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On Quasi-Poisson Homogeneous Spaces of Quasi-Poisson Lie Groups

Eugene Karolinsky and Kolya Muzykin

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Abstract. Drinfeld showed that if G is a Poisson Lie group with corresponding Lie bialgebra $\mathfrak{g}$, then the isomorphism classes of Poisson homogeneous G-spaces are essentially in a 1-1 correspondence with the G-orbits of Lagrangian subalgebras in $\mathfrak{g} \oplus \mathfrak{g}^*$. The main goal of this paper is to generalize this result to the quasi-Poisson case. We also study the behavior of quasi-Poisson homogeneous spaces under twisting. Some examples of quasi-Poisson homogeneous spaces and corresponding Lagrangian subalgebras are also provided.

1. Introduction

The notion of Poisson Lie group and its infinitesimal counterpart, Lie bialgebra, was introduced by Drinfeld [4]. Later it was explained that these objects are quasiclassical limits of Hopf QUE algebras. In [5] the more general objects, quasi-Hopf QUE algebras, were introduced along with their quasiclassical limits, Lie quasi-bialgebras. The corresponding geometric objects, quasi-Poisson Lie groups, were first studied by Kosmann-Schwarzbach [8].

It is well known that Lie bialgebra structures on $\mathfrak{g}$ are in a natural 1-1 correspondence with Lie algebra structures on $\mathcal{D}(\mathfrak{g}) = \mathfrak{g} \oplus \mathfrak{g}^*$ such that $\mathfrak{g}$ and $\mathfrak{g}^*$ are subalgebras in $\mathcal{D}(\mathfrak{g})$ and the natural bilinear form on $\mathcal{D}(\mathfrak{g})$ is invariant. Respectively, in order to get a Lie quasi-bialgebra structure on $\mathfrak{g}$, one should drop the condition that $\mathfrak{g}^*$ is a subalgebra in $\mathcal{D}(\mathfrak{g})$.

Along with (quasi-)Poisson Lie groups it is natural to study their (quasi-) Poisson actions [1, 2] and, in particular, (quasi-)Poisson homogeneous spaces. Drinfeld in [6] presented an approach to the classification of Poisson homogeneous spaces. Namely, he showed that if G is a Poisson Lie group, $\mathfrak{g}$ is the corresponding Lie bialgebra, then the isomorphism classes of Poisson homogeneous G-spaces are essentially in a 1-1 correspondence with the G-orbits of Lagrangian subalgebras in $\mathcal{D}(\mathfrak{g})$.

The main goal of this paper is to generalize this result to the quasi-Poisson case (see Theorem 3.2). We also study the behavior of quasi-Poisson homogeneous spaces under twisting. Some examples of quasi-Poisson homogeneous spaces and

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corresponding Lagrangian subalgebras are also provided.

It also turns out that quasi-Poisson homogeneous spaces, as well as Poisson ones, are related to solutions of the classical dynamical Yang-Baxter equation (see [7, 10] for the Poisson case). This topic will be discussed in a forthcoming paper.

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2. Preliminaries

2.1. Notation. We will use the following normalization of the wedge product of multivector fields on a smooth manifold. If v is an m-vector field, w is an n-vector field, then

$$v \wedge w = \frac{1}{n!m!}\operatorname{Alt}(v \otimes w),$$

where

$$\operatorname{Alt}(x_1 \otimes x_2 \otimes \cdots \otimes x_k) = \sum_{\sigma \in S_k} \operatorname{sign}(\sigma) x_{\sigma(1)} \otimes x_{\sigma(2)} \otimes \cdots \otimes x_{\sigma(k)}.$$

We will denote by $[\![\ ,\ ]\!]$ the Schouten bracket of multivector fields.

Let G be a Lie group, $\mathfrak{g} = \operatorname{Lie} G$ its Lie algebra. For any $v \in \bigwedge^\bullet \mathfrak{g}$ denote by v^λ (resp. v^ρ) the left (resp. right) invariant multivector field that corresponds to v, i.e., $v^\lambda(g) = (l_g)_* v$, $v^\rho(g) = (r_g)_* v$ for all $g \in G$, where l_g (resp. r_g) is the left (resp. right) translation by g.

Suppose that G acts smoothly on a smooth manifold X. Then for any $v \in \mathfrak{g}$ we denote by v_X the corresponding vector field on X, i.e.,

$$(v_X f)(x) = \frac{d}{dt}\Big|_{t=0} f(\exp tv \cdot x)$$

for any $x \in X$. Similarly, for $v \in \bigwedge^\bullet \mathfrak{g}$ one can define the multivector field v_X. For any $x \in X$ consider the map $\rho_x : G \to X$, $\rho_x(g) = g \cdot x$. Then $(\rho_x)_* v = v_X(x)$ for $v \in \mathfrak{g}$.

For any point $x \in X$ we denote by $H_x = \{g \in G \mid g \cdot x = x\}$ its stabilizer. Let $\mathfrak{h}_x = \operatorname{Lie} H_x \subset \mathfrak{g}$.

Suppose now that X is a homogeneous G-space. In this case we will identify $T_x X$ with $\mathfrak{g}/\mathfrak{h}_x$ for all $x \in X$. Fix $x \in X$ and for any $f \in C^\infty X$ define $f^G \in C^\infty G$ by the formula $f^G(g) = (f \circ \rho_x)(g) = f(g \cdot x)$. Note that the mapping $f \mapsto f^G$ is an isomorphism between the spaces of smooth functions on X and right H_x-invariant smooth functions on G.

2.2. Quasi-Poisson Lie groups and quasi-Poisson actions. Following [9] and [1], we define the notions of quasi-Poisson Lie group and quasi-Poisson action.

