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Some linear groups virtually having a free quotient

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Abstract. It is proved that some discrete automorphism groups of convex cones have a finite index subgroup that maps onto a free group of rank 2. This generalizes recent results of A. Lubotzky.

We say that a group Γ virtually has some property, if some subgroup of finite index in Γ has this property. A group virtually having a non-abelian free quotient is called *large* [4]. In this paper we prove that some linear groups are large. This generalizes results of A. Lubotzky [5]. Except for some additions and simplifications, we essentially follow the lines of his proof.

Note that if a group Γ has a non-abelian free quotient, then any subgroup of finite index in Γ has such a quotient. It follows that if a group Γ is large, then any group commensurable with Γ shares this property. Clearly, if a group Γ has a virtually non-abelian free quotient, it is large. (The converse is not true.)

Let V be a finite-dimensional real vector space and V^* its dual space. If $K \subset V$ is an open convex cone, the *dual cone* $K^* \subset V^*$ is defined by

$$K^* = \{\alpha \in V^* : \alpha(v) > 0 \quad \forall v \in K\}.$$

Obviously, $K^* \cup \{0\}$ is a closed strictly convex cone in V^*. For a linear group $\Gamma \subset \mathrm{GL}(V)$ leaving invariant an open convex cone $K \subset V$, we consider the condition

(∗) Γ has no finite orbits in the projectivization PK^* of K^*.

Main Theorem. *Let $\Gamma \subset \mathrm{GL}(V)$ be a finitely generated linear group leaving invariant some open convex cone $K \subset V$ and satisfying the condition (∗). Suppose that there exists a subspace U of codimension 1 in V such that*

- (H1) $H = U \cap K \neq \varnothing$;
- (H2) *for any $\gamma \in \Gamma$, we have either $\gamma H = H$, or $\gamma H \cap H = \varnothing$;*
- (H3) *any compact subset of K meets only finitely many sets of the form $\gamma H, \gamma \in \Gamma$;*
- (H4) *both connected components of $K \backslash H$ contain some $\gamma H, \gamma \in \Gamma$.*

Then Γ has a subgroup of index $\leqslant 2$ that has a virtually non-abelian free quotient.

We shall deduce from this the theorems stated below, in which the term "reflection" always means a reflection in a hyperplane.

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Theorem 1. *Let $\Gamma \subset GL(V)$ be a finitely generated irreducible linear group leaving invariant some open convex cone $K \subset V$ and acting on it discretely. Suppose that Γ contains a linear reflection. Then one of the following alternatives takes place:*

(1) *$K = V$ and Γ is finite;*

(2) *K is a simplicial cone and Γ is virtually abelian;*

(3) *Γ is large.*

Corollary 1. *Let Γ be a finitely generated discrete group of motions of n-dimensional Lobachevsky space $L^n (n > 1)$, containing a reflection and leaving invariant no plane (of any dimension) or point at infinity. Then Γ is large.*

Corollary 2. *Any non-affine infinite indecomposable finitely generated Coxeter group is large.*

Let f be a quadratic form of signature $(n, 1)$ over $\mathbb{Q}$. Then the group $\mathrm{O}(f, \mathbb{Z})$ is a lattice in $\mathrm{O}(f, \mathbb{R})$.

Theorem 2. *Any finitely generated subgroup of $\mathrm{O}(f, \mathbb{Z})$ is either virtually abelian, or large.*

Before we started this work, we had been informed that C. Gonciulea had proved the assertion of Corollary 2 above. But unfortunately his proof was not available to us at the time we wrote this paper. (See [3] and [2] for a proof of a weaker theorem that any infinite finitely generated Coxeter group virtually maps onto $\mathbb{Z}$.)

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1. Residually Finite Actions

Definition 1. An action of a group Γ on a set X is called *residually finite*, if for any different points $x, x' \in X$ there exist an action of Γ on a finite set Y and a Γ-equivariant map $f : X \to Y$ such that $f(x) \neq f(x')$.

