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The CR Structure of Minimal Orbits in Complex Flag Manifolds

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Abstract. Let $\widehat{\mathbf{G}}$ be a complex semisimple Lie group, $\mathbf{Q}$ a parabolic subgroup and $\mathbf{G}$ a real form of $\widehat{\mathbf{G}}$. The flag manifold $\widehat{\mathbf{G}}/\mathbf{Q}$ decomposes into finitely many $\mathbf{G}$-orbits; among them there is exactly one orbit of minimal dimension, which is compact. We study these minimal orbits from the point of view of CR geometry. In particular we characterize those minimal orbits that are of finite type and satisfy various nondegeneracy conditions, compute their fundamental group and describe the space of their global CR functions. Our main tool are parabolic CR algebras, which give an infinitesimal description of the CR structure of minimal orbits.
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1. Introduction

In the last decades the study of CR manifolds grew to become an increasingly important theme of research (see e.g. [5], [6], [9], [16], [37]). Particularly important was the work of N.Tanaka ([35], [36]). He considered the CR manifolds as generalized contact manifolds carrying a partial complex structure, and showed that under some regularity and strict nondegeneracy assumptions the study of their differential geometrical invariants fits into the scheme of Cartan geometry. Later Chern and Moser ([12]) deeply investigated the CR invariants of strictly Levi nondegenerate hypersurfaces.

More recently, there has been an increasing interest in the study of *homogeneous* CR manifolds, both of the hypersurface type and of arbitrary CR codimension ([1], [2], [3], [8], [23], [24], [34]). They provide the natural examples that suggest and motivate also the directions in which it is reasonable to pursue the analysis on the general abstract CR manifolds. In fact the generalization in [17] of the classical notion of (local) pseudoconcavity of [16] was largely motivated

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by the work on homogeneous models of [4], [25], [26], [27], [28], [29].

In [26] we associated to the graded Lie algebras introduced by N.Tanaka in [36] and [37], and that we called in [26] *Levi-Tanaka algebras*, some *standard* CR manifolds, showing in [29] that they are characterized by special rigidity properties. These standard CR manifolds are compact if and only if they are minimal orbits for the action of a real form in a complex flag manifold. In turn, all the minimal orbits of the action of real forms in complex flag manifolds are compact homogeneous CR manifolds. However, not all of them are standard. Some, which are strictly Levi nondegenerate, correspond to non compact standard models. Beside, there are others (see §12 for the complete classification) which have an *irreducible* CR structure, but are not strictly Levi nondegenerate. These considerations led us to investigate in [30] more general objects, that we called CR *algebras*. They are canonically associated to homogeneous CR manifolds, as will be explained in §4. Unlike the Levi-Tanaka algebras, they are not required to be graded. Indeed, the existence of a CR compatible $\mathbb{Z}$ or $\mathbb{Z}_2$-gradation of the CR algebra was shown in [25] to be related to that of a Riemannian CR-symmetric structure (in the sense of [24]) of the associated CR manifold. In particular in [30] we discussed weaker nondegeneracy assumptions to ensure finite dimensionality of the Lie algebra of infinitesimal CR isomorphism, and hence the possibility of utilizing the homogeneous model as the 0-curvature object of a Cartan geometry. Among the different nondegeneracy conditions for the partial complex structure of a CR manifold M, we discussed in [30] the concept of *weak nondegeneracy*. A homogeneous CR manifold M which is not weakly nondegenerate is locally the product of a CR manifold and of a complex manifold of positive dimension. A complete classification of the minimal orbits that are weakly nondegenerate is obtained in §11.

In this paper we concentrate on the CR structure of the minimal orbit M of the action of a real form $\mathbf{G}$ in a complex flag manifold $\hat{\mathbf{G}}/\mathbf{Q}$. These orbits, especially the open ones, have been studied in connection with representation theory (cf. [14], [21], [22], [41], [43]). Our point of view here is strictly that of CR geometry.

The arguments are organized as follows. In §2, 3, 4 we rehearse the essential definitions and notions to prepare the general setting for the study of the CR geometry of the minimal orbits.

In §5, 6 we give the notion of *parabolic* CR *algebras* and we associate to each minimal orbit M a special parabolic CR algebras that we call *minimal*. Minimal orbits and parabolic minimal CR algebras are in a one to one correspondence. We classify parabolic minimal CR algebras, and thus the minimal orbits, by attaching to each of them a *cross-marked Satake diagram*.

In §7 we study some special morphisms of CR algebras, which are infinitesimal analogues of smooth $\mathbf{G}$ equivariant fibrations. They will be an essential tool in the following sections.

In §8 we compute the fundamental group of M and we show that, under a condition (F) that is shared by all minimal orbits that are *fundamental*, (i.e. those in which the Cauchy-Riemann distribution generates the full tangent space), all $\mathbf{G}$-homogeneous CR manifolds that are locally CR-diffeomorphic to M are simply connected and globally CR-diffeomorphic to M.

In §9 we read off the cross-marked Satake diagrams the property of being fundamental and prove that every $\mathbf{G}$-homogeneous CR manifolds that is locally CR-diffeomorphic to M admits a *fundamental reduction* which is a CR fibration on a totally real base with a connected and simply connected fiber.

