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Low Order Terms of the Campbell-Hausdorff Series and the Kashiwara-Vergne Conjecture

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Abstract. The Kashiwara-Vergne conjecture states that the Campbell-Hausdorff series of a Lie algebra can be written using a certain couple of functions. In this paper we consider universal solutions which apply to every real finite-dimensional Lie algebra. We prove that a universal solution of the Kashiwara-Vergne conjecture verifying a natural symmetry condition is unique up to order one. In the Appendix by the second author, this result is used to show that the solutions of the Kashiwara-Vergne conjecture for a quadratic Lie algebra which exist in literature are not universal.
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1. Introduction

Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{K} = \mathbb{R}$ (or $\mathbb{K} = \mathbb{C}$). Recall (see e.g. [3] Chap. 2, §6, Thm. 2, page 56) that the Campbell-Hausdorff series

$$Z(X, Y) = X + Y + \frac{1}{2}[X, Y] + \cdots$$

defines an associative multiplication for $X, Y \in \mathfrak{g}$ sufficiently small. Here $\dots$ stands for a series in multiple Lie brackets between X and Y. If $\mathfrak{g}$ is an abelian Lie algebra, the Campbell-Hausdorff series reduces to $Z(X, Y) = X + Y$. In the general case, there is a classical Dynkin formula [4] for $Z(X, Y)$ expressing the coefficients in terms of ratios of factorials. Recently, Kathotia [6] gave a new formula for $Z(X, Y)$ using Kontsevich's diagrammatic technique [7].

In [5], Kashiwara and Vergne put forward the following conjecture on the properties of the Campbell-Hausdorff series which sometimes is referred to as the "combinatorial Kashiwara-Vergne conjecture" (see for instance [8]). To state it, we introduce the notation

$$\varphi(t) := \frac{t}{\exp(t) - 1} = 1 - \frac{1}{2}t + o(t)$$

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for the generating series of Bernoulli numbers, and a separate notation for the analytic function

$$\psi(t) := -(\varphi(t) - 1)/2\,.$$

Kashiwara-Vergne conjecture. *Let $\mathfrak{g}$ be a finite dimensional Lie algebra over $\mathbb{K}$. There exists a pair of $\mathfrak{g}$-valued analytic functions A and B defined on an open subset $U \subset \mathfrak{g} \times \mathfrak{g}$ containing $(0,0)$, such that $A(0,0) = B(0,0) = 0$, and for any $(X,Y) \in U$ one has*

$$Z(Y,X) - X - Y = (\mathrm{id} - \exp(-\operatorname{ad} X))A(X,Y) + (\exp(\operatorname{ad} Y) - \mathrm{id})B(X,Y), \quad (1)$$

$$\operatorname{tr}(\operatorname{ad} X \circ \delta_1 A(X,Y) + \operatorname{ad} Y \circ \delta_2 B(X,Y)) = \operatorname{tr}\left(\psi(\operatorname{ad} X) + \psi(\operatorname{ad} Y) - \psi(\operatorname{ad} Z)\right), \quad (2)$$

where $Z = Z(X,Y)$, and $\delta_1 A(X,Y)$, $\delta_2 B(X,Y) \in \mathrm{End}(\mathfrak{g})$ are defined as follows:

$$\delta_1 A(X,Y) : U \mapsto \left.\frac{d}{dt}\right|_{t=0} A(X + tU, Y), \quad \delta_2 B(X,Y) : U \mapsto \left.\frac{d}{dt}\right|_{t=0} B(X, Y + tU).$$

This conjecture was established for solvable Lie algebras in [5] and for quadratic Lie algebras in [9]. Recently, the general case was settled in [2] based on the earlier works [7, 6, 8].