Obviously, an action is residually finite if an only if its restriction to each orbit is residually finite. If an action $\Gamma : X$ is residually finite, then for any finite number of points of X there exists a Γ-equivariant map of X to a finite set separating these points.

Let us also note that an action $\Gamma : X$ is residually finite if and only if for any different points $x, x' \in X$ there exists a normal subgroup Δ of finite index in Γ such that x and x' belong to different Δ-orbits. Indeed, let x and x' be different points of X. If the action $\Gamma : X$ is residually finite, then, in the above notation, one can take for Δ the kernel of the action $\Gamma : Y$. Conversely, if the action $\Gamma : X$ is transitive and $\Delta x \neq \Delta x'$, one can take for Y the quotient X/Δ.

The following fact is known, but we did not find a reference for it.

Lemma 1. *Let A be a finitely generated integral domain. Then for any nonzero element a of A there exists a homomorphism φ of A to a finite field such that $\varphi(a) \neq 0$.*

Proof. Let k denote the field of fractions of A, and let F be the minimal subfield of k. By Hilbert's Nullstellensatz there exists a homomorphism φ of A into a finite extension of F such that $\varphi(a) \neq 0$. If char $k > 0$ this finite extension is a finite field and φ is the desired homomorphism. Let char $k = 0$. Then k is an algebraic number field. There exists a (multiplicative) non-archimedian valuation v of k such that A belongs to the ring $\mathcal{O} = \{x \in k \mid v(x) \leq 1\}$ and $v(\varphi(a)) = 1$. Let $\mathfrak{m} = \{x \in k : v(x) < 1\}$ be the maximal ideal of $\mathcal{O}$ and $\pi : \mathcal{O} \to \mathcal{O}/\mathfrak{m}$ the natural projection. Then $\pi \circ \varphi$ is the desired homomorphism.■

Let V be a finite-dimensional vector space over a field k. Denote by PV the projective space associated to V, and, for any non-zero vector $v \in V$, denote by $[v]$ the corresponding point of PV.

Proposition 1. *Any linear action of a finitely generated group Γ in the space V and the induced action $\Gamma : PV$ are residually finite.*

Proof. Let $\gamma_1, \ldots, \gamma_s$ be generators of Γ, and let v be a vector in V. Consider the subring A of k generated by the matrix coefficients of $\gamma_1, \ldots, \gamma_s, \gamma_1^{-1}, \ldots, \gamma_s^{-s}$ and the coordinates of v in some fixed basis of V. Any homomorphism φ of A to a field K defines a group homomorphism $\Gamma \to \mathrm{GL}_n(K)$, where $n = \dim V$, and thereby a linear action $\Gamma : K^n$. At the same time it defines a Γ-equivariant map $f : A^n \to K^n$. According to Lemma 1, for any $a \in A, a \neq 0$, there exists a homomorphism φ of A to a finite field K such that $\varphi(a) \neq 0$. In particular, if $v' \in \Gamma v$ is different from v, one can choose φ in such a way that $f(v) \neq f(v')$. Thus, the action $\Gamma : V$ is residually finite.

Let now $v \neq 0$. A vector $v' \in \Gamma v$ is proportional to v if and only if all the minors of order 2 of the matrix $M(v, v')$ of the co-ordinates of v and v' are equal to 0. In this case $f(v)$ and $f(v')$ are also proportional. We can choose φ in such a way that $f(v) \neq 0$. Then f gives rise to a Γ-equivariant map $\bar{f} : \Gamma[v] \to \Gamma[f(v)]$. For any $v' \in \Gamma v$, which is not proportional to v, we can choose φ in such a way that $\varphi(D) \neq 0$ for some minor D of order 2 of $M(v, v')$. Then $\bar{f}([v]) \neq \bar{f}([v'])$. Thus, the action $\Gamma : PV$ is also residually finite. ■

Remark 1. The proposition implies that, for any finitely generated subgroup Γ of an algebraic group G and any action of G on an algebraic variety X, the induced action $\Gamma : X$ is residually finite. Indeed, it suffices to prove this for transitive G-actions. But due to a theorem of Chevalley any transitive G-action is embedded into the natural G-action on PV, where V is a G-module. ■