Definition 2.1. Let G be a Lie group, $\mathfrak{g}$ its Lie algebra, P_G a bivector field on G, and $\varphi \in \bigwedge^3 \mathfrak{g}$. A triple (G, P_G, φ) is called a *quasi-Poisson Lie group* if

$$P_G \text{ is } \textit{multiplicative}\text{, i.e., } P_G(gg') = (l_g)_* P_G(g') + (r_{g'})_* P_G(g), \tag{1}$$

$$\frac{1}{2}[\![P_G, P_G]\!] = \varphi^\rho - \varphi^\lambda, \tag{2}$$

$$[\![P_G, \varphi^\rho]\!] = 0. \tag{3}$$

The notion of Poisson Lie group is a special case of the notion of quasi-Poisson Lie group. Namely, for any Poisson Lie group (G, P_G) the triple $(G, P_G, 0)$ is a quasi-Poisson Lie group.

Consider the mapping $\eta : G \to \mathfrak{g}\wedge\mathfrak{g}$ defined by $\eta(g) = (r_g^{-1})_* P_G(g)$. It is a $\mathfrak{g}\wedge\mathfrak{g}$-valued 1-cocycle of G with respect to the adjoint action of G on $\mathfrak{g}\wedge\mathfrak{g}$, i.e.,

$$\eta(g_1 g_2) = \eta(g_1) + \operatorname{Ad}_{g_1}\eta(g_2).$$

Here $\operatorname{Ad}_g(x\otimes y) = (\operatorname{Ad}_g x)\otimes(\operatorname{Ad}_g y)$. The fact that η is a 1-cocycle is equivalent to the multiplicativity condition (1).

Consider $\delta = d_e\eta : \mathfrak{g} \to \mathfrak{g}\wedge\mathfrak{g}$. It is a 1-cocycle of $\mathfrak{g}$ with respect to the adjoint action of $\mathfrak{g}$ on $\mathfrak{g}\wedge\mathfrak{g}$, i.e.,

$$\delta([x, y]) = \operatorname{ad}_x\delta(y) - \operatorname{ad}_y\delta(x),$$

where $\operatorname{ad}_x(y\otimes z) = [x\otimes 1 + 1\otimes x, y\otimes z] = \operatorname{ad}_x y\otimes z + y\otimes\operatorname{ad}_x z$.

Definition 2.2. Suppose (G, P_G, φ) is a quasi-Poisson Lie group, and X is a smooth manifold equipped with a bivector field P_X. A smooth action of G on X is called *quasi-Poisson* if

$$P_X(gx) = (l_g)_* P_X(x) + (\rho_x)_* P_G(g), \tag{4}$$

$$\frac{1}{2}[\![P_X, P_X]\!] = \varphi_X \tag{5}$$

(here l_g denotes the mapping $x \mapsto g\cdot x$).

Let us consider the case $\varphi = 0$, i.e., G is a Poisson Lie group. Then condition (5) means that X is a Poisson manifold, and from (4) it follows that the action of G on X is Poisson.

Definition 2.3. A *(quasi-Poisson) isomorphism* between two quasi-Poison actions of a quasi-Poisson Lie group (G, P_G, φ) on manifolds (X, P_X) and (Y, P_Y) is a G-equivariant diffeomorphism $u : X \to Y$ such that $u_*(P_X) = P_Y$.

Definition 2.4. Suppose that (G, P_G, φ) is a quasi-Poisson group, G acts smoothly on a manifold X equipped with a bivector field P_X, and this action is quasi-Poisson. We call X a *quasi-Poisson homogeneous G-space* if the action of G on X is transitive.

Lemma 2.5. *Suppose that (G, P_G, φ) is a quasi-Poisson group, X is a homogeneous G-space, P_X is a bivector field on X. Then condition* (4) *is equivalent to*

$$P_X(gx) = \operatorname{Ad}_g P_X(x) + \overline{\eta(g)}, \tag{6}$$

where $\operatorname{Ad}_g : \bigwedge^2(\mathfrak{g}/\mathfrak{h}_x) \to \bigwedge^2(\mathfrak{g}/\mathfrak{h}_{gx})$ *is the isomorphism of the vector spaces induced by the automorphism* $\operatorname{Ad}_g : \mathfrak{g}\to\mathfrak{g}$, *and* $\overline{\eta(g)}$ *is the image of* $\eta(g)$ *in* $\bigwedge^2(\mathfrak{g}/\mathfrak{h}_{gx})$. ■

2.3. Lie quasi-bialgebras. Recall that a Poisson Lie structure on a Lie group G induces the structure of a Lie bialgebra on the Lie algebra $\mathfrak{g} = \operatorname{Lie} G$. A quasi-Poisson structure on a Lie group G induces a similar structure on $\mathfrak{g}$. We follow [5] in defining the notion of Lie quasi-bialgebra.

Definition 2.6. Let $\mathfrak{g}$ be a Lie algebra, δ a $\mathfrak{g}\wedge\mathfrak{g}$-valued 1-cocycle of $\mathfrak{g}$, and $\varphi\in\bigwedge^3\mathfrak{g}$. A triple $(\mathfrak{g},\delta,\varphi)$ is called a *Lie quasi-bialgebra* if

$$\frac{1}{2}\operatorname{Alt}(\delta\otimes\mathrm{id})\delta(x)=\operatorname{ad}_x\varphi\quad\text{for any } x\in\mathfrak{g}, \tag{7}$$

$$\operatorname{Alt}(\delta\otimes\mathrm{id}\otimes\mathrm{id})\varphi=0, \tag{8}$$

where $\operatorname{ad}_x(a\otimes b\otimes c)=[x\otimes 1\otimes 1+1\otimes x\otimes 1+1\otimes 1\otimes x, a\otimes b\otimes c]$.

Equation (7) is called the quasi co-Jacobi identity.

If we set $\varphi=0$, then the notion of Lie quasi-bialgebra coincides with the notion of Lie bialgebra. In this case equation (7) becomes the ordinary co-Jacobi identity, and condition (8) is obviously satisfied.

For any quasi-Poisson Lie group (G,P_G,φ) there exists a Lie quasi-bialgebra structure on $\mathfrak{g}$ given by the 1-cocycle $\delta=d_e\eta$ and φ. Conversely, to any Lie quasi-bialgebra there corresponds a unique connected and simply connected quasi-Poisson Lie group (see [9]).

