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Brownian Motion and the Heat Kernels of Iwasawa NA-Type Groups

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Abstract. Let G be a semisimple Lie group with a finite centre and Iwasawa decomposition NAK. We shall consider the heat equation on NA and prove a structure theorem for a diffusion on the group which is typically called a Brownian motion. This theorem then shows how we may calculate the heat kernel of real hyperbolic space in a relatively easy fashion.

1. Introduction

Throughout this paper we shall consider a semisimple Lie group G with finite centre and Iwasawa decomposition NAK. The Lie algebra $\mathfrak{g}$ of G then has the decomposition $\mathfrak{n} \oplus \mathfrak{a} \oplus \mathfrak{k}$. We recall that the tangent vectors at the identity correspond to unique left invariant vector fields. Using the Killing form and a Cartan involution we may construct a Riemannian metric on G and hence on the component NA. We may then consider the heat equation on NA:

$$\Delta u = \frac{\partial}{\partial t} u, \tag{1}$$

where Δ is the Laplace–Beltrami operator associated with the metric. Using the root space decomposition of $\mathfrak{n}$ we may form a distinguished Laplacian L on NA,

$$L = \frac{1}{2}\left(\sum_{k=1}^{n} Y_k^2 + \sum_{k=1}^{m} H_k^2\right),$$

where $\{Y_1, \ldots, Y_n\}$ is an orthonormal basis of $\mathfrak{n}$ compatible with its root space decomposition and $\{H_1, \ldots, H_m\}$ is an orthonormal basis of $\mathfrak{a}$ (see [2]). To solve the heat equation on NA, it is then enough to solve the equation

$$Lu = \frac{\partial}{\partial t} u. \tag{2}$$

That we may consider this equation rather than (1) follows from the geometric relationship between them. The following discussion about this relationship was communicated to the author by Professor J. C. Taylor.

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For the solvable group NA that occurs in the Iwasawa decomposition of a semisimple Lie group we always have $\Delta = L + Y$ where $Y = -2H_\rho$, ρ is the half sum of the positive roots and H_ρ is the unique element of $\mathfrak{a}$ associated with ρ (see [13, Cor. A.1.2]). Now for any Riemannian manifold M there is an intrinsic number determined by the spectrum of $-\Delta$ in L^2. This number $\lambda_0(M)$ is the minimum of the L^2-spectrum. If $\lambda \geq \lambda_0(M)$, there are positive global solutions of the equation $\Delta u + \lambda u = 0$ (see [11, 14]). When $M = G/K$, a symmetric space of noncompact type, this constant $\lambda_0(M)$ is $\|H_\rho\| = \|\rho\|$. Consequently on NA we have (see [1])

$$e^{-\rho(\log(a))}\tfrac{1}{2}\Delta(fe^{\rho(\log(a))}) + \|\rho\|^2 f = Lf.$$

Thus with $u = fe^{\rho(\log(a))}$,

$$\tfrac{1}{2}\Delta u + \|\rho\| u = \frac{\partial u}{\partial t} \quad \text{if and only if} \quad Lf = \frac{\partial f}{\partial t}.$$

Hence if $v(x,t) = e^{ct}u(x,t)$ and $g(x,t) = e^{ct}f(x,t)$ we have

$$\tfrac{1}{2}\Delta v + (\|\rho\| - c)v = \frac{\partial v}{\partial t} \quad \text{if and only if} \quad Lg - cg = \frac{\partial g}{\partial t},$$

and we may use the heat kernel for one value of c to determines all the others.

In this paper we shall consider the stochastic process $(\xi_t)_{t\geq 0}$ which has L as its generator and satisfies the stochastic differential equation

$$d\xi_t = \sum_{k=1}^{n} Y_k(\xi_t) \circ dB_t^{(k)} + \sum_{k=1}^{m} H_k(\xi_t) \circ dB_t^{(n+k)}, \tag{3}$$

where $(B_t)_{t\geq 0}$ is a standard $(n+m)$-dimensional Brownian motion and $\circ$ indicates a Stratonovich stochastic integral. This process is often called a Brownian motion on NA or an L-diffusion (see [5]). Using the structure of the Lie algebra $\mathfrak{n} \oplus \mathfrak{a}$ we show it is possible to write down the solution of (3) in a way which is compatible with the root space decomposition. Consequently since the density of ξ_t is the kernel p_t of (2) we may write p_t compatibly with this decomposition.

2. Review of Results

In this section we shall present some results for stochastic differential equations on manifolds that we shall require later. These results have been proved in [7, 8] and are collected here for convenience.

A key result in stochastic calculus is Itô's lemma. In [8], Kunita extends the Itô formula and gives a general discussion about its formulation for processes on manifolds. We will now summarise this discussion by stating a proposition found in [8].

