Journal of Lie Theory
Volume **14** (2004) 583–617

Automorphisms of Normalizers of Maximal Tori and First Cohomology of Weyl Groups

J.-F. Hämmerli*, M. Matthey† and U. Suter

Communicated by A. Valette

Dedicated to Prof. Alain Robert on the occasion of his 60th birthday

Abstract. Let T be a maximal torus in a connected compact Lie group G, and let W be the corresponding Weyl group with its natural action on T as a reflection group. The cohomology group $H^1(W;T)$ is computed for all simple Lie groups, and the general case is studied. The method is based on a suitable interpretation of $H^1(W;T)$ as a group of (outer) automorphisms of the normalizer of T.

1. Introduction

Let G be a non-abelian connected compact Lie group. Fix a maximal torus T in G and consider its normalizer $N = N_G(T)$. Let $W = N/T$ be the associated Weyl group. Since W acts naturally on T, as a finite reflection group, one can consider the usual cohomology group $H^1(W;T)$. The purpose of this paper is to compute this group for all simple Lie groups G and to describe the situation in the general – non-simple – case. One of our motivations lies in the fact that $H^1(W;T)$ plays a key role in the understanding of the automorphisms of N. Indeed the first named author has shown that the outer automorphism group $\mathrm{Out}(N)$ of the normalizer N canonically decomposes as the semidirect product $\mathrm{Out}(N) \cong H^1(W;T) \rtimes \mathrm{Out}(G)$, where $\mathrm{Out}(G)$ denotes the outer automorphism group of G (see [9, 10]). As $\mathrm{Out}(G)$ is well understood, and essentially given by the group of automorphisms of the Dynkin diagram of G, the computation of $H^1(W;T)$ gives an explicit description of $\mathrm{Out}(N)$. This description is a key ingredient in [9] for a generalization to the nonconnected setting of the remarkable theorem of Curtis, Wiederhold, and Williams saying that *two connected compact Lie groups are isomorphic if and only if the normalizers of their maximal tori are*

*The first author was supported by the Japan Society for the Promotion of Science (JSPS) and by the Grant-in-Aid for Scientific Research No. 12000751-00 of the JSPS

†The second author was partially supported by the Swiss National Fund for Scientific Research and partially by the SFB 478 in Münster (Germany)

ISSN 0949–5932 / \$2.50

isomorphic (see [4] and the papers by Notbohm [19] and by Osse [20] for a general proof, valid in the non-semisimple case; see also Dwyer-Wilkerson [5, 6, 8, 7], Andersen [1] and Møller [14, 15, 16] for related ideas for p-compact groups). This generalization provides a new proof in the nonconnected case of the fact that *a compact Lie group is, up to isomorphism, characterized by its classifying space* [9, 11]. Another motivation has its origin in the work of the second named author [17, 18], which gives a detailed analysis of the cohomology groups $H^*(W;T)$ in degree 0, 1 and 2, together with their close relationships to normalizers of maximal tori.

We prove that $H^1(W;T)$ is a finite elementary abelian 2-group, i.e. a finite $\mathbb{F}_2$-vector space (see Proposition 2.6 (iv) below). The results of our computations for all simple Lie groups are collected in the following theorem. When referring to a specific group G, we denote $H^1(W;T)$ by $H^1(W_G)$ to keep track of the group.

Main Theorem.

Type A_ℓ, $\ell \geqslant 1$ *:*

Let G be a group of type A_ℓ such that $G \ncong \mathrm{SU}(2)$ and $G \ncong \mathrm{PSU}(4)$. Then $H^1(W_G) \cong \mathbb{Z}/2$ if the center of G contains an element of order 2, and $H^1(W_G) = 0$ otherwise. One has:

(i) $H^1(W_{\mathrm{SU}(2)}) = 0$

(ii) $H^1(W_{\mathrm{SU}(2n+1)}) = 0$, $n \geqslant 1$

(iii) $H^1(W_{\mathrm{SU}(2n)}) \cong \mathbb{Z}/2$, $n \geqslant 2$

(iv) $H^1(W_{\mathrm{PSU}(4)}) \cong \mathbb{Z}/2$

(v) $H^1(W_{\mathrm{PSU}(n)}) = 0$, $n \neq 4$

Type B_ℓ, $\ell \geqslant 2$ *:*

(i) $H^1(W_{\mathrm{Spin}(4n+1)}) \cong \mathbb{Z}/2$, $n \geqslant 1$

(ii) $H^1(W_{\mathrm{Spin}(4n+3)}) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$, $n \geqslant 1$

(iii) $H^1(W_{\mathrm{SO}(5)}) \cong \mathbb{Z}/2$

(iv) $H^1(W_{\mathrm{SO}(2n+1)}) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$, $n \geqslant 3$

Type C_ℓ, $\ell \geqslant 3$ *:*

(i) $H^1(W_{\mathrm{Sp}(n)}) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$, $n \geqslant 3$

(ii) $H^1(W_{\mathrm{PSp}(3)}) \cong \mathbb{Z}/2$

(iii) $H^1(W_{\mathrm{PSp}(4)}) \cong \mathbb{Z}/2$

(iv) $H^1(W_{\mathrm{PSp}(n)}) = 0$, $n \geqslant 5$

Type D_ℓ, $\ell \geqslant 4$ *:*

(i) $H^1(W_{\mathrm{Spin}(4n)}) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$, $n \geqslant 2$

(ii) $H^1(W_{\mathrm{Spin}(4n+2)}) \cong \mathbb{Z}/2$, $n \geqslant 2$

(iii) $H^1(W_{\mathrm{SO}(2n)}) \cong \mathbb{Z}/2$, $n \geqslant 4$

(iv) $H^1(W_{\mathrm{sSpin}(4n)}) \cong \mathbb{Z}/2$, $n \geqslant 3$

(v) $H^1(W_{\mathrm{PSO}(8)}) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$

(vi) $H^1(W_{\mathrm{PSO}(2n)}) = 0$, $n \geqslant 5$

Types G_2, F_4, E_6, E_7 *and* E_8 *:*

One has $H^1(W_{\mathrm{E}_7}) \cong \mathbb{Z}/2$, and $H^1(W_G) = 0$ for $G = \mathrm{G}_2$, F_4, E_6, PE_6, PE_7 and E_8.

Remark 1.1. In [17], the center $Z(N)$ of N is computed for all Lie groups G in terms of $Z(G)$, namely $Z(N) \cong Z(G) \oplus (\mathbb{Z}/2)^u$, where u denotes the number of direct factors of G isomorphic to an odd orthogonal group. Explicitly, for $G = \mathrm{SO}(2n+1)$ with its usual upper-left maximal torus, the diagonal matrix $\mathrm{diag}(-1, -1, \ldots, -1, 1)$ lies in the center of N. We also quote from [17] that $Z(N) = T^W$. Now, we observe from the Main Theorem that for G simple, $H^1(W_G)$ is non-trivial if and only if $Z(N)$ contains an element of order 2, with only five exceptions (all of rank $\leqslant 4$), namely G isomorphic to one of $\mathrm{SU}(2)$, $\mathrm{PSO}(6) \cong \mathrm{PSU}(4)$, $\mathrm{PSp}(3)$, $\mathrm{PSp}(4)$ and $\mathrm{PSO}(8)$. (Compare with Theorem 1.2, and with Propositions 5.6 and 5.11 below.)

We briefly describe the method used for these computations. Even though the group $H^1(W;T)$ has an intrinsic definition, independent from the normalizer N, we will use their close relationship to carry out the computations for the classical groups. Namely we will consider $H^1(W;T)$ as a subgroup of $\mathrm{Out}(N)$, as mentioned above and more precisely recalled in Section 2. Our main tool will then be a famous theorem of Tits giving a presentation of the group N, which relies on the fact that N sits in the ambient group G. This presentation will allow us to actually compute the automorphisms of N that correspond to classes in $H^1(W;T) \subseteq \mathrm{Out}(N)$, and to determine when two such automorphisms produce the same class. This method is well suited to direct calculations and to treating the infinite classical families. For the case of exceptional Lie groups the method will be different. We will use an intrinsic description of $H^1(W;T)$ as the kernel of a homomorphism described in [17] (see Proposition 2.6 below for a simplified version, which is sufficient for our purpose). The former method can also be applied to perform computations for the exceptional Lie groups, as illustrated in [9] for E_6.

Eventually, every non-trivial class in $H^1(W;T)$ will be explicitly realized by an automorphism of N, for every simple connected compact Lie group G.

By the well-known classification theorem, for a compact connected Lie group G, there exists a 1-connected (i.e. connected and simply connected) compact Lie group $\widetilde{G}$, a torus $\mathbb{T}^k$ (possibly $k=0$) and a finite central subgroup K of $\widetilde{G} \times \mathbb{T}^k$, such that $G \cong (\widetilde{G} \times \mathbb{T}^k)/K$. The semisimple group $\widetilde{G}$ admits the product decomposition $\widetilde{G} \cong \widetilde{G}_1 \times \ldots \times \widetilde{G}_r$, where each $\widetilde{G}_j$ is a simple 1-connected compact Lie group. Recall also that $\mathrm{Spin}(6) \cong \mathrm{SU}(4)$. In Section 5, we establish the following result.

Theorem 1.2. *Let $G \cong (\widetilde{G} \times \mathbb{T}^k)/K$ be a connected compact Lie group. Denote by $Z_2(N)$ the subgroup of elements of order dividing 2 in the center $Z(N)$ of N. Then, there is a canonical homomorphism*

$$\vartheta\colon \mathrm{Hom}(W, Z_2(N)) \longrightarrow H^1(W;T)$$

satisfying the following properties:

(i) it is injective if $\widetilde{G}$ does not contain direct factors isomorphic to $\mathrm{Spin}(2n+1)$ *with $n \geqslant 1$; in this case, $Z(N) = Z(G)$;*

(ii) it is surjective if $\widetilde{G}$ does not contain direct factors isomorphic to $\mathrm{SU}(4)$, $\mathrm{Sp}(3)$, $\mathrm{Sp}(4)$, $\mathrm{Spin}(8)$ *nor* $\mathrm{Spin}(2n+1)$ *with $n \geqslant 1$.*

In Section 5, the homomorphism ϑ is explicitly constructed (cf. Remark 5.3) and finer results related to surjectivity are established, see in particular Propositions 5.5, 5.6 and 5.11. Theorem 5.16 provides an explicit computation of the kernel of ϑ for all Lie groups G. Note that to prove Theorem 1.2, we first need to know the situation for the simple Lie groups, that is, the proof relies on the Main Theorem. Letting $Z_2(G)$ denote the group of elements of order dividing 2 in $Z(G)$, and W_{ab} the abelianization of W, Theorem 1.2 has the following immediate corollary.

Corollary 1.3. *If $\widetilde{G}$ does not contain direct factors isomorphic to* $\mathrm{SU}(4)$, $\mathrm{Sp}(3)$, $\mathrm{Sp}(4)$, $\mathrm{Spin}(8)$ *nor* $\mathrm{Spin}(2n+1)$ *with $n \geqslant 1$, then*

$$H^1(W;T) \cong \mathrm{Hom}(W_{ab}, Z_2(G)).$$

As is readily checked, the abelianization W_{ab} of a Weyl group is a finite elementary abelian 2-group of rank $|\pi_0(\mathcal{C}_{odd}(W))|$, where $\mathcal{C}_{odd}(W)$ denotes the "odd Coxeter diagram of W", namely the graph obtained from the Coxeter diagram $\mathcal{C}(W)$ by removing the multiple edges, i.e. those connecting two simple roots α and β with $\ell_{\alpha\beta}$ even (and $\geqslant 4$). Thus Corollary 1.3 allows the explicit determination of $H^1(W;T)$ for every group G satisfying the hypothesis.

Let us mention that our results are used in Møller's article [16].

We end up this introduction with a brief overview of the contents of the paper. Although it should really be seen as a companion paper of [9, 10, 17], Section 2 gives the necessary material to make the present work reasonably self-contained. It also explains in details the strategy for the computations of $H^1(W;T)$ for the classical groups. Section 3 describes the actual computations for these groups, focusing on the type A_ℓ and then explaining how to adapt this case to the types B_ℓ, C_ℓ and D_ℓ. Section 4 treats the case of exceptional Lie groups. Finally, Section 5 explains how to deal with the semisimple and general cases and presents some examples, particularly the unitary groups.

Acknowledgements: The authors wish to express their deep gratitude to Jesper Michael Møller for pointing out an error in an earlier version of the paper and for generously providing very helpful comments and examples.

2. Description of $H^1(W;T)$ and strategy for the computations

This section blends materials from [9], [10], [17] and [18], in order to explain the strategy of our computations. It also highlights the importance of Tits' presentation of N. For a more detailed analysis and more precise results, we refer to [17, 18].

We start by briefly recalling how $H^1(W;T)$ can be seen as a subgroup of the outer automorphism group of the normalizer. Let $\mathrm{Aut}(N,T)$ be the subgroup of the automorphism group of N consisting of elements ψ that are the identity on T (as the action of W on T is faithful, such automorphisms automatically induce the identity on the quotient $W = N/T$). Since T is a maximal abelian normal subgroup of N, a result established in [21] implies that the cohomology group $H^1(W;T)$ can be identified with a subgroup of $\mathrm{Out}(N)$ as follows:

$$H^1(W;T) \cong \mathrm{Aut}(N,T)/\,\mathrm{Inn}(N,T),$$

where $\mathrm{Inn}(N,T)$ consists of the conjugations of N by elements in T (see [9, 10] for details). We make this identification for the rest of the paper.

Before recalling a remarkable theorem of Tits, we introduce some notations. Let $\mathrm{L}T$ denote the Lie algebra of the maximal torus T and $\exp\colon \mathrm{L}T \longrightarrow T$ designate the exponential homomorphism. Let $R = R(G,\,T) \subset \mathrm{L}T^*$ be the root system of G associated to T, and $R^\vee \subset \mathrm{L}T$ the corresponding coroot system. The Weyl group W acts on $\mathrm{L}T$ and is generated by the reflections s_α $(\alpha \in R)$, explicitly given by

$$s_\alpha(X) = X - 2\tfrac{(X,\alpha^\vee)}{(\alpha^\vee,\alpha^\vee)} \cdot \alpha^\vee\,,$$

for $X \in \mathrm{L}T$ (we have fixed a W-invariant inner product $(.\,,.)$ on $\mathrm{L}T$). For a root α of G, define $T_\alpha := \exp(\mathbb{R}\cdot\alpha^\vee)$; this is a closed subgroup of T isomorphic to the circle $\mathbb{S}^1$ and it is easily seen to be the connected component of the subgroup $Q_\alpha := \{t \in T \mid s_\alpha(t) = t^{-1}\}$. We also define $h_\alpha := \exp(\frac{\alpha^\vee}{2}) \in T_\alpha$. Later, we will need other subgroups of T, namely $F_\alpha := \{t \in T \mid s_\alpha(t) = t\}$, the fixed point set of s_α in T, and the subgroup $S := \{t \in T \mid t^2 = e\}$ of elements of order dividing 2. For a root α, let U_α be the kernel of the associated global root $\rho_\alpha\colon T \longrightarrow \mathbb{S}^1$, characterized by $\mathrm{L}\rho_\alpha = 2\pi\cdot\alpha$; denote the centralizer of U_α in G by $Z_G(U_\alpha)$. Consider the group $\mathbb{S}^3$ of quaternions of unit norm; the normalizer of its 'standard' maximal torus $\mathbb{S}^1$ is given by $N_{\mathbb{S}^3}(\mathbb{S}^1) = \mathbb{S}^1 \amalg j\mathbb{S}^1$. Denote by $\pi\colon N \twoheadrightarrow W$ the projection map. Recall that there exists a homomorphism $\nu_\alpha\colon \mathbb{S}^3 \longrightarrow G$ satisfying the following properties:

- $\nu_\alpha(\mathbb{S}^3) \subseteq Z_G(U_\alpha)$
- $\nu_\alpha(\mathbb{S}^1) = T_\alpha$
- $\nu_\alpha(j\mathbb{S}^1) \subseteq \pi^{-1}(s_\alpha)$
- $\nu_\alpha(-1) = h_\alpha$;

it is unique up to composition with a conjugation by an element of $\mathbb{S}^1 \amalg j\mathbb{S}^1$ (see [3, §4, N^0 5]). Note that the set $\mathrm{C}_\alpha := \nu_\alpha(j\mathbb{S}^1)$ does *not* depend on the choice of ν_α. Fix a basis B of R; for $\alpha,\,\beta \in B$, denote by $\ell_{\alpha\beta}$ the order of the product $s_\alpha s_\beta$ in W (recall that for two distinct roots $\alpha \neq \beta$, the only possible values are $\ell_{\alpha\beta} = 2$, 3, 4 or 6). Finally, we let ℓ denote the rank of G.

Theorem 2.1. Tits *For every root $\alpha \in B$, let q_α be a fixed element in C_α. Then for all $\alpha,\,\beta \in B$, with $\alpha \neq \beta$, and for all $t \in T$, the following relations hold in N:*

$$\text{(R1)} \qquad q_\alpha^2 = h_\alpha$$

$$\text{(R2)} \qquad \underbrace{q_\alpha q_\beta q_\alpha \cdots}_{\ell_{\alpha\beta}\,\textit{factors}} = \underbrace{q_\beta q_\alpha q_\beta \cdots}_{\ell_{\alpha\beta}\,\textit{factors}}$$

$$\text{(R3)} \qquad q_\alpha t q_\alpha^{-1} = s_\alpha(t)\,.$$

Furthermore, the group N is generated by the set $T \cup \{q_\alpha\}_{\alpha\in B}$ and is defined by relations (R1), (R2) *and* (R3)*, added to the fact that T is a subgroup of N. More precisely, any relation between elements of N is a consequence of* (R1), (R2), (R3) *and of relations among elements of T.*

Tits proved this theorem for split reductive semisimple algebraic groups in his original papers [22, 23], however his proof adapts to compact Lie groups (see [4] and [17]). Related to this theorem, we introduce some definitions that will play a crucial role in our approach.

Definition 2.2. (i) A subset $A = \{q_\alpha\}_{\alpha\in B}$ of N such that $q_\alpha \in \mathrm{C}_\alpha$ for all $\alpha \in B$ is called a *Tits configuration*, and each coset $\mathrm{C}_\alpha = q_\alpha T_\alpha$ a *Tits circle.*

(ii) A subset $\tilde{A} = \{\tilde{q}_\alpha\}_{\alpha\in B}$ of N such that

- $\tilde{q}_\alpha \in \pi^{-1}(s_\alpha)$, for all $\alpha \in B$
- $\tilde{q}_\alpha^2 = h_\alpha$, for all $\alpha \in B$
- $\tilde{q}_\beta \notin \mathrm{C}_\beta$, for some $\beta \in B$

is called a *fake Tits configuration* and the corresponding coset $\tilde{q}_\beta T_\beta$ a *fake Tits circle.*

Next lemma shows that, for a fixed root α , the union of the corresponding Tits circle and of the fake ones is in bijection with the subgroup Q_α .

Lemma 2.3. *Let $\alpha \in B$ and $t \in T$. Then $t \cdot \mathrm{C}_\alpha$ is a fake Tits circle if and only if $t \in Q_\alpha \setminus T_\alpha$.*

Proof. Let $q_\alpha \in \mathrm{C}_\alpha$ and suppose that tq_α is on a fake Tits circle. Then

$$h_\alpha = (tq_\alpha)^2 = tq_\alpha tq_\alpha = ts_\alpha(t)q_\alpha^2 = ts_\alpha(t)h_\alpha\,,$$

which amounts to $s_\alpha(t) = t^{-1}$. As $q_\alpha T_\alpha = T_\alpha q_\alpha = \mathrm{C}_\alpha$, the result follows. ■

Next, we show that the number of fake Tits circles is finite.

Lemma 2.4. *Let $\alpha \in R$. Every coset in Q_α/T_α has a representative element in $F_\alpha \cap S$ and there is an isomorphism*

$$Q_\alpha/T_\alpha \cong \frac{F_\alpha \cap S}{\mathbb{Z}/2}\,.$$

In particular Q_α/T_α is a finite elementary abelian 2-group.