Given any linear map $\delta:\mathfrak{g}\to\mathfrak{g}\wedge\mathfrak{g}\subset\mathfrak{g}\otimes\mathfrak{g}$ we can define the skew-symmetric bilinear operation on $\mathfrak{g}^*$: for all $l,m\in\mathfrak{g}^*$ set $[l,m]_\delta=\delta^*(l\otimes m)$.

Recall that for any Lie quasi-bialgebra $(\mathfrak{g},\delta,\varphi)$ one can construct the so-called *double Lie algebra* $\mathcal{D}(\mathfrak{g})$ (see [3]):

let $\mathcal{D}(\mathfrak{g})=\mathfrak{g}\oplus\mathfrak{g}^*$ as a vector space;

define the bilinear operation $[\ ,\ ]_{\mathcal{D}(\mathfrak{g})}$ on $\mathcal{D}(\mathfrak{g})$ by the following conditions:

1. $[a,b]_{\mathcal{D}(\mathfrak{g})}=[a,b]$ for $a,b\in\mathfrak{g}$;
2. $[l,m]_{\mathcal{D}(\mathfrak{g})}=[l,m]_\delta-(l\otimes m\otimes\mathrm{id})\varphi$ for $l,m\in\mathfrak{g}^*$;
3. $[a,l]_{\mathcal{D}(\mathfrak{g})}=\operatorname{coad}_a l-\operatorname{coad}_l a$ for $a\in\mathfrak{g}, l\in\mathfrak{g}^*$.

where $\operatorname{coad}_l:\mathfrak{g}\to\mathfrak{g}$ is defined by $\langle\operatorname{coad}_l a,m\rangle=-\langle[l,m]_\delta,a\rangle=-\langle l\otimes m,\delta(a)\rangle$, and $\operatorname{coad}_a:\mathfrak{g}^*\to\mathfrak{g}^*$ is defined by $\langle\operatorname{coad}_a l,b\rangle=-\langle l,[a,b]\rangle$. Here and below $\langle\ ,\ \rangle$ denotes the standard pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$.

We denote by $Q(\ ,\ )$ the following invariant symmetric bilinear form on $\mathcal{D}(\mathfrak{g})$:

$$Q(a+l,b+m)=\langle l,b\rangle+\langle m,a\rangle.$$

Suppose G is a quasi-Poisson Lie group, $\mathfrak{g}$ is the corresponding Lie quasi-bialgebra, $\mathcal{D}(\mathfrak{g})$ is its double Lie algebra. Then the adjoint action of G on $\mathfrak{g}$ can be extended to the action of G on $\mathcal{D}(\mathfrak{g})$ defined by

$$g\cdot(a+l)=\operatorname{Ad}_g a+(l'\otimes\mathrm{id})\eta(g)+l',$$

where $l'=(\operatorname{Ad}_g^{-1})^*l$. The differential of this action is the adjoint action of $\mathfrak{g}$ on $\mathcal{D}(\mathfrak{g})$.

3. Main results

In [6] a characterization of all Poisson homogeneous structures on a given homogeneous G-space in terms of Lagrangian subalgebras in $\mathcal{D}(\mathfrak{g})$ is presented. We generalize this result to the quasi-Poisson case.

Suppose G is a quasi-Poisson Lie group, X is a quasi-Poisson homogeneous G-space. Recall that we identify T_xX and $\mathfrak{g}/\mathfrak{h}_x$ for all $x \in X$. For any $x \in X$ define

$$L_x = \left\{a + l \mid a \in \mathfrak{g},\ l \in (\mathfrak{g}/\mathfrak{h}_x)^* = \mathfrak{h}_x^\perp \subset \mathfrak{g}^*,\ (l \otimes \mathrm{id})P_X(x) = \overline{a}\right\}, \tag{9}$$

where $\overline{a}$ is the image of a in $\mathfrak{g}/\mathfrak{h}_x$.

Recall that a subspace $L \subset \mathcal{D}(\mathfrak{g})$ is called *Lagrangian* if L is a maximal isotropic subspace with respect to Q. We will say that a subalgebra of $\mathcal{D}(\mathfrak{g})$ is *Lagrangian* if it is a Lagrangian subspace.

Lemma 3.1. *L_x is a Lagrangian subspace in $\mathcal{D}(\mathfrak{g})$, and $L_x \cap \mathfrak{g} = \mathfrak{h}_x$.* ■

Denote by Λ the set of all Lagrangian subalgebras in $\mathcal{D}(\mathfrak{g})$.

Theorem 3.2. *Suppose (G, P_G, φ) is a quasi-Poisson Lie group, (X, P_X) is a quasi-Poisson homogeneous G-space. Then the following statements hold:*

1. *L_x is a subalgebra in $\mathcal{D}(\mathfrak{g})$ for all $x \in X$;*
2. *$L_{gx} = g \cdot L_x$;*
3. *There is a bijection between the set of all G-quasi-Poisson structures on X and the set of G-equivariant maps $x \mapsto L_x$ from X to Λ such that $L_x \cap \mathfrak{g} = \mathfrak{h}_x$ for all $x \in X$.*

Corollary 3.3. *There is a bijection between the set of all isomorphism classes of quasi-Poisson homogeneous G-spaces and the set of G-conjugacy classes of pairs (L, H), where $L \subset \mathcal{D}(\mathfrak{g})$ is a Lagrangian subalgebra, H is a closed subgroup in $G_L = \{g \in G \mid g \cdot L = L\}$, and $L \cap \mathfrak{g} = \mathrm{Lie}\, H$.* ■

The rest of this section is devoted to the proof of Theorem 3.2. We start with a technical lemma.