Proposition 2.1. Extended Itô Formula *(see [8]) Suppose that M is a smooth n-dimensional manifold and $(F_t(p))_{t\geq 0}$ is an $\mathbb{R}$-valued process which is continuous in (t,p) almost surely and satisfies*

1. *for each $t > 0$, $p \mapsto F_t(p)$ is a smooth map from M into $\mathbb{R}$ almost surely*

2. for each p, $F_t(p)$ is a continuous semimartingale with the representation

$$F_t(p) = F_0(p) + \sum_{j=1}^{m} \int_0^t f_s^{(j)}(p) \circ dN_s^{(j)},$$

where $(N_s)_{s\geq 0} = ((N_s^{(1)}, \ldots, N_s^{(m)}))_{s\geq 0}$ is a continuous semimartingale and for each j, $(f_s^{(j)}(p))_{s\geq 0}$ is a process, continuous in (s,p), such that

(a) for each s, $p \mapsto f_s^{(j)}(p)$ is a smooth map of M into $\mathbb{R}$

(b) for each p, $(f_s^{(j)}(p))_{s\geq 0}$ the stochastic integral with respect to $N^{(j)}$ is well defined.

Then if $(\xi_t)_{t\geq 0}$ satisfies the stochastic differential equation

$$d\xi_t = \sum_{i=1}^{d} X_i(\xi_t) \circ dM_t^{(i)},$$

where $(M_t)_{t\geq 0}$ is a continuous semimartingale, we have

$$F_t(\xi_t) = F_0(\xi_0) + \sum_{j=1}^{m} \int_0^t f_s^{(j)}(\xi_s) \circ dN_s^{(j)} + \sum_{i=1}^{d} \int_0^t (X_i F_s)(\xi_s) \circ dM_s^{(i)}.$$

This result underlies the main result of [7] on the decomposition of solutions. Its proof is an easy application of Proposition 2.1, once we know that the stochastic process generates a stochastic flow. For then we may consider the notion of a "pullback" of a stochastic process and its inverse. Kunita shows in [7] that the processes we are considering do indeed generate stochastic flows. For a process $(\eta_t)_{t\geq 0}$ on the manifold satisfying the stochastic differential equation

$$d\eta_t = \sum_{i=1}^{m} X_i(\eta_t) \circ dW_t^{(i)},$$

we define the "pullback" to be the process $(\eta_{t\star})_{t\geq 0}$, where for each vector field X and smooth function f on the manifold we have

$$(\eta_{t*}(X))f(p) = X(f \circ \eta_t)(\eta_t^{-1}(p))$$

for each $p \in M$. The inverse of the pullback $(\eta_{t\star}^{-1})_{t\geq 0}$ satisfies the stochastic differential equation

$$(\eta_{t*})^{-1} = \mathrm{Id} + \sum_{i=1}^{m} \int_0^t (\eta_{s*})^{-1} \operatorname{ad}(X_i) \circ dW_s^{(i)}.$$

Theorem 2.2. *(see [7, Theorem 4.3]) Suppose that $(\eta_t)_{t\geq 0}$ and $(\zeta_t)_{t\geq 0}$ are processes on M which satisfy the stochastic differential equations*

$$d\eta_t = \sum_{i=1}^{m} X_i(\eta_t) \circ dM_t^{(i)}$$

and

$$d\zeta_t = \sum_{j=1}^{d} (\eta_{t*})^{-1} Y_j(\zeta_t) \circ dN_t^{(j)}.$$

Then the process $(\eta_t \circ \zeta_t)_{t\geq 0}$ *satisfies the stochastic differential equation*

$$d(\eta_t \circ \zeta_t) = \sum_{i=1}^{m} X_i(\eta_t \circ \zeta_t) \circ dM_t^{(i)} + \sum_{j=1}^{d} Y_j(\eta_t \circ \zeta_t) \circ dN_t^{(j)}.$$

We may use this theorem to reflect the structure of the Lie algebra generated by the vector fields of the manifold in the solution of the stochastic differential equation.

3. Solutions of Stochastic Differential Equations on Symmetric Spaces.

By repeatedly applying the theorem on the decomposition of solutions and the Dambis, Dubins–Schwarz theorem (see [6, Theorem 3.4.6]) we may write down the solution of a stochastic differential equation on a symmetric space as the combination of a solution of an stochastic differential equation on N and a solution of an stochastic differential equation on A.

Firstly let G be a real semisimple Lie group with a finite centre K and an Iwasawa decomposition NAK. We shall denote the Lie algebras by $\mathfrak{g}$, $\mathfrak{k}$ and $\mathfrak{n} \oplus \mathfrak{a} \oplus \mathfrak{k}$. Let Σ^+ denote a set of positive roots.

