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On Inverse Limits of Finite Dimensional Lie Groups

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Abstract. We give a short proof of the Hofmann–Morris Theorem characterizing inverse limits of finite dimensional Lie groups [Hofmann, K. H., and S. A. Morris: Projective limits of finite dimensional Lie groups, Proc. Lond. Math. Soc. 87 (2003), 647–676, Theorem 4.7]. The proof depends on the Gleason–Palais characterization of finite dimensional Lie groups [Gleason, A., and R. Palais: On a class of transformation groups, Amer. J. Math. 79 (1957), 631–648, Theorem 7.2].
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In [4], Theorem 4.7, Hofmann and Morris showed that if a topological group G is an inverse limit of finite dimensional Lie groups, then every neighborhood of 1 in G contains a closed normal subgroup of G with a finite dimensional Lie quotient group G/N. (See also [5], Chapter 3.) The proof of this theorem is quite involved. In this paper we obtain a short proof of this theorem using the Gleason–Palais characterization of finite dimensional Lie groups [3], Theorem 7.2, which in turn depends on the solution of Hilbert's Fifth Problem.

We shall use use the following basic functorial construction: For any topological group K, we let $\mathfrak{L}(K)$ be the set of all continuous group homomorphisms $f\colon \mathbb{R} \to K$ equipped with the compact open topology where $0 \in \mathfrak{L}(K)$ denotes the trivial homomorphism. For $r \in \mathbb{R}$, $f \in \mathfrak{L}(K)$, let $rf \in \mathfrak{L}(K)$ be defined by $rf(t) = f(rt)$ for all $t \in \mathbb{R}$. Define the continuous map $\exp\colon \mathfrak{L}(K) \to K$ by $\exp(f) = f(1)$ for $f \in \mathfrak{L}(K)$. If K_1 and K_2 are two topological groups and if $\phi\colon K_1 \to K_2$ is any continuous group homomorphism, we define the continuous map $\mathfrak{L}(f)\colon \mathfrak{L}(K_1) \to \mathfrak{L}(K_2)$ by $\mathfrak{L}(\phi)(f) = \phi \circ f$. Then $\mathfrak{L}(\phi)(rf) = r\mathfrak{L}(\phi)(f)$ for all $r \in \mathbb{R}$, $f \in \mathfrak{L}(K_1)$ and $\exp_{K_1} \circ \mathfrak{L}(\phi) = \phi \circ \exp_{K_1}$.

Now we shall prove

Theorem. (Hofmann-Morris [4], Theorem 4.7) *For a Hausdorff topological group G, the following two statements are equivalent:*

1. *G is the inverse limit of finite dimensional Lie groups.*
2. *G is a complete topological group that has a filter basis of closed normal subgroups N converging to $1 \in G$ such that all quotient groups G/N are finite dimensional Lie groups.*

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Proof. Since the implication $2 \Rightarrow 1$ is straightforward (see e.g. [2], TGIII. 60, Proposition 2) we have to concentrate on the proof of the implication $1 \Rightarrow 2$.

We assume that $G = \lim_i G_i$ for finite dimensional Lie groups G_i, $i \in I$ and we write $\mathfrak{g}$ for $\mathfrak{L}(G)$ and $\mathfrak{g}_i$ for $\mathfrak{L}(G_i)$. For $i \in I$ let $p_i: G \to G_i$ be the canonical projection and for $i < j$, $i, j \in I$, let $\phi_{ij}: G_j \to G_i$ be the connecting homomorphism. We have $\mathfrak{g} = \lim_i \mathfrak{g}_i$ and for each $i \in I$, the space $\mathfrak{g}_i$ has a finite dimensional real vector space structure given by $(X+Y)(t) = \lim_{n\to\infty} X(\frac{t}{n})Y(\frac{t}{n})$ and $(rX)(t) = X(rt)$ for all $r, t \in \mathbb{R}$, $X, Y \in \mathfrak{g}_i$ such that $\exp_{G_i}: \mathfrak{g}_i \to G_i$ is a local homeomorphism at 0. Further, $\mathfrak{L}(\phi_{ij}): \mathfrak{L}(G_j) \to \mathfrak{L}(G_i)$ is a real vector space homomorphism for all $i < j$, $i, j \in I$ and $\mathfrak{g}$ inherits a real topological vector space structure such that $\mathfrak{L}(p_i)$ is a continuous real vector space homomorphism for all $i \in I$ and $\exp_G = \lim_i \exp_{G_i}$ (see [6], p. 118—124 and [2], TGI.28). The proof now proceeds through several lemmas.