Lemma 3.4. *Let P be a bivector field on a smooth manifold X. Define $\{f_1, f_2\} = P(df_1, df_2)$ for all $f_1, f_2 \in C^\infty X$. Then*

$$\oint \{\{f_1, f_2\}, f_3\} = -\frac{1}{2}[\![P, P]\!](df_1, df_2, df_3), \tag{10}$$

where $\oint$ denotes the sum over all cyclic permutations of f_1, f_2, f_3

Proof. Straightforward computation. ■

Lemma 3.5. *$L_{gx} = g \cdot L_x$ iff (6) holds.*

Proof. By definition,

$$L_x = \{a + l \mid a \in \mathfrak{g}, l \in (\mathfrak{g}/\mathfrak{h}_x)^*, (l \otimes \mathrm{id})P_X(x) = \overline{a}\},$$

$$L_{gx} = \left\{a' + l' \mid a' \in \mathfrak{g}, l' \in (\mathfrak{g}/\mathfrak{h}_{gx})^*, (l' \otimes \mathrm{id})P_X(gx) = \overline{a'}\right\}.$$

It is enough to check that

$$g \cdot L_x = \left\{a' + l' \mid a' \in \mathfrak{g}, l' \in (\mathfrak{g}/\mathfrak{h}_{gx})^*, (l' \otimes \mathrm{id})\big(\mathrm{Ad}_g P_X(x) + \overline{\eta(g)}\big) = \overline{a'}\right\}.$$

Consider $a' + l' = g \cdot (a + l), a \in \mathfrak{g}, l \in (\mathfrak{g}/\mathfrak{h}_x)^*$, that is,

$$l' = (\mathrm{Ad}_g^{-1})^* l, \ a' = \mathrm{Ad}_g a + (l' \otimes \mathrm{id})\eta(g).$$

We have

$$(l' \otimes \mathrm{id})\big(\mathrm{Ad}_g P_X(x) + \overline{\eta(g)}\big) =$$
$$(l \otimes \mathrm{id})(\mathrm{Ad}_g^{-1} \otimes \mathrm{id})(\mathrm{Ad}_g \otimes \mathrm{Ad}_g)P_X(x) + (l' \otimes \mathrm{id})\overline{\eta(g)} =$$
$$\mathrm{Ad}_g(l \otimes \mathrm{id})P_X(x) + (l' \otimes \mathrm{id})\overline{\eta(g)}.$$

So $(l' \otimes \mathrm{id})\big(\mathrm{Ad}_g P_X(x) + \overline{\eta(g)}\big) = \overline{a'}$ if and only if $a + l \in L_x$. This proves the required equality. ■

Now we are heading for the first statement of the theorem.

Let e_i form a basis in $\mathfrak{g}$, ∂_i (resp. ∂_i') be the right (resp. left) invariant vector field on G that corresponds to e_i.

Suppose $\eta(g) = \eta^{ij}(g)e_i \wedge e_j$. Then $P_G = \eta^{ij}\partial_i \wedge \partial_j$. Choose any $r \in \bigwedge^2 \mathfrak{g}$ such that the image of r in $\bigwedge^2(\mathfrak{g}/\mathfrak{h}_x)$ equals $P_X(x)$. Define

$$\mathrm{CYB}(r) = [r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, r^{23}].$$

Lemma 3.6. *Assume that* (4) *holds. Then the image of*

$$\varphi - \mathrm{CYB}(r) + \frac{1}{2}\,\mathrm{Alt}(\delta \otimes \mathrm{id})(r)$$

in $\bigwedge^3(\mathfrak{g}/\mathfrak{h}_x)$ *vanishes iff* (5) *holds.*

Proof. From (4) it follows that

$$P_X(gx)(d_{gx}f_1, d_{gx}f_2) = ((l_g)_* P_X(x) + (\rho_x)_* P_G(g))\,(d_{gx}f_1, d_{gx}f_2) =$$
$$P_X(x)(d_x(f_1 \circ l_g), d_x(f_2 \circ l_g)) + P_G(g)(d_g(f_1 \circ \rho_x), d_g(f_2 \circ \rho_x)) =$$
$$r(d_e(f_1 \circ l_g)^G, d_e(f_2 \circ l_g)^G) + P_G(g)(d_g f_1^G, d_g f_2^G) = (r^\lambda(g) + P_G(g))(d_g f_1^G, d_g f_2^G).$$

For any $f_1, f_2 \in C^\infty G$ define the bracket

$$\{f_1, f_2\}(g) = (r^\lambda(g) + P_G(g))(d_g f_1, d_g f_2).$$

Using Lemma 3.4 we see that

$$\oint \{\{f_1, f_2\}_X, f_3\}_X(g \cdot x) = \oint \{\{f_1^G, f_2^G\}, f_3^G\}(g) =$$
$$-\frac{1}{2}[\![P_G + r^\lambda, P_G + r^\lambda]\!](df_1^G, df_2^G, df_3^G)(g).$$

Lemma 3.7. $[\![P_G + r^\lambda, P_G + r^\lambda]\!] = 2\left(\varphi^\rho - \varphi^\lambda + \mathrm{CYB}(r)^\lambda - \frac{1}{2}\,\mathrm{Alt}(\delta \otimes \mathrm{id})(r)^\lambda\right).$

Proof. Using the graded anticommutativity of the Schouten bracket, we get

$$[\![P_G + r^\lambda, P_G + r^\lambda]\!] = [\![P_G, P_G]\!] + 2[\![P_G, r^\lambda]\!] + [\![r^\lambda, r^\lambda]\!].$$

From (2) it follows that $[\![P_G, P_G]\!] = 2(\varphi^\rho - \varphi^\lambda)$. We will calculate the rest of the terms on the right-hand side using coordinates. Let $r = r^{ij} e_i \wedge e_j$. Then $r^\lambda = r^{ij}\partial'_i \wedge \partial'_j$, and

$$[\![r^\lambda, r^\lambda]\!] = -4r^{\mu\nu}r^{ij}[\![\partial'_\mu, \partial'_i]\!] \wedge \partial'_j \wedge \partial'_\nu =$$
$$-4r^{\mu\nu}r^{ij}\operatorname{Alt}\left([\![\partial'_\mu, \partial'_i]\!] \otimes \partial'_j \otimes \partial'_\nu\right) = -\operatorname{Alt}\left([r^{13}, r^{12}]\right)^\lambda = 2\operatorname{CYB}(r)^\lambda.$$