Theorem 3.1. *Suppose that* $\{Y_1, \ldots, Y_n,\ H_1, \ldots, H_r\}$ *is an orthonormal basis of* $\mathfrak{n} \oplus \mathfrak{a}$ *such that* $\{H_1, \ldots, H_r\}$ *is an orthonormal basis of* $\mathfrak{a}$ *and* $\{Y_1, \ldots, Y_n\}$ *is an orthonormal basis of* $\mathfrak{n}$ *which is compatible with its root space decomposition. Consider the stochastic differential equation*

$$\begin{aligned} d\xi_t &= \sum_{i=1}^{r} H_i(\xi_t) \circ dB_t^{(i)} + \sum_{j=1}^{n} Y_j(\xi_t) \circ dB_t^{(r+j)} \\ \xi_0 &= e, \end{aligned}$$

and suppose that $\Sigma^+ = \{\lambda_1, \ldots, \lambda_p\}$, *where the multiplicity of* λ_i *is* m_i. *Then* $(\xi_t)_{t\geq 0}$ *may be decomposed as* $\xi_t = \eta_t \cdot \alpha_t$, *where* $\alpha_t = \exp(H_t)$ *with*

$$H_t = \sum_{i=1}^{r} B_t^{(i)} H_i,$$

and there is an n*-dimensional Brownian motion* $(W_t)_{t\geq 0}$ *such that* $(\eta_t)_{t\geq 0}$ *may be decomposed as*

$$\eta_t = \widetilde{\zeta}^1_{A_t^{\lambda_1}} \circ \cdots \circ \widetilde{\zeta}^p_{A_t^{\lambda_p}},$$

where

$$\begin{aligned} d\widetilde{\zeta}_t^1 &= \sum_{j=1}^{m_1} Y_i(\widetilde{\zeta}_t^1) \circ dW_t^{(j)}, \\ d\widetilde{\zeta}_t^i &= \sum_{j=k_i+1}^{k_i+m_i} (\widetilde{\zeta}^{i-1}(T_t^{i,i-1})_*)^{-1} \cdots (\widetilde{\zeta}^1(T_t^{i,1})_*)^{-1} Y_j(\widetilde{\zeta}_t^i) \circ dW_t^{(j)} \end{aligned}$$

for $i = 2, \ldots, p$,

$$A_t^\lambda = \int_0^t \exp(2\lambda(H_s))\, ds,$$
$$T_t^\lambda = \inf\{s : A_s^\lambda > t\}$$

for a root λ, *and for* $q = 1, \ldots, i-1$,

$$T_t^{i,q} = t + \int_{T_t^{\lambda_q}}^{T_t^{\lambda_i}} \exp(2\lambda_q(H_s))\, ds.$$

Proof. By Theorem 2.2, we may decompose $(\xi_t)_{t\geq 0}$ as $\xi_t = \alpha_t \circ \eta_t(e)$, where

$$d\alpha_t = \sum_{i=1}^r H_i(\alpha_t) \circ dB_t^{(i)}$$
$$d\eta_t = \sum_{j=1}^n (\alpha_{t*})^{-1} Y_i(\eta_t) \circ dB_t^{(r+j)}. \tag{4}$$

Since $H_1, \ldots, H_r$ commute, $\alpha_t = \exp(H_t)$ where

$$H_t = \sum_{i=1}^r B_t^{(i)} H_i.$$

Note that then

$$(\alpha_{t*})^{-1} = e^{\mathrm{ad}(H_t)}.$$

Thus we may write (4) as

$$d\eta_t = \sum_{i=1}^p \sum_{j=k_i+1}^{k_i+m_i} e^{\mathrm{ad}(H_t)} Y_j(\eta_t) \circ dB_t^{(r+j)}, \tag{5}$$

where $k_1 = 0$ and $k_i = m_1 + \cdots + m_{i-1}$ for $i = 2, \ldots, p$. Since each Y_j is an eigenvector we may rewrite (5) as

$$d\eta_t = \sum_{i=1}^p \sum_{j=k_i+1}^{k_i+m_i} Y_j(\eta_t) e^{\lambda_i(H_t)} \circ dB_t^{(r+j)}.$$

Now for each j we may apply the Dambis, Dubins–Schwarz theorem to yield a standard Brownian motion $(W_t^{(j)})_{t\geq 0}$ such that

$$d\eta_t = \sum_{i=1}^p \sum_{j=k_i+1}^{k_i+m_i} Y_j(\eta_t) \circ dW_{A_t^{\lambda_i}}^{(j)},$$

where

$$A_t^{\lambda_i} = \int_0^t \exp(2\lambda_i(H_s))\, ds.$$

For notational convenience we shall now restrict our attention to the case where $p=2$. By Theorem 2.2, we may write $(\eta_t)_{t\geq 0}$ as the decomposition of the solutions to the stochastic differential equations

$$d\zeta_t^1 = \sum_{j=1}^{m_1} Y_j(\zeta_t^1) \circ dW^{(j)}_{A_t^{\lambda_1}}$$
$$d\zeta_t^2 = \sum_{j=m_1+1}^{m_1+m_2} (\zeta_{t*}^1)^{-1} Y_j(\zeta_t^2) \circ dW^{(j)}_{A_t^{\lambda_2}}.$$