2. Some Group Actions on Trees

For any graph S (possibly with loops and multiple edges) denote by $[S]$ its topological space. If S is connected, set $\pi_1(S) = \pi_1([S])$. If a group Γ acts on a graph T without reversing edges, one can naturally define a topological action of Γ on $[T]$ in such a way that the topological quotient $[T]/\Gamma$ is identified with $[T/\Gamma]$. In particular, if T is a tree, this gives rise to a surjective homomorphism $\varphi : \Gamma \to \pi_1(T/\Gamma)$, taking each $\gamma \in \Gamma$ to the homotopy class of the image in $[T/\Gamma]$ of a path l connecting the base vertex o with γo in $[T]$.

Lemma 2. *The kernel Γ_0 of φ does not depend on the choice of the base vertex and is normalized by any automorphism of T normalizing Γ.*

Proof. Let o' be any other vertex of T and m a path connecting o' with o. In the above notation, $l' = ml(\gamma m)^{-1}$ is a path connecting o' with $\gamma o'$. If the image of l in $[T/\Gamma]$ is homotopic to a point, then the image of l' is also homotopic to a point. This proves the first assertion of the lemma. The second assertion immediately follows. ■

Remark 2. In fact it is easy to show that Γ_0 is the subgroup of Γ generated by all the stabilizers of vertices of T. ■

It is useful to have in mind the following

Lemma 3. *Any group Γ acting on a tree T contains a subgroup Γ_+ of index 1 or 2 that does not reverse edges.*

Proof. Note that if the group Γ preserves some orientation of the edges of T, it cannot reverse edges. There are two distinguished opposite orientations of the edges of T, under which any vertex is either a source or a sink. Any automorphism of T can only permute them. It follows that the subgroup Γ_+ of Γ preserving one of these orientations has index 1 or 2, and it does not reverse edges. ■

A tree will be called a *star* if all its edges have a common vertex, and a *line* if its topological space is homeomorphic to $\mathbb{R}$. A graph will be called a *cycle* if its topological space is homeomorphic to a circle.

Denote by $V(S)$ (resp. $E(S)$) the set of vertices (resp. edges) of a graph S. For a connected finite graph S, we consider the Euler characteristic

$$e(S) = \#V(S) - \#E(S).$$

Obviously, $e(S) \leq 1$. If $e(S) = 1$, the graph S is acyclic (i.e. a tree); if $e(S) = 0$, it contains exactly one cycle; if $e(S) < 0$, it contains at least two different cycles. In the latter case $[S]$ is mapped onto the figure eight, whose fundamental group is a free group of rank 2 which implies that $\pi_1(S)$ has a non-abelian free quotient.

Lemma 4. *Let S be a connected finite graph whose automorphism group acts transitively on $E(S)$. Then*

1) *if $e(S) = 1$, S is a star;*

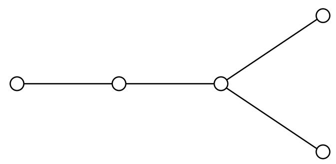
2) *if* $e(S) = 0$, S *is a cycle.*

Proof. If S has an extreme vertex, then each edge contains an extreme vertex. The other ends of the edges must coincide, so S is a star.

If $e(S) = 1$, S has an extreme vertex and hence is a star. If $e(S) = 0, S$ contains exactly one cycle and no extreme vertices, so it is a cycle. ■

Proposition 2. *Let a group* Γ *act on a tree* T *which is not a star or a line. Suppose that* Γ *does not reverse edges and the action* $\Gamma : E(T)$ *is transitive and residually finite. Then* Γ *has a virtually non-abelian free quotient.*

Proof. Since the tree T is not a star or a line, it contains a subgraph T_0 of the form



There exists a normal subgroup Δ of finite index in Γ such that the edges of T_0 belong to different Δ-orbits and thereby go to different edges of T/Δ (but some vertices of T_0 may glue together). This configuration of edges does not allow T/Δ to be a star or a cycle. Since Γ acts transitively on $E(T/\Delta)$, Lemma 4 shows that under such a choice of Δ we have $e(T/\Delta) < 0$. Let $\varphi : \Delta \to \pi_1(T/\Delta)$ be the homomorphism defined by the action $\Delta : T$. It follows from Lemma 2 that $\operatorname{Ker}\varphi = \Delta_0$ is a normal subgroup of Γ. Since $\Delta/\Delta_0 \simeq \pi_1(T/\Delta)$ is a non-abelian free group, Γ/Δ_0 is a virtually non-abelian free group. ■

3. Some Virtually Abelian Linear Groups

Let V be a finite-dimensional vector space.