Now we prove that $[\![P_G, r^\lambda]\!] = -\frac{1}{2}\operatorname{Alt}(\delta \otimes \mathrm{id})(r)^\lambda$. We have

$$[\![P_G, r^\lambda]\!] = [\![\eta^{\mu\nu}\partial_\mu \wedge \partial_\nu, r^{ij}\partial'_i \wedge \partial'_j]\!] =$$
$$r^{ij}\left([\![\partial'_i, \eta^{\mu\nu}\partial_\mu]\!] \wedge \partial'_j \wedge \partial_\nu - [\![\partial'_j, \eta^{\mu\nu}\partial_\mu]\!] \wedge \partial'_i \wedge \partial_\nu\right) =$$
$$2r^{ij}[\![\partial'_i, \eta^{\mu\nu}\partial_\mu]\!] \wedge \partial'_j \wedge \partial_\nu = -2r^{ij}(\partial'_i\eta^{\mu\nu})\partial_\mu \wedge \partial_\nu \wedge \partial'_j.$$

Using the fact that η is a 1-cocycle, we get

$$\partial'_i\eta^{\mu\nu}(g)e_\mu \wedge e_\nu = \frac{d}{dt}\Big|_{t=0}\eta^{\mu\nu}(g\exp te_i)e_\mu \wedge e_\nu =$$
$$\frac{d}{dt}\Big|_{t=0}(\eta^{\mu\nu}(g)e_\mu \wedge e_\nu + \operatorname{Ad}_g(\eta^{\mu\nu}(\exp te_i)e_\mu \wedge e_\nu)) =$$
$$\frac{d}{dt}\Big|_{t=0}\eta^{kl}(\exp te_i)(\operatorname{Ad}_g)^\mu_k(\operatorname{Ad}_g)^\nu_l e_\mu \wedge e_\nu = \partial'_i\eta^{kl}(e)(\operatorname{Ad}_g)^\mu_k(\operatorname{Ad}_g)^\nu_l e_\mu \wedge e_\nu,$$

where $\operatorname{Ad}_g e_k = (\operatorname{Ad}_g)^\mu_k e_\mu$. So $\partial'_i\eta^{\mu\nu}(g) = \partial'_i\eta^{kl}(e)(\operatorname{Ad}_g)^\mu_k(\operatorname{Ad}_g)^\nu_l$.

Continuing our calculations, we have

$$[\![P_G, r^\lambda]\!](g) = -2r^{ij}(\partial'_i\eta^{\mu\nu})(g)\partial_\mu(g) \wedge \partial_\nu(g) \wedge \partial'_j(g) =$$
$$-2r^{ij}\partial'_i\eta^{kl}(e)(\operatorname{Ad}_g)^\mu_k(\operatorname{Ad}_g)^\nu_l\partial_\mu(g) \wedge \partial_\nu(g) \wedge \partial'_j(g) =$$
$$-2r^{ij}\partial'_i\eta^{\mu\nu}(e)\partial'_\mu(g) \wedge \partial'_\nu(g) \wedge \partial'_j(g) =$$
$$-2r^{ij}\partial'_i\eta^{\mu\nu}(e)\operatorname{Alt}\left(\partial'_\mu(g) \otimes \partial'_\nu(g) \otimes \partial'_j(g)\right) =$$
$$-r^{ij}\operatorname{Alt}\left((d_e\eta(e_i))^\lambda(g) \otimes \partial'_j(g)\right) = -r^{ij}\operatorname{Alt}\left(\delta(e_i) \otimes e_j\right)^\lambda(g) =$$
$$-\tfrac{1}{2}\left(\operatorname{Alt}(\delta \otimes \mathrm{id})r\right)^\lambda(g). \qquad \blacksquare$$

Now we finish the proof of Lemma 3.6. From the definition of a quasi-Poisson action it follows that

$$\oint\{\{f_1, f_2\}_X, f_3\}_X(g \cdot x) = -\varphi_X(df_1, df_2, df_3)(g \cdot x) = -\varphi^\rho(df_1^G, df_2^G, df_3^G)(g).$$

It means that for all $f_1, f_2, f_3 \in C^\infty X$ we have

$$\left(\varphi - \operatorname{CYB}(r) + \frac{1}{2}\operatorname{Alt}(\delta \otimes \mathrm{id})r\right)^\lambda(df_1^G, df_2^G, df_3^G) = 0.$$

Consequently, for all $l, m, n \in \mathfrak{h}_x^\perp$ we get

$$\left\langle\varphi - \operatorname{CYB}(r) + \frac{1}{2}\operatorname{Alt}(\delta \otimes \mathrm{id})r, l \otimes m \otimes n\right\rangle = 0,$$

which proves the statement of the lemma. $\blacksquare$

Lemma 3.8. *Assume that* (4) *holds. Then* L_x *is a subalgebra in* $\mathcal{D}(\mathfrak{g})$ *if and only if the image of the tensor* $\varphi+\frac{1}{2}\operatorname{Alt}(\delta\otimes\mathrm{id})(r)-\operatorname{CYB}(r)$ *in* $\bigwedge^3(\mathfrak{g}/\mathfrak{h}_x)$ *vanishes.*

Proof. Consider the mapping $R:\mathfrak{g}^*\to\mathfrak{g}$ that corresponds to $r\in\bigwedge^2\mathfrak{g}$:

$$R(l)=(l\otimes\mathrm{id})r=\sum_i l(r_i')r_i'',$$

where $r=\sum_i r_i'\otimes r_i''$.