By a corollary to the Dambis, Dubins–Schwarz Theorem (see [6, Proposition 3.4.8]) we have

$$\zeta_t^1 = \zeta_0^1 + \sum_{j=1}^{m_1} \int_0^{A_t^{\lambda_1}} Y_j(\zeta^1_{T_s^{\lambda_1}}) \circ dW_t^{(j)},$$

where $T_s^{\lambda_1} = \inf\{u : A_u^{\lambda_1} > s\}$. Therefore

$$\zeta^1_{T_t^{\lambda_1}} = \zeta_0^1 + \sum_{j=1}^{m_1} \int_0^{t} Y_j(\zeta^1_{T_s^{\lambda_1}}) \circ dW_t^{(j)},$$

and hence if $(\widetilde{\zeta}_t^1)_{t\geq 0}$ is the unique solution to the stochastic differential equation

$$d\widetilde{\zeta}_t^1 = \sum_{j=1}^{m_1} Y_j(\widetilde{\zeta}_t^1) \circ dW_t^{(j)},$$

then $\zeta^1_{T_s^{\lambda_1}} = \widetilde{\zeta}_t^1$, or in other words, $\widetilde{\zeta}_{A_t^{\lambda_1}} = \zeta_t^1$. Applying the same corollary to $(\zeta_t^2)_{t\geq 0}$, we get

$$\zeta_t^2 = \zeta_0^2 + \sum_{j=k_2+1}^{k_2+m_2} \int_0^{A_t^{\lambda_2}} (\zeta^1_{T_s^{\lambda_2}*})^{-1} Y_j(\zeta^2_{T_s^{\lambda_2}}) \circ dW_t^{(j)}.$$

Now $\zeta^1(T_s^{\lambda_2}) = \widetilde{\zeta}^1(A^{\lambda_1}_{T_s^{\lambda_2}})$, and

$$\begin{aligned} A^{\lambda_1}_{T_s^{\lambda_2}} &= \int_0^{T_s^{\lambda_1}} e^{2\lambda_1(H_u)}\,du + \int_{T_s^{\lambda_1}}^{T_s^{\lambda_2}} e^{2\lambda_1(H_u)}\,du \\ &= s + \int_{T_s^{\lambda_1}}^{T_s^{\lambda_2}} e^{2\lambda_1(H_u)}\,du \\ &= T_s^{2,1} \end{aligned}$$

(say). Thus if we consider the unique solution to the stochastic differential equation

$$d\widetilde{\zeta}_t^2 = \sum_{j=k_2+1}^{k_2+m_2} (\widetilde{\zeta}^1(T_t^{2,1})_*)^{-1} Y_j(\widetilde{\zeta}_t^2) dW_t^{(j)},$$

we have $\widetilde{\zeta}^2_{A_t^{\lambda_2}} = \zeta_t^2$. For $p>2$, a similar argument gives the other $\widetilde{\zeta}_t^i$ and we are done. ∎

Corollary 3.2. *If $(\xi_t^i)_{t\geq 0}$ is the solution to the stochastic differential equation*

$$d\xi_t^i = \sum_{j=1}^{r} H_j(\xi_t^i) \circ dB_t^{(j)} + \sum_{j=k_i+1}^{k_i+m_i} Y_j(\xi_t^i) \circ dB_t^{(j)},$$

then there is an m_i-dimensional Brownian motion $(W_t)_{t\geq 0}$ such that $(\xi_t^i)_{t\geq 0}$ has the decomposition $\xi_t^i = \eta_{A_t^{\lambda_i}} \cdot \alpha_t$ where $\alpha_t = \exp(B_t^{(1)} H_1 + \cdots + B_t^{(r)} H_r)$ and

$$d\eta_t = \sum_{j=k_i+1}^{k_i+m_i} Y_j(\eta_t) \circ dW_t^{(j)}.$$

It follows from the above corollary that we may immediately write down the density of the random variable $\eta_t \cdot \alpha_t$ once we have the densities of η_t and α_t. For the density of α_t we have the following result from [15].