We denote by $\mathbb{K}[[t]]$ the ring of formal power series, and we call a solution of the Kashiwara-Vergne conjecture *universal* if A and B are given by series in Lie polynomials of the variables X and Y:

$$A(X,Y) = \rho X + \beta(\operatorname{ad} X)(Y) + o(Y)$$
$$B(X,Y) = \alpha X + \gamma(\operatorname{ad} X)(Y) + o(Y)$$

with $\beta(t)$, $\gamma(t) \in \mathbb{K}[[t]]$, $\alpha, \rho \in \mathbb{K}$, and both $o(Y)$ are of type

$$o(Y) \in \sum_{k \geq 2} \sum_{\substack{j_1, \dots, j_k \geq 0 \\ j_{k-1} < j_k}} \mathbb{K}\, \operatorname{ad}_{(\operatorname{ad} X)^{j_1} Y} \circ \cdots \circ \operatorname{ad}_{(\operatorname{ad} X)^{j_{k-1}} Y} \circ (\operatorname{ad} X)^{j_k}(Y).$$

If (A, B) is a universal solution, the coefficients of the Taylor expansions of A and B are the same for all Lie algebras over $\mathbb{K}$.

The set of solutions of the Kashiwara-Vergne conjecture carries a natural $\mathbb{Z}/2\mathbb{Z}$-action,

$$(A(X,Y), B(X,Y)) \mapsto (B(-Y,-X), A(-Y,-X)). \quad (3)$$

A solution is called *symmetric* if it is stable with respect to this action. Averaging of any solution produces a symmetric solution. Hence, without loss of generality we can restrict our attention to symmetric solutions. It is well-known (see for instance [8]) that α, ρ and $\beta(t)$ are uniquely determined by the Kashiwara-Vergne equations and by the symmetry condition. In this note we prove the uniqueness statement for the function $\gamma(t)$ (see Theorem 5.2). Thus, the symmetric universal solution of the Kashiwara-Vergne conjecture is unique up to order one in Y.

In the Appendix by the second author, this result is applied to show that solutions of the Kashiwara-Vergne conjecture for quadratic Lie algebras obtained in [9] and [1] are not universal.

2. Preliminaries

In this section we recall some elementary properties of Lie algebras.

Remark 2.1. (Free Lie algebra with two generators). We denote by $L_{\mathbb{K}}(x,y)$ the free Lie $\mathbb{K}$-algebra with generators x and y. In this section we use the Hall basis H of $L_{\mathbb{K}}(x,y)$ defined in [3] (Definition 2, page 27). Recall that H consists of Lie words with the following order relation: $x, y \in H$ and $x < y$; if the number of Lie brackets in $a \in H$ is smaller than the number of Lie brackets in $b \in H$ then $a < b$; and we omit the description of the order relation for a and b of equal length. The basis H is built inductively starting with $x, y, [x,y]$, and one adds the elements of the form $[a,[b,c]]$ such that $a, b, c, [b,c] \in H$, $b \le a \le [b,c]$, and $b < c$. In particular, one easily proves by induction that

$$\forall n \ge 0, \quad (\operatorname{ad} x)^n(y) \in H.$$

Remark 2.2. (Finite-dimensional Lie algebras $\mathfrak{g}_N$). Let $N \ge 2$, and I_N be the ideal of $L_{\mathbb{K}}(x,y)$ generated by $\{\operatorname{ad} Z_1 \circ \cdots \circ \operatorname{ad} Z_N(Z); Z_1, ..., Z_N, Z \in L_{\mathbb{K}}(x,y)\}$. Then $\mathfrak{g}_N := L_{\mathbb{K}}(x,y)/I_N$ is an N-nilpotent Lie algebra with basis H/I_N. In particular, $\mathfrak{g}_N$ is a finite-dimensional Lie $\mathbb{K}$-algebra.

Proposition 2.3. *Let $\xi(t) \in \mathbb{K}[[t]]$. The following statements are equivalent:*

i) *for any Lie $\mathbb{K}$-algebra $\mathfrak{g}$ and all $X, Y \in \mathfrak{g}$ we have $\xi(\operatorname{ad} X)(Y) = 0$,*

ii) $\xi(t) = 0$,

iii) *for any finite-dimensional Lie $\mathbb{K}$-algebra $\mathfrak{g}$ and all $X, Y \in \mathfrak{g}$ we have $\xi(\operatorname{ad} X)(Y) = 0$.*

Proof. First we show that i) implies ii). Let $n \in \mathbb{N}$. By rescaling $X \mapsto tX$ and applying $\frac{d^n}{dt^n}\big|_{t=0}$ we get $\xi_n(\operatorname{ad} X)^n(Y) = 0$. Choosing $\mathfrak{g} = L_{\mathbb{K}}(x,y)$, $X = x$ and $Y = y$ we get $\xi_n = 0$.