Lemma 1. ([4], Lemma 3.2) *For all $i \in I$ there is a $k_i > i$ in I such that $\mathfrak{L}(p_i)(\mathfrak{g}) = \mathfrak{L}(\phi_{ik})(\mathfrak{g}_k)$ for all $k \geq k_i$.*

Proof. Note that $\mathfrak{L}(p_i)(\mathfrak{g}) \subseteq \bigcap_{k\geq i} \mathfrak{L}(\phi_{ik}(\mathfrak{g}_k)$. Choose a $k_i \geq i$ such that $\dim \mathfrak{L}\big(\phi_{ik_i}(\mathfrak{g}_{k_i})\big) = \min\{\mathfrak{L}\big(\phi_{ik}(\mathfrak{g}_k)\big) : k \geq i\}$. Then $\mathfrak{L}(\phi_{ik})(\mathfrak{g}_k) = \mathfrak{L}(\phi_{ik_i})(\mathfrak{g}_{k_i})$ for $k \geq k_i$.

Let $x \in \mathfrak{L}(\phi_{ik_i})(\mathfrak{g}_{k_i})$. We have to show that $x \in \mathfrak{L}(p_i)(\mathfrak{g})$. Now $\{\mathfrak{L}(\phi_{ik})^{-1}(x) : k \geq k_i\}$ is an inverse system of affine sets. Note that for $k \geq k_i$, the family

$$S_k = \{\emptyset\} \cup \{y + M : y \in \mathfrak{L}(\phi_{ik})^{-1}(x), M \leq \ker \mathfrak{L}(\phi_{ik})\}$$

satisfies all the conditions of [1], p. 198, Theorem 1. Condition (ii) is the only non-trivial condition to check. So suppose that $\{y_\alpha + M_\alpha \in S_k : \alpha \in J\}$ be a family of sets with the non-trivial finite intersection property. Provide every finite dimensional real vector space by its unique Hausdorff vector space topology and consider the natural homomorphism $\psi: \mathfrak{g}_k \to \prod_{\alpha\in J} \mathfrak{g}_k/M_\alpha$. We have $(y_\alpha + M_\alpha)_{\alpha\in J} \in \overline{\psi(\mathfrak{g}_k)} = \psi(\mathfrak{g}_k)$, hence $\bigcap_{\alpha\in J}(y_\alpha + M_\alpha) \neq \emptyset$. It follows that the inverse limit S of $\{\mathfrak{L}(\phi_{ik})^{-1}(x) : k \geq k_i\}$ is a nonempty subset of the limit $\mathfrak{g}$ of the $\mathfrak{g}_k$. Let $y \in S$. Then $x = \mathfrak{L}(p_i)(y)$, and this had to be shown. ■

Lemma 2. *For all $i \in I$, the quotient $G/\ker p_i$ is locally path connected.*

Proof. Let U be an open identity neighborhood of $G/\ker p_i$. We have to show that U contains a path connected identity neighborhood. Each morphism $p_j: G \to G_j$ factors in the form

$$G \xrightarrow{q_j} G/\ker p_j \xrightarrow{p_j'} G_i$$

with a quotient morphism q_j and an injective morphism p_i' into a Lie group.

We may assume that there is an $j \in I$, $i \leq j$ and an open identity neighborhood V in G_j such that $U = q_i\big(p_j^{-1}(V)\big)$. Let $k_j \in I$ be determined as in Lemma 1 and consider any $k \geq k_j$. We have a commutative diagram

$$\begin{array}{ccc}
G & \xleftarrow{\text{id}} & G \\
{\scriptstyle q_i}\big\downarrow & & \big\downarrow{\scriptstyle q_k} \\
G/\ker p_i & \xleftarrow{\pi_{ik}} & G/\ker p_k \\
{\scriptstyle p_i'}\big\downarrow & & \big\downarrow{\scriptstyle p_k'} \\
G_i & \xleftarrow[\phi_{ik_i}]{} & G_k,
\end{array}$$