Proposition 3.3. *(see [15, Section 6, Proposition 2]) Suppose $(B_t)_{t\geq 0}$ is a real Brownian motion starting from 0 and*

$$A_t = \int_0^t e^{2B_s}\, ds.$$

If we let $\mathrm{P}(A_t \in du \mid B_t = x) = a_t(x,u)\, dx$, then we have

$$\frac{1}{\sqrt{2\pi t}} e^{-x^2/2t} a_t(x,u) = \frac{1}{u} \exp(-(1+e^{2x})/2u) \Theta_{e^x/u}(t),$$

where

$$\Theta_{e^x/u}(t) = \frac{e^x}{u\sqrt{2\pi^3 t}} \int_0^\infty e^{-(y^2-\pi^2)/2t} \exp(-e^x \cosh(y)/u) \sinh(y) \sin(\pi y/t)\, dy.$$

Thus from Corollary 3.2 and Proposition 3.3, we get the following result.

Proposition 3.4. *Under the assumptions of Theorem 3.1, suppose $\dim \mathfrak{a} = 1$ and $(\xi_t)_{t\geq 0}$ is the solution of the stochastic differential equation*

$$d\xi_t = H_1(\xi_t) \circ dB_t^{(1)} + \sum_{j=1}^{n} Y_j(\xi_t) \circ dB_t^{j+1}$$

$$\xi_0 = e,$$

with $\lambda_1(H_1) = 1$. Then there is an n-dimensional Brownian motion $(W_t)_{t\geq 0}$ such that $\xi_t = \eta_{A_t} \cdot \alpha_t$, where $\alpha_t = \exp(B_t^{(1)} H_1)$ and

$$d\eta_t = \sum_{j=1}^{n} Y_j(\eta_t) \circ dW_t^j.$$

Moreover ξ_t has the density

$$p_t(n, e^x) = \frac{1}{\sqrt{2\pi t}} \int_0^\infty q_u(n) a_t(x,u) e^{-x^2/2t}\, du,$$

where $q_t(n)$ is the density of η_t.

We shall now use this proposition to calculate heat kernels of the real hyperbolic spaces.

4. The Heat Kernel of the NA-component of $\mathrm{SO}_0(n,1)$

The NA-component of $\mathrm{SO}_0(n,1)$ may be identified with $\mathbb{R}^{n-1} \rtimes \mathbb{R}$ (see [1]). We may choose $H \in \mathfrak{a}$ such that, in local coordinates, we have

$$(Hf)(\mathbf{x}, a) = a\frac{\partial f}{\partial a}(\mathbf{x}, a).$$

Thus the A-component, $((\mathbf{0}, \alpha_t))_{t\geq 0}$ satisfies the stochastic differential equation

$$d(\mathbf{0}, \alpha_t) = H(\mathbf{0}, \alpha_t) \circ dB_t,$$

where $(B_t)_{t\geq 0}$ is a standard one-dimensional Brownian motion. Therefore $\alpha_t = \exp(B_t)$.

If we consider N in isolation it is just $\mathbb{R}^{n-1}$ and so the left Brownian motion on N is $((W_t, 1))_{t\geq 0}$, where $(W_t)_{t\geq 0}$ is a standard $(n-1)$-dimensional Brownian motion. Thus the N-component $(\eta_t)_{t\geq 0}$ of the Brownian motion on $\mathbb{R}^{n-1} \rtimes \mathbb{R}^+$ is

$$\eta_t = (W_{A_t}, 1),$$

where

$$A_t = \int_0^t e^{2B_s}\, ds,$$

and hence

$$\eta_t \cdot \alpha_t = (W_{A_t}, e^{B_t})$$

is the Brownian motion on $\mathbb{R}^{n-1} \rtimes \mathbb{R}^+$. This stochastic process has also been considered in [1]. The expression given by Bougerol for the Brownian motion $(\zeta_t)_{t\geq 0}$ is

$$\zeta_t = \left(\int_0^t e^{B_s}\, dW_s, e^{B_t}\right),$$

where $(B_t)_{t\geq 0}$ is a standard one-dimensional Brownian motion and $(W_t)_{t\geq 0}$ is a standard $(n-1)$-dimensional Brownian motion. However by the Dambis, Dubins–Schwarz theorem we have equality of the random variables in his expression and our own. In [1, Proposition 2.3.2] the expression for the density of $\eta_t \cdot \alpha_t$ is also given. The proof given uses an expression derived by Lohoué and Rychener [9], which is ultimately based on pages 329 and 330 of the thesis of Takahashi [12]. Bougerol notes that the expression in [9] has some minor mistakes in the constants and corrects these.