This proof is easily modified to get that iii) implies ii). In fact, it is sufficient to replace the infinite-dimensional Lie algebra $L_{\mathbb{K}}(x,y)$ with any $\mathfrak{g}_N$ such that $n \le N-1$. ■

3. The Campbell-Hausdorff series

In this section we derive a formula for $\beta(t)$. This requires a rather standard manipulation with the Campbell-Hausdorff series which we include for completeness of the presentation.

The free Lie algebra $L_{\mathbb{K}}(x,y)$ with generators x, y is graded with degrees of x and y equal to 1. We consider the degree completion $\overline{U}(L_{\mathbb{K}}(x,y))$ of its universal enveloping algebra. Then, for any $z \in L_{\mathbb{K}}(x,y)$ the expression

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \in \overline{U}(L_{\mathbb{K}}(x,y))$$

defines a group-like element in $\overline{U}(L_{\mathbb{K}}(x,y))$. Moreover, there is an inverse function ln mapping group-like elements to $L_{\mathbb{K}}(x,y)$, and the Campbell-Hausdorff series is the expression for

$$\ln(e^x e^y) = x + y + \frac{1}{2}[x,y] + \ldots$$

Proposition 3.1. *The following formula holds true in the free Lie algebra with two generators x, y*

$$\left.\frac{d}{dt}\right|_{t=0} Z(ty,x) = \varphi(\operatorname{ad} x)(y).$$

Proof. Let's denote $z_t = Z(ty,x) = \ln\left(e^{ty}e^x\right)$ and introduce $g_{s,t} = e^{sz_t} \in \overline{U}(L_{\mathbb{K}}(x,y))$. We use the identity

$$\partial_s\left((\partial_t g_{s,t})g_{s,t}^{-1}\right) = g_{s,t}\,\partial_t\left(g_{s,t}^{-1}(\partial_s g_{s,t})\right)\, g_{s,t}^{-1}$$

and integrate over $[0,1]$ with respect to s:

$$\begin{array}{rcl} \left(\frac{d}{dt}\, e^{z_t}\right) e^{-z_t} & = & \left(\frac{d}{dt}\, g_{1,t}\right) g_{1,t}^{-1} \\ & = & \int_0^1 \partial_s\left((\partial_t g_{s,t})g_{s,t}^{-1}\right)\, ds \\ & = & \int_0^1 g_{s,t}\,\partial_t\left(g_{s,t}^{-1}(\partial_s g_{s,t})\right)\, g_{s,t}^{-1}\, ds \\ & = & \int_0^1 \left(e^{s\,\operatorname{ad} z_t}\,\frac{dz_t}{dt}\right)\, ds \\ & = & \frac{\exp(\operatorname{ad} z_t)-1}{\operatorname{ad} z_t}\,\frac{dz_t}{dt}. \end{array}$$

By putting $t=0$ one obtains

$$\left.\frac{dz_t}{dt}\right|_{t=0} = \frac{\operatorname{ad} x}{\exp(\operatorname{ad} x)-1}\, y,$$

where we have used $z_0 = x$ and

$$\left(\frac{d}{dt}\, e^{z_t}\right) e^{-z_t} = \left(\frac{d}{dt}\, e^{ty}e^x\right) e^{-x}e^{-ty} = y. \blacksquare$$

Proposition 3.2. *In a universal solution of the Kashiwara-Vergne conjecture,*

$$\beta(t) = \beta_\alpha(t) := \varphi(-t)\left(\frac{\varphi(t)-1}{t} + \alpha\right).$$