Proof. As clearly $F_\alpha \cap S \subset Q_\alpha$ and $|F_\alpha \cap S \cap T_\alpha| = 2$, it is enough to show that $F_\alpha \cap S$ intersects every connected component of Q_α . Let tT_α be such a component. Considering the situation in the Lie algebra of T , it is easy to see that $T = T_\alpha \cdot F_\alpha$. So, we can write $t = ru$ with $r \in T_\alpha$ and $u \in F_\alpha$; thus $uT_\alpha = tT_\alpha \subseteq Q_\alpha$ and $u \in F_\alpha \cap Q_\alpha$. We deduce that $u \in F_\alpha \cap S$. The final statement follows from the isomorphism $S \cong (\mathbb{Z}/2)^\ell$, where ℓ is the rank of G . ■

We will also need the following lemma.

Lemma 2.5. *(i) Let $A = \{q_\alpha\}_{\alpha\in B}$ and $A' = \{q'_\alpha\}_{\alpha\in B}$ be two Tits configurations. Then A and A' are termwise conjugate by an element of the maximal torus, i.e. there exists $t \in T$ such that $tq_\alpha t^{-1} = q'_\alpha$ for all $\alpha \in B$.*

(ii) Let $\tilde{A} = \{\tilde{q}_\alpha\}_{\alpha\in B}$ and $\tilde{A}' = \{\tilde{q}'_\alpha\}_{\alpha\in B}$ be two fake Tits configurations such that $\tilde{q}_\alpha$ and $\tilde{q}'_\alpha$ lie on the same Tits or fake Tits circle for all $\alpha \in B$. Then $\tilde{A}$ and $\tilde{A}'$ are termwise conjugate by an element of T .

Before the proof, and for the rest of the paper, we fix a linear ordering "$<$" on the basis B , i.e. we write $B = \{\alpha_1 < \alpha_2 < \ldots < \alpha_{\ell_S}\}$, where ℓ_S is the semisimple rank of G , in other words, the rank of W . For $\alpha, \beta \in B$, the relation $\alpha < \beta$ will always understand that α and β are distinct. Now we begin the proof of Lemma 2.5.

Proof. Obviously, by Lemma 2.3, (ii) is implied by (i). Now, it is enough to check (i) in the semisimple case (then, $\ell = \ell_S$). By definition there exist elements $r_1 \in T_{\alpha_1}, \ldots, r_\ell \in T_{\alpha_\ell}$ such that $q'_{\alpha_1} = r_1 q_{\alpha_1}, \ldots, q'_{\alpha_\ell} = r_\ell q_{\alpha_\ell}$. We have to construct an element $t \in T$ such that, for all j, $t q_{\alpha_j} t^{-1} = r_j q_{\alpha_j}$, or equivalently such that

$$t \cdot s_{\alpha_j}(t)^{-1} = r_j \,. \qquad (*)$$

We show that there exists an element $X \in \mathrm{L}T$ such that $t = \exp(X)$ has the desired property. Fix elements $\lambda_j \in \mathbb{R}$ such that $\exp(\lambda_j \alpha_j^\vee) = r_j$. Equality $(*)$ is then successively equivalent to

$$\exp(X) \exp\big(s_{\alpha_j}(X)\big)^{-1} = \exp(\lambda_j \alpha_j^\vee) \iff \exp\big(X - s_{\alpha_j}(X)\big) = \exp(\lambda_j \alpha_j^\vee)\,.$$

As $s_{\alpha_j}(X) = X - 2\frac{(X,\alpha_j^\vee)}{(\alpha_j^\vee,\alpha_j^\vee)} \cdot \alpha_j^\vee$, the latter expression becomes

$$\exp\left(2\frac{(X,\alpha_j^\vee)}{(\alpha_j^\vee,\alpha_j^\vee)} \cdot \alpha_j^\vee - \lambda_j \cdot \alpha_j^\vee\right) = e$$

for all j. Since $B^\vee = \{\alpha_1^\vee, \ldots, \alpha_\ell^\vee\}$ is a basis of $\mathrm{L}T$, the ℓ scalars defined by

$$(X,\, \alpha_j^\vee) = \tfrac{\lambda_j \cdot (\alpha_j^\vee,\alpha_j^\vee)}{2}$$

determine the covariant coordinates of a vector $X \in \mathrm{L}T$, and, by construction, $t = \exp(X)$ satisfies $(*)$ for all j. ■

We are almost ready to present a description of $H^1(W;T)$ that will allow explicit computations. We fix once and for all a Tits configuration $A = \{q_\alpha\}_{\alpha \in B}$. Let $\psi \in \mathrm{Aut}(N,T)$. From Tits' presentation 2.1, it is clear that ψ is completely determined by the images $\psi(q_\alpha)$ for $\alpha \in B$, and these are of the form $t_\alpha \cdot q_\alpha$, with $t_\alpha \in T$. Relations (R1) and (R3) imply that $t_\alpha \cdot q_\alpha$ is on a Tits or fake Tits circle and therefore, by Lemma 2.3, $t_\alpha \in Q_\alpha$. From relation (R2), for $\alpha \neq \beta$, one has successively

$$\begin{array}{rrcll}
& \psi(q_\alpha q_\beta q_\alpha \cdots) & = & \psi(q_\beta q_\alpha q_\beta \cdots) & \\
\iff & \psi(q_\alpha)\psi(q_\beta)\psi(q_\alpha)\cdots & = & \psi(q_\beta)\psi(q_\alpha)\psi(q_\beta)\cdots & \\
\iff & t_\alpha q_\alpha t_\beta q_\beta t_\alpha q_\alpha \cdots & = & t_\beta q_\beta t_\alpha q_\alpha t_\beta q_\beta \cdots & \\
\iff & t_\alpha \cdot s_\alpha(t_\beta) \cdot s_\alpha s_\beta(t_\alpha) \cdot q_\alpha q_\beta q_\alpha \cdots & = & t_\beta \cdot s_\beta(t_\alpha) \cdot s_\beta s_\alpha(t_\beta) \cdot q_\beta q_\alpha q_\beta \cdots & \\
\iff & t_\alpha \cdot s_\alpha(t_\beta) \cdot s_\alpha s_\beta(t_\alpha) \cdots & = & t_\beta \cdot s_\beta(t_\alpha) \cdot s_\beta s_\alpha(t_\beta) \cdots & (\spadesuit) \\
\iff & t_\alpha \cdot s_\beta(t_\alpha)^{-1} \cdot s_\alpha s_\beta(t_\alpha) \cdots & = & t_\beta \cdot s_\alpha(t_\beta)^{-1} \cdot s_\beta s_\alpha(t_\beta) \cdots & \\
\iff & w_{\alpha\beta}(t_\alpha) & = & w_{\beta\alpha}(t_\beta) & (\clubsuit)
\end{array}$$

where $w_{\alpha\beta}$ is the element in the integral group algebra $\mathbb{Z}W$ defined by

$$w_{\alpha\beta} = 1 - s_\beta + s_\alpha s_\beta - + \ldots + (-1)^{\ell_{\alpha\beta}-1} \underbrace{\cdots s_\beta s_\alpha s_\beta}_{\ell_{\alpha\beta}-1 \text{ factors}}$$

with its obvious action on T (this element was already defined in [4]). Consider the injective map

$$\bar\Lambda\colon\ \mathrm{Aut}(N,T) \longrightarrow \mathrm{Ker}(\bar\Theta) \subseteq \bigoplus_{\alpha \in B} Q_\alpha\,, \quad \psi \longmapsto \big(\psi(q_\alpha) q_\alpha^{-1}\big)_\alpha\,.$$

It follows from the above computation that its image coincides with the subgroup of elements $(t_\alpha)_\alpha$ satisfying $w_{\alpha\beta}(t_\alpha)\cdot w_{\beta\alpha}(t_\beta)^{-1}=e$ for all $\alpha\neq\beta$ in the basis B. For indices $1\leqslant i,\, j,\, k\leqslant \ell_S$, we write $t_i=t_{\alpha_i}$, $w_{ij}=w_{\alpha_i\alpha_j}$ and $T_{jk}=T$. The preceding consideration leads to the introduction of the following homomorphism

$$\bar\Theta\colon \bigoplus_{1\leqslant i\leqslant \ell_S} Q_{\alpha_i}\longrightarrow \bigoplus_{1\leqslant j<k\leqslant \ell_S} T_{jk}\,,\quad (t_i)_i\longmapsto (w_{jk}(t_j)\cdot w_{kj}(t_k)^{-1})_{(j<k)}\,.$$

We can now state an important result for the sequel. It is proved in [17], but, for sake of completeness and the ease of the reader, we provide a proof that differs slightly from that of [17] and suits perfectly in the framework that has just been settled.

Proposition 2.6. *(i) The following map is an isomorphism:*

$$\bar\Lambda\colon \operatorname{Aut}(N,T)\xrightarrow{\cong}\operatorname{Ker}(\bar\Theta)\subseteq \bigoplus_{1\leqslant i\leqslant \ell_S} Q_{\alpha_i}\,,\quad \psi\longmapsto \big(\psi(q_{\alpha_i})q_{\alpha_i}^{-1}\big)_i\,.$$

(ii) The homomorphism $\bar\Theta$ factorizes through the map

$$\Theta\colon \bigoplus_{1\leqslant i\leqslant \ell_S} Q_{\alpha_i}/T_{\alpha_i}\longrightarrow \bigoplus_{1\leqslant j<k\leqslant \ell_S} T_{jk}\,,\quad (t_iT_{\alpha_i})_i\longmapsto \big(w_{jk}(t_j)\cdot w_{kj}(t_k)^{-1}\big)_{(j<k)}\,.$$

(iii) The following map is an isomorphism:

$$\Lambda\colon H^1(W;T)\xrightarrow{\cong}\operatorname{Ker}(\Theta)\subseteq \bigoplus_{1\leqslant i\leqslant \ell_S} Q_{\alpha_i}/T_{\alpha_i}\,,\quad [\psi]\longmapsto \big(\psi(q_{\alpha_i})q_{\alpha_i}^{-1}\cdot T_{\alpha_i}\big)_i\,.$$

(iv) The cohomology group $H^1(W;T)$ is an elementary abelian 2-group.

Proof. By straightforward computation, $\bar\Lambda$ is a homomorphism. It is clearly into, and it is onto by the very definition of $\bar\Theta$, establishing (i). For (ii), consider first an arbitrary element $(r_i)_i\in\bigoplus T_{\alpha_i}$. Note that the set $A':=\{r_iq_{\alpha_i}\}_i$ is a Tits configuration. By Lemma 2.5, we find $t\in T$ such that for the conjugation c_t, we have $c_t(q_{\alpha_i})=r_iq_{\alpha_i}$. We get $(r_i)_i=\bar\Lambda(c_t)$ and $\bigoplus T_{\alpha_i}\subseteq\bar\Lambda(\operatorname{Inn}(N,T))\subseteq\operatorname{Ker}(\bar\Theta)$. In particular, $\bar\Theta$ factors through Θ, proving (ii). Conversely, conjugation by any element $t\in T$ preserves the Tits circles, which means that $\bar\Lambda(c_t)\in\bigoplus T_{\alpha_i}$. It follows, with part (i), that $\bar\Lambda$ maps $\operatorname{Inn}(N,T)$ isomorphically onto $\bigoplus T_{\alpha_i}$. Therefore,

$$\Lambda\colon H^1(W;T)\cong\operatorname{Aut}(N,T)/\operatorname{Inn}(N,T)\longrightarrow\operatorname{Ker}(\bar\Theta)/\bigoplus T_{\alpha_i}=\operatorname{Ker}(\Theta)$$

is an isomorphism. This proves (iii). Part (iv) follows from Lemma 2.4. ∎

Note. ¿From now on, we always identify an automorphism $\psi\in\operatorname{Aut}(N,T)$ with the element $(t_\alpha)_\alpha:=\big(\psi(q_\alpha)q_\alpha^{-1}\big)_\alpha$ in $\operatorname{Ker}(\bar\Theta)$, via $\bar\Lambda$.

Remark 2.7. In their famous work on maps between classifying spaces of connected compact Lie groups [12, 13], Jackowski, McClure and Oliver encounter the cohomology group $H^1(W;T)$ and mention in [13] that it is a 2-group. The fact that this group is a finite $\mathbb{F}_2$-vector space was already established by the first named author in [9] by a slightly different method.

Proposition 2.6 and its proof show that non-trivial elements in $H^1(W;T)$ arise from the existence of *fake* Tits circles. Roughly speaking, an element in $H^1(W;T)$ is given by a 'compatible choice' of a Tits or fake Tits circle in each component of N corresponding to a root in B. Proposition 2.6 also gives a precise, intrinsic description of the cohomology group $H^1(W;T)$ in terms of the W-module T only. In the case of classical simple Lie groups, we will however not directly compute $\mathrm{Ker}(\Theta)$, but rather use a method that we now describe. ¿From Proposition 2.6, we know that an element $\psi \in \mathrm{Aut}(N,T)$ corresponds to a tuple $(t_\alpha)_\alpha \in \bigoplus Q_\alpha$ satisfying the compatibility condition $(\clubsuit)$. As we want to compute $H^1(W;T)$, we are only interested in the class $[\psi] \in H^1(W;T)$ and can thus replace each component t_α by the coset $t_\alpha T_\alpha$ (in other words, we use the isomorphism Λ in place of $\bar{\Lambda}$). By Lemma 2.4, we know that each representative t_α can be chosen in $F_\alpha \cap S$, and that in each coset $t_\alpha T_\alpha$, there are exactly two such elements. To compute $H^1(W;T)$, we will thus compute automorphisms in $\mathrm{Aut}(N,T)$ corresponding to $(t_\alpha)_\alpha \in \bigoplus F_\alpha \cap S$, and identify those for which $t'_\alpha = k_\alpha t_\alpha$ for some $\alpha \in B$, where k_α denotes the unique element of order 2 in T_α. We will write $t'_\alpha \sim t_\alpha$ if $t'_\alpha = k_\alpha t_\alpha$. Choosing for every $\alpha \in B$ a homomorphic cross-section $\underline{s}^{(\alpha)}$ of the projection $F_\alpha \cap S \twoheadrightarrow Q_\alpha / T_\alpha$ of $\mathbb{F}_2$-vector spaces produces a homomorphism $\bigoplus \underline{s}^{(\alpha)} \colon \bigoplus Q_\alpha/T_\alpha \longrightarrow \bigoplus F_\alpha \cap S \subseteq \bigoplus Q_\alpha$. As is readily checked, the restricted homomorphism $(\bigoplus \underline{s}^{(\alpha)})|_{\mathrm{Ker}(\Theta)}$ maps $\mathrm{Ker}(\Theta)$ into $\mathrm{Ker}(\bar{\Theta})$. Therefore it corresponds, via the isomorphisms Λ and $\bar{\Lambda}$, to a homomorphism

$$\underline{s} \colon H^1(W;T) \longrightarrow \mathrm{Aut}(N,T)\,, \quad [\psi] \longmapsto \underline{s}([\psi]) = \varphi\,,$$

with φ being identified with an element $(t_\alpha)_\alpha \in \mathrm{Ker}(\bar{\Theta})$ and satisfying $[\varphi] = [\psi]$ in $H^1(W;T)$. To remain concise, abusing notation, we will often write $\underline{s}([\psi]) = \psi$. From this discussion, we deduce a proposition.

Proposition 2.8. *The canonical projection $p_N \colon \mathrm{Aut}(N,T) \longrightarrow H^1(W;T)$ is split by the homomorphism $\underline{s}$ (for every choice of $\{\underline{s}^{(\alpha)}\}_{\alpha \in B}$). In particular, there is a (non-canonical) isomorphism.*

$$\mathrm{Aut}(N,T) \cong \mathrm{Inn}(N,T) \times H^1(W;T)\,.$$

Proof. After what has just been said, it only remains to show that the semi-direct product $\mathrm{Inn}(N,T) \rtimes H^1(W;T)$ is a direct one. But this is clear, since $\mathrm{Aut}(N,T)$ is isomorphic to the group $Z^1(W;T)$ of 1-cocycles, and is therefore abelian (see Remark 3.4 in [10]). ∎

Summarizing, in the classical cases our strategy for computing $H^1(W;T)$ consists in finding the image of such a splitting $\underline{s}$ of $p_N \colon \mathrm{Aut}(N,T) \longrightarrow H^1(W;T)$.

3. Computations for the classical Lie groups

In the present section, we follow the strategy described in the previous one. We will simultaneously perform our computations in the simply connected group and its quotients by (finite) central subgroups working with representative elements, a strategy that will prove especially useful for the A_ℓ family. In this section, except where otherwise stated, for each type, G denotes the corresponding simply-connected group, T the 'standard' maximal torus in G, and N its normalizer; ℓ is the rank of G. Let K be a (possibly trivial) subgroup of the center $Z = Z(G)$, which is finite, and let

$$p\colon G \longrightarrow \bar{G} = G/K\,, \quad g \longmapsto p(g) = \bar{g} = \big[g\big]$$

denote the canonical projection. In $\bar{G}$, the image $\bar{T} = p(T)$ is a maximal torus, and $\bar{N} = p(N)$ is its normalizer. Identifying the Weyl group $\bar{N}/\bar{T}$ with W, note that p is W-equivariant. We identify the Lie algebras of G and $\bar{G}$, the exponential map $\overline{\exp}$ for $\bar{G}$ being obtained by composing that of G with p. Let $B = \{\alpha_1, \ldots, \alpha_\ell\}$ be a fixed basis of the root system of G, and thus of $\bar{G}$, and let us specify a Tits configuration $A := \{q_1 = q_{\alpha_1}, \ldots, q_\ell = q_{\alpha_\ell}\}$ of G (we will not need to make it explicit). By 'uniqueness' of the homomorphism ν_{α_j}, p takes the Tits circle C_{α_j} onto the corresponding Tits circle in $\bar{N}$, so that $\bar{A} = \{\bar{q}_1 = p(q_1), \ldots, \bar{q}_\ell = p(q_\ell)\}$ is a Tits configuration of $\bar{G}$. The subgroup $T_j = T_{\alpha_j}$ is mapped onto the subgroup $\bar{T}_j = \overline{\exp}(\mathbb{R}\cdot\alpha_j^\vee)$. For each family, we will start by recalling the standard maximal torus of G, its center, its Dynkin diagram, and the way s_{α_j} acts on T, the subgroup T_j, and the non-trivial element $\bar{k}_{\alpha_j} \in \bar{F}_{\alpha_j} \cap \bar{S} \cap \bar{T}_j$, with the obvious notations (note that in general, $\bar{F}_{\alpha_j} \subseteq p(F_{\alpha_j})$ and $p(S) \subseteq \bar{S}$). Then, we will determine a section $\underline{s}$ (as in Proposition 2.8), and, for every class $[\psi] \in H^1(W;\bar{T})$, determine $\psi = \underline{s}([\psi])$ explicitly, in order to prove the Main Theorem. Recall that ψ is entirely determined by $\psi(\bar{q}_j) = \bar{t}_j \cdot \bar{q}_j$, with $j = 1, \ldots, \ell$, for suitable elements $\bar{t}_j \in \bar{F}_{\alpha_j} \cap \bar{S}$. We write $\ell_{i,j} = \ell_{\alpha_i\alpha_j}$ (the order of the product $s_{\alpha_i}s_{\alpha_j}$ in W). The trivial element in G is denoted by e and in $\bar{G}$ by $\bar{e}$.

Proof of the Main Theorem for the type A_ℓ, $\ell \geqslant 1$.

Put $n = \ell + 1$. The standard maximal torus T in $G = \mathrm{SU}(n)$ consists of the subgroup of diagonal matrices of determinant 1, namely

$$T = \big\{\, \mathrm{diag}(z_1, \ldots, z_n) \,\big|\, z_k \in \mathbb{S}^1\,,\ \forall\, k\,,\ \text{and}\ z_1 \cdots z_n = 1 \big\}\,.$$

We denote an element of T simply by $t = (z_1, \ldots, z_n)$. The center is given by

$$Z = \big\{(\zeta, \ldots, \zeta) \,\big|\, \zeta^n = 1\big\} \cong C_n\,,$$

where C_n denotes the cyclic group of n-th roots of unity, and the isomorphism is the obvious one. To the subgroup $K \subseteq Z$ corresponds a subgroup $C_K \subseteq C_n$. The Dynkin diagram is

$$\overset{\alpha_1}{\circ}\!-\!\!-\!\!-\overset{\alpha_2}{\circ}\!-\cdots-\!\overset{\alpha_{n-2}}{\circ}\!-\!\!-\!\!-\overset{\alpha_{n-1}}{\circ}$$

and the reflection s_{α_j} exchanges the entries z_j and z_{j+1} on the diagonal. The subgroup T_j is given by

$$T_j = \big\{(1, \ldots, 1,\, z_j = z,\, z_{j+1} = z^{-1},\, 1, \ldots, 1) \,\big|\, z \in \mathbb{S}^1\big\}\,.$$

Therefore, for $n \geqslant 3$, the non-trivial element in $\bar{F}_{\alpha_j} \cap \bar{S} \cap \bar{T}_j$ is

$$\bar{k}_{\alpha_j} = \big[1,\, \ldots,\, z_j = -1,\, z_{j+1} = -1,\, 1,\, \ldots,\, 1\big]$$

for all $j = 1, \ldots, n-1$.