Rather than follow the route of [9], we shall now give our own derivation of the heat kernel. The density of W_t is

$$\mathrm{P}(W_t \in d\mathbf{x}) = \frac{e^{-||\mathbf{x}||/2t}}{(2\pi t)^{(n-1)/2}}\, d\mathbf{x}.$$

Consequently, by Proposition 3.4, the density of (W_{A_t}, e^{B_t}) is given by

$$p_t(\mathbf{z}, e^x) = \frac{1}{\sqrt{2\pi t}} \int_0^\infty \frac{e^{-||\mathbf{z}||/2u}}{(2\pi u)^{(n-1)/2}}\, a_t(x, u)\, e^{-x^2/2t}\, du.$$

This expression for $p_t(\mathbf{z}, e^x)$ differs from that which appears in [1]. We shall now show how they are in fact the same by transforming our expression into the one given in [1].

Substituting in the expression for $a_t(x,u)$ given in Proposition 3.3 we get

$$p_t(\mathbf{z}, e^x) = \frac{e^x}{\sqrt{2\pi^3 t}} \int_0^\infty \frac{e^{-(y^2-\pi^2)/2t}}{(2\pi)^{(n-1)/2}} \sinh(y) \sin(\pi y/t)\, F(y; \mathbf{z}, e^x)\, dy,$$

where

$$\begin{aligned} F(y;\mathbf{z},e^x) &= \int_0^\infty \frac{e^{-(||\mathbf{z}||+1+e^{2x})/2u}}{u^{(n-1)/2}} \frac{e^{-e^x \cosh(y)/u}}{u^2}\, du \\ &= \int_0^\infty u^{(n-1)/2} e^{-ue^x(\frac{1}{2}e^{-x}||\mathbf{z}||+\cosh(x)+\cosh(y))}\, du \\ &= \frac{\Gamma\left((n+1)/2\right)}{(e^x(\frac{1}{2}e^{-x}||\mathbf{z}|| + \cosh(x) + \cosh(y)))^{(n+1)/2}}. \end{aligned}$$

Hence

$$p_t(\mathbf{z}, e^x) = \kappa(x,t) \int_0^\infty e^{-(y^2-\pi^2)/2t} \frac{\sinh(y)\sin(\pi y/t)}{(\cosh(r)+\cosh(y))^{(n+1)/2}}\, dy,$$

where

$$\begin{aligned} \kappa(x,t) &= (2\pi)^{-(n-1)/2} \frac{e^{-(n+1)x/2}}{\sqrt{2\pi^3 t}} \Gamma((n+1)/2) \\ \cosh(r) &= \frac{1}{2} e^{-x} \|\mathbf{z}\|^2 + \cosh(x). \end{aligned}$$

We shall now consider

$$\begin{aligned} I &= \int_0^\infty e^{-(y^2-\pi^2)/2t} \frac{\sinh(y)\sin(\pi y/t)}{(\cosh(r)+\cosh(y))^{(n+1)/2}}\, dy \\ &= \frac{1}{2}\mathrm{Im}\left(\int_{-\infty}^\infty e^{(y-i\pi)^2/2t} \frac{\sinh(y)}{(\cosh(r)+\cosh(y))^{(n+1)/2}}\, dy \right), \end{aligned}$$

and suppose that $n-1$ is of the form $2k+1$. On integrating by parts, we see that I is equal to

$$\frac{1}{2\Gamma\left((2k+1)/2\right)} \mathrm{Im}\left(\int_{-\infty}^\infty \left(\frac{1}{\sinh(y)} \frac{d}{dy} \right)^{k+1} (e^{(y-i\pi)^2/2t}) \frac{\sinh(y)}{\sqrt{\cosh(r)+\cosh(y)}}\, dy \right).$$

To find the imaginary part of the integral we let R be large and ϵ be small and consider

$$\int_{\gamma_1} f(\xi)\, d\xi - \sum_{i=2}^{8} \int_{\gamma_i} f(\xi)\, d\xi,$$

where

$$f : \mathbb{C} \ni \xi \mapsto \left(\frac{1}{\sinh(\xi)} \frac{d}{d\xi} \right)^{k+1} (e^{(\xi-i\pi)^2/2t}) \frac{\sinh(\xi)}{\sqrt{\cosh(r)+\cosh(\xi)}}$$

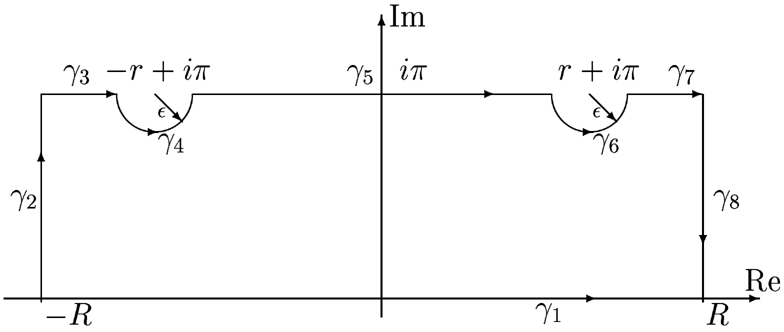

Figure 1: Odd case Contour

and each γ_i is as in Figure 1.