Lemma 5. *Any abelian linear group* $\Gamma \subset \mathrm{GL}(V)$ *leaving invariant a non-trivial strictly convex cone* $K \subset V$, *has a weight vector in* $\overline{K}$.

Proof. If all the operators of Γ are scalar, there is nothing to prove. Otherwise, let $\gamma \in \Gamma$ be a non-scalar operator. By the Brouwer theorem on a fixed point γ has an eigenvector in $\overline{K}$, with some eigenvalue $\lambda > 0$. The eigenspace V_1 of γ corresponding to λ, is invariant under Γ and does not coincide with V. Moreover, $K_1 = \overline{K} \cap V_1$ is a non-trivial strictly convex cone in V_1 invariant under Γ. Proceeding by induction in $\dim V$, we may conclude that Γ has a weight vector in K_1 and, hence, in $\overline{K}$. ■

Let now $\Gamma \subset \mathrm{GL}(V)$ be an irreducible linear group.

Lemma 6. *If* Γ *has a finite orbit in* PV *(or* PV^**), then it is virtually abelian.*

Proof. Let $e \in V$ be a non-zero vector such that the orbit of the corresponding point $[e] \in PV$ is finite. The vectors γe, $\gamma \in \Gamma$, span a Γ-invariant subspace, which must coincide with V. The kernel Δ of the action of Γ on the orbit of $[e]$ is a (normal) subgroup of finite index in Γ. All the vectors γe, $\gamma \in \Gamma$, are weight vectors of Δ. This implies that Δ is abelian. ■

Note that if K is a Γ-invariant convex cone in V,then $\overline{K} \cap (-\overline{K})$ is a Γ-invariant subspace. If follows that any Γ-invariant convex cone either is strictly convex, or coincides with V.

Let $\mathbb{R}_+^*$ denote the subgroup of $\mathrm{GL}(V)$ consisting of the multiplications by positive scalars.

Lemma 7. *If Γ leaves invariant a convex cone $K \neq V$ and $\Gamma/(\Gamma \cap \mathbb{R}_+^*)$ is finite, then* $\dim V = 1$.

Proof. Multiplying the elements of Γ by suitable positive numbers, we may assume that $\det \gamma = \pm 1$ for any $\gamma \in \Gamma$. Then Γ is finite and, hence, has a fixed non-zero vector in K. The linear span of this vector must coincide with V. ∎

Lemma 8. *If Γ is virtually abelian, then every Γ-invariant convex cone $K \neq V$ is simplicial and the one-dimensional subspaces spanned by its edges are the weight subspaces of any normal abelian subgroup Δ of finite index in Γ.*

Proof. According to Lemma 5, Δ has a weight subspace W such that $\overline{K} \cap W \neq \{0\}$. It follows that the corresponding weight χ of Δ is positive valued, and the space V decomposes into a direct sum of weight subspaces $\gamma W, \gamma \in \Gamma$, of Δ. Moreover, the stabilizer of the subspace W acts on it irreducibly. Applying Lemma 7, we get that $\dim\, W = 1$.