Let us start with an easy, but useful observation. Given two elements $t = (z_1, \ldots, z_n)$ and $s = (w_1, \ldots, w_n)$ in T, one obviously has $\bar{t} = \bar{s}$ if and only if there exists $\zeta \in C_K$ such that $z_k = \zeta w_k$ for all k; in particular, if $\bar{t} = \bar{s}$ and $z_k = w_k$ for some k, then $t = s$. Clearly $\bar{t}_j^2 = \bar{e}$ implies that $t_j^2 = (\zeta, \ldots, \zeta) \in K$ and thus $t_j = (\mu_1, \ldots, \mu_n)$; here, $\zeta = \zeta(j)$, and $\mu_k = \mu_k(j)$ satisfies $\mu_k^2 = \zeta$, that is, $\mu_k = \pm\mu$ for all k, for a choice of $\mu = \mu(j)$ such that $\mu^2 = \zeta$. Since $\bar{t}_j \in \bar{F}_{\alpha_j} \cap \bar{S}$, one has

$$\begin{aligned}
\bar{e} = s_{\alpha_j}(\bar{t}_j)\bar{t}_j &= \big[\mu_1,\, \ldots,\, \mu_{j-1},\, \mu_{j+1},\, \mu_j,\, \mu_{j+2},\, \ldots,\, \mu_n\big] \cdot \\
&\qquad\qquad \cdot \big[\mu_1,\, \ldots,\, \mu_{j-1},\, \mu_j,\, \mu_{j+1},\, \mu_{j+2},\, \ldots,\, \mu_n\big] \\
&= \big[\mu_1^2,\, \ldots,\, \mu_{j-1}^2,\, \mu_j\mu_{j+1},\, \mu_j\mu_{j+1},\, \mu_{j+2}^2,\, \ldots,\, \mu_n^2\big] \\
&= \big[\zeta,\, \ldots,\, \zeta,\, \mu_j\mu_{j+1},\, \mu_j\mu_{j+1},\, \zeta,\, \ldots,\, \zeta\big],
\end{aligned}$$

and therefore $\mu_j\mu_{j+1} = \zeta = \mu_j^2$, yielding $\mu_j = \mu_{j+1}$. Now, supposing that $n \geqslant 5$, let $\bar{t}_1 = \big[\lambda,\, \lambda,\, \lambda_3,\, \ldots,\, \lambda_n\big]$ and $\bar{t}_3 = \big[\omega_1,\, \omega_2,\, \omega,\, \omega,\, \omega_5,\, \ldots,\, \omega_n\big]$. Since $\ell_{1,3} = 2$, condition (♠) gives

$$\begin{aligned}
&\big[\lambda,\, \lambda,\, \lambda_3,\, \lambda_4,\, \lambda_5,\, \ldots,\, \lambda_n\big] \cdot \big[\omega_2,\, \omega_1,\, \omega,\, \omega,\, \omega_5,\, \ldots,\, \omega_n\big] = \\
&\qquad\qquad = \big[\omega_1,\, \omega_2,\, \omega,\, \omega,\, \omega_5,\, \ldots,\, \omega_n\big] \cdot \big[\lambda,\, \lambda,\, \lambda_4,\, \lambda_3,\, \lambda_5,\, \ldots,\, \lambda_n\big]
\end{aligned}$$

and consequently,

$$\big[\lambda\omega_2,\, \lambda\omega_1,\, \lambda_3\omega,\, \lambda_4\omega,\, \lambda_5\omega_5,\, \ldots,\, \lambda_n\omega_n\big] = \big[\lambda\omega_1,\, \lambda\omega_2,\, \lambda_4\omega,\, \lambda_3\omega,\, \lambda_5\omega_5,\, \ldots,\, \lambda_n\omega_n\big],$$

which implies $\lambda_3\omega = \lambda_4\omega$ and $\lambda_3 = \lambda_4$. In the same way, with $\ell_{1,j} = 2$ for $j \geqslant 4$, we get $\bar{t}_1 = \big[\lambda,\, \lambda,\, \lambda_3,\, \ldots,\, \lambda_3\big]$ with $\lambda_3 = \pm\lambda$. We proceed by considering $\bar{t}_2 = \big[\nu_1,\, \nu,\, \nu,\, \nu_4,\, \ldots,\, \nu_n\big]$. Since $\ell_{1,2} = 3$, condition (♠) now yields

$$\big[\lambda\lambda_3\nu,\, \lambda^2\nu_1,\, \lambda\lambda_3\nu,\, \lambda_3^2\nu_4,\, \ldots,\, \lambda_3^2\nu_4\big] = \big[\lambda\nu\nu_1,\, \lambda_3\nu^2,\, \lambda\nu\nu_1,\, \lambda_3\nu_4^2,\, \ldots,\, \lambda_3\nu_4^2\big].$$

In particular, we extract $\lambda\lambda_3\nu\zeta = \lambda\nu\nu_1$ and $\lambda_3^2\nu_4\zeta = \lambda_3\nu_4^2$ for some $\zeta \in C_K$, and therefore $\nu_1 = \lambda_3\zeta = \nu_4$. So, we conclude that $\bar{t}_2 = \big[\nu_1,\, \nu,\, \nu,\, \nu_1,\, \ldots, \nu_1\big]$. More generally, by the same arguments, we get

$$\bar{t}_j = \big[\mu,\, \ldots,\, \mu,\, \mu_j = \epsilon_j\mu,\, \mu_{j+1} = \epsilon_j\mu,\, \mu,\, \ldots,\, \mu\big]$$

with $\mu = \mu(j)$ such that $\mu^2 \in C_K$, and $\epsilon_j = \pm 1$, for all j. Finally, from the explicit description of $\bar{k}_{\alpha_j}$, choosing a section $\underline{s}$ (as in Proposition 2.8) amounts to selecting ϵ_j for each j (up to replacing $\mu = \mu(j)$ by $-\mu$). So, setting $\epsilon_j = 1$, we get, for every j, $\bar{t}_j = \big[\mu,\, \ldots,\, \mu\big]$ and $\bar{t}_j^2 = \bar{e}$; therefore, $\bar{t}_j$ is a central element. As a final step, applying condition (♠) with $\ell_{j,j+1} = 3$ yields $\bar{t}_j\bar{t}_{j+1}\bar{t}_j = \bar{t}_{j+1}\bar{t}_j\bar{t}_{j+1}$, so that $\bar{t}_j = \bar{t}_{j+1} =: \bar{z}$ for all $j \leqslant n-1$, where $\bar{z} \in \bar{G}$ is a central element of order dividing 2. Summarizing, we have constructed a section $\underline{s}$, whose image consists precisely of all the automorphisms $\psi \in \mathrm{Aut}(\bar{N}, \bar{T})$ determined by

$$\psi(\bar{q}_j) = \bar{z}\cdot\bar{q}_j\,, \quad \bar{z} \in Z(\bar{G})\,, \quad \bar{z}^2 = \bar{e}\,, \quad \text{for all } j\,. \tag{A$_\ell$}$$

This establishes the theorem for $n \geqslant 5$.

The low dimensional cases are treated case-by-case. The absence of fake Tits circle in $\mathrm{SU}(2)$, $\mathrm{SU}(3)$ and their quotients, implies that $H^1(W_{\bar{G}})$ is trivial. For $\mathrm{PSU}(4)$, one gets precisely one non-trivial element in $H^1(W_{\mathrm{PSU}(4)})$ with an explicit representative described in Corollary 3.1 (ii) below. This automorphism cannot be lifted to $\mathrm{SU}(4)$ and $\mathrm{SU}(4)/\mathbb{Z}/2$, for which (A_ℓ) also holds. ■

Keeping the same ordering for B , from this proof, we deduce the next corollary.

Corollary 3.1. *(i) For $\ell \geqslant 1$, let G be a Lie group of type A_ℓ such that $G \ncong \mathrm{PSU}(4)$, and possessing a central element z of order 2 . Then the only non-trivial class $[\psi] \in H^1(W_G)$ has a representative $\psi \in \mathrm{Aut}(N,T)$ defined by*

$$\psi(q_j) = z{\cdot}q_j \,, \text{ for all } j = 1, \ldots, \ell \,.$$

(ii) The unique non-trivial class $[\psi] \in H^1(W_{\mathrm{PSU}(4)})$ has a representative element $\psi \in \mathrm{Aut}(\bar{N},\bar{T})$ defined by

$$\begin{cases} \psi(\bar{q}_1) = \big[\zeta,\, \zeta,\, \zeta,\, -\zeta\big] \cdot \bar{q}_1 \\ \psi(\bar{q}_2) = \big[-\zeta,\, \zeta,\, \zeta,\, \zeta\big] \cdot \bar{q}_2 \\ \psi(\bar{q}_3) = \big[\zeta,\, -\zeta,\, \zeta,\, \zeta\big] \cdot \bar{q}_3 \,, \end{cases}$$

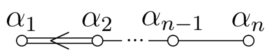
where $\zeta := e^{2\pi i/8}$ and the 4-tuples indicate diagonal matrices.

Proof of the Main Theorem for the type B_ℓ , $\ell \geqslant 2$.

Put $n = \ell$. The standard maximal torus in $G = \mathrm{Spin}(2n+1)$, considered, as usual, as a subset of the Clifford algebra $\mathrm{Cliff}(\mathbb{R}^{2n+1})$, is

$$T = \big\{(\cos x_1 + e_1 e_2 \sin x_1)\cdots(\cos x_n + e_{2n-1}e_{2n} \sin x_n) \,\big|\, x_k \in \mathbb{R} \text{ for all } k\big\} \,.$$

The center is given by $Z = \{\pm 1\} \cong \mathbb{Z}/2$. The Dynkin diagram is



The reflection s_{α_1} changes the sign of x_1 ; for $j \geqslant 2$, the reflection s_{α_j} permutes x_{j-1} and x_j . The subgroup T_j is given by

$$T_1 = \big\{(\cos x + e_1 e_2 \sin x) \,\big|\, x \in \mathbb{R}\big\}$$

for $j = 1$, and, for $j \geqslant 2$, by

$$T_j = \big\{(\cos x + e_{2j-3}e_{2j-2} \sin x){\cdot}(\cos x - e_{2j-1}e_{2j} \sin x) \,\big|\, x \in \mathbb{R}\big\} \,.$$

Therefore the non-trivial element in $\bar{F}_{\alpha_j} \cap \bar{S} \cap \bar{T}_j$, is, for $j = 1$,

$$\bar{k}_{\alpha_1} = \begin{cases} -1\,, & \text{if } K = \{e\}\,, \text{ that is, } \bar{G} = G = \mathrm{Spin}(2n+1) \\ \big[e_1 e_2\big]\,, & \text{if } K = Z\,, \text{ that is, } \bar{G} \cong \mathrm{SO}(2n+1) \end{cases}$$

and $\bar{k}_{\alpha_j} = \left[-e_{2j-3}e_{2j-2}e_{2j-1}e_{2j}\right]$, for $j = 2, \ldots, n$. This dichotomy will force us to distinguish both cases in the determination of the image under ψ of the first element of the Tits configuration. Let

$$T_I = \left\{t \in T \,\middle|\, t^2 = \pm 1\right\} = \left\{t = \pm \prod_{k \in J} e_{2k-1}e_{2k} \,\middle|\, J \subseteq \{1, 2, \ldots, n\}\right\}$$

denote the subgroup of elements in T whose square is central. An element $t \in T_I$ is a (reduced) word in the $e_{2k-1}e_{2k}$'s; we denote it as $t = \mu(\mu_1, \ldots, \mu_n) = \mu x$, where $\mu = \pm 1$, and, for $k = 1, \ldots, n$,

$$\mu_k = \begin{cases} \ \ 1\,, & \text{if } e_{2k-1}e_{2k} \text{ does not appear in } t \\ -1\,, & \text{if } e_{2k-1}e_{2k} \text{ does appear in } t\,. \end{cases}$$

For example, $t = -e_1e_2e_5e_6$ is encoded by $t = -(-1, 1, -1, 1, 1, \ldots, 1)$. In this notation, the product of two elements $\mu x,\ \lambda y \in T_I$, with $y = (\lambda_1, \ldots, \lambda_n)$, reads

$$(\mu x) \cdot (\lambda y) = (-1)^{|xy|} \mu\lambda xy\,,$$

where $|xy| := \left|\{k \,|\, \mu_k = \lambda_k = -1\}\right|$, and all the products on the right-hand side are performed in the cyclic group with two elements. The action of W is then as follows: s_{α_1} changes the leading sign of elements in which e_1e_2 appears (in particular it acts trivially on $\bar{S}$ for $\bar{G} \cong \mathrm{SO}(2n+1)$); for $j \geqslant 2$, s_{α_j} permutes the $(j-1)$-st and the j-th coordinates. Let $\bar{t} = \left[\mu(\mu_1, \ldots, \mu_n)\right]$ and $\bar{s} = \left[\lambda(\lambda_1, \ldots, \lambda_n)\right]$ be two elements with $s, t \in T_I$. Obviously, if $\bar{t} = \bar{s}$ and $\mu_k = \lambda_k$ for some k , then $\mu_k = \lambda_k$ for all k , that is, $t = s$ for $G = \mathrm{Spin}(2n+1)$, and $t = \pm s$ otherwise.

We are now ready to start the computations. To be able to mimic the arguments for the type A_ℓ , we suppose that $n \geqslant 5$. For $j = 2, \ldots, n$, the situation is almost the same as that for $\mathrm{SU}(n)$ (for $\mathrm{Spin}(2n+1)$, the only possible non-trivial K is the whole center, but the rank is $n = \mathrm{rank}(\mathrm{SU}(n)) + 1$, which does not alter the argument), and our encoding of elements in T_I allows to follow the argument for the type A_ℓ . We immediately get that

$$\bar{t}_j = \left[\mu(\mu_1, \ldots, \mu_1)\right] \quad \text{and} \quad \bar{t}_j^{\,2} = \bar{e}\,, \tag{$\star$}$$

with $\mu = \mu(j)$ and $\mu_1 = \mu_1(j)$, for all $j \geqslant 2$. From here on, it is easier to treat $\mathrm{Spin}(2n+1)$ and $\mathrm{SO}(2n+1)$ separately. (It might seem to be a waste to have tried to treat both together, but the encoding we have introduced here will also be useful for the type D_ℓ , where we will be able to treat all groups together more efficiently.)

We start with $\mathrm{Spin}(2n+1)$. First, from ($\star$), we deduce, for $j \geqslant 2$, that $t_j = t$ and $t \in \{\pm 1, \pm a = \pm e_1e_2 \cdots e_{2n-1}e_{2n}\}$ for n even; and $t_j = t$ and $t \in Z(G) = \{\pm 1\}$ for n odd. Now, let $t_1 = \nu(\nu_1, \ldots, \nu_n)$ be such that $t_1^2 = 1$ (that is, an even number of ν_k's are equal to -1). The condition $t_1 = s_{\alpha_1}(t_1)$ implies that e_1e_2 cannot appear in t_1 , so that $t_1 = \nu(1, \nu_2, \ldots, \nu_n)$. Applying condition (♣) with $\ell_{1,3} = 2$ yields

$$\begin{aligned} \pm 1 = t \cdot (\pm t) &= \nu(1, \nu_2, \nu_3, \nu_4, \ldots, \nu_n) \cdot \nu(1, \nu_3, \nu_2, \nu_4, \ldots, \nu_n) \\ &= (1, \nu_2\nu_3, \nu_2\nu_3, 1, \ldots, 1)\,, \end{aligned}$$

which implies the equality $\nu_2 = \nu_3$. Repeating the argument successively for $j = 4, \ldots, n$, we get $t_1 = \nu(1, \nu_2, \ldots, \nu_2)$, and therefore, $t_1 \in Z(G) = \{\pm 1\}$ for n even, and $t_1 \in \{\pm 1, \pm b = \pm e_3 e_4 \cdots e_{2n-1} e_{2n}\}$ for n odd. We then distinguish two subcases according to the parity of n. Suppose first that n is even. If $t_j = t = \pm a$ for all $j \geqslant 2$, then, condition (♣) with $\ell_{1,3} = 2$ gives

$$1 = t_1 \cdot s_{\alpha_3}(t_1) = t_3 \cdot s_{\alpha_1}(t_3) = t \cdot (-t) = -1\,,$$

a contradiction showing that we have necessarily $t_j = \pm 1$ for $j \geqslant 2$. Since we have $t_1 = 1 \sim -1$, and as (♠) for $\ell_{1,2} = 4$ holds even for $t_2 = -1$ (in contrast to the A_ℓ case), we have in fact constructed a section $\underline{s}$, whose image consists precisely of all the automorphisms $\psi \in \mathrm{Aut}(N, T)$ determined by

$$\psi(q_1) = q_1 \quad \text{and} \quad \psi(q_j) = z \cdot q_j\,, \text{ for } j = 2, \ldots, n\,,$$

with $z \in Z(\mathrm{Spin}(2n+1))$. Secondly, suppose that n is odd. Then $t_j = \pm 1$ holds for $j \geqslant 2$. Two straightforward computations, for $\ell_{1,2} = 4$ and $\ell_{1,j} = 2$ with $j \geqslant 3$, show that (♣) holds for all possible values of t_1. Since $t_1 = b \sim -b$, we have a section $\underline{s}$, whose image consists precisely of all the automorphisms $\psi \in \mathrm{Aut}(N, T)$ determined by

$$\psi(q_1) = t_1 \cdot q_1 \quad \text{and} \quad \psi(q_j) = z \cdot q_j\,, \text{ for } j = 2, \ldots, n\,,$$

with $t_1 \in \{1, e_3 e_4 \cdots e_{2n-1} e_{2n}\}$, and $z \in Z(\mathrm{Spin}(2n+1))$. This completes the proof for $\mathrm{Spin}(2n+1)$, with $n \geqslant 5$.

We next treat $\mathrm{SO}(2n+1)$. Since $\bar{t} = \big[\mu(\mu_1, \ldots, \mu_n)\big] = \pm\mu(\mu_1, \ldots, \mu_n)$, we can drop the leading sign and write $\bar{t} = \big[\mu_1, \ldots, \mu_n\big]$ (which is a diagonal matrix $\mathrm{diag}(d_1, \ldots, d_{2n}, 1)$ in $\mathrm{SO}(2n+1)$, with each pair (d_{2k-1}, d_{2k}) either equal to $(1, 1)$ or to $(-1, -1)$; for example $\big[1, \ldots, 1, \mu_k = -1, 1, \ldots, 1\big]$ designates the matrix $\mathrm{diag}(1, \ldots, 1, d_{2k-1} = -1, d_{2k} = -1, 1, \ldots, 1)$. From $(\star)$, we know that

$$\bar{t}_j = \bar{t} \text{ and } \bar{t} \in \Big\{\bar{1},\, \bar{a} = \big[e_1 e_2 \cdots e_{2n-1} e_{2n}\big] = \big[-1, \ldots, -1\big]\Big\}$$

for $j \geqslant 2$. For $\bar{t}_1 = \big[\nu_1, \ldots, \nu_n\big]$, the equality $\bar{t}_1 = s_{\alpha_1}(\bar{t}_1)$ does not impose any condition, because s_{α_1} acts trivially on $\bar{t}_1$. Then, following the argument for $\mathrm{Spin}(2n+1)$, condition (♣) for $\ell_{1,j} = 2$ with $j \neq 2$ implies that

$$\bar{t}_1 = \big[\nu_1, \nu_2, \ldots, \nu_2\big] \sim \big[\nu_2, \nu_2, \ldots, \nu_2\big]\,,$$

in other words, we can choose $\bar{t}_1 \in \{\bar{1}, \bar{a}\}$. Finally, the same argument as the one used for $\mathrm{Spin}(2n+1)$ with n odd, shows that there is a section $\underline{s}$, whose image consists precisely of the automorphisms $\psi \in \mathrm{Aut}(\bar{N}, \bar{T})$ determined by

$$\psi(\bar{q}_1) = \bar{t}_1 \cdot \bar{q}_1 \quad \text{and} \quad \psi(\bar{q}_j) = \bar{t} \cdot \bar{q}_j\,, \text{ for } j = 2, \ldots, n\,,$$

with $\bar{t}_1, \bar{t} \in \{\bar{1}, \bar{a}\}$. This establishes the theorem for $n \geqslant 5$.