It is straightforward to show that the integrals along γ_2 and γ_8 tend to zero as R tends to infinity, and that

$$\begin{aligned}\int_{\gamma_3} f(\xi)\,d\xi &= \int_{\gamma_7} f(\xi)\,d\xi \\ &= i\int_{r+\epsilon}^{R} (-1)^k \left(\frac{1}{\sinh(y)}\frac{d}{dy}\right)^{k+1} (e^{-y^2/2t}) \frac{\sinh(y)}{\sqrt{\cosh(y)-\cosh(r)}}\,dy.\end{aligned}$$

It is also easy to show that the integrals along γ_4 and γ_6 tend to zero as ϵ tends to zero. Finally the integral along γ_5 is real, and hence we need not consider it further.

By letting R tend to ∞ and ϵ tend to zero the above considerations allow us to conclude that

$$I = \frac{1}{\Gamma\left((2k+1)/2\right)} \int_r^{\infty} (-1)^k \left(\frac{1}{\sinh(y)}\frac{d}{dy}\right)^{k+1} (e^{-y^2/2t}) \frac{\sinh(y)}{\sqrt{\cosh(y)-\cosh(r)}}\,dy,$$

and hence

$$p(\mathbf{z}, e^x) = \kappa_2(x,t) \int_r^{\infty} \left(\frac{1}{\sinh(y)}\frac{d}{dy}\right)^{k+1} (e^{-y^2/2t}) \frac{\sinh(y)}{\sqrt{\cosh(y)-\cosh(r)}}\,dy,$$

where

$$\kappa_2(x,t) = (-1)^k \frac{(\sqrt{2\pi})^{-2k-1}}{\sqrt{\pi}} \frac{e^{-(2k+1)x/2}}{\sqrt{2\pi t}}.$$

The above expression is that given by [1] for the heat kernel of $\mathbb{R}^{2k+1} \rtimes \mathbb{R}^+$.

When $n-1$ is of the form $2k$, we may follow a similar procedure.

After integrating by parts we consider the contour integral

$$\int_\gamma f(\xi)\,d\xi,$$

where

$$f : \mathbb{C} \ni \xi \mapsto \left(\frac{1}{\sinh(\xi)}\frac{d}{d\xi}\right)^k (e^{-(\xi-i\pi)^2/2t}) \frac{\sinh(\xi)}{\cosh(r)+\cosh(\xi)}$$

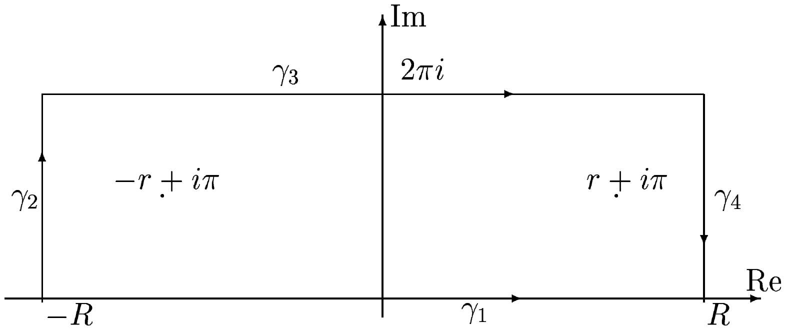

Figure 2: Even case Contour

and γ is the contour in Figure 2.

After finding the imaginary part, we can show that

$$p_t(\mathbf{z}, e^x) = (-2\pi)^{-k}\frac{e^{-kx}}{\sqrt{2\pi t}}\left(\frac{1}{\sinh(r)}\frac{d}{dr}\right)^k (e^{-r^2/2t}),$$

which is the expression given in [1] for the heat kernel of $\mathbb{R}^{2k} \rtimes \mathbb{R}$.

5. Some sub-Laplacians and heat kernels from H-type groups

For the NA-component of $\mathrm{SO}_0(n,1)$ the Laplacian and the sub-Laplacian coincide. However we may also apply Proposition 3.4 for groups where N is an H-type group and consider the sub-Laplacian. Initially let us consider the Heisenberg group H_{2n+1} of real dimension $2n+1$. Accordingly we may realise H_{2n+1} as the group $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ with

$$(\mathbf{x}, \mathbf{y}, t)\cdot(\mathbf{u}, \mathbf{v}, s) = \Big(\mathbf{x}+\mathbf{u}, \mathbf{y}+\mathbf{v}, t+s-\sum_{i=1}^n (y_i u_i - v_i x_i)\Big).$$