Let χ_γ denote the weight of Δ corresponding to γW. There exists an element $\delta_0 \in \Delta$ such that all the weights χ_γ take different values on it. By symmetry, we may assume that the maximum of these values is taken by $\chi = \chi_e$ itself. Then for any $v \in V$ we have

$$\lim \frac{\delta_0^n v}{\chi(\delta_0)^n} = w,$$

where w is the projection of v to W. It follows that the cone $\overline{K}$ contains its projection to W and, by symmetry , its projection to each subspace γW. This means that $\overline{K}$ is spanned by these projections and, hence, is simplicial. ∎

4. Proof of Main Theorem

The sets $\gamma H, \gamma \in \Gamma$, which we shall call *walls*, decompose K into some (closed in K) convex bodies, which we shall call *chambers*. Consider the graph T, whose vertices (resp. edges) are the chambers (resp. the walls) and the incidence is defined by inclusion. Clearly, T is a tree. The group Γ naturally acts on it. The action $\Gamma : E(T)$ is transitive and residually finite, as it follows from the definition of walls and Proposition 1. Possibly passing to a subgroup of index 2, we may assume that Γ does not reverse edges (see Lemma 3). Condition 4) provides that T is not a star. So, if T is not a line, the group Γ has a virtually non-abelian free quotient according to Proposition 2.

Suppose T is a line. Possibly passing to a subgroup of index 2 and deleting half of the walls, we may assume that Γ acts on T just by shifts. Let

Γ_1 be the kernel of this action. Then $\Gamma/\Gamma_1 \simeq \mathbb{Z}$. The linear form α defining the subspace U is a weight vector of Γ_1 in the dual space V^* of V. Since there are only finitely many weight subspaces of Γ_1 in V^* and Γ can only permute them, there is a subgroup Γ_0 of finite index in Γ containing Γ_1 such that $\Gamma_0\alpha$ lies in one weight subspace. Denote by A the linear span of $\Gamma_0\alpha$. Obviously, it is a Γ_0-invariant subspace. Take any $\gamma_0 \in \Gamma_0 \backslash \Gamma_1$. Since

$$(\gamma_0 U \cap U) \cap K = \varnothing,$$

some non-trivial linear combination of α and $\gamma_0\alpha$ belongs to K^* (here we use one of the versions of the Hahn–Banach theorem). Hence,

$$K^* \cap A \neq \varnothing.$$

By the Brouwer theorem γ_0 has a fixed point in $P(K^* \cap A)$. The Γ-orbit of this point is finite, which contradicts our condition $(*)$. ■

5. Proof of Theorem 1

If $K = V$, then Γ is finite as the stabilizer of $0 \in K$. In the further consideration, let us assume that $K \neq V$ (and, hence, K is strictly convex). By the Selberg lemma [6] Γ has a torsion free normal subgroup of finite index, say, Γ_1. Apply the Main Theorem to Γ_1, taking for U the mirror of a linear reflection r contained in Γ. We are to prove conditions 1)-4) and $(*)$.

To prove 1), take any point $x \in K$. Then $x + r(x) \in H$.

To prove 2), note that for $\gamma \in \Gamma$ the subspace γU is the mirror of the reflection $\gamma r \gamma^{-1}$. If $\gamma H \cap H \neq \varnothing$, then $\delta = (\gamma r \gamma^{-1}) r$ has a fixed point in K and, hence, is of finite order. Let $\gamma \in \Gamma_1$. Then $\delta = \gamma (r \gamma r)^{-1} \in \Gamma_1$ and hence, $\delta = id$; this means that $\gamma U = U$ or, equivalently, $\gamma H = H$.

Let us prove 3). By the definition of a discrete action, for any compact set $C \subset K$ we have $\#\{\gamma \in \Gamma : \gamma C \cap C \neq \varnothing\} < \infty$. But if $\gamma H \cap C \neq \varnothing$, the reflection $r_\gamma = \gamma r \gamma^{-1}$ has a fixed point in C so $r_\gamma C \cap C \neq \varnothing$. Therefore, C meets only finitely many sets of the form $\gamma H, \gamma \in \Gamma$.