Then

$$L_x=\{a+l\mid a\in\mathfrak{g},l\in(\mathfrak{g}/\mathfrak{h}_x)^*,(l\otimes\mathrm{id})\overline{r}=\overline{a}\}=$$
$$\left\{a+l\mid a\in\mathfrak{g},l\in\mathfrak{h}_x^\perp,\overline{R(l)}=\overline{a}\right\}=\left\{l+R(l)\mid l\in\mathfrak{h}_x^\perp\right\}+\mathfrak{h}_x.$$

From Lemma 3.5 it follows that $h\cdot L_x=L_{hx}=L_x$ for any $h\in H_x$. Consequently, for all $a\in\mathfrak{h}_x$ we have $\operatorname{ad}_a(L_x)\subset L_x$. So L_x is a Lie subalgebra in $\mathcal{D}(\mathfrak{g})$ if and only if $[l_1+R(l_1),l_2+R(l_2)]\in L_x$ for any $l_1,l_2\in\mathfrak{h}_x^\perp$.

Choose any $l_1,l_2,l_3\in\mathfrak{h}_x^\perp$. We are going to check that

$$Q([l_1+R(l_1),l_2+R(l_2)],l_3+R(l_3))=$$
$$\left\langle l_1\otimes l_2\otimes l_3,-\varphi+\operatorname{CYB}(r)-\frac{1}{2}\operatorname{Alt}(\delta\otimes\mathrm{id})r\right\rangle.$$

Indeed,

$$\left\langle l_1\otimes l_2\otimes l_3,[r^{12},r^{13}]\right\rangle=\left\langle l_1\otimes l_2\otimes l_3,\sum_{i,j}[r_i',r_j']\otimes r_i''\otimes r_j''\right\rangle=$$
$$\left\langle l_1,\sum_{i,j}[\langle l_2,r_i''\rangle r_i',\langle l_3,r_j''\rangle r_j']\right\rangle=Q(l_1,[R(l_2),R(l_3)])=Q([l_1,R(l_2)],R(l_3)).$$

Similarly,

$$\left\langle l_1\otimes l_2\otimes l_3,[r^{12},r^{23}]\right\rangle=Q([R(l_1),l_2],R(l_3)),$$
$$\left\langle l_1\otimes l_2\otimes l_3,[r^{13},r^{23}]\right\rangle=Q([R(l_1),R(l_2)],l_3).$$

It is easy to see that $\frac{1}{2}\operatorname{Alt}(\delta\otimes\mathrm{id})r=(\delta\otimes\mathrm{id})r+\tau(\delta\otimes\mathrm{id})r+\tau^2(\delta\otimes\mathrm{id})r$, where $\tau(x\otimes y\otimes z)=z\otimes x\otimes y$. We have

$$\left\langle l_1\otimes l_2\otimes l_3,(\delta\otimes\mathrm{id})r\right\rangle=\sum_i\left\langle l_1\otimes l_2,\delta(r_i')\right\rangle\left\langle l_3,r_i''\right\rangle=$$
$$\sum_i\left\langle[l_1,l_2]_\delta,\langle l_3,r_i''\rangle r_i'\right\rangle=-Q([l_1,l_2],R(l_3)),$$

$$\left\langle l_1\otimes l_2\otimes l_3,\tau(\delta\otimes\mathrm{id})r\right\rangle=-Q([R(l_1),l_2],l_3),$$
$$\left\langle l_1\otimes l_2\otimes l_3,\tau^2(\delta\otimes\mathrm{id})r\right\rangle=-Q([l_1,R(l_2)],l_3),$$
$$\left\langle l_1\otimes l_2\otimes l_3,\varphi\right\rangle=-Q([l_1,l_2],l_3).$$

Adding up all the terms on the right-hand side and using the fact that $Q([R(l_1),R(l_2)],R(l_3))=0$ we see that

$$Q([l_1+R(l_1),l_2+R(l_2)],l_3+R(l_3))=$$
$$\left\langle l_1\otimes l_2\otimes l_3,-\varphi+\operatorname{CYB}(r)-\frac{1}{2}\operatorname{Alt}(\delta\otimes\mathrm{id})r\right\rangle.$$

The right-hand side of this equality vanishes for any $l_1, l_2, l_3 \in (\mathfrak{g}/\mathfrak{h}_x)^*$ iff the image of $\varphi - \mathrm{CYB}(r) + \frac{1}{2}\mathrm{Alt}(\delta \otimes \mathrm{id})r$ in $\bigwedge^3(\mathfrak{g}/\mathfrak{h}_x)$ vanishes.

The left-hand side vanishes for any $l_1, l_2, l_3 \in (\mathfrak{g}/\mathfrak{h}_x)^*$ iff $Q([l_1 + R(l_1), l_2 + R(l_2)], L_x)$ vanishes, i.e., since L_x is maximal isotropic, iff $[l_1 + R(l_1), l_2 + R(l_2)] \in L_x$.

This finishes the proof of the lemma. ■

Suppose $v \in \bigwedge^2(\mathfrak{g}/\mathfrak{h}_x)$. Consider the mapping $v \mapsto L_v$, where

$$L_v = \{a + l \mid a \in \mathfrak{g},\ l \in \mathfrak{g}/\mathfrak{h}_x,\ (l \otimes \mathrm{id})v = \overline{a}\}.$$

This is a bijection between $\bigwedge^2(\mathfrak{g}/\mathfrak{h}_x)$ and the set of all Lagrangian subspaces $L \subset \mathcal{D}(\mathfrak{g})$ such that $L \cap \mathfrak{g} = \mathfrak{h}_x$.

Further, there is a bijection between bivector fields P_X on X and smooth maps $x \mapsto L_x$ from X to the set of all Lagrangian subspaces in $\mathcal{D}(\mathfrak{g})$ such that $L_x \cap \mathfrak{g} = \mathfrak{h}_x$ for all $x \in X$.

From Lemmas 3.5, 3.6 and 3.8 it follows that (X, P_X) is a quasi-Poisson homogeneous G-space iff the corresponding map $x \mapsto L_x$ is G-equivariant, subalgebra-valued, and $L_x \cap \mathfrak{g} = \mathfrak{h}_x$ for all $x \in X$.

This finishes the proof of Theorem 3.2.