The low dimensional cases are treated case-by-case. The previous results hold except for $\mathrm{SO}(5)$. In this case, as $\bar{t}_2 = \bar{1} \sim \bar{a} = \big[e_1 e_2 e_3 e_4\big]$, we only get one non-trivial element in $H^1(W_{\mathrm{SO}(5)})$, with a representative automorphism ψ defined by $\psi(\bar{q}_1) = \big[e_1 e_2 e_3 e_4\big] \cdot \bar{q}_1$ and $\psi(\bar{q}_2) = \bar{q}_2$. ∎

Before we state a direct corollary of this proof, recall, for $\bar{G} = \mathrm{SO}(2n+1)$, that the center of the normalizer $\bar{N}$ is $Z(\bar{N}) = \{\mathbb{I}_{2n+1}, \mathrm{diag}(-1, \ldots, -1, 1)\}$, see for instance [17]. (In the statement, we order B as in the proof above, and still consider $\mathrm{Spin}(2n+1)$ as a subset of the Clifford algebra $\mathrm{Cliff}(\mathbb{R}^{2n+1})$.)

Corollary 3.2. *(i) For* $4m+1 \geqslant 5$ *, every class* $[\psi] \in H^1(W_{\mathrm{Spin}(4m+1)})$ *has a representative element* $\psi \in \mathrm{Aut}(N, T)$ *defined by*

$$\begin{cases} \psi(q_1) = q_1 \,, \\ \psi(q_j) = z \cdot q_j \,, \ \textit{for } j = 2, \ldots, 2m \,, \end{cases}$$

where z *is a central element in* $\mathrm{Spin}(4m+1)$.

(ii) For $4m+3 \geqslant 7$ *, every class* $[\psi] \in H^1(W_{\mathrm{Spin}(4m+3)})$ *has a representative element* $\psi \in \mathrm{Aut}(N, T)$ *defined by*

$$\begin{cases} \psi(q_1) = t \cdot q_1 \,, \\ \psi(q_j) = z \cdot q_j \,, \ \textit{for } j = 2, \ldots, 2m+1 \,, \end{cases}$$

where z *is central in* $\mathrm{Spin}(4m+1)$ *, and* $t \in \{1, e_3 e_4 \cdots e_{4m+1} e_{4m+2}\}$.

(iii) The only non-trivial class $[\psi] \in H^1(W_{\mathrm{SO}(5)})$ *has a representative element* $\psi \in \mathrm{Aut}(\bar N, \bar T)$ *defined by*

$$\begin{cases} \psi(\bar q_1) = \bar z \cdot \bar q_1 \\ \psi(\bar q_2) = \bar q_2 \,, \end{cases}$$

where $\bar z = \mathrm{diag}(-1, -1, -1, -1, 1)$ *is the central element of order* 2 *in* $\bar N$.

(iv) For $n \geqslant 3$ *, every class* $[\psi] \in H^1(W_{\mathrm{SO}(2n+1)})$ *has a representative element* $\psi \in \mathrm{Aut}(\bar N, \bar T)$ *defined by*

$$\begin{cases} \psi(\bar q_1) = \bar z_1 \cdot \bar q_1 \\ \psi(\bar q_j) = \bar z \cdot \bar q_j \,, \ \textit{for } j = 2, \ldots, n \,, \end{cases}$$

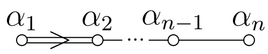
where $\bar z_1$ *and* $\bar z$ *are central elements in* $\bar N$.

Proof of the Main Theorem for the type C_ℓ **,** $\ell \geqslant 3$ **.**

Put $n = \ell$. The standard maximal torus T in $G = \mathrm{Sp}(n) \subset \mathrm{GL}_n(\mathbb{H})$ consists of the subgroup of diagonal matrices

$$T = \left\{ \mathrm{diag}(z_1, \ldots, z_n) \,\middle|\, z_k \in \mathbb{S}^1 \right\}.$$

We denote an element of T simply by $t = (z_1, \ldots, z_n)$. The center is given by $Z = \{\pm(1, \ldots, 1)\} \cong C_2$. The Dynkin diagram is



The reflection s_{α_1} takes z_1 to z_1^{-1} ; for $j \geqslant 2$, the reflection s_{α_j} exchanges the entries z_{j-1} and z_j on the diagonal. The subgroup T_j is given by

$$T_1 = \left\{ (z, 1, \ldots, 1) \,\middle|\, z \in \mathbb{S}^1 \right\}$$

for $j = 1$, and, for $j \geqslant 2$, by

$$T_j = \left\{ (1, \ldots, 1, z_j = z, z_{j+1} = z^{-1}, 1, \ldots, 1) \,\middle|\, z \in \mathbb{S}^1 \right\}.$$

Therefore, the non-trivial element in $\bar F_{\alpha_j} \cap \bar S \cap \bar T_j$ is $\bar k_{\alpha_1} = \left[-1, 1, \ldots, 1\right]$ for $j = 1$, and

$$\bar k_{\alpha_j} = \left[1, \ldots, z_{j-1} = -1, z_j = -1, 1, \ldots, 1\right]$$

for $j = 2, \dots, n$.

For $t,\, s \in T$, one obviously has $\bar{t} = \bar{s}$ if and only if $t = s$ for $\bar{G} = \mathrm{Sp}(n)$, and $t = \pm s$ for $\bar{G} = \mathrm{PSp}(n)$. For $j = 2, \dots, n$, the situation is almost the same as that of $\mathrm{SU}(n)$ (for $\mathrm{Sp}(n)$, the only possible non-trivial K is the whole center, but the rank is $n = \mathrm{rank}(\mathrm{SU}(n)) + 1$, which does not alter the argument). Consequently, for $n \geqslant 5$, we immediately deduce that

$$\bar{t}_j = \bar{z}\,, \quad \bar{z} \in Z(\bar{G})\,, \quad \bar{z}^2 = \bar{e}\,, \quad \text{for } j \geqslant 2\,.$$

It remains to determine the possible values of $\bar{t}_1 = \big[\nu_1, \dots, \nu_n\big]$. Now, from $\bar{t}_1 \in \bar{F}_{\alpha 1} \cap \bar{S}$, we easily deduce that $\nu_k = \pm 1$, for all k . Applying condition (♠) with $\ell_{1,j} = 2$ and $j \geqslant 3$, we get $\bar{t}_1 = \big[\nu_1, \nu, \dots, \nu\big]$. We then check that (♠) for $\ell_{1,2} = 4$ holds, imposing no further conditions on $\bar{t}_1$ (in contrast to the A_ℓ case). Finally, from the explicit description of $\bar{k}_{\alpha_1}$ given above, we have $\big[\nu_1, \nu, \dots, \nu\big] \sim \big[\nu, \nu, \dots, \nu\big]$, and therefore, we can choose $\bar{t}_1 = \big[\nu, \dots, \nu\big]$, that is, the element $\bar{t}_1$ is central. For $n \geqslant 5$, we have constructed a section $\underline{s}$, whose image consists precisely of all the automorphisms $\psi \in \mathrm{Aut}(\bar{N}, \bar{T})$ determined by

$$\psi(\bar{q}_1) = \bar{z}_1 {\cdot} \bar{q}_1 \quad \text{and} \quad \psi(\bar{q}_j) = \bar{z} {\cdot} \bar{q}_j\,, \ \text{ for } j = 2, \dots, n\,, \tag{C_ℓ}$$

with $\bar{z}_1,\, \bar{z} \in Z(\bar{G})$. This gives the conclusion of the theorem for $n \geqslant 5$.

The low dimensional cases are treated case-by-case. One gets a non-trivial element in $H^1(W_{\mathrm{PSp}(3)})$, respectively $H^1(W_{\mathrm{PSp}(4)})$, with an explicit representative given as in Corollary 3.3 (ii) below. These automorphisms cannot be lifted to $\mathrm{Sp}(3)$, respectively $\mathrm{Sp}(4)$, for which equation (C_ℓ) also holds. ■

Keeping the same ordering for B , from this proof, we deduce the next corollary.

Corollary 3.3. *(i) For $n \geqslant 3$, every class $[\psi] \in H^1(W_{\mathrm{Sp}(n)})$ has a representative element $\psi \in \mathrm{Aut}(N, T)$ defined by*

$$\begin{cases} \psi(q_1) = z_1 {\cdot} q_1\,, \\ \psi(q_j) = z {\cdot} q_j\,, \ \text{for } j = 2, \dots, n\,, \end{cases}$$

where z_1 and z are central elements in $\mathrm{Sp}(n)$.

(ii) The only non-trivial class $[\psi]$ in $H^1(W_{\mathrm{PSp}(3)})$ and in $H^1(W_{\mathrm{PSp}(4)})$ has a representative $\psi \in \mathrm{Aut}(\bar{N}, \bar{T})$ defined by

$$\begin{cases} \psi(\bar{q}_1) = \big[1,\, 1,\, -1\big] \cdot \bar{q}_1 \\ \psi(\bar{q}_2) = \big[i,\, i,\, i\big] \cdot \bar{q}_2 \\ \psi(\bar{q}_3) = \big[i,\, i,\, i\big] \cdot \bar{q}_3 \end{cases} \qquad \text{and} \qquad \begin{cases} \psi(\bar{q}_1) = \bar{q}_1 \\ \psi(\bar{q}_2) = \big[1,\, 1,\, 1,\, -1\big] \cdot \bar{q}_2 \\ \psi(\bar{q}_3) = \big[-1,\, 1,\, 1,\, 1\big] \cdot \bar{q}_3 \\ \psi(\bar{q}_4) = \big[-1,\, 1,\, 1,\, 1\big] \cdot \bar{q}_4 \end{cases}$$

respectively, where the 3- and 4-tuples indicate diagonal matrices over $\mathbb{H}$.

Remark 3.4. There is a more efficient way to prove the Main Theorem for the group $\mathrm{Sp}(n)$ (but *not* for $\mathrm{PSp}(n)$): it is well-known that the W-modules T for $\mathrm{Sp}(n)$ and $\mathrm{SO}(2n+1)$ are isomorphic (see [4]); as a consequence, one has an isomorphism $H^1(W_{\mathrm{Sp}(n)}) \cong H^1(W_{\mathrm{SO}(2n+1)})$.

Proof of the Main Theorem for the type D_ℓ **,** $\ell \geqslant 4$ **.**

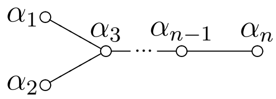
Put $n = \ell$. The standard maximal torus in $G = \mathrm{Spin}(2n)$, considered, as usual, as a subset of the Clifford algebra $\mathrm{Cliff}(\mathbb{R}^{2n})$, is

$$T = \left\{ (\cos x_1 + e_1 e_2 \sin x_1) \cdots (\cos x_n + e_{2n-1} e_{2n} \sin x_n) \,\middle|\, x_k \in \mathbb{R}\,,\ \forall\, k \right\}.$$

The center is given by

$$Z = \left\{ \pm 1,\ \pm a = \pm e_1 e_2 \cdots e_{2n-1} e_{2n} \right\} \cong \begin{cases} \mathbb{Z}/2 \oplus \mathbb{Z}/2\,, & n \text{ even} \\ \mathbb{Z}/4\,, & n \text{ odd.} \end{cases}$$

The Dynkin diagram is



The reflection s_{α_1} permutes x_1 and x_2 , changing both signs; for $j \geqslant 2$, the reflection s_{α_j} permutes x_{j-1} and x_j . The subgroup T_j is given by

$$T_1 = \left\{ (\cos x + e_1 e_2 \sin x) \cdot (\cos x + e_3 e_4 \sin x) \,\middle|\, x \in \mathbb{R} \right\}$$

for $j = 1$, and, for $j \geqslant 2$, by

$$T_j = \left\{ (\cos x + e_{2j-3} e_{2j-2} \sin x) \cdot (\cos x - e_{2j-1} e_{2j} \sin x) \,\middle|\, x \in \mathbb{R} \right\}.$$

Therefore, the non-trivial element in $\bar{F}_{\alpha_j} \cap \bar{S} \cap \bar{T}_j$ is $\bar{k}_{\alpha_1} = \left[e_1 e_2 e_3 e_4 \right]$ for $j = 1$, and

$$\bar{k}_{\alpha_j} = \left[-e_{2j-3} e_{2j-2} e_{2j-1} e_{2j} \right],\ \text{for } j = 2,\ \ldots,\ n\,.$$

We start with some easy observations about the possible elements t_j such that $\bar{t}_j^{\,2} = \bar{1}$. We will consider two different cases for t_j . We first define

$$\begin{aligned} T_I &:= \left\{ t \in T \,\middle|\, t^2 = \pm 1 \right\} = \left\{ t = \pm \textstyle\prod_{k \in J} e_{2k-1} e_{2k} \,\middle|\, J \subseteq \{1,\, 2,\, \ldots,\, n\} \right\} \\ T_{II} &:= \left\{ t \in T \,\middle|\, t^2 = \pm a \right\} = \left\{ t = \pm \tfrac{1}{\sqrt{2}} (1 \pm e_1 e_2) \cdots \tfrac{1}{\sqrt{2}} (1 \pm e_{2n-1} e_{2n}) \right\}. \end{aligned}$$

Clearly T_I is a subgroup, and so is $T_I \amalg T_{II}$. Since T_I and T_{II} are both stable under multiplication by a central element, we can define the *type* of $\bar{t}$ saying that $\bar{t}$ is of type T_I , respectively of type T_{II} , if $t \in T_I$, respectively $t \in T_{II}$. It is obvious that the action of W preserves the type and, for $\bar{t}_j,\ \bar{t}'_j \in \bar{F}_{\alpha_j} \cap \bar{S}$, that $\bar{t}_j \sim \bar{t}'_j$ if and only if $\bar{t}_j$ and $\bar{t}'_j$ have the same type.

Claim. The elements $\bar{t}_j$, with $j = 1,\ \ldots,\ n$, are all of the same type.

It suffices to check this for the representative elements t_j's. By connectedness of the Dynkin diagram, it is enough to check that elements corresponding to two neighboring roots α_j and α_{j+1} have the same type. Now $\ell_{j,j+1} = 3$, so that

$$w_{\alpha_j \alpha_{j+1}}(t_j) = t_j \cdot s_{\alpha_{j+1}}(t_j) \cdot s_{\alpha_j} s_{\alpha_{j+1}}(t_j)$$

has the same type as t_j , because T_{II} is of index 2 in $T_I \amalg T_{II}$. Symmetrically $w_{\alpha_{j+1} \alpha_j}(t_{j+1})$ has the same type as t_{j+1} . Therefore, condition (♣) tells that t_j and t_{j+1} have the same type, proving the claim.

We proceed with the description of an encoding of the elements in T_I and T_{II} , which will allow to reduce the present proof to mimicking that for the type A_ℓ . For T_I , we keep the one introduced in the proof for the type (B_ℓ) . The action of W is then as follows: s_{α_1} permutes the first two coordinates, but we loose control of the leading sign if the element is not given explicitly (this minor drawback has no effect on the forthcoming computations); for $j \geqslant 2$, s_{α_j} permutes the $(j-1)$-st and the j-th coordinates. Let $\bar{t} = \big[\mu(\mu_1, \ldots, \mu_n)\big]$ and $\bar{s} = \big[\lambda(\lambda_1, \ldots, \lambda_n)\big]$ be two elements in T_I . A quick checking shows that the following useful property still holds: if $\bar{t} = \bar{s}$ and $\mu_k = \lambda_k$ for some k , then $\mu_k = \lambda_k$ for all k , so that $t = \pm s$.

For the second type, an element

$$t = \varepsilon \tfrac{1}{\sqrt{2}}(1 + \varepsilon_1 e_1 e_2) \cdots \tfrac{1}{\sqrt{2}}(1 + \varepsilon_n e_{2n-1} e_{2n}) \in T_{II} \,,$$

with $\varepsilon, \varepsilon_k = \pm 1$ for $k = 1, \ldots, n$, is written as $t = \varepsilon(\varepsilon_1, \ldots, \varepsilon_n) = \varepsilon u$. Not bothering about the leading sign, the product

$$(\varepsilon u) \cdot (\delta v) = \pm(\varepsilon_1 * \delta_1, \ldots, \varepsilon_n * \delta_n)$$

of two elements in T_{II} is an element in T_I , with

$$\varepsilon_k * \delta_k := \begin{cases} \;\;1 \,, \text{ if } \varepsilon_k \neq \delta_k \\ -1 \,, \text{ if } \varepsilon_k = \delta_k \,. \end{cases}$$

The reflection s_{α_1} permutes the first two coordinates, and changes both signs; for $j \geqslant 2$, as before, s_{α_j} permutes the $(j-1)$-st and the j-th coordinates.

We are now ready to start the computations; we suppose that $n \geqslant 5$. According to the above claim, we begin with the case in which all t_j's are of type T_I . For $\bar{t}_j$ with $j \geqslant 2$, our encoding of elements in T_I allows to follow the arguments for the type A_ℓ , with only minor modifications. We get that

$$\bar{t}_j = \bar{z} \,, \quad \bar{z} \in Z(\bar{G}) \,, \quad \bar{z}^2 = \bar{e} \,, \quad \text{for } j \geqslant 2 \,.$$

Now let $\bar{t}_1 = \big[\nu(\nu_1, \ldots, \nu_n)\big]$. Then $\bar{t}_1 = s_{\alpha_1}(\bar{t}_1)$ is equivalent to

$$\big[\nu(\nu_1, \nu_2, \nu_3, \ldots, \nu_n)\big] = \big[\pm\nu(\nu_2, \nu_1, \nu_3, \ldots, \nu_n)\big] \,,$$

which implies that $\nu_1 = \nu_2$. Applying condition ($\spadesuit$) with $\ell_{1,4} = 2$ yields

$$\begin{aligned} \bar{e} = \bar{z}^2 &= \big[\nu(\nu_1, \nu_1, \nu_3, \nu_4, \ldots, \nu_n)\big] \cdot \big[\nu(\nu_1, \nu_3, \nu_1, \nu_4, \ldots, \nu_n)\big] \\ &= \big[1, \nu_1\nu_3, \nu_1\nu_3, 1, \ldots, 1\big] \,, \end{aligned}$$

from which we deduce that $\nu_3 = \nu_1$. Repeating the argument for $j = 5, \ldots, n$, we get that $\bar{t}_1 = \big[\nu(\nu_1, \ldots, \nu_1)\big]$, so that $\bar{t}_1$ is central. Finally, condition ($\spadesuit$) with $\ell_{1,3} = 3$ implies that $\bar{t}_1 = \bar{t}_3 = \bar{z}$, and therefore

$$\bar{t}_j = \bar{z} \,, \quad \bar{z} \in Z(\bar{G}) \,, \quad \bar{z}^2 = \bar{e} \,, \quad \text{for all } j \,.$$

This concludes the first case.