If one of the connected components of $K \backslash H$ contains some γH, $\gamma \in \Gamma_1$, then the other one contains $r\gamma H = (r\gamma r)H$, where $r\gamma r \in \Gamma_1$, so condition 4) is satisfied. If no one of the connected components of $K \backslash H$ contains any γH, $\gamma \in \Gamma_1$, then $\gamma H = H$ for any $\gamma \in \Gamma_1$. This implies that the Γ-orbit of the point of PV^* corresponding to U, is finite, so according to Lemmas 6 and 8 the second alternative of the theorem takes place. The same is true, if Γ has a finite orbit in PK^*. Thus, either the second alternative of the theorem takes place, or all the conditions of the Main Theorem are satisfied for Γ_1. In the latter case Γ is large. ■

6. Proof of Corollary 1

The space L^n can be realized as the projectivization of the cone K of the future in the Minkowski space $E^{n,1}$ in such a way that the motions of L^n be induced by pseudo-orthogonal operators. Under this realization the planes of L^n correspond to subspaces of $E^{n,1}$ intersecting K, while the points at infinity correspond to isotropic one-dimensional subspaces. The conditions of Corollary 1 mean that Γ is an irreducible subgroup of the pseudo-orthogonal group $O_{n,1}$ acting on K discretely and containing a reflection. By Theorem 1 it is large.

7. Proof of Corollary 2

It is known [7] that any non-affine indecomposable finitely generated Coxeter group Γ can be realized as a linear group in a finite-dimensional real vector space V in such a way that

1) Γ is irreducible;
2) Γ leaves invariant a non-degenerate symmetric bilinear form f in V;
3) the generators of Γ are represented by orthogonal reflections (with respect to f);
4) Γ leaves invariant an open convex cone $K \subset V$ and acts on it discretely.

Applying Theorem 1, we see that if Γ is infinite, one of alternatives 2) and 3) takes place, so we only need to exclude 2).

Suppose that the cone K is simplicial. The group Γ can only permute the edges of K. Let Δ be the kernel of the action of Γ on the set of edges of K. Then Δ is abelian and the one-dimensional subspaces $V_1, \dots, V_m$ spanned by the edges of K, are the weight subspaces of Δ. They are isotropic with respect to f and decompose in pairs in such a way that the subspaces of one pair correspond to opposite weights and are not orthogonal, while any subspaces of different pairs are orthogonal. Any reflection of Γ permutes the subspaces of one pair, leaving all the others invariant. It follows that the group Γ leaves any pair of the subspaces $V_1, \dots, V_m$ invariant, which contradicts its irreducibility, unless $m = 2$. But the case $m = 2$ occurs only for dihedral groups, which are finite or affine. ■

8. Proof of Theorem 2

Consider the space $V = \mathbb{R}^{n+1}$ with the scalar product defined by the quadratic form f. Denote by K one of two connected components of the cone $f < 0$. The subgroup

$$\mathrm{O}'(f, \mathbb{R}) = \{g \in \mathrm{O}(f, \mathbb{R}) : gK = K\}$$

has index 2 in $\mathrm{O}(f, \mathbb{R})$. Set

$$\mathrm{O}'(f, \mathbb{Z}) = \mathrm{O}(f, \mathbb{Z}) \cap \mathrm{O}'(f, \mathbb{R}).$$

Take any vector $e \in \mathbb{Z}^{n+1}$ with $f(e) = m > 0$ and set

$$\begin{aligned} Q &= \{v \in V : \quad f(v) = m\}, \\ U &= \{v \in V : \quad (v, e) = 0\}. \end{aligned}$$

Obviously,

$$H = U \cap K \neq \varnothing.$$

Lemma 9. *The sets $\gamma H, \gamma \in \mathrm{O}'(f,\mathbb{Z})$, constitute a discrete family in K, i.e. any compact subset $C \subset K$ meets only finitely many of them.*

Proof. We have

$$\gamma H = \{v \in K : (v, \gamma e) = 0\}.$$

Clearly, $\gamma e \in Q \cap \mathbb{Z}^{n+1}$. The orthogonal subspace of any vector $v \in K$ is Euclidean, so its intersection with Q is compact. It follows that the set

$$Q_C = \{v \in Q : (v, u) = 0 \text{ for some } v \in C\},$$

is also compact and, hence, contains only finitely many integral vectors. Therefore, C meets only finitely many sets γH, $\gamma \in \mathrm{O}'(f, \mathbb{Z})$. ■