Proof. In (1) we rescale Y by tY, compute the derivative at $t=0$, and use Proposition 3.1 to get a formula for $\left.\frac{d}{dt}\right|_{t=0} Z(tY,X)$:

$$\varphi(\operatorname{ad} X)(Y) - Y = (\mathrm{id} - \exp(-\operatorname{ad} X)) \circ \beta(\operatorname{ad} X)(Y) - \alpha(\operatorname{ad} X)(Y).$$

By definition, $(\mathrm{id} - \exp(-t)) \equiv \varphi(-t)^{-1}t$, so we get

$$\varphi(-\operatorname{ad} X) \circ \big(\varphi(\operatorname{ad} X) - \mathrm{id} + \alpha\, \operatorname{ad} X\big)(Y) = \operatorname{ad} X \circ \beta(\operatorname{ad} X)(Y). \tag{4}$$

Equation (4) is verified for $X, Y \in \mathfrak{g}$ sufficiently small. Since φ is an entire function, (4) is verified for all $X, Y \in \mathfrak{g}$. As we look for a universal solution of the Kashiwara-Vergne conjecture, (4) is verified for every finite-dimensional $\mathbb{K}$-Lie algebra $\mathfrak{g}$ and for all $X, Y \in \mathfrak{g}$. Then Proposition 2.3 applies and we get $\varphi(-t)(\varphi(t) - 1 + \alpha t) = t\beta(t)$. $\blacksquare$

4. The equation with traces

In this section we derive formulas for ρ and $\gamma(t)$ from the Kashiwara-Vergne equation (2). We begin with a technical remark, where $\mathbb{K}[t]$ is the ring of polynomials.

Remark 4.1. Let $\lambda, \mu \in \mathbb{K} \setminus \{0\}$ be two distinct numbers, and $\mathfrak{g}_{\lambda,\mu} = \mathbb{K}a \oplus \mathbb{K}b \oplus \mathbb{K}c$ be the 3-dimensional Lie algebra with Lie brackets $[a,b]=0$, $[a,c]=\lambda c$, $[b,c]=\mu c$.

The expression $Z(\epsilon_1 a, \epsilon_2 b)$ is well-defined if $\epsilon_1, \epsilon_2 \in \mathbb{K}$ are sufficiently close to $0 \in \mathbb{K}$. As $[a,b]=0$, the Campbell-Hausdorff series gives

$$Z(\epsilon_1 a, \epsilon_2 b) = \epsilon_1 a + \epsilon_2 b.$$

Moreover, for all polynomial $\xi(t,u) \in \mathbb{K}[t,u]$ we have

$$\operatorname{tr}(\xi(\operatorname{ad} a, \operatorname{ad} b)) = \xi(\lambda,\mu) + 2\xi(0,0). \tag{5}$$

Proposition 4.2. *In a universal solution of the Kashiwara-Vergne conjecture,*

i) $\rho = 0$,

ii) $\gamma(t) = \gamma_\alpha(t) := \beta_\alpha(t) - \beta_\alpha(0) + \psi'(0) - \psi'(t)$.

Proof. Here we write $\beta(t) = \sum_{n\ge 0} \beta_n t^n$ and we use the analogue notations also for $\gamma(t)$ and $\psi(t)$. We consider Equation (2) and we rescale $Y \mapsto \epsilon Y$. Then $(\operatorname{ad} Y\epsilon) \circ \delta_2 B(X, Y\epsilon) = \operatorname{ad} Y \circ \gamma(\operatorname{ad} X)\epsilon + o(\epsilon)$, and

$$\delta_1 A(X, Y\epsilon) = \rho \operatorname{id} - \sum_{n\ge 1} \beta_n \sum_{j=0}^{n-1} (\operatorname{ad} X)^j \circ \operatorname{ad}\big((\operatorname{ad} X)^{n-1-j} Y\big)\epsilon + o(\epsilon).$$