Still for $n \geqslant 5$, it remains to treat the second case, in which all $\bar{t}_j$'s are of type T_{II} ; we show that this case is in fact impossible. Let $\bar{t}_1 = \big[\varepsilon(\varepsilon_1, \ldots, \varepsilon_n)\big]$,

$\bar{t}_2 = \big[\delta(\delta_1, \ldots, \delta_n)\big]$ and $\bar{t}_4 = \big[\theta(\theta_1, \ldots, \theta_n)\big]$ be all of type T_{II}. Then $\bar{t}_1 s_{\alpha_1}(\bar{t}_1) = \bar{e}$ is equivalent to

$$\begin{aligned}\bar{e} = \big[-1, \ldots, -1\big] &= \big[\varepsilon(\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots, \varepsilon_n)\big] \cdot \big[\varepsilon(-\varepsilon_2, -\varepsilon_1, \varepsilon_3, \ldots, \varepsilon_n)\big] \\ &= \big[\pm(-\varepsilon_1 * \varepsilon_2, -\varepsilon_1 * \varepsilon_2, -1, -1, \ldots, -1)\big],\end{aligned}$$

so that $\varepsilon_1 * \varepsilon_2 = 1$, that is, $\varepsilon_2 = -\varepsilon_1$. Applying (♣) with $\ell_{1,4} = 2$ gives

$$\begin{aligned}&\big[\pm(-1, -1, \varepsilon_3 * \varepsilon_4, \varepsilon_3 * \varepsilon_4, -1, \ldots, -1)\big] = \\ &\qquad = \big[\pm(-\theta_1 * \theta_2, -\theta_1 * \theta_2, -1, -1, -1, \ldots, -1)\big],\end{aligned}$$

and thus, in particular, $\theta_2 = \theta_1$. Now, applying (♣) with $\ell_{2,4} = 2$ yields

$$\big[\pm(-1, -1, \delta_3 * \delta_4, \delta_3 * \delta_4, -1, \ldots, -1)\big] = \big[\pm(1, 1, -1, -1, -1, \ldots, -1)\big],$$

which is impossible, and rules out the second case.

Summarizing, for $n \geqslant 5$, we have constructed a section $\underline{s}$, whose image consists precisely of all the automorphisms $\psi \in \mathrm{Aut}(\bar{N}, \bar{T})$ determined by

$$\psi(\bar{q}_j) = \bar{z} \cdot \bar{q}_j, \text{ for all } j, \qquad (\mathrm{D}_\ell)$$

with $\bar{z} \in Z(\bar{G})$, of order dividing 2. This completes the proof for $n \geqslant 5$.

The case $n = 4$ is treated separately. For PSO(8), we get three non-trivial elements with explicit representative automorphisms given as in Corollary 3.5 (ii) below. These automorphisms cannot be lifted to Spin(8) and SO(8), for which equation (D_ℓ) also holds. ■

Keeping the same ordering for B, from this proof, we deduce the next corollary.

Corollary 3.5. *(i) For a Lie group G of type D_ℓ, $\ell \geqslant 4$, with $G \ncong \mathrm{PSO}(8)$, every non-trivial class $[\psi] \in H^1(W_G)$ has a representative $\psi \in \mathrm{Aut}(N, T)$ defined by*

$$\psi(q_j) = z \cdot q_j, \text{ for all } j = 1, \ldots, \ell,$$

where z is a non-trivial central element of order 2 in G.

(ii) The three non-trivial elements $[\psi_k] \in H^1(W_{\mathrm{PSO}(8)})$, with $1 \leqslant k \leqslant 3$, have, up to reordering, a representative element $\psi_k \in \mathrm{Aut}(\bar{N}, \bar{T})$ defined by

$$\begin{cases}\psi_1(\bar{q}_1) = \big[e_7 e_8\big] \cdot \bar{q}_1 \\ \psi_1(\bar{q}_2) = \big[e_7 e_8\big] \cdot \bar{q}_2 \\ \psi_1(\bar{q}_3) = \big[e_1 e_2\big] \cdot \bar{q}_3 \\ \psi_1(\bar{q}_4) = \big[e_1 e_2\big] \cdot \bar{q}_4,\end{cases} \quad \text{and} \quad \begin{cases}\psi_2(\bar{q}_1) = \bar{u} \cdot \bar{q}_1 \\ \psi_2(\bar{q}_2) = \bar{v} \cdot \bar{q}_2 \\ \psi_2(\bar{q}_3) = \bar{v} \cdot \bar{q}_3 \\ \psi_2(\bar{q}_4) = \bar{v} \cdot \bar{q}_4,\end{cases}$$

and $\psi_3 = \psi_2 \circ \psi_1$, respectively, where

$$\begin{aligned}\bar{u} &:= \big[\tfrac{1}{4}(1 + e_1 e_2)(1 - e_3 e_4)(1 - e_5 e_6)(1 + e_7 e_8)\big] \\ \bar{v} &:= \big[\tfrac{1}{4}(1 + e_1 e_2)(1 + e_3 e_4)(1 + e_5 e_6)(1 + e_7 e_8)\big].\end{aligned}$$

Remark 3.6. The elements $[e_1e_2]$, $[e_7e_8]$, $\bar{u}$ and $\bar{v}$ occurring in part (ii) above are explicitly given by the following classes of diagonal-by-block matrices in $\mathrm{PSO}(8)$ viewed as $\mathrm{SO}(8)/\{\pm 1\!\!1_8\}$:

$$\begin{aligned}
[e_1e_2] &= \pm\,\mathrm{diag}\left(\left(\begin{smallmatrix}-1&0\\0&-1\end{smallmatrix}\right),\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right),\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right),\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)\right)\\
[e_7e_8] &= \pm\,\mathrm{diag}\left(\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right),\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right),\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right),\left(\begin{smallmatrix}-1&0\\0&-1\end{smallmatrix}\right)\right)\\
\bar{u} &= \pm\,\mathrm{diag}\left(\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right),\left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right),\left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right),\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)\right)\\
\bar{v} &= \pm\,\mathrm{diag}\left(\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right),\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right),\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right),\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)\right).
\end{aligned}$$

4. Computations for the exceptional Lie groups

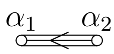
In this section, we present our computation of the group $H^1(W;T)$ for the seven exceptional Lie groups. The method adopted is to calculate the kernel of the map $\Theta\colon \bigoplus_{\alpha\in B} Q_\alpha/T_\alpha \longrightarrow \bigoplus_{\alpha<\beta} T$ of Proposition 2.6, invoking the isomorphism $H^1(W;T)\cong \mathrm{Ker}(\Theta)$ of this proposition.

We let ℓ be the rank of G and write $B=\{\alpha_1<\alpha_2<\ldots<\alpha_\ell\}$ with the same ordering as in the tables of Bourbaki [2]. We will work in the Lie algebra of T , modulo the integral lattice $\mathrm{Ker}(\exp)$. The latter coincides with the coroot lattice $\bigoplus_{1\leqslant j\leqslant \ell}\mathbb{Z}\cdot\alpha_j^\vee$ for simply connected groups, and in the other two cases, we will indicate a $\mathbb{Z}$-basis of it. We will also express the chosen $\mathbb{F}_2$-basis of $Q_{\alpha_j}/T_{\alpha_j}$ (ordered as indicated) in terms of the fixed $\mathbb{Z}$-basis of $\mathrm{Ker}(\exp)$; recall, by Lemma 2.4, that $Q_{\alpha_j}/T_{\alpha_j}$ indeed is an $\mathbb{F}_2$-vector space. In each case, we have computed that the $\mathbb{F}_2$-dimension of $Q_{\alpha_j}/T_{\alpha_j}$ is $\ell-2$ (see [18] for a general result). Since the range of Θ consists only of elements of order dividing 2 , we work in the $\mathbb{F}_2$-basis $\left\{\frac{1}{2}\alpha_1^\vee,\,\ldots,\,\frac{1}{2}\alpha_\ell^\vee\right\}$ of S for all the copies of T, that we order by the lexicographical ordering, that is,

$$\Big\{(\alpha_1,\,\alpha_2),\,(\alpha_1,\,\alpha_3),\,\ldots,\,(\alpha_1,\,\alpha_\ell),\,(\alpha_2,\,\alpha_3),\,\ldots,\,(\alpha_{\ell-1},\,\alpha_\ell)\Big\}.$$

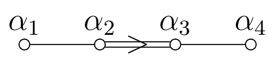
We finally indicate the corresponding matrix of Θ , as an $\left(\frac{\ell(\ell-1)}{2}\times\ell\right)$-by block-matrix. Each block is an $\big(\ell\times(\ell-2)\big)$-matrix. To simplify the notations, we denote by $\mathbb{O}$ the zero $\big(\ell\times(\ell-2)\big)$-matrix, and by E_{ij} the 'elementary $\big(\ell\times(\ell-2)\big)$-matrix' with all entries zero, *except* the $(i,\,j)$-th, which is 1 . We also let E_{ij}^{kl} , $E_{ij,kl}^{st}$ and $E_{ij,kl}^{st,uv}$ denote $E_{ij}+E_{kl}$, $E_{ij}+E_{kl}+E_{st}$ and $E_{ij}+E_{kl}+E_{st}+E_{uv}$ respectively. The matrix of Θ for the E-family has been obtained using a very simple algorithm written for Mathematica (see also [18]).

1) Type G_2 :

The Dynkin diagram is  . We have $Q_{\alpha_1}/T_{\alpha_1}=0$ and $Q_{\alpha_2}/T_{\alpha_2}=0$, so that Θ is an operator with the zero vector space as domain, and $\mathrm{Ker}(\Theta)=0$.

2) Type F_4 :

The Dynkin diagram is



The chosen basis of $Q_{\alpha_j}/T_{\alpha_j}$ is

$$\begin{aligned}
j=1 &: \ \tfrac{1}{2}\cdot\{\alpha_3^\vee,\, \alpha_4^\vee\}\,;\\
j=2 &: \ \tfrac{1}{2}\cdot\{\alpha_3^\vee,\, \alpha_4^\vee\}\,;\\
j=3 &: \ \tfrac{1}{2}\cdot\{\alpha_1^\vee,\, \alpha_2^\vee+\alpha_4^\vee\}\,;\\
j=4 &: \ \tfrac{1}{2}\cdot\{\alpha_1^\vee,\, \alpha_2^\vee\}\,.
\end{aligned}$$

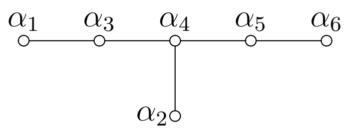
The matrix of Θ is

$$\left(\begin{array}{cccc}
E_{31}^{42} & E_{31}^{42} & \mathbb{O} & \mathbb{O}\\
E_{32} & \mathbb{O} & E_{12} & \mathbb{O}\\
E_{41} & \mathbb{O} & \mathbb{O} & E_{12}\\
\hline
\mathbb{O} & \mathbb{O} & \mathbb{O} & \mathbb{O}\\
\mathbb{O} & E_{41} & \mathbb{O} & E_{21}\\
\hline
\mathbb{O} & \mathbb{O} & E_{11,22}^{42} & E_{11,22}^{42}
\end{array}\right)$$

Row reduction (over $\mathbb{F}_2$) yields $\mathrm{Ker}(\Theta)=0$.

3) Type E_6 **:**

The Dynkin diagram is



The chosen basis of $Q_{\alpha_j}/T_{\alpha_j}$ is

$$\begin{aligned}
j=1 &: \ \tfrac{1}{2}\cdot\{\alpha_2^\vee,\, \alpha_4^\vee,\, \alpha_5^\vee,\, \alpha_6^\vee\}\,;\\
j=2 &: \ \tfrac{1}{2}\cdot\{\alpha_1^\vee,\, \alpha_3^\vee,\, \alpha_5^\vee,\, \alpha_6^\vee\}\,;\\
j=3 &: \ \tfrac{1}{2}\cdot\{\alpha_1^\vee+\alpha_4^\vee,\, \alpha_2^\vee,\, \alpha_5^\vee,\, \alpha_6^\vee\}\,;\\
j=4 &: \ \tfrac{1}{2}\cdot\{\alpha_1^\vee,\, \alpha_2^\vee+\alpha_3^\vee,\, \alpha_2^\vee+\alpha_5^\vee,\, \alpha_6^\vee\}\,;\\
j=5 &: \ \tfrac{1}{2}\cdot\{\alpha_1^\vee,\, \alpha_2^\vee,\, \alpha_3^\vee,\, \alpha_4^\vee+\alpha_6^\vee\}\,;\\
j=6 &: \ \tfrac{1}{2}\cdot\{\alpha_1^\vee,\, \alpha_2^\vee,\, \alpha_3^\vee,\, \alpha_4^\vee\}\,.
\end{aligned}$$

The matrix of Θ is

$$\left(\begin{array}{cccccc}
E_{22} & E_{12} & \mathbb{O} & \mathbb{O} & \mathbb{O} & \mathbb{O}\\
A & \mathbb{O} & B & \mathbb{O} & \mathbb{O} & \mathbb{O}\\
E_{41}^{43} & \mathbb{O} & \mathbb{O} & E_{12} & \mathbb{O} & \mathbb{O}\\
E_{52}^{54} & \mathbb{O} & \mathbb{O} & \mathbb{O} & E_{13} & \mathbb{O}\\
E_{63} & \mathbb{O} & \mathbb{O} & \mathbb{O} & \mathbb{O} & E_{13}\\
\hline
\mathbb{O} & E_{31} & E_{21} & \mathbb{O} & \mathbb{O} & \mathbb{O}\\
\mathbb{O} & C & \mathbb{O} & C & \mathbb{O} & \mathbb{O}\\
\mathbb{O} & E_{54} & \mathbb{O} & \mathbb{O} & E_{24} & \mathbb{O}\\
\mathbb{O} & E_{63} & \mathbb{O} & \mathbb{O} & \mathbb{O} & E_{24}\\
\hline
\mathbb{O} & \mathbb{O} & D & E & \mathbb{O} & \mathbb{O}\\
\mathbb{O} & \mathbb{O} & E_{51}^{54} & \mathbb{O} & E_{31}^{34} & \mathbb{O}\\
\mathbb{O} & \mathbb{O} & E_{63} & \mathbb{O} & \mathbb{O} & E_{31}^{34}\\
\hline
\mathbb{O} & \mathbb{O} & \mathbb{O} & F & G & \mathbb{O}\\
\mathbb{O} & \mathbb{O} & \mathbb{O} & E_{63} & \mathbb{O} & E_{42}^{43}\\
\hline
\mathbb{O} & \mathbb{O} & \mathbb{O} & \mathbb{O} & H & H
\end{array}\right)$$

where

$$A=\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\quad B=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\\1&0&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\quad C=\begin{pmatrix}1&0&0&0\\0&1&1&0\\0&1&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\quad D=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&1&1&0\\1&0&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$$

$$E = \begin{pmatrix} 1&0&0&0\\ 0&1&1&1\\ 0&1&0&0\\ 1&0&0&0\\ 0&0&1&0\\ 0&0&0&1 \end{pmatrix} \quad F = \begin{pmatrix} 1&0&0&0\\ 0&1&1&0\\ 0&1&0&0\\ 0&0&0&1\\ 0&0&1&0\\ 0&0&0&1 \end{pmatrix} \quad G = \begin{pmatrix} 1&0&0&0\\ 0&1&0&0\\ 0&0&1&0\\ 0&0&0&1\\ 0&1&1&0\\ 0&0&0&1 \end{pmatrix} \quad H = \begin{pmatrix} 1&0&0&0\\ 0&1&0&0\\ 0&0&1&0\\ 0&0&0&1\\ 0&0&0&0\\ 0&0&0&1 \end{pmatrix}$$

Row reduction yields $\mathrm{Ker}(\Theta) = 0$.

4) Type PE_6 **:**

The Dynkin diagram is as for E_6 . The integral lattice Ker(exp) is obtained from the coroot lattice by adjoining the vector $\frac{2\alpha_1^\vee}{3} + \alpha_2^\vee + \frac{4\alpha_3^\vee}{3} + 2\alpha_4^\vee + \frac{5\alpha_5^\vee}{3} + \frac{4\alpha_6^\vee}{3}$; its selected basis is

$$\begin{aligned} \nu_1^\vee &= \frac{\alpha_1^\vee}{3} + \frac{2\alpha_3^\vee}{3} + \frac{\alpha_5^\vee}{3} - \frac{\alpha_6^\vee}{3} & \nu_2^\vee &= \alpha_2^\vee \\ \nu_3^\vee &= \alpha_4^\vee & \nu_4^\vee &= -\frac{\alpha_1^\vee}{3} + \frac{\alpha_3^\vee}{3} + \frac{2\alpha_5^\vee}{3} + \frac{\alpha_3^\vee}{3} \\ \nu_5^\vee &= -\frac{\alpha_1^\vee}{3} + \frac{\alpha_3^\vee}{3} - \frac{\alpha_5^\vee}{3} + \frac{\alpha_6^\vee}{3} & \nu_6^\vee &= -\frac{\alpha_1^\vee}{3} + \frac{\alpha_3^\vee}{3} - \frac{\alpha_5^\vee}{3} - \frac{2\alpha_6^\vee}{3} . \end{aligned}$$

The chosen basis of $Q_{\alpha_j}/T_{\alpha_j}$ is

$$\begin{aligned} j = 1 :\ & \tfrac{1}{2} \cdot \{\nu_2^\vee, \nu_3^\vee, \nu_4^\vee + \nu_5^\vee, \nu_4^\vee + \nu_6^\vee\}; \\ j = 2 :\ & \tfrac{1}{2} \cdot \{\nu_1^\vee, \nu_4^\vee, \nu_5^\vee, \nu_6^\vee\}; \\ j = 3 :\ & \tfrac{1}{2} \cdot \{\nu_1^\vee + \nu_3^\vee, \nu_1^\vee + \nu_4^\vee, \nu_1^\vee + \nu_6^\vee, \nu_2^\vee\}; \\ j = 4 :\ & \tfrac{1}{2} \cdot \{\nu_1^\vee + \nu_2^\vee, \nu_1^\vee + \nu_4^\vee, \nu_5^\vee, \nu_6^\vee\}; \\ j = 5 :\ & \tfrac{1}{2} \cdot \{\nu_1^\vee + \nu_3^\vee, \nu_1 + \nu_5^\vee, \nu_2^\vee, \nu_6^\vee\}; \\ j = 6 :\ & \tfrac{1}{2} \cdot \{\nu_1^\vee + \nu_6^\vee, \nu_2^\vee, \nu_3^\vee, \nu_4^\vee\}. \end{aligned}$$

The matrix of Θ is

$$\left(\begin{array}{cccccc} E_{22} & A & \mathbb{O} & \mathbb{O} & \mathbb{O} & \mathbb{O} \\ B & \mathbb{O} & C & \mathbb{O} & \mathbb{O} & \mathbb{O} \\ E_{31,33}^{34} & \mathbb{O} & \mathbb{O} & A & \mathbb{O} & \mathbb{O} \\ E_{42,44}^{52,54} & \mathbb{O} & \mathbb{O} & \mathbb{O} & D & \mathbb{O} \\ E_{53,54}^{63,64} & \mathbb{O} & \mathbb{O} & \mathbb{O} & \mathbb{O} & E \\ \hline \mathbb{O} & F & E_{21} & \mathbb{O} & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & G & \mathbb{O} & H & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & I & \mathbb{O} & \mathbb{O} & E_{21} & \mathbb{O} \\ \mathbb{O} & J & \mathbb{O} & \mathbb{O} & \mathbb{O} & E_{23} \\ \hline \mathbb{O} & \mathbb{O} & K & L & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & E_{43}^{53} & \mathbb{O} & E_{14}^{54} & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & E_{51,52}^{61,62} & \mathbb{O} & \mathbb{O} & E_{13,14}^{53,54} \\ \hline \mathbb{O} & \mathbb{O} & \mathbb{O} & M & N & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & \mathbb{O} & P & \mathbb{O} & E_{31,32}^{34} \\ \hline \mathbb{O} & \mathbb{O} & \mathbb{O} & \mathbb{O} & Q & R \end{array}\right)$$

where

$$A = \begin{pmatrix} 0&1&1&1\\ 0&0&0&0\\ 0&0&0&0\\ 0&1&1&1\\ 0&0&0&0\\ 0&1&1&1 \end{pmatrix} \; B = \begin{pmatrix} 0&1&0&0\\ 1&0&0&0\\ 0&1&0&0\\ 0&1&1&1\\ 0&0&1&0\\ 0&1&0&1 \end{pmatrix} \; C = \begin{pmatrix} 1&0&0&0\\ 0&0&0&1\\ 1&0&0&0\\ 0&1&0&0\\ 0&1&1&0\\ 0&0&1&0 \end{pmatrix} \; D = \begin{pmatrix} 0&1&0&1\\ 0&0&0&0\\ 0&0&0&0\\ 0&1&0&1\\ 0&0&0&0\\ 0&1&0&1 \end{pmatrix} \; E = \begin{pmatrix} 1&0&0&1\\ 0&0&0&0\\ 0&0&0&0\\ 1&0&0&1\\ 0&0&0&0\\ 1&0&0&1 \end{pmatrix}$$

$$F = \begin{pmatrix} 1&1&1&1\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 1&1&1&1\\ 0&0&0&0 \end{pmatrix} \; G = \begin{pmatrix} 1&0&0&0\\ 1&1&0&0\\ 0&0&0&0\\ 0&1&0&0\\ 0&0&1&0\\ 0&0&0&1 \end{pmatrix} \; H = \begin{pmatrix} 1&1&0&0\\ 1&0&0&0\\ 0&0&0&0\\ 0&1&0&0\\ 0&0&1&0\\ 0&0&0&1 \end{pmatrix} \; I = \begin{pmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 1&1&1&0\\ 1&1&1&0\\ 0&0&0&0 \end{pmatrix} \; J = \begin{pmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 1&0&1&1\\ 1&0&1&1 \end{pmatrix}$$