Lemma 10. *There is a subgroup Θ of finite index in $\mathrm{O}'(f,\mathbb{Z})$ such that for any $\gamma \in \Theta$ we have either $\gamma H = H$, or $\gamma H \cap H = \varnothing$.*

Proof. For $\gamma \in \mathrm{O}'(f,\mathbb{R})$ the sets H and γH intersect but do not coincide if and only if the vectors e and γe span a two-dimensional Euclidean subspace in V, i.e. if

$$|(e, \gamma e)| < m. \tag{1}$$

Let Θ be the congruence subgroup of $\mathrm{O}'(f,\mathbb{Z})$ modulo $2m$. For $\gamma \in \Theta$ we have $\gamma e \equiv e \pmod{2m}$, whence

$$(e, \gamma e) \equiv m \pmod{2m}.$$

This makes (1) impossible. ■

Now let Γ be a finitely generated subgroup of $\mathrm{O}(f,\mathbb{Z})$. Passing to a subgroup of finite index, we may assume that $\Gamma \subset \Theta$. Then conditions 1)–3) of the Main Theorem hold. To ensure condition 4), we choose the vector e depending on Γ as follows.

The group $\mathrm{O}'(f,\mathbb{R})$ can be considered as the group of motions of Lobachevsky space L^n modelled on the projectivization of the cone K. In this interpretation, an element $\gamma \in \mathrm{O}'(f,\mathbb{R})$ is called *elliptic*, if it has a fixed point in L^n, *hyperbolic*, if it has no fixed point in L^n but has an invariant line, and *parabolic*, if it has no fixed points or invariant lines in L^n, but has a fixed point at infinity. Any element of $\mathrm{O}'(f,\mathbb{R})$ is one of these three types (see, e.g., [1]).

Passing to a subgroup of finite index, we may assume that Γ is torsion-free and, hence, contains no elliptic elements but the identity.

Let $\gamma_0 \in \Gamma$ be a hyperbolic element and l the (unique) invariant line of it in L^n. Since the projectivization of $\mathbb{Z}^{n+1}$ is dense in PV, we can choose e in such a way that H intersects the line l but does not contain it. Then $\gamma_0^d H$ and $\gamma_0^{-d} H$ are in different connected components of $K \backslash H$ for any $d \in \mathbb{N}$. Some γ_0^d survives after possible passing to a subgroup of finite index in Γ, so condition 4) will hold for the obtained group.

Now let $\gamma_0 \in \Gamma$ be a parabolic element and p the (unique) fixed point at infinity of it. The point p is represented by an isotropic vector with rational co-ordinates. Any horosphere of L^n centered at p is invariant under γ_0, and there exists an invariant Euclidean line l on it. We can choose e in such a way that (the closure of) H passes through p and intersects the line l but does not contain it. Then again $\gamma_0^d H$ and $\gamma_0^{-d} H$ are in different connected components of $K \backslash H$, and we can apply the above argument.

Thus, we can conclude that if the group Γ satisfies the condition $(*)$, it is large. If it does not satisfy $(*)$, then a subgroup of finite index in Γ has a fixed point in L^n or in ∂L^n. In both cases Γ is virtually abelian. ∎

Remark 3. Theorem 2 implies the following generalization of its own. Let $k \subset \mathbb{R}$ be a totally real algebraic number field and f a quadratic form of signature $(n,1)$ over k that becomes positive definite under any non-identity embedding $\sigma : k \to \mathbb{R}$. Let $\mathcal{O}$ be the ring of integers of k. Then any finitely generated subgroup of $\mathrm{O}(f,\mathcal{O})$ is either virtually abelian, or large. Indeed, if $[k:\mathbb{Q}] = d$, then there are exactly d embeddings $\sigma : k \to \mathbb{R}$, and the restriction of scalars permits us to embed $\mathrm{O}(f,\mathcal{O})$ into $\mathrm{O}(F,\mathbb{Z})$, where F is a rational quadratic form of signature $(d(n+1)-1, 1)$. ∎

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