Choosing the Lie algebra in Remark 4.1 as $\mathfrak{g}$, $Y = b$, $X = a_1 := \epsilon_1 a$ with $\epsilon_1 \neq 0$, and ϵ, ϵ_1 sufficently closed to $0 \in \mathbb{K}$, we get

$$\begin{aligned} \operatorname{ad} a_1 \circ \delta_1 A(a_1, b\epsilon) &= \rho \operatorname{ad} a_1 - \sum_{n\ge 1} \epsilon \beta_n (\operatorname{ad} a_1)^n \circ \operatorname{ad} b + o(\epsilon), \\ \psi(\operatorname{ad} a_1) + \psi(\epsilon \operatorname{ad} b) - \psi(\operatorname{ad} a_1 + \epsilon \operatorname{ad} b) &= \epsilon(\psi'(0)\mathrm{id} - \psi'(\operatorname{ad} a_1)) \circ \operatorname{ad} b + o(\epsilon). \end{aligned}$$

Then Equation (2) gives

$$\operatorname{tr}(\rho \operatorname{ad} a_1) + \epsilon \operatorname{tr}\Big(\Big(- \sum_{n\ge 1} \beta_n (\operatorname{ad} a_1)^n + \gamma(\operatorname{ad} a_1) - \psi'(0)\mathrm{id} + \psi'(\operatorname{ad} a_1)\Big) \circ \operatorname{ad} b\Big) + o(\epsilon) = 0. \tag{6}$$

i) Setting $\epsilon = 0$ we get $\operatorname{tr}(\rho \operatorname{ad} a_1) = 0$. Then property (5) gives $\rho = 0$.

ii) Applying $\frac{d}{d\epsilon}\big|_{\epsilon=0}$ to Equation (6) we get

$$\operatorname{tr}\Big(\big(- \beta(\operatorname{ad} a_1) + \beta(0) + \gamma(\operatorname{ad} a_1) - \psi'(0)\operatorname{id} + \psi'(\operatorname{ad} a_1)\big) \circ \operatorname{ad} b\Big) = 0. \tag{7}$$

Let $n \in \mathbb{N}$. Applying $\frac{d^n}{d\epsilon_1^n}\Big|_{\epsilon_1=0}$ to (7) and using (5) one can show that that $-\beta_n + \beta(0)\delta_{n.0} + \gamma_n + \psi'_n - \psi'(0)\delta_{n,0}$ is zero. In particular $-\beta(t) + \beta(0) + \gamma(t) + \psi'(t) - \psi'(0) = 0$. ■

5. Symmetric solutions

In the previous sections we did not determine the value of the constant α.

Proposition 5.1. *In a symmetric solution of the Kashiwara-Vergne conjecture, one has* $\alpha = \frac{1}{4}$.

Proof. By imposing the symmetry condition (3) on a universal solution of the Kashiwara-Vergne conjecture, we obtain

$$\beta_\alpha(\operatorname{ad} X)Y + o(Y) = -\alpha Y - \gamma_\alpha(-\operatorname{ad} Y)X + o(X).$$

Setting $X = 0$ gives $\beta_\alpha(0) = -\alpha$. Since

$$\beta_\alpha(0) \equiv \varphi(0)(\varphi'(0) + \alpha) = -\frac{1}{2} + \alpha,$$

we get $\alpha - 1/2 = -\alpha$ and $\alpha = 1/4$. ■

We summarize our results in the following Theorem:

Theorem 5.2. *In a universal symmetric solution of the Kashiwara-Vergne conjecture the lower order terms are of the following form,*

$$\begin{aligned} A(X,Y) &= \beta_{\frac{1}{4}}(\operatorname{ad} X)(Y) + o(Y), \\ B(X,Y) &= \frac{1}{4}X + \Big(\underbrace{\beta_{\frac{1}{4}}(\operatorname{ad} X) - \psi'(\operatorname{ad} X) + \frac{1}{2}\operatorname{id}}_{\gamma_{\frac{1}{4}}(\operatorname{ad} X)}\Big)(Y) + o(Y). \end{aligned}$$

with $\beta_\alpha(t)$ *given in* Proposition 3.2.