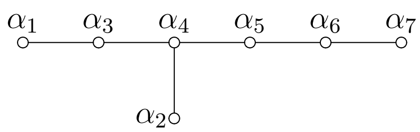
$$K = \begin{pmatrix} 0&1&0&1\\ 0&0&0&1\\ 1&0&0&0\\ 0&1&0&0\\ 1&0&1&1\\ 0&0&1&0 \end{pmatrix} \; L = \begin{pmatrix} 1&1&0&0\\ 1&0&0&0\\ 1&0&1&1\\ 0&1&0&0\\ 0&0&1&0\\ 0&0&0&1 \end{pmatrix} \; M = \begin{pmatrix} 1&1&0&0\\ 1&0&0&0\\ 1&0&1&0\\ 0&1&0&0\\ 0&0&1&0\\ 0&0&0&1 \end{pmatrix} \; N = \begin{pmatrix} 1&1&0&0\\ 0&0&1&0\\ 1&0&0&0\\ 1&1&1&0\\ 1&0&1&0\\ 0&0&0&1 \end{pmatrix} \; P = \begin{pmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 1&1&1&1\\ 1&1&1&1 \end{pmatrix}$$

$$Q = \begin{pmatrix} 1&1&0&0\\ 0&0&1&0\\ 1&0&0&0\\ 1&0&0&1\\ 1&1&0&1\\ 0&0&0&1 \end{pmatrix} \; R = \begin{pmatrix} 1&0&0&0\\ 0&1&0&0\\ 0&0&1&0\\ 0&0&0&1\\ 1&0&1&1\\ 0&0&1&1 \end{pmatrix}$$

Row reduction yields $\mathrm{Ker}(\Theta) = 0$. In this case, another method consists in applying the long exact cohomology sequence associated to the short exact sequence of W-modules $0 \longrightarrow \mathbb{Z}/3 \longrightarrow T_{\mathrm{E}_6} \longrightarrow T_{\mathrm{PE}_6} \longrightarrow 0$ and then use the fact that $H^1(W; T_{\mathrm{PE}_6})$ is an $\mathbb{F}_2$-vector space, to get an isomorphism between the latter and $H^1(W; T_{\mathrm{E}_6}) = 0$.

5) Type E_7 **:**

The Dynkin diagram is



The chosen basis of $Q_{\alpha_j}/T_{\alpha_j}$ is

$$\begin{aligned}
j = 1 &: \; \tfrac{1}{2} \cdot \{\alpha_2^\vee, \, \alpha_4^\vee, \, \alpha_5^\vee, \, \alpha_6^\vee, \, \alpha_7^\vee\}\,; \\
j = 2 &: \; \tfrac{1}{2} \cdot \{\alpha_1^\vee, \, \alpha_3^\vee, \, \alpha_5^\vee, \, \alpha_6^\vee, \, \alpha_7^\vee\}\,; \\
j = 3 &: \; \tfrac{1}{2} \cdot \{\alpha_1^\vee + \alpha_4^\vee, \, \alpha_2^\vee, \, \alpha_5^\vee, \, \alpha_6^\vee, \, \alpha_7^\vee\}\,; \\
j = 4 &: \; \tfrac{1}{2} \cdot \{\alpha_1^\vee, \, \alpha_2^\vee + \alpha_3^\vee, \, \alpha_2^\vee + \alpha_5^\vee, \, \alpha_6^\vee, \, \alpha_7^\vee\}\,; \\
j = 5 &: \; \tfrac{1}{2} \cdot \{\alpha_1^\vee, \, \alpha_2^\vee, \, \alpha_3, \, \alpha_4^\vee + \alpha_6^\vee, \, \alpha_7^\vee\}\,; \\
j = 6 &: \; \tfrac{1}{2} \cdot \{\alpha_1^\vee, \, \alpha_2^\vee, \, \alpha_3^\vee, \, \alpha_4^\vee, \, \alpha_5^\vee + \alpha_7^\vee\}\,; \\
j = 7 &: \; \tfrac{1}{2} \cdot \{\alpha_1^\vee, \, \alpha_2^\vee, \, \alpha_3^\vee, \, \alpha_4^\vee, \, \alpha_5^\vee\}\,.
\end{aligned}$$

The matrix of Θ is

$$\left(\begin{array}{ccccccc}
E_{22} & E_{12} & \mathbb{O} & \mathbb{O} & \mathbb{O} & \mathbb{O} & \mathbb{O} \\
A & \mathbb{O} & B & \mathbb{O} & \mathbb{O} & \mathbb{O} & \mathbb{O} \\
E_{41}^{43} & \mathbb{O} & \mathbb{O} & E_{12} & \mathbb{O} & \mathbb{O} & \mathbb{O} \\
E_{52}^{54} & \mathbb{O} & \mathbb{O} & \mathbb{O} & E_{13} & \mathbb{O} & \mathbb{O} \\
E_{63}^{65} & \mathbb{O} & \mathbb{O} & \mathbb{O} & \mathbb{O} & E_{13} & \mathbb{O} \\
E_{74} & \mathbb{O} & \mathbb{O} & \mathbb{O} & \mathbb{O} & \mathbb{O} & E_{13} \\
\hline
\mathbb{O} & E_{31} & E_{21} & \mathbb{O} & \mathbb{O} & \mathbb{O} & \mathbb{O} \\
\mathbb{O} & C & \mathbb{O} & C & \mathbb{O} & \mathbb{O} & \mathbb{O} \\
\mathbb{O} & E_{54} & \mathbb{O} & \mathbb{O} & E_{24} & \mathbb{O} & \mathbb{O} \\
\mathbb{O} & E_{63}^{65} & \mathbb{O} & \mathbb{O} & \mathbb{O} & E_{24} & \mathbb{O} \\
\mathbb{O} & E_{74} & \mathbb{O} & \mathbb{O} & \mathbb{O} & \mathbb{O} & E_{24} \\
\hline
\mathbb{O} & \mathbb{O} & D & E & \mathbb{O} & \mathbb{O} & \mathbb{O} \\
\mathbb{O} & \mathbb{O} & E_{51}^{54} & \mathbb{O} & E_{31}^{34} & \mathbb{O} & \mathbb{O} \\
\mathbb{O} & \mathbb{O} & E_{63}^{65} & \mathbb{O} & \mathbb{O} & E_{31}^{34} & \mathbb{O} \\
\mathbb{O} & \mathbb{O} & E_{74} & \mathbb{O} & \mathbb{O} & \mathbb{O} & E_{31}^{34} \\
\hline
\mathbb{O} & \mathbb{O} & \mathbb{O} & F & G & \mathbb{O} & \mathbb{O} \\
\mathbb{O} & \mathbb{O} & \mathbb{O} & E_{63}^{65} & \mathbb{O} & E_{42,43}^{45} & \mathbb{O} \\
\mathbb{O} & \mathbb{O} & \mathbb{O} & E_{74} & \mathbb{O} & \mathbb{O} & E_{42,43}^{45} \\
\hline
\mathbb{O} & \mathbb{O} & \mathbb{O} & \mathbb{O} & H & H & \mathbb{O} \\
\mathbb{O} & \mathbb{O} & \mathbb{O} & \mathbb{O} & E_{74} & \mathbb{O} & E_{54} \\
\hline
\mathbb{O} & \mathbb{O} & \mathbb{O} & \mathbb{O} & \mathbb{O} & I & I
\end{array}\right)$$

where

$$A = \begin{pmatrix} 0&1&0&0&0\\ 1&0&0&0&0\\ 0&0&0&0&0\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1 \end{pmatrix} \quad B = \begin{pmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&0&0&0\\ 1&0&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1 \end{pmatrix} \quad C = \begin{pmatrix} 1&0&0&0&0\\ 0&1&1&0&0\\ 0&1&0&0&0\\ 0&0&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1 \end{pmatrix}$$

$$D = \begin{pmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&1&1&0&0\\ 1&0&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1 \end{pmatrix} \quad E = \begin{pmatrix} 1&0&0&0&0\\ 0&1&1&0&0\\ 0&1&0&0&0\\ 1&0&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1 \end{pmatrix} \quad F = \begin{pmatrix} 1&0&0&0&0\\ 0&1&1&0&0\\ 0&1&0&0&0\\ 0&0&0&1&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1 \end{pmatrix}$$

$$G = \begin{pmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&1&1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1 \end{pmatrix} \quad H = \begin{pmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1\\ 0&0&0&1&0\\ 0&0&0&0&1 \end{pmatrix} \quad I = \begin{pmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 1&0&0&1&0\\ 0&0&0&0&1\\ 0&0&0&0&0\\ 0&0&0&0&1 \end{pmatrix}$$

One computes that the kernel of Θ is of $\mathbb{F}_2$-dimension one, with generator having the following components in the specified basis:

$$\tfrac{1}{2} \cdot \bigl(1,0,1,0,1|0,0,1,0,1|0,1,1,0,1|0,0,1,0,1|0,1,0,0,1|0,1,0,0,1|0,1,0,0,1\bigr)\,.$$

This means that the non-trivial element in $H^1(W;T) \hookrightarrow \mathrm{Out}(N)$ is represented by the automorphism $\tilde{\psi}$ of N uniquely determined by $\tilde{\psi}|_T = \mathrm{id}_T$ and taking the elements of a prescribed Tits configuration $A := \{q_1, \ldots, q_7\}$ of E_7 to

$$z_0 \cdot q_1\,,\ h_5 h_7 \cdot q_2\,,\ z_0 \cdot q_3\,,\ z_0 \cdot q_4\,,\ h_2 h_7 \cdot q_5\,,\ z_0 \cdot q_6\,,\ h_2 h_5 \cdot q_7\,,$$

respectively, where $h_j := \exp\bigl(\frac{\alpha_j^\vee}{2}\bigr)$ and $z_0 := h_2 h_5 h_7$. Since

$$A' := \{q_1,\, h_2 \cdot q_2,\, q_3,\, q_4,\, h_5 \cdot q_5,\, q_6,\, h_7 \cdot q_7\}$$

is also a Tits configuration of E_7 , it is termwise conjugate to A by an element of T (see Proposition 2.5 (i)), so that the non-trivial element in $H^1(W;T)$ is also represented by the automorphism uniquely determined by

$$\psi \colon N \longrightarrow N\,, \quad \psi|_T = \mathrm{id}_T \ \text{ and } \ q_{\alpha_j} \longmapsto z_0 \cdot q_{\alpha_j}\,, \ \text{for all } j = 1, \ldots, 7\,.$$

Note that since $Z_0 := \frac{\alpha_2^\vee + \alpha_5^\vee + \alpha_7^\vee}{2}$ is an element of the co-weight lattice of E_7 , i.e. $\beta(Z_0) \in \mathbb{Z}$ for all $\beta \in B$ (as is readily checked), $z_0 = \exp(Z_0)$ is the only non-trivial central element of E_7 .

6) Type PE_7 **:**

The Dynkin diagram is as for E_7 . The integral lattice is obtained from the coroot lattice by adjoining the vector $\alpha_1^\vee + \frac{3\alpha_2^\vee}{2} + 2\alpha_3^\vee + 3\alpha_4^\vee + \frac{5\alpha_5^\vee}{2} + 2\alpha_6^\vee + \frac{3\alpha_7^\vee}{2}$; its selected basis is

$$\nu_1^\vee = \alpha_1^\vee \qquad \nu_2^\vee = \frac{\alpha_2^\vee}{2} + \frac{\alpha_5^\vee}{2} + \frac{\alpha_7^\vee}{2} \qquad \nu_3^\vee = \alpha_3^\vee \qquad \nu_4^\vee = \alpha_4^\vee$$

$$\nu_5^\vee = \alpha_6^\vee \qquad \nu_6^\vee = -\frac{\alpha_2^\vee}{2} + \frac{\alpha_5^\vee}{2} - \frac{\alpha_7^\vee}{2} \qquad \nu_7^\vee = -\frac{\alpha_2^\vee}{2} - \frac{\alpha_5^\vee}{2} + \frac{\alpha_7^\vee}{2}\,.$$

The chosen basis of $Q_{\alpha_j}/T_{\alpha_j}$ is

$$
\begin{aligned}
j=1 &: \quad \tfrac{1}{2}\cdot\{\nu_2^\vee,\,\nu_4^\vee,\,\nu_5^\vee,\,\nu_6^\vee,\,\nu_7^\vee\}\,;\\
j=2 &: \quad \tfrac{1}{2}\cdot\{\nu_1^\vee,\,\nu_2^\vee+\nu_4^\vee,\,\nu_2^\vee+\nu_7^\vee,\,\nu_3^\vee,\,\nu_5^\vee\}\,;\\
j=3 &: \quad \tfrac{1}{2}\cdot\{\nu_1^\vee+\nu_4^\vee,\,\nu_2^\vee,\,\nu_5^\vee,\,\nu_6^\vee,\,\nu_7^\vee\}\,;\\
j=4 &: \quad \tfrac{1}{2}\cdot\{\nu_1^\vee,\,\nu_2^\vee+\nu_3^\vee,\,\nu_2^\vee+\nu_7^\vee,\,\nu_5^\vee,\,\nu_6^\vee\}\,;\\
j=5 &: \quad \tfrac{1}{2}\cdot\{\nu_1^\vee,\,\nu_2^\vee+\nu_4^\vee,\,\nu_2^\vee+\nu_5^\vee,\,\nu_2^\vee+\nu_7^\vee,\,\nu_3^\vee\}\,;\\
j=6 &: \quad \tfrac{1}{2}\cdot\{\nu_1^\vee,\,\nu_3^\vee,\,\nu_4^\vee,\,\nu_6^\vee,\,\nu_7^\vee\}\,;\\
j=7 &: \quad \tfrac{1}{2}\cdot\{\nu_1^\vee,\,\nu_2^\vee+\nu_5^\vee,\,\nu_2^\vee+\nu_6^\vee,\,\nu_3^\vee,\,\nu_4^\vee\}\,.
\end{aligned}
$$

The matrix of Θ is

$$
\left(\begin{array}{ccccccc}
A & E_{14} & \mathbb{O} & \mathbb{O} & \mathbb{O} & \mathbb{O} & \mathbb{O}\\
B & \mathbb{O} & C & \mathbb{O} & \mathbb{O} & \mathbb{O} & \mathbb{O}\\
E_{41}^{45} & \mathbb{O} & \mathbb{O} & E_{12} & \mathbb{O} & \mathbb{O} & \mathbb{O}\\
D & \mathbb{O} & \mathbb{O} & \mathbb{O} & E_{15} & \mathbb{O} & \mathbb{O}\\
E_{51} & \mathbb{O} & \mathbb{O} & \mathbb{O} & \mathbb{O} & E_{12} & \mathbb{O}\\
E & \mathbb{O} & \mathbb{O} & \mathbb{O} & \mathbb{O} & \mathbb{O} & E_{14}\\
\hline
\mathbb{O} & E_{31}^{32} & A & \mathbb{O} & \mathbb{O} & \mathbb{O} & \mathbb{O}\\
\mathbb{O} & F & \mathbb{O} & G & \mathbb{O} & \mathbb{O} & \mathbb{O}\\
\mathbb{O} & E_{25}^{65} & \mathbb{O} & \mathbb{O} & E_{63}^{73} & \mathbb{O} & \mathbb{O}\\
\mathbb{O} & E_{52}^{53} & \mathbb{O} & \mathbb{O} & \mathbb{O} & H & \mathbb{O}\\
\mathbb{O} & E_{22,25}^{72,75} & \mathbb{O} & \mathbb{O} & \mathbb{O} & \mathbb{O} & E_{62,65}^{72,75}\\
\hline
\mathbb{O} & \mathbb{O} & I & J & \mathbb{O} & \mathbb{O} & \mathbb{O}\\
\mathbb{O} & \mathbb{O} & K & \mathbb{O} & E_{31}^{32} & \mathbb{O} & \mathbb{O}\\
\mathbb{O} & \mathbb{O} & E_{52} & \mathbb{O} & \mathbb{O} & E_{31}^{33} & \mathbb{O}\\
\mathbb{O} & \mathbb{O} & L & \mathbb{O} & \mathbb{O} & \mathbb{O} & E_{31}^{35}\\
\hline
\mathbb{O} & \mathbb{O} & \mathbb{O} & M & N & \mathbb{O} & \mathbb{O}\\
\mathbb{O} & \mathbb{O} & \mathbb{O} & E_{52}^{53} & \mathbb{O} & E_{42}^{45} & \mathbb{O}\\
\mathbb{O} & \mathbb{O} & \mathbb{O} & P & \mathbb{O} & \mathbb{O} & E_{42,43}^{44}\\
\hline
\mathbb{O} & \mathbb{O} & \mathbb{O} & \mathbb{O} & Q & R & \mathbb{O}\\
\mathbb{O} & \mathbb{O} & \mathbb{O} & \mathbb{O} & E_{22}^{72} & \mathbb{O} & E_{25}^{65}\\
\hline
\mathbb{O} & \mathbb{O} & \mathbb{O} & \mathbb{O} & \mathbb{O} & S & T
\end{array}\right)
$$

where

$$
A=\begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\1&1&0&1&1\\1&1&0&1&1\end{pmatrix}\quad
B=\begin{pmatrix}0&1&0&0&0\\1&0&0&0&0\\0&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix}\quad
C=\begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&0&0&0\\1&0&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix}\quad
D=\begin{pmatrix}0&0&0&0&0\\1&1&1&1&1\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\1&1&1&1&1\\0&0&0&0&0\end{pmatrix}
$$

$$
E=\begin{pmatrix}0&0&0&0&0\\1&0&1&1&1\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\1&0&1&1&1\end{pmatrix}\quad
F=\begin{pmatrix}1&0&0&0&0\\0&1&1&0&0\\0&0&0&1&0\\0&1&0&0&0\\0&0&0&0&1\\0&1&0&1&0\\0&1&1&1&0\end{pmatrix}\quad
G=\begin{pmatrix}1&0&0&0&0\\0&1&1&0&0\\0&1&0&0&0\\0&1&0&0&1\\0&0&0&1&0\\0&0&0&0&1\\0&0&1&0&0\end{pmatrix}\quad
H=\begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&1&1&1\\0&0&1&1&1\end{pmatrix}
$$

$$
I=\begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&1&0&0&1\\1&0&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix}\quad
J=\begin{pmatrix}1&0&0&0&0\\0&1&1&0&0\\0&1&0&0&0\\1&0&0&0&0\\0&0&0&1&0\\0&0&0&0&1\\0&0&1&0&0\end{pmatrix}\quad
K=\begin{pmatrix}0&0&0&0&0\\1&1&1&1&1\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\1&1&1&1&1\\0&0&0&0&0\end{pmatrix}\quad
L=\begin{pmatrix}0&0&0&0&0\\0&1&1&1&1\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&1&1&1&1\end{pmatrix}
$$

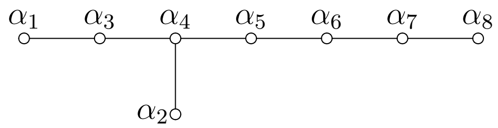
$$
M=\begin{pmatrix}1&0&0&0&0\\0&1&1&0&0\\0&1&0&0&0\\0&1&0&1&1\\0&0&0&1&0\\0&0&0&0&1\\0&0&1&0&0\end{pmatrix}\quad
N=\begin{pmatrix}1&0&0&0&0\\0&0&0&1&1\\0&0&0&0&1\\0&1&0&0&0\\0&0&1&0&0\\0&1&1&0&1\\0&0&0&1&0\end{pmatrix}\quad
P=\begin{pmatrix}0&0&0&0&0\\0&1&0&1&1\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&1&0&1&1\end{pmatrix}\quad
Q=\begin{pmatrix}1&0&0&0&0\\0&0&0&0&0\\0&0&0&0&1\\0&1&0&0&0\\0&0&1&0&0\\0&1&1&1&0\\0&0&0&1&0\end{pmatrix}
$$

$$R=\begin{pmatrix}1&0&0&0&0\\0&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&1&1&1\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix}\quad S=\begin{pmatrix}1&0&0&0&0\\0&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&1\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix}\quad T=\begin{pmatrix}1&0&0&0&0\\0&0&0&0&0\\0&0&0&1&0\\0&0&0&0&1\\0&1&0&0&0\\0&0&1&0&0\\0&1&1&0&0\end{pmatrix}$$

One computes that the kernel of Θ is zero.

