \text{int}\,(Tx) \neq \emptyset$ which shows that there exists $g \in T$ such that $gx = y$ and hence $T$ is transitive. ■

Finally, we have the following hereditary property for admissible subgroups.

**Proposition 6.4.** *Suppose that $L$ is $S$-admissible and let $H \subset L$ be a closed subgroup such that $S \cap L$ is transitive in $L/H$. Then $S$ is transitive in $G/H$.*

**Proof.** Consider the canonical fibration

$$\pi : G/H \longrightarrow G/L$$

with fiber $L/H$. Let $x_0 = L$ be the origin in $G/L$. The assumption that $S \cap L$ is transitive in $L/H$ implies that $S$ is transitive along the fiber $\pi^{-1}(x_0)$. Since $S$ is transitive in $G/L$ it follows that $S$ is transitive in $G/H$. ■

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