4. Twisting

Let G be a Lie group. Suppose (P_G, φ) and (P'_G, φ') are quasi-Poisson structures on G. According to [9], we say that (G, P'_G, φ') is *obtained by twisting* (by $r \in \bigwedge^2 \mathfrak{g}$) from (G, P_G, φ) if

$$P'_G = P_G + r^\lambda - r^\rho,$$

$$\varphi' = \varphi + \frac{1}{2}\mathrm{Alt}(\delta \otimes \mathrm{id})r - \mathrm{CYB}(r).$$

There is a similar relation on Lie quasi-bialgebras. Let $\mathfrak{g}$ be a Lie algebra, (δ, φ) and (δ', φ') are Lie quasi-bialgebra structures on $\mathfrak{g}$. According to [5, 9], we say that $(\mathfrak{g}, \delta', \varphi')$ is *obtained by twisting* (by $r \in \bigwedge^2 \mathfrak{g}$) from $(\mathfrak{g}, \delta, \varphi)$ if

$$\delta'(x) = \delta(x) + \mathrm{ad}_x r \quad \text{for all } x \in \mathfrak{g},$$

$$\varphi' = \varphi + \frac{1}{2}\mathrm{Alt}(\delta \otimes \mathrm{id})r - \mathrm{CYB}(r).$$

Twisting is an equivalence relation.

If (G, P'_G, φ') is obtained by twisting from (G, P_G, φ) then the corresponding Lie quasi-bialgebra $(\mathfrak{g}, \delta', \varphi')$ is obtained by twisting from $(\mathfrak{g}, \delta, \varphi)$. The converse holds if G is connected.

Denote by $\mathcal{D}(\mathfrak{g}, \delta, \varphi)$ and $\mathcal{D}(\mathfrak{g}, \delta', \varphi')$ the double Lie algebras of Lie quasi-bialgebras $(\mathfrak{g}, \delta, \varphi)$ and $(\mathfrak{g}, \delta', \varphi')$ respectively. The following result is obtained in [5].

Theorem 4.1. *$(\mathfrak{g}, \delta', \varphi')$ is obtained by twisting from $(\mathfrak{g}, \delta, \varphi)$ if and only if there exists a Lie algebra isomorphism $f_r : \mathcal{D}(\mathfrak{g}, \delta, \varphi) \to \mathcal{D}(\mathfrak{g}, \delta', \varphi')$ fixing all the elements of $\mathfrak{g}$ and preserving the canonical bilinear forms on the doubles.*

Suppose that (G, P'_G, φ') is obtained by twisting from (G, P_G, φ). Let $r \in \bigwedge^2 \mathfrak{g}$ be the corresponding bivector. Then $f_r : \mathcal{D}(\mathfrak{g}, \delta, \varphi) \to \mathcal{D}(\mathfrak{g}, \delta', \varphi')$, $f_r(a + l) = a + l + (l \otimes \mathrm{id})r$ is the corresponding Lie algebra isomorphism.

Using f_r we can identify $\mathcal{D}(\mathfrak{g}, \delta, \varphi)$ and $\mathcal{D}(\mathfrak{g}, \delta', \varphi')$. Since f_r preserves the canonical bilinear forms, the sets of Lagrangian subalgebras under this identification are the same.

Theorem 4.2. *Let (X, P_X) be a homogeneous quasi-Poisson (G, P_G, φ)-space. Then $(X, P_X - r_X)$ is a homogeneous quasi-Poisson (G, P'_G, φ')-space, and the map $P_X \mapsto P_X - r_X$ is a bijection between the set of all (G, P_G, φ)- and (G, P'_G, φ')-quasi-Poisson structures on X.*

Proof. Denote by Λ (resp. Λ') the set of all Lagrangian Lie subalgebras in $\mathcal{D}(\mathfrak{g}, \delta, \varphi)$ (resp. $\mathcal{D}(\mathfrak{g}, \delta', \varphi')$).

Theorem 3.2 gives us the G-equivariant map $x \mapsto L_x$ from X to Λ such that $L_x \cap \mathfrak{g} = \mathfrak{h}_x$ defined by (9). On the other hand, consider the map $x \mapsto L'_x$ from X to the set of subspaces in $\mathcal{D}(\mathfrak{g}, \delta', \varphi')$ corresponding to $P_X - r_X$:

$$L'_x = \{a + l \mid a \in \mathfrak{g},\ l \in \mathfrak{h}_x^\perp,\ (l \otimes \mathrm{id})(P_X(x) - r_X) = \overline{a}\}.$$

It is easy to see that $f_r(L_x) = L'_x$. Since f_r is a Lie algebra isomorphism, preserves the canonical bilinear forms on the doubles and commutes with the action of G on the doubles, the map $x \mapsto L'_x$ is a G-equivariant map from X to Λ'. Since f_r fixes all the points of $\mathfrak{g}$, we have $L'_x \cap \mathfrak{g} = \mathfrak{h}_x$. From Theorem 3.2 it follows that $P_X - r_X$ defines a (G, P'_G, φ')-quasi-Poisson structure on X.

Obviously, the map $P_X \mapsto P_X - r_X$ from the set of all (G, P_G, φ)-quasi-Poisson structures on X to the set of all (G, P'_G, φ')-quasi-Poisson structures on X is injective. Similarly, the map $P'_X \mapsto P'_X + r_X$ transforms a (G, P'_G, φ')-structure to a (G, P_G, φ)-structure. Thus we have a bijection. ∎

5. Examples

Recall that if (G, P_G) is a Poisson Lie group, then the homogeneous G-spaces $X = \{x\}$ and $Y = G$ admit the structure of Poisson homogeneous (G, P_G)-spaces. Here we consider the quasi-Poisson case.

Example 5.1. Let (G, P_G, φ) be a quasi-Poisson Lie group, $X = \{x\}$ is a homogeneous G-space, $P_X = 0$ is the only bivector field on X. Then the (trivial) action of G on X is quasi-Poisson. The corresponding Lagrangian subalgebra is $\mathfrak{g}$.