7) Type E_8 **:**

The Dynkin diagram is



The chosen basis of $Q_{\alpha_j}/T_{\alpha_j}$ is

$$\begin{aligned}
j=1 &: \ \tfrac{1}{2}\cdot\{\alpha_2^\vee,\,\alpha_4^\vee,\,\alpha_5^\vee,\,\alpha_6^\vee,\,\alpha_7^\vee,\,\alpha_8^\vee\}\,;\\
j=2 &: \ \tfrac{1}{2}\cdot\{\alpha_1^\vee,\,\alpha_3^\vee,\,\alpha_5^\vee,\,\alpha_6^\vee,\,\alpha_7^\vee,\,\alpha_8^\vee\}\,;\\
j=3 &: \ \tfrac{1}{2}\cdot\{\alpha_1^\vee+\alpha_4^\vee,\,\alpha_2^\vee,\,\alpha_5^\vee,\,\alpha_6^\vee,\,\alpha_7^\vee,\,\alpha_8^\vee\}\,;\\
j=4 &: \ \tfrac{1}{2}\cdot\{\alpha_1^\vee,\,\alpha_2^\vee+\alpha_3^\vee,\,\alpha_2^\vee+\alpha_5^\vee,\,\alpha_6^\vee,\,\alpha_7^\vee,\,\alpha_8^\vee\}\,;\\
j=5 &: \ \tfrac{1}{2}\cdot\{\alpha_1^\vee,\,\alpha_2^\vee,\,\alpha_3^\vee,\,\alpha_4^\vee+\alpha_6^\vee,\,\alpha_7^\vee,\,\alpha_8^\vee\}\,;\\
j=6 &: \ \tfrac{1}{2}\cdot\{\alpha_1^\vee,\,\alpha_2^\vee,\,\alpha_3^\vee,\,\alpha_4^\vee,\,\alpha_5^\vee+\alpha_7^\vee,\,\alpha_8^\vee\}\,;\\
j=7 &: \ \tfrac{1}{2}\cdot\{\alpha_1^\vee,\,\alpha_2^\vee,\,\alpha_3^\vee,\,\alpha_4^\vee,\,\alpha_5^\vee,\,\alpha_6^\vee+\alpha_8^\vee\}\,;\\
j=8 &: \ \tfrac{1}{2}\cdot\{\alpha_1^\vee,\,\alpha_2^\vee,\,\alpha_3^\vee,\,\alpha_4^\vee,\,\alpha_5^\vee,\,\alpha_6^\vee\}\,.
\end{aligned}$$

The matrix of Θ is

$$\left(\begin{array}{cccccccc}
E_{22} & E_{12} & \mathbb{O} & \mathbb{O} & \mathbb{O} & \mathbb{O} & \mathbb{O} & \mathbb{O}\\
A & \mathbb{O} & B & \mathbb{O} & \mathbb{O} & \mathbb{O} & \mathbb{O} & \mathbb{O}\\
E_{41}^{43} & \mathbb{O} & \mathbb{O} & E_{12} & \mathbb{O} & \mathbb{O} & \mathbb{O} & \mathbb{O}\\
E_{52}^{54} & \mathbb{O} & \mathbb{O} & \mathbb{O} & E_{13} & \mathbb{O} & \mathbb{O} & \mathbb{O}\\
E_{63}^{65} & \mathbb{O} & \mathbb{O} & \mathbb{O} & \mathbb{O} & E_{13} & \mathbb{O} & \mathbb{O}\\
E_{74}^{76} & \mathbb{O} & \mathbb{O} & \mathbb{O} & \mathbb{O} & \mathbb{O} & E_{13} & \mathbb{O}\\
E_{85} & \mathbb{O} & \mathbb{O} & \mathbb{O} & \mathbb{O} & \mathbb{O} & \mathbb{O} & E_{13}\\
\hline
\mathbb{O} & E_{31} & E_{21} & \mathbb{O} & \mathbb{O} & \mathbb{O} & \mathbb{O} & \\
\mathbb{O} & C & \mathbb{O} & C & \mathbb{O} & \mathbb{O} & \mathbb{O} & \mathbb{O}\\
\mathbb{O} & E_{54} & \mathbb{O} & \mathbb{O} & E_{24} & \mathbb{O} & \mathbb{O} & \mathbb{O}\\
\mathbb{O} & E_{63}^{65} & \mathbb{O} & \mathbb{O} & \mathbb{O} & E_{24} & \mathbb{O} & \mathbb{O}\\
\mathbb{O} & E_{74}^{76} & \mathbb{O} & \mathbb{O} & \mathbb{O} & \mathbb{O} & E_{24} & \mathbb{O}\\
\mathbb{O} & E_{85} & \mathbb{O} & \mathbb{O} & \mathbb{O} & \mathbb{O} & \mathbb{O} & E_{24}\\
\hline
\mathbb{O} & \mathbb{O} & D & E & \mathbb{O} & \mathbb{O} & \mathbb{O} & \mathbb{O}\\
\mathbb{O} & \mathbb{O} & E_{51}^{54} & \mathbb{O} & E_{31}^{34} & \mathbb{O} & \mathbb{O} & \mathbb{O}\\
\mathbb{O} & \mathbb{O} & E_{63}^{65} & \mathbb{O} & \mathbb{O} & E_{31}^{34} & \mathbb{O} & \mathbb{O}\\
\mathbb{O} & \mathbb{O} & E_{74}^{76} & \mathbb{O} & \mathbb{O} & \mathbb{O} & E_{31}^{34} & \mathbb{O}\\
\mathbb{O} & \mathbb{O} & E_{85} & \mathbb{O} & \mathbb{O} & \mathbb{O} & \mathbb{O} & E_{31}^{34}\\
\hline
\mathbb{O} & \mathbb{O} & \mathbb{O} & F & G & \mathbb{O} & \mathbb{O} & \mathbb{O}\\
\mathbb{O} & \mathbb{O} & \mathbb{O} & E_{63}^{65} & \mathbb{O} & E_{42,43}^{45} & \mathbb{O} & \mathbb{O}\\
\mathbb{O} & \mathbb{O} & \mathbb{O} & E_{74}^{76} & \mathbb{O} & \mathbb{O} & E_{42,43}^{45} & \mathbb{O}\\
\mathbb{O} & \mathbb{O} & \mathbb{O} & E_{85} & \mathbb{O} & \mathbb{O} & \mathbb{O} & E_{42,43}^{45}\\
\hline
\mathbb{O} & \mathbb{O} & \mathbb{O} & \mathbb{O} & H & H & \mathbb{O} & \mathbb{O}\\
\mathbb{O} & \mathbb{O} & \mathbb{O} & \mathbb{O} & E_{74}^{76} & \mathbb{O} & E_{54}^{56} & \mathbb{O}\\
\mathbb{O} & \mathbb{O} & \mathbb{O} & \mathbb{O} & E_{85} & \mathbb{O} & \mathbb{O} & E_{54}^{56}\\
\hline
\mathbb{O} & \mathbb{O} & \mathbb{O} & \mathbb{O} & \mathbb{O} & I & I & \mathbb{O}\\
\mathbb{O} & \mathbb{O} & \mathbb{O} & \mathbb{O} & \mathbb{O} & E_{85} & \mathbb{O} & E_{65}\\
\hline
\mathbb{O} & \mathbb{O} & \mathbb{O} & \mathbb{O} & \mathbb{O} & \mathbb{O} & J & J
\end{array}\right)$$

where

$$A = \begin{pmatrix} 0&1&0&0&0&0\\ 1&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{pmatrix} \quad B = \begin{pmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&0&0&0&0\\ 1&0&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{pmatrix} \quad C = \begin{pmatrix} 1&0&0&0&0&0\\ 0&1&1&0&0&0\\ 0&1&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{pmatrix} \quad D = \begin{pmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&1&1&0&0&0\\ 1&0&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{pmatrix}$$

$$E = \begin{pmatrix} 1&0&0&0&0&0\\ 0&1&1&0&0&0\\ 0&1&0&0&0&0\\ 1&0&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{pmatrix} \quad F = \begin{pmatrix} 1&0&0&0&0&0\\ 0&1&1&0&0&0\\ 0&1&0&0&0&0\\ 0&0&0&1&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{pmatrix} \quad G = \begin{pmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&1&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{pmatrix} \quad H = \begin{pmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{pmatrix}$$

$$I = \begin{pmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{pmatrix} \quad J = \begin{pmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1\\ 0&0&0&0&0&0\\ 0&0&0&0&0&1 \end{pmatrix}$$

One checks that the equality $\mathrm{Ker}(\Theta) = 0$ holds. Finally, this completes the proof of the Main Theorem. ■

Let us single out the following consequence of the above computations for the group E_7 (we still number the simple roots as in 5) above and write $h_j = \exp\big(\frac{\alpha_j^\vee}{2}\big)$).

Corollary 4.1. *The unique non-trivial class $[\psi] \in H^1(W_{\mathrm{E}_7})$ has a representative element $\psi \in \mathrm{Aut}(N, T)$ defined by*

$$\psi(q_j) = z_0 \cdot q_j \,, \text{ for all } j = 1, \ldots, 7 \,,$$

where $z_0 = h_2 h_5 h_7$ is the non-trivial central element (of order 2) in E_7 .

5. On the general case

In this final section, we treat the semisimple and the general cases. As before, G denotes a non-abelian connected compact Lie group, T a fixed maximal torus, N its normalizer, and $\pi \colon N \twoheadrightarrow W$ the canonical map. Otherwise, we assume the notations previously introduced. We show that 'in most cases', the group $H^1(W;T)$ is a quotient of the group of homomorphisms from the Weyl group to the subgroup $Z_2(N) = Z(N) \cap S$ of central elements of order dividing 2 in the normalizer. Finally, we prove Theorem 1.2.

Definition 5.1. (i) An automorphism $\psi \in \mathrm{Aut}(N,T)$ is called *special* if the corresponding element $(t_\alpha)_\alpha = \big(\psi(q_\alpha)q_\alpha^{-1}\big)_\alpha \in \bigoplus Q_\alpha$ lies in $\bigoplus F_\alpha \cap S$ (i.e. if there exists, in the procedure described before Proposition 2.8, a choice for the maps $\{\underline{s}^{(\alpha)}\}_{\alpha \in B}$ such that $\psi \in \mathrm{Im}(\underline{s}) \subseteq \mathrm{Aut}(N,T)$ for the corresponding section $\underline{s} \colon H^1(W;T) \longrightarrow \mathrm{Aut}(N,T)$).

(ii) We say that two automorphisms in $\mathrm{Aut}(N,T)$ are T-*conjugate* if they represent the same class in $H^1(W;T) \subseteq \mathrm{Out}(N)$, that is, if they differ by a conjugation with an element in T .

Let $\tau \colon W \longrightarrow Z(N)$ be a group homomorphism and consider the composition $\bar\tau := \tau \circ \pi \colon N \longrightarrow Z(N)$. Note that the Weyl group W being generated by elements of order 2 , τ factorizes (uniquely) through the inclusion $Z_2(N) \hookrightarrow Z(N)$; in particular, $\mathrm{Hom}(W, Z(N)) = \mathrm{Hom}(W, Z_2(N))$.

Lemma 5.2. *(i) Every automorphism in* $\operatorname{Aut}(N,T)$ *is* T*-conjugate to a special one.*

(ii) For $\tau \in \operatorname{Hom}(W, Z_2(N))$ *, the map* $\psi_\tau \colon N \longrightarrow N$ *,* $x \longmapsto \bar{\tau}(x)\cdot x$ *is a special automorphism of* N *.*

(iii) For every automorphism $\psi \in \operatorname{Aut}(N,T)$ *with* $t_\alpha \in Z(N)$ *for all* $\alpha \in B$ *, there exists a homomorphism* $\tau \in \operatorname{Hom}(W, Z_2(N))$ *such that* $\psi = \psi_\tau$ *.*

Proof. ¿From the discussion preceding Proposition 2.8, (i) follows. Part (ii) is obvious. For (iii), first note that for every $\alpha \in B$, $q_\alpha^2 = \psi(q_\alpha^2) = t_\alpha^2 q_\alpha^2$, i.e. $t_\alpha \in Z_2(N)$. Recall that the Weyl group admits the Coxeter presentation

$$W = \big\langle \{s_\alpha\}_{\alpha\in B} \,\big|\, s_\alpha^2 = 1,\ \underbrace{s_\alpha s_\beta s_\alpha \cdots}_{\ell_{\alpha\beta} \text{ factors}} = \underbrace{s_\beta s_\alpha s_\beta \cdots}_{\ell_{\alpha\beta} \text{ factors}} \big\rangle \,.$$

Now, as t_α is central for all $\alpha \in B$, relation (♠) of Section 2 gives

$$\underbrace{t_\alpha t_\beta t_\alpha \cdots}_{\ell_{\alpha\beta} \text{ factors}} = \underbrace{t_\beta t_\alpha t_\beta \cdots}_{\ell_{\alpha\beta} \text{ factors}} \,.$$

Therefore, the map $\{s_\alpha \,|\, \alpha \in B\} \longrightarrow Z_2(N)$, $s_\alpha \longmapsto t_\alpha$ uniquely extends to a group homomorphism $\tau \colon W \longrightarrow Z_2(N)$. Clearly, $\psi = \psi_\tau$ and the proof is complete. ∎

Remark 5.3. Assigning to $\tau \in \operatorname{Hom}(W, Z_2(N))$ the special automorphism ψ_τ depicted in Lemma 5.2 (ii) defines a group homomorphism

$$\vartheta \colon \operatorname{Hom}(W, Z_2(N)) \longrightarrow H^1(W;T)\,, \quad \tau \longmapsto [\psi_\tau]\,.$$

Definition 5.4. (i) A special automorphism ψ is called *regular* if $\psi = \psi_\tau$ for some $\tau \in \operatorname{Hom}(W, Z_2(N))$.

(ii) The Lie group G is called *regular* if every automorphism in $\operatorname{Aut}(N,T)$ is T-conjugate to a regular automorphism.

Lemma 5.2 (i) and the isomorphism $H^1(W;T) \cong \operatorname{Aut}(N,T)/\operatorname{Inn}(N,T)$ directly imply the following characterization.

Proposition 5.5. *The Lie group* G *is regular if and only if* ϑ *is surjective.* ∎

An immediate consequence of the computations performed in Sections 3 and 4 is the classification of regular simple Lie groups; before we state the result, recall that $\mathrm{PSO}(6) \cong \mathrm{PSU}(4)$.

Proposition 5.6. *Let* G *be a simple connected compact Lie group. Then,* G *is regular if and only if* G *is* not *isomorphic to* $\mathrm{PSU}(4)$ *,* $\mathrm{PSp}(3)$ *,* $\mathrm{PSp}(4)$ *,* $\mathrm{PSO}(8)$ *nor to* $\mathrm{Spin}(4n+3)$ *with* $n \geqslant 1$ *.* ∎

As before, we write $B = \{\alpha_1, \dots, \alpha_{\ell_S}\}$ with ℓ_S the semisimple rank of G, and similarly for a second connected compact Lie group G'. Let $N = N_G(T)$ and $N' = N_{G'}(T')$ be normalizers of maximal tori in G and G', with Tits configurations $A = \{q_1, \dots, q_{\ell_S}\} \subset N$ and $A' = \{p_1, \dots, p_{\ell'_S}\} \subset N'$. Let $\varphi \in \mathrm{Aut}(N,T)$ and $\rho \in \mathrm{Aut}(N', T')$ be special automorphisms. Consider two group homomorphisms $\gamma \in \mathrm{Hom}(W, Z_2(N'))$ and $\delta \in \mathrm{Hom}(W', Z_2(N))$, and set $\bar\gamma := \gamma \circ \pi$ and $\bar\delta := \delta \circ \pi'$. Then, the map

$$\psi = \psi_{\varphi,\rho,\gamma,\delta}\colon N \times N' \longrightarrow N \times N', \quad (x, y) \longmapsto \big(\bar\delta(y)\cdot\varphi(x),\, \bar\gamma(x)\cdot\rho(y)\big)$$

is a special automorphism of $N \times N'$, which is completely determined by

$$\psi(q_i, e) = \big(\varphi(q_i),\, \gamma(q_i)\big) \text{ and } \psi(e, p_j) = \big(\delta(p_j),\, \rho(p_j)\big)$$

with $i = 1, \dots, \ell_S$ and $j = 1, \dots, \ell'_S$.

Note that to take the general case into account, we allow G' to be a torus, namely $G' = \mathbb{T}^k$. In this case, $N' = \mathbb{T}^k$, A' is empty, ρ is the identity, W' is trivial, and both δ and $\bar\delta$ are zero. By convention, $G' = \mathbb{T}^k$ is regular.

Lemma 5.7. *Any special automorphism of $N \times N'$ is given by $\psi_{\varphi,\rho,\gamma,\delta}$, for some φ, ρ, γ and δ as above.*

Proof. Let ψ be a special automorphism of $N \times N'$. This implies that for all i and j, we can write $\psi(q_i, e) = (t_i q_i, t'_i)$ and $\psi(e, p_j) = (u_j, u'_j p_j)$, with $t_i, u_j \in S$ and $t'_i, u'_j \in S'$. Since the elements $\psi(q_i, e)$ and $\psi(e, p_j)$ commute, we get $q_i u_j = u_j q_i$ and $t'_i p_j = p_j t'_i$ for all i and j. This implies that $u_j \in Z(N)$ for all j, and $t'_i \in Z(N')$ for all i. Now, let $\iota^{(\prime)}\colon N^{(\prime)} \hookrightarrow N \times N'$ denote the canonical inclusion, and $p^{(\prime)}\colon N \times N' \twoheadrightarrow N^{(\prime)}$ the canonical projection. Let us define $\varphi := p \circ \psi \circ \iota$, $\rho := p' \circ \psi \circ \iota'$, $\bar\gamma := p' \circ \psi \circ \iota$ and $\bar\delta := p \circ \psi \circ \iota'$. Then, $\psi = \psi_{\varphi,\rho,\gamma,\delta}$ holds. ∎

We get the following corollary.