6. Appendix: Comparison with quadratic solutions

by E. Petracci

Vergne and Alekseev-Meinrenken both considered a quadratic Lie algebra and obtained symmetric solutions. We ask whether these solutions are universal. In fact, quadratic Lie algebras have the special property that $\operatorname{tr}((\operatorname{ad} X)^{2n} \circ \operatorname{ad} Y) = 0$ for any $n \in \mathbb{N}$ and any pair of vectors X, Y. This simplifies the Kashiwara-Vergne equation (2).

In the following remarks we use the notation $\gamma(t)_{\mathrm{odd}} := (\gamma(t) - \gamma(t))/2$ for any power series $\gamma(t) \in \mathbb{K}[[t]]$.

Remark 6.1. (Vergne's solution for quadratic Lie algebras). We denote by $B_V(X,Y)$ the B found by Vergne in her paper [9]. Following this article we find $B_V(X,Y) = \frac{1}{4}X + \gamma_V(\operatorname{ad} X)(Y) + o(Y)$. Let

$$R(t) := \frac{e^t - e^{-t} - 2t}{t^2}.$$

after a long calculation we find that the power series $\gamma_V(t)$ is given by

$$\begin{aligned} t{\gamma_V}'(t) + 2\gamma_V(t) &= \frac{1}{8}t - \frac{1}{2}t\varphi(-t)R(t)\varphi'(t) = \\ &= \frac{1}{8}t + \frac{1}{12}t^2 + \frac{1}{72}t^3 - \frac{1}{360}t^4 + o(t^4). \end{aligned}$$

This differential equation gives $\gamma_V(t)_{\text{odd}} = \gamma_{\frac{1}{4}}(t)_{\text{odd}}$. For a symmetric universal solution we have (see Theorem 5.2)

$$t\gamma'_{\frac{1}{4}}(t) + 2\gamma_{\frac{1}{4}}(t) = \frac{1}{8}t + \frac{1}{12}t^2 + \frac{1}{72}t^3 - \frac{1}{480}t^4 + o(t^4).$$

In particular, the symmetric solution found by Vergne for a quadratic Lie algebra is not universal.

Remark 6.2. (Alekseev-Meinrenken's solution for quadratic Lie algebras). Let $B_{AM}(X,Y) = \alpha_{AM} + \gamma_{AM}(\operatorname{ad} X)(Y) + o(Y)$ the B found by Alekseev and Meinrenken in [1], $\beta_{AM}(t)$ their power series $\beta(t)$, etc.

Following their paper [1] and the paper [9] of Vergne, after some efforts we find the following formulas. Let $g(t) = \frac{1}{2}R(t)$, and let $\Pi(t)$ be the power series such that $t\Pi'(t) + 2\Pi(t) = \frac{1}{2}\varphi(t)^{-1} - g(t)\varphi(-t)(1-\varphi(t))$. Then

$$\begin{aligned} \beta_{AM}(t) &= \Pi(t) - \frac{1}{4}\left(g(t)\varphi(-t) - \frac{1}{2}\varphi(t)^{-1}\right)t - \frac{1}{2}\varphi(t)^{-1} + \\ &\qquad + g(t)\varphi(-t)(1-\varphi(t)), \\ \rho_{AM} &= 0, \\ \gamma_{AM}(-t) + \gamma_V(t) &= \varphi(-t)g(t)t\left(\beta_{AM}(-t) - \frac{1}{4}\varphi(t) - \varphi'(t)\right) + \\ &\qquad - \frac{1}{2}\varphi(t)^{-1}t\beta_{AM}(-t) - \frac{1}{8}t, \\ \alpha_{AM} &= \frac{1}{4}. \end{aligned}$$

Using Maple, we get $\beta_{AM}(t) = \beta_{\frac{1}{4}}(t)$, $\gamma_{AM}(t)_{\text{odd}} = \gamma_V(t)_{\text{odd}}$, and

$$t\gamma'_{AM}(t) + 2\gamma_{AM}(t) = \frac{1}{8}t + \frac{1}{12}t^2 + \frac{1}{72}t^3 - \frac{1}{720}t^4 + o(t^4).$$

In particular, the symmetric solution of Alekseev and Meinrenken is not universal, and it is different from the solution of Vergne.

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