The process $(\eta_t)_{t\geq 0}$ on H_{2n+1} associated with the sub-Laplacian adapted to the root space decomposition is well known (see [3, 4]), namely

$$\eta_t = (\mathbf{X}_t, \mathbf{Y}_t, \sum_{i=1}^n \int_0^t X_s^{(i)} dY_s^{(i)} - \int_0^t Y_s^{(i)} dX_s^{(i)}),$$

where $(\mathbf{X}_t)_{t\geq 0}$ and $(\mathbf{Y}_t)_{t\geq 0}$ are independent n-dimensional Brownian motions. In [3, 4], we also find the density $q_t(\mathbf{x}, \mathbf{y}, \tau)$ of η_t,

$$q_t(\mathbf{x}, \mathbf{y}, \tau) = \int_{-\infty}^{\infty}\left(\frac{2s}{\sinh(2s)}\right)^n \exp\left(\frac{2is\tau}{t} - \frac{\|\mathbf{x}+\mathbf{y}\|^2}{2t}\frac{2s}{\tanh(2s)}\right) ds.$$

If we now consider $H_{2n+1} \rtimes \mathbb{R}^+$, which is the NA-component of $\mathrm{SU}(n,1)$, then the process on $\mathbb{R}^+$ is again $((\mathbf{0}, \alpha_t))_{t\geq 0}$, where $\alpha_t = \exp(B_t)$. As before the density of the process on $H_{2n+1} \rtimes \mathbb{R}^+$ is

$$p_t(\mathbf{x}, \mathbf{y}, \tau, e^v) = \frac{1}{\sqrt{2\pi t}}\int_{-\infty}^{\infty} q_u(\mathbf{x}, \mathbf{y}, \tau) a_t(v,u) e^{-v^2/2t}\, du.$$

We may write this explicitly as

$$\frac{2}{(2\pi)^{n+3/2}}\int_{-\infty}^{\infty}\left(\frac{2s}{\sinh(2s)}\right)^n\frac{e^v}{t^{1/2}}\int_0^{\infty}e^{-(\tilde{y}^2-\pi^2)/2t}\sinh(\tilde{y})\sin(\pi\tilde{y}/t)\,F(\tilde{y},s)\,d\tilde{y}\,ds,$$

where

$$F(y',s;\mathbf{x},\mathbf{y},\tau,e^v)=\frac{\Gamma(n+2)}{(s\|\mathbf{x}+\mathbf{y}\|^2/2\tanh(2s)+e^v(\cosh(v)+\cosh(y'))-is\tau)^{n+2}},$$

by taking the required Laplace transform.

Suppose now that N is a generalized Heisenberg or H-type group. Then the heat kernel $h_t(\nu,\zeta)$ of N is known (see [10]) to be

$$\frac{1}{2^d}\frac{1}{(2\pi t)^{k/2+d}}\int_0^{\infty}\frac{\sigma^{k/2}}{|\zeta|^{k/2-1}}J_{k/2-1}(\sigma|\zeta|/t)\left(\frac{\sigma}{\sinh(\sigma)}\right)^d\exp\left(\frac{-|\nu|\sigma}{4t\tanh(\sigma)}\right)d\sigma,$$

where $J_{k/2-1}$ is a Bessel function. Again by applying Proposition 3.4, we may write down the heat kernel $p_t(\nu,\zeta,e^x)$ of $N\rtimes\mathbb{R}^+$, namely,

$$p_t(\nu,\zeta,e^x)=\frac{1}{\sqrt{2\pi t}}\int_{-\infty}^{\infty}h_u(\nu,\zeta)a_t(x,u)e^{-x^2/2t}\,du.$$

This may be written more explicitly as

$$\frac{e^x}{\sqrt{2\pi^3 t}}\int_0^{\infty}\frac{\sigma^{k/2}}{|\zeta|^{k/2-1}}\left(\frac{\sigma}{2\sinh(\sigma)}\right)^d\int_0^{\infty}\frac{e^{-(y^2-\pi^2)/2t}}{(2\pi)^{k/2+d}}\sinh(y)\sin(\pi y/t)F(y,\sigma)\,dy\,d\sigma,$$

where

$$F(y,\sigma;\nu,\zeta,e^x)=\frac{\Gamma(k+d)}{(\sqrt{p^2+\sigma^2|\zeta|^2})^{k/2+d+1}}P^{1-k/2}_{k/2+d}\left(\frac{p}{\sqrt{p^2+\sigma^2|\zeta|^2}}\right),$$

where $P^{1-k/2}_{k/2+d}$ is a Legendre function and

$$p=\frac{|\nu|^2\sigma}{4\tanh(\sigma)}+e^x(\cosh(y)+\cosh(x)).$$

The function F is a Laplace transform.

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