Example 5.2. Consider the action of a connected quasi-Poisson Lie group (G, P_G, φ) on $Y = G$ by left translations. By Theorem 3.2, there is a bijection between the set of G-quasi-Poisson structures on Y and the set of G-conjugacy classes of Lagrangian subalgebras $L \subset \mathcal{D}(\mathfrak{g})$ such that $L \cap \mathfrak{g} = 0$.

The map $r \mapsto L_r = \{a + l \in \mathcal{D}(\mathfrak{g}) \mid (l \otimes \mathrm{id})r = a\}$ from $\bigwedge^2 \mathfrak{g}$ to the set of Lagrangian subspaces in $\mathcal{D}(\mathfrak{g})$ transversal to $\mathfrak{g}$ is a bijection. On the other hand, L_r is a Lie subalgebra iff $\varphi + \frac{1}{2}\,\mathrm{Alt}(\delta \otimes \mathrm{id})r - \mathrm{CYB}(r) = 0$.

Thus Y can be a quasi-Poisson homogeneous G-space if and only if G is obtained by twisting from a Poisson Lie group. In this case there is a 1-1 correspondence between the solutions of the equation $\mathrm{CYB}(r)-\frac{1}{2}\operatorname{Alt}(\delta\otimes\mathrm{id})r=\varphi$ and (G,P_G,φ)-quasi-Poisson structures on Y given by $P_Y=P_G+r^\lambda$.

Let us also consider the quasi-Poisson analogue of dressing orbits.

Example 5.3. Consider a connected Lie group D such that its Lie algebra $\mathfrak{d}=\operatorname{Lie}D$ is equipped with a non-degenerate invariant symmetric bilinear form $(\,|\,)$. Suppose that $\mathfrak{d}=\mathfrak{g}\oplus\mathfrak{m}$, where $\mathfrak{g}$ and $\mathfrak{m}$ are isotropic subspaces and $\mathfrak{g}$ is a subalgebra (i.e., the triple $(\mathfrak{d},\mathfrak{g},\mathfrak{m})$ is a *Manin quasi-triple*, see [1]). Let G be the connected Lie subgroup in D corresponding to $\mathfrak{g}$. Let $\{e_i\}$ form a basis in $\mathfrak{g}$, and $\{f_i\}$ be its dual basis in $\mathfrak{m}$. Set $t=\sum_i e_i\otimes f_i$. The triple (G,P_G,φ), where $P_G=t^\lambda-t^\rho$, $\varphi=-\mathrm{CYB}(t)$, is a quasi-Poisson Lie group (see [1]).

Now consider the manifold $S=D/G$ equipped with the bivector field

$$P_S=-t_S. \tag{11}$$

It is shown in [1] that the natural (G,P_G,φ)-action on (S,P_S) is quasi-Poisson, and its orbits are quasi-Poisson homogeneous (G,P_G,φ)-spaces. Consider an orbit $X\subset S$, let $s\in X$, $s=dG$, where $d\in D$. It is straightforward to calculate that L_s (the Lagrangian subalgebra corresponding to $s\in S$) equals $\operatorname{Ad}_d\mathfrak{g}$.

Example 5.4. Suppose $\mathfrak{g}$ is a finite-dimensional Lie algebra with a non-degenerate invariant symmetric bilinear form $(\,|\,)$. Let G be a connected Lie group such that $\operatorname{Lie}G=\mathfrak{g}$. Consider the Manin quasi-triple $(\mathfrak{d},\mathfrak{a}_1,\mathfrak{a}_2)$, where $\mathfrak{d}=\mathfrak{g}\times\mathfrak{g}$,

$$\mathfrak{a}_1=\{(x,x)\mid x\in\mathfrak{g}\}\simeq\mathfrak{g},\ \ \mathfrak{a}_2=\{(x,-x)\mid x\in\mathfrak{g}\},$$

and $\mathfrak{d}$ is equipped with the non-degenerate invariant symmetric bilinear form

$$((a,b),(c,d))\mapsto\frac{1}{2}\left((a|c)-(b|d)\right).$$

It is easy to calculate that the corresponding Lie quasi-bialgebra structure on $\mathfrak{g}$ is given by $\delta=0$, $\varphi=[\Omega^{12},\Omega^{23}]=-\mathrm{CYB}(\Omega)$, where $\Omega\in(S^2\mathfrak{g})^{\mathfrak{g}}$ corresponds to $(\,|\,)$. This Lie quasi-bialgebra gives rise to the quasi-Poisson Lie group $(G,0,\varphi)$.

Pick any $g\in G$, and consider the Lagrangian subalgebra

$$L_g=\{(x,y)\mid y=\operatorname{Ad}_g x\}\subset\mathfrak{d}.$$

It can be shown that it corresponds to the quasi-Poisson homogeneous space (C_g,P), where $C_g\subset G$ is the conjugacy class of g, and

$$P(g)=(r_g\otimes l_g-l_g\otimes r_g)(\Omega). \tag{12}$$

Moreover, one can show that (G,P) is a quasi-Poisson G-manifold with respect to the action by conjugation, and (C_g,P) are "quasi-Poisson G-submanifolds" of (G,P) (see [2], where this example was introduced and studied for a compact Lie group G).

Actually this example is a special case of the previous one. To see this, set $D=G\times G$, embed G diagonally into D, and identify $S=D/G$ with G via $(x,y)\cdot G\mapsto yx^{-1}$. It is routine to check that, for example, under this identification the bivector field (11) coincides with (12), etc.

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Eugene Karolinsky
Department of Mathematics
Kharkov National University
4 Svobody Sq.
Kharkov, 61077, Ukraine
and
Institute for Low Temperature
Physics & Engineering
47 Lenin Avenue
Kharkov, 61103, Ukraine
eugene.a.karolinsky@univer.kharkov.ua

Kolya Muzykin
Department of Mathematics
Kharkov National University
4 Svobody Sq.
Kharkov, 61077, Ukraine

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