where $\pi_{ik}(g\ker p_k) = g\ker p_i$. Let B be an open ball around 0 in $\mathfrak{g}_k$ chosen so small that $\exp_{G_k}\colon \mathfrak{g}_k \to G_k$ maps B homeomorphically onto an open identity neighborhood W of the Lie group G_k and that $\phi_{jk}(W) \subseteq V$. Then $p_k^{-1}(W) \subseteq p_j^{-1}(V)$, and thus $U' = q_i\big(p_k^{-1}(W)\big)$ is an open identity neighborhood of $G/\ker p_i$ that is contained in U. The Lemma will be proved if we show that U' is ruled by local one-parameter subgroups. So let $u \in U'$. Then there is a $z \in p_k^{-1}(W)$ such that $u = q_i(z)$. Since $p_k(z) \in W = \exp_{G_k}(B)$ there is an $X \in B \subseteq \mathfrak{g}_k$ such that $p_k(z) = \exp_{G_k} X$. Now we apply Lemma 1 in order to observe that $\mathfrak{L}(\phi_{jk})(\mathfrak{g}_k) = \mathfrak{L}(p_j)(\mathfrak{g})$, and therefore $\mathfrak{L}(\phi_{ik})(\mathfrak{g}_k) = \mathfrak{L}(\phi_{ij}\phi_{jk})(\mathfrak{g}_k) = \mathfrak{L}(\phi_{ij})\mathfrak{L}(p_j)(\mathfrak{g}) = \mathfrak{L}(p_i)(\mathfrak{g})$. Hence we find a $Y \in \mathfrak{g}$ such that $\mathfrak{L}(\phi_{ik})(X) = \mathfrak{L}(p_i)(Y)$. For all $t \in \mathbb{R}$ we have $\phi_{ik}(\exp_{G_k} tX) = \exp_{G_i} t\mathfrak{L}(\phi_{ik})(X) = \exp_{G_i} t\mathfrak{L}(p_i)(Y) = p_i(\exp_G tY) = p_i'q_i(\exp_G tY)$. For $t = 1$ we get $p_i'(u) = p_i(z) = \phi_{ik}\big(p_k(z)\big) = \phi_{ik}(\exp_{G_k} X) = p_i'\big(q_i(\exp_G Y)$. Since p_i' is injective, $u = q_i(\exp_G Y)$.

As B is convex, $[0,1]\cdot X \subseteq B$ and thus $\exp_{G_k}[0,1]\cdot X \subseteq W$. So for $t \in [0,1]$, we have $p_k(\exp_G tY) = \exp_{G_k} t\mathfrak{L}(p_k)Y = \exp_{G_k} tX \in W$, whence $\exp_G tY \in p_k^{-1}(W)$ and thus $q_i(\exp_G tY) \in U'$. Hence $t \mapsto q_i(\exp_G tY) : [0,1] \to U'$ is a path connecting the identiy and u in U', proving our claim. ∎

Lemma 3. *If C is a compact subset of $G/\ker p_i$, then $\dim C \le \dim G_i$.*

Proof. The injective morphism $p_i'\colon G/\ker p_i \to G_i$ of topological groups maps the compact space C homeomorphically to a subspace of the Lie group G_i. Thus $\dim C = \dim p_i'(C) \le \dim G_i$. ∎

We now record the

Gleason–Palais Theorem. ([4], Theorem 7.2) *A locally arcwise connected topological group G in which the compact metrizable subspaces are of bounded dimension is a Lie group.* ∎

Finally we finish the proof of the theorem. By Lemmas 2 and 3 and the Gleason–Palais Theorem, each $G/\ker p_i$ is a Lie group. Since the filterbasis $\{\ker p_i : i \in I\}$ converges to 1, this completes the proof. ∎

The two conditions of the Gleason–Palais Theorem characterising a Lie group are sufficient and necessary. Neither condition alone is sufficient: The character group $\widehat{\mathbb{Q}}$ of the discrete additive group of rationals is a one dimensional compact abelian non-Lie group (failing to be locally connected); the additive group $L^1([0,1],\mathbb{Z})$ of all (equivalence classes modulo null functions of) integrally valued Lebesgue integrable functions is a complete, contractible and locally contractible metric topological abelian group that has no one parameter subgroups whatsoever (except the constant one).

Corollary. *Let $G = \lim_i G_i$ be a projective limit of finite dimensional Lie groups G_i, and let $p_i\colon G \to G_i$ be the canonical limit projections. Then $\{\ker p_i : i \in I\}$ is a filter base of closed normal subgroups of G that converges to $1 \in G$, and all quotient groups $G/\ker p_i$ are finite dimensional Lie groups.* ∎

Conjecture. *For a Hausdorff topological group G, the following two statements are equivalent:*

1. *G is the inverse limit of locally compact groups.*
2. *G is a complete topological group that has a filter basis of closed normal subgroups N converging to $1 \in G$ such that all quotient groups G/N are locally compact.*

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