Corollary 5.8. *The product $G \times G'$ is regular if and only if so are G and G'.*

Proof. Let us first suppose that G and G' are regular. Let ψ be a special automorphism of $N \times N'$; by Lemma 5.7, we have $\psi = \psi_{\varphi,\rho,\gamma,\delta}$. By hypothesis, and as conjugation is carried out in each factor separately, we can suppose that φ and ρ are regular, that is $\varphi = \varphi_\sigma$ for some $\sigma \in \mathrm{Hom}(W, Z_2(N))$, and $\rho = \rho_\nu$ for some $\nu \in \mathrm{Hom}(W', Z_2(N'))$. Explicitly, ψ is thus given by

$$\begin{aligned}\psi(x, y) &= \big(\bar\delta(y)\cdot\varphi(x),\, \bar\gamma(x)\cdot\rho(y)\big)\\ &= \big(\bar\delta(y)\cdot\bar\sigma(x)\cdot x,\, \bar\gamma(x)\cdot\bar\nu(y)\cdot y\big)\\ &= \big(\bar\delta(y)\cdot\bar\sigma(x),\, \bar\gamma(x)\cdot\bar\nu(y)\big)\cdot(x, y).\end{aligned}$$

Then, noting that $Z_2(N \times N') = Z_2(N) \times Z_2(N')$, one checks that the map

$$\bar\tau\colon N \times N \longrightarrow Z_2(N \times N'), \quad (x, y) \longmapsto \big(\bar\delta(y)\cdot\bar\sigma(x),\, \bar\gamma(x)\cdot\bar\nu(y)\big)$$

is a homomorphism that factorizes through $\tau\colon W \times W' \longrightarrow Z_2(N \times N')$. Since we clearly have $\psi = \psi_\tau$, the automorphism ψ is regular, and therefore $G \times G'$ is regular. To prove the converse, let $\varphi\colon N \longrightarrow N$ be a special automorphism. Then the special automorphism $\psi(x, y) := (\varphi(x), y)$ of $N \times N'$ is easily seen to be regular if and only if φ is regular. This concludes the proof. ∎

Example 5.9. Let G and G' be both equal to $\mathrm{SU}(2)$, so that their product $G \times G'$ is $\mathrm{SU}(2) \times \mathrm{SU}(2) \cong \mathrm{Spin}(4)$. We easily deduce from Lemma 5.7 that $H^1(W_{\mathrm{SU}(2)\times\mathrm{SU}(2)}) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$ (even though $H^1(W_{\mathrm{SU}(2)}) = 0$). More generally, for $n \geqslant 1$, $H^1(W_{\mathrm{SU}(2)^n}) \cong (\mathbb{Z}/2)^{n(n-1)}$ holds.

We now discuss the quotients of $\mathrm{Spin}(4)$.

Example 5.10. For $G = \mathrm{SU}(2) \times \mathrm{SU}(2)$, $\mathrm{SU}(2) \times \mathrm{SO}(3)$ and $\mathrm{SO}(3) \times \mathrm{SO}(3)$ (all having isomorphic W-modules T), we have $H^1(W_G) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$. On the other hand, for $G = \mathrm{SO}(4)$, we have computed that $H^1(W_{\mathrm{SO}(4)}) = 0$.

For a connected compact Lie group G , there exists, by classical Lie theory, a minimal central subgroup K' of G (not necessarily finite) such that $\bar{G}' := G/K'$ is the (largest) quotient of G that decomposes as a product of simple connected compact Lie groups and of a torus; let $\bar{G}' \cong \bar{G}'_1 \times \cdots \times \bar{G}'_r \times \mathbb{T}^m$ denote this decomposition. Recall that the quotient $\bar{N}' := N/K'$ is the normalizer of the maximal torus $\bar{T}' := T/K'$ in $\bar{G}'$. We are now ready for one of the main results of the present section.

Proposition 5.11. *Let G be a connected compact Lie group such that all the factors $\bar{G}'_j$ of $\bar{G}'$ as above are regular. Suppose further that every $\bar{G}'_j$ which is isomorphic to an odd orthogonal group lifts to a direct factor of G. Then, G is regular and in particular, ϑ is onto:*

$$\vartheta\colon\ \mathrm{Hom}(W, Z_2(N)) \twoheadrightarrow H^1(W;T)\,.$$

For the proof, we need the following result that explains the origin of the second assumption on the factors $\bar{G}'_j$.

Lemma 5.12. *Keep notations as above and let $\bar{\pi}\colon G \twoheadrightarrow \bar{G}' = G/K'$ be the projection map. Then $\bar{\pi}^{-1}(Z(\bar{N}')) = Z(N)$ if and only if every direct factor $\bar{G}'_j$ of $\bar{G}'$ which is isomorphic to an odd orthogonal group lifts to a direct factor of G .*

Proof. Clearly, the result for those Lie groups G having no direct factor isomorphic to an odd orthogonal group implies the result for the general case. We therefore make this assumption. In particular, $Z(N) = Z(G)$ holds by Remark 1.1.

We start with the "if" part. By our assumptions, no $\bar{G}'_j$ is isomorphic to an odd orthogonal group. So, by Remark 1.1, we have $Z(\bar{N}') = Z(\bar{G}')$, and from the equality $Z(\bar{G}') = Z(G)/K'$, we deduce that $\bar{\pi}^{-1}(Z(\bar{N}')) = Z(G) = Z(N)$.

We pass to the "only if" part. By our assumptions and by surjectivity of $\bar{\pi}$ onto $\bar{G}'$, we have

$$Z(\bar{N}') = \bar{\pi}\big(\bar{\pi}^{-1}(Z(\bar{N}'))\big) = \bar{\pi}(Z(N)) = \bar{\pi}(Z(G)) = Z(\bar{G}')\,.$$

Applying Remark 1.1 to $\bar{G}'$, we see that none of its direct factors $\bar{G}'_j$ is isomorphic to an odd orthogonal group. ■

Now we enter the proof of Proposition 5.11.

Proof. Let $\psi \in \operatorname{Aut}(N,T)$; we can suppose that ψ is special. As ψ is the identity on T , it preserves K' and therefore induces a special automorphism $\bar{\psi}$ of $\bar{N}'$. By the first hypothesis on the $\bar{G}'_j$'s and by Corollary 5.8, $\bar{G}'$ is regular, so that $\bar{\psi}(\bar{q}_j) = \bar{z}_j \cdot \bar{q}_j$ with $\bar{z}_j \in Z(\bar{N}')$ for all j . One readily deduces that $\psi(q_j) = z_j \cdot q_j$ with $z_j \in \bar{\pi}^{-1}(\bar{z}_j) \subseteq \bar{\pi}^{-1}(Z(\bar{N}'))$ for all j. By our second assumption on the $\bar{G}'_j$'s, Lemma 5.12 applies to give that $z_j \in Z(N)$ for all j. By Lemma 5.2, ψ is regular. The statement about surjectivity follows from Proposition 5.5. ∎

The following example is due to J. M. Møller (private communication).

Example 5.13. Here is an example of a non-regular Lie group G, for which all direct factors of $\bar{G}'$ are regular. Consider $G = \big(\mathrm{SO}(4) \times \mathrm{SO}(4)\big)/\Delta\mathbb{Z}/2$, where $\Delta\mathbb{Z}/2$ designates the diagonal central copy of $\mathbb{Z}/2$, that is, the central subgroup generated by $(-\mathbb{1}_4, -\mathbb{1}_4)$. One has $\bar{G}' = \bar{G} \cong \mathrm{SO}(3)^4$, and therefore $\bar{G}'$ has only regular direct factors, see Proposition 5.6. In this case, $T^W = Z(N) = Z(G) \cong \mathbb{Z}/2$ holds by Remark 1.1, so that $T/T^W = \bar{T}$. One has

$$H^1(W; T/T^W) \cong H^1(W_{\mathrm{SO}(3)^4}) \cong H^1(W_{\mathrm{SU}(2)^4}) \cong (\mathbb{Z}/2)^{12} ,$$

by Example 5.9, and, using the Künneth Theorem, one computes that

$$H^2(W; T^W) \cong H^2((\mathbb{Z}/2)^4; \mathbb{Z}/2) \cong (\mathbb{Z}/2)^{10} .$$

It follows from the long exact sequence presented in Remark 5.15 below that ϑ is not surjective, so that G is not regular. In fact, both $(T/T^W)^W$ and $\operatorname{Hom}(W; T^W)$ are isomorphic to $(\mathbb{Z}/2)^4$ and, by computer computations performed by Møller, one has $H^1(W;T) \cong (\mathbb{Z}/2)^3$. (This example also shows that one cannot replace "odd orthogonal group" by "odd orthogonal group different from SO(3)" in the statement of Proposition 5.11, even though $H^1(W_{\mathrm{SO}(3)}) = 0$.) Møller has performed other computations and showed in particular that the Lie groups $\big(\mathrm{SO}(6)\times\mathrm{SO}(8)\big)/\Delta\mathbb{Z}/2$ and $\big(\mathrm{SO}(6)\times\mathrm{SO}(10)\big)/\Delta\mathbb{Z}/2$ are regular, although PSO(6) and PSO(8) are not regular by Proposition 5.6, so that Proposition 5.11 does not apply in either cases.

Next, we discuss injectivity of ϑ. We begin with an example.

Example 5.14. As the computations performed in Section 3 show, for the odd orthogonal group $\mathrm{SO}(2n+1)$, the homomorphism ϑ is injective if and only if $n \geqslant 3$. For SO(3) and for SO(5) , the kernel of ϑ is isomorphic to $\mathbb{Z}/2$.

The regular automorphisms form a subgroup of $\operatorname{Aut}(N,T)$ which we denote by $\operatorname{Reg}(N,T)$. The injective homomorphism

$$\varrho\colon \operatorname{Hom}(W, Z_2(N)) \longrightarrow \operatorname{Aut}(N,T) , \quad \tau \longmapsto \psi_\tau$$

identifies $\operatorname{Hom}(W, Z_2(N))$ with $\operatorname{Reg}(N,T)$. Under this identification, the kernel of the map ϑ of Remark 5.3 corresponds to the subgroup $\operatorname{Reg}(N,T) \cap \operatorname{Inn}(N,T)$:

$$\varrho\colon \operatorname{Ker}(\vartheta) \xrightarrow{\cong} \operatorname{Reg}(N,T) \cap \operatorname{Inn}(N,T) .$$

Let $\bar{G} = G/Z(G)$ be the *adjoint group* of G, whose center is trivial. Suppose that $\bar{G}$ does not contain direct factors isomorphic to an odd orthogonal group. Then, the center $Z(\bar{N})$ is trivial (see Remark 1.1) and it is not difficult to deduce that the latter intersection is reduced to $\{\mathrm{id}_N\}$, so that ϑ is injective in this case. We will however get a better result by exploiting the following interpretation of ϑ kindly brought to our attention by J. M. Møller.

Remark 5.15. The homomorphism ϑ has the following interpretation in usual cohomological terms. First, consider the short exact sequence

$$0 \longrightarrow T^W \longrightarrow T \longrightarrow T/T^W \longrightarrow 0$$

of W-modules. This sequence induces the usual long exact sequence in cohomology. Recall that there is an isomorphism of functors $H^0(W;-) \cong (-)^W$ and that $H^1(W;T^W) \cong \operatorname{Hom}(W,T^W)$, because T^W is a trivial W-module. Recall also from Remark 1.1 that $T^W = Z(N)$. Finally, W being generated by reflections, we have $\operatorname{Hom}(W,Z(N)) = \operatorname{Hom}(W,Z_2(N))$. Now, we see that this long exact sequence leads to the following exact sequence:

$$0 \to (T/T^W)^W \to \operatorname{Hom}(W,Z_2(N)) \xrightarrow{\vartheta'} H^1(W;T) \to H^1(W;T/T^W) \to H^2(W;T^W) \to \dots$$

Tracing back the above identifications, one easily checks that $\vartheta' = \vartheta$. It follows in particular that

$$\operatorname{Ker}(\vartheta) \cong (T/T^W)^W = (T/Z(N))^W .$$

As a special case, if G has no direct factor isomorphic to an odd orthogonal group, then $Z(N) = Z(G)$ (see Remark 1.1), so that $\operatorname{Ker}(\vartheta) \cong \bar{T}^W \cong Z(\bar{N}) \cong (\mathbb{Z}/2)^{\bar{u}}$, where $\bar{T}$ is the maximal torus in the adjoint group $\bar{G}$ of G, $\bar{N}$ is its normalizer, and $\bar{u}$ is the number of direct factors of $\bar{G}$ isomorphic to an odd orthogonal group (for the last two indicated isomorphisms, see Remark 1.1).

Theorem 5.16. *Let G be a connected compact Lie group. Let d be the number of direct factors of G isomorphic either to* $\mathrm{SO}(3)$ *or to* $\mathrm{SO}(5)$. *Let d' be the number of direct factors of the adjoint group $\bar{G} = G/Z(G)$ that are isomorphic to an odd orthogonal group and that do* not *lift to a direct factor of G. Then, one has*

$$\operatorname{Ker}(\vartheta) \cong (\mathbb{Z}/2)^{d+d'} .$$

In particular, ϑ is injective if and only if G has no direct factor isomorphic to $\mathrm{SO}(3)$ *nor to* $\mathrm{SO}(5)$ *and every direct factor of $\bar{G}$ that is isomorphic to an odd orthogonal group lifts to a direct factor of G.*

Proof. Up to isomorphism, we can suppose that $G = G_1 \times G_2$, where G_1 has no direct factor isomorphic to an odd orthogonal group, and G_2 is a product of odd orthogonal groups, d of which are $\mathrm{SO}(3)$ or $\mathrm{SO}(5)$. We correspondingly write $T = T_1 \times T_2$ and similarly for N, W, and for the adjoint group $\bar{G}$. We have

$$(T/T^W)^W = (T_1/T_1^{W_1})^{W_1} \times (T_2/T_2^{W_2})^{W_2} .$$

To compute both factors, we apply Remark 1.1 to see that $T_i^{W_i} = Z(N_i)$ and also to compare $Z(N_i)$ with $Z(G_i)$, with $i = 1,2$. We first get

$$(T_1/T_1^{W_1})^{W_1} = (T_1/Z(G_1))^{W_1} = \bar{T}_1^{W_1} = Z(\bar{N}_1) \cong (\mathbb{Z}/2)^{d'} ,$$

since the number of direct factors of $\bar{G}_1$ isomorphic to an odd orthogonal group is d' by definition. Now, to treat the other factor, it suffices to verify that when $G = \mathrm{SO}(2n+1)$, the group $(T/T^W)^W$ is trivial if $n \geqslant 3$ and isomorphic to $\mathbb{Z}/2$ if $n = 1,2$. One can either compute this directly or invoke Example 5.14 combined with the isomorphism $\operatorname{Ker}(\vartheta) \cong (T/T^W)^W$ of Remark 5.15. ∎

Proposition 5.11 and Theorem 5.16 show that $\mathrm{Hom}(W, Z_2(N))$ completely controls the group $H^1(W;T)$, except if very specific factors of $\bar{G}'$ or of $\bar{G}$ occur as in Proposition 5.6 and Theorem 5.16. In the latter case, Lemma 5.7 may still provide some information.

Example 5.17. For $G = \mathrm{SU}(2)$, one has $Z_2(N) \cong \mathbb{Z}/2$ (see Remark 1.1) and $W \cong \mathbb{Z}/2$. It follows that $\mathrm{Hom}(W, Z_2(N)) \cong \mathbb{Z}/2$, whereas $H^1(W_{\mathrm{SU}(2)}) = 0$, by the Main Theorem. So, in this case, ϑ is *not* injective. One checks similarly that neither it is for $G = \mathrm{SO}(5)$ and $\mathrm{Spin}(4n+1)$ with $n \geqslant 1$.

We now prove Theorem 1.2 of the introduction.

Proof. OfTheorem 1.2 Write $G = (\widetilde{G} \times \mathbb{T}^k)/K$ as in the statement. If $\widetilde{G}$ contains no direct factor of type B_n, the same holds for the adjoint group $\bar{G}$, and then, Theorem 5.16 implies injectivity of ϑ. If $\widetilde{G}$ contains no direct factor isomorphic to $\mathrm{SU}(4)$, $\mathrm{Sp}(3)$, $\mathrm{Sp}(4)$, $\mathrm{Spin}(8)$ and $\mathrm{Spin}(4n+3)$ ($n \geqslant 1$), then, by Proposition 5.6, all the factors $\bar{G}_j'$ mentioned in Proposition 5.11 are regular and it follows from the latter proposition that ϑ is onto. For the equality between $Z(N)$ and $Z(G)$ under the hypothesis of (i), see Remark 1.1. This establishes the theorem. ∎

We conclude with the most familiar examples of non-semisimple groups: the unitary groups.

Example 5.18. For $n \geqslant 2$, one has an isomorphism $\mathrm{U}(n) \cong (\mathbb{S}^1 \times \mathrm{SU}(n))\big/\mathbb{Z}/n$, with $W_{ab} \cong \mathbb{Z}/2$ and $Z_2(N) = Z_2(G) \cong \mathbb{Z}/2$. We can apply Corollary 1.3 for $n \neq 2, 4$, so that $\mathrm{U}(n)$ is regular for $n \neq 2, 4$; treating the two remaining cases separately, we obtain

$$H^1(W_{\mathrm{U}(2)}) = 0 \quad \text{and} \quad H^1(W_{\mathrm{U}(n)}) \cong \mathbb{Z}/2\,, \text{ for } n \geqslant 3\,.$$

By Propositions 5.6 and 5.11, $\mathrm{U}(2)$ is regular. Our explicit computation shows that $\mathrm{U}(4)$ regular as well, although Proposition 5.11 does not apply in this case.

References

[1] Andersen, K. K. S., *The normalizer splitting conjecture for p-compact groups*, Fund. Math. **161** (1999), 1–16.

[2] Bourbaki, N., “Groupes et algèbres de Lie (Chapitres 4, 5 et 6),” Masson, Paris, 1983.

[3] —, “Groupes et algèbres de Lie (Chapitre 9),” Masson, Paris, 1983.

[4] Curtis, M., A. Wiederhold, and B. Williams, *Normalizers of maximal tori*, In: Localization in group theory and homotopy theory, Lecture Notes in Math. (Springer) **418** (1974), 31–47.

[5] Dwyer, W. G., and C. W. Wilkerson, *Homotopy fixed-point methods for Lie groups and finite loop spaces*, Ann. of Math. **139** (1994), 395–442.

[6] —, *The center of a p-compact group*, Contemp. Math. **181** (1995), 395–442.

[7] —, *Centers and Coxeter elements*, Contemp. Math. **271** (2001), 53–75.

[8] —, *Cartan involutions and normalizers of maximal tori*, Preprint (2001).

[9] Hämmerli, J.-F., “Normalizers of maximal tori and classifying spaces of compact Lie groups,” PhD Thesis, Université de Neuchâtel, Switzerland, 2000.

[10] —, *The outer automorphism group of normalizers of maximal tori in connected compact Lie groups*, J. Lie Theory **12** (2002), 357–368.

[11] —, *Normalizers of maximal tori and classifying spaces of nonconnected compact Lie groups*, In preparation.

[12] Jackowski, S., J. McClure, and B. Oliver, *Homotopy classification of self-maps of BG via G-actions I*, Ann. of Math. **135** (1992), 183–226.

[13] —, *Homotopy theory of classifying spaces of compact Lie groups*, In: Algebraic Topology and its Applications, Springer-Verlag, Berlin etc., Math. Sci. Res. Inst. Publ. **27** (1994), 81–123.

[14] Møller, J. M., *Normalizers of maximal tori*, Math. Z. **231** (1999), 51–74.

[15] —, *N-determined p-compact groups*, Fund. Math. **173** (2002), 201–300.

[16] —, *The p-compact groups of the A-family are N-determined*, Preprint, 2003.

[17] Matthey, M., *Normalizers of maximal tori and cohomology of Weyl groups*, Trans. Amer. Math. Soc., to appear.

[18] —, *Computing the second cohomology of Weyl groups acting on maximal tori*, Preprint, 2002.

[19] Notbohm, D., *On the “classifying space” functor for compact Lie groups*, J. London Math. Soc. **52** (1995), 185–198.

[20] Osse, A., *λ-structures and representation rings of compact connected Lie groups*, J. Pure Appl. Algebra **121** (1997), 69–93.

[21] Robinson, D. J. S., *Automorphisms of group extensions*, In: H. L. Manocha and J. B. Srivastava, Eds., Algebra and its Applications, Lecture Notes in Pure and Applied Math. (Marcel Dekker) **91** (1984), 163–167.

[22] Tits, J., *Normalisateurs de tores I. Groupes de Coxeter étendus*, J. Algebra **4** (1966), 96–116.

[23] —, *Sur les constantes de structure et le théorème d'existence des algèbres de Lie semi-simples*, Inst. Hautes Études Sci. Publ. Math. **31** (1966), 21–58.

Jean-François Hämmerli
University of Lausanne
IGAT (Institute for Geometry,
Algebra and Topology)
BCH, EPFL
CH-1015 Lausanne
Switzerland
jean-francois.haemmerli@ima.unil.ch

Michel Matthey
University of Lausanne
IGAT (Institute for Geometry,
Algebra and Topology)
BCH, EPFL
CH-1015 Lausanne
Switzerland
michel.matthey@ima.unil.ch

Ulrich Suter
Institute for Mathematics
University of Neuchâtel
Rue Émile-Argand 11
CH-2007 Neuchâtel
Switzerland
ulrich.suter@unine.ch

Received September 1, 2003
and in final form November 18, 2003