In §10 we characterize totally real and totally complex minimal orbits, and in §11, 12 we read off the cross-marked Satake diagram the property of weak and strong nondegeneracy.

In §13 we classify all minimal orbits that are essentially pseudoconcave, justifying the experimental claim made in [HN2] that "*the vast majority of them are essentially pseudoconcave*".

Finally, in §14, we study the space of global smooth CR functions on M.

A Satake diagram gives a graphic representation of the conjugation defined by a real form on the Dynkin diagram of the corresponding complex simple Lie algebra. We largely utilize Satake diagrams in the presentation of our results. Thus we found expedient, to fix the notation and for identifying specific sets of simple roots used to define special parabolic subalgebras, to add, at the end of the paper, the table of the Satake diagrams of all the non compact real forms of simple real Lie algebras of the real type. In all statements concerning cross-marked Satake diagrams, we understand that the notation refers to that table.

2. Preliminaries on CR manifolds

We briefly rehearse some basic notions for CR manifolds (see e.g. [5], [9], [16], [26]).

An (abstract) almost CR manifold of type (n,k) is a triple (M,HM,J), consisting of a paracompact smooth manifold M of real dimension $(2n+k)$, of a smooth subbundle HM of TM of even rank $2n$, its *holomorphic tangent space*, and of a smooth partial complex structure $J : HM \longrightarrow HM$, $J^2 = -\mathbf{1}$, on the fibers of HM. The integer $n \geq 0$ is the CR *dimension* and k the CR *codimension* of (M,HM,J).

Let $T^{1,0}M$ and $T^{0,1}M$ be the complex subbundles of the complexification $\mathbb{C}HM$ of HM, which correspond to the i- and $(-i)$-eigenspaces of J:

$$(2.1)\quad T^{1,0}M = \{X - i\,J\,X \,\big|\, X \in HM\}\ ,\ T^{0,1}M = \{X + i\,J\,X \,\big|\, X \in HM\}\,.$$

We say that (M,HM,J) is a CR manifold if the formal integrability condition

$$(2.2)\qquad \big[\mathcal{C}^\infty(M,T^{0,1}M),\,\mathcal{C}^\infty(M,T^{0,1}M)\big] \subset \mathcal{C}^\infty(M,T^{0,1}M)$$

holds [we get an equivalent condition by substituting $T^{1,0}$ for $T^{0,1}$ in (2.2)]. When $k=0$, we have $HM = TM$ and, via the Newlander-Nirenberg theorem, we recover the definition of a complex manifold. A smooth real manifold of real dimension k can always be considered as a *totally real* CR manifold, i.e. a CR manifold of CR dimension 0 and CR codimension k.

Let (M_1, HM_1, J_1), (M_2, HM_2, J_2) be two abstract smooth CR manifolds. A smooth map $f : M_1 \to M_2$, with differential $f_* : TM_1 \to TM_2$, is a CR *map* if $f_*(HM_1) \subset HM_2$, and $f_*(J_1 v) = J_2 f_*(v)$ for every $v \in HM_1$. We say that f is a CR *diffeomorphism* if $f : M_1 \to M_2$ is a smooth diffeomorphism and both f and f^{-1} are CR maps.

A CR *function* is a CR map $f : M \to \mathbb{C}$ of a CR manifold (M, HM, J) in $\mathbb{C}$, endowed with the standard complex structure.

Let M be a CR manifold. Denote by $T^{*1,0}M$ the annihilator of $T^{0,1}M$ in the complexified cotangent bundle $\mathbb{C}T^*M$ and by $Q^{0,1}M$ the quotient bundle $\mathbb{C}T^*M/T^{*1,0}M$, with projection π_Q. It is a rank n complex vector bundle on M, dual to $T^{0,1}M$. The $\bar\partial_M$-operator acts on smooth complex valued functions by: $\bar\partial_M = \pi_Q \circ d$. The CR functions on M are the smooth solutions u of $\bar\partial_M u = 0$, i.e. of $Lu = 0$ for all $L \in T^{0,1}M$. We shall denote by $\mathcal{O}_M(M)$ the space of smooth CR functions on M.

A CR *embedding* ϕ of an abstract CR manifold (M, HM, J) into a complex manifold $\mathfrak{M}$, with complex structure $J_{\mathfrak{M}}$, is a CR map which is a smooth embedding and satisfies $\phi_*(H_pM) = \phi_*(T_pM) \cap J_{\mathfrak{M}}(\phi_*(T_pM))$ for every $p \in M$. We say that the embedding is *generic* if the complex dimension of $\mathfrak{M}$ is $(n+k)$, where (n,k) is the type of M. A real analytic CR manifold (M, HM, J) always admits an embedding into a complex manifold $\mathfrak{M}$ (see [5]).

If $\phi : M \to \mathfrak{M}$ is a smooth embedding of a paracompact smooth manifold M into a complex manifold $\mathfrak{M}$, for each point $p \in M$ we can define H_pM to be the set of tangent vectors $v \in T_pM$ such that $J_{\mathfrak{M}}\phi_*(v) \in \phi_*(T_pM)$. For $v \in H_pM$, let $J_M v$ be the unique tangent vector $w \in H_pM$ satisfying $\phi_*(w) = J_{\mathfrak{M}}\phi_*(v)$. If the dimension of H_pM is constant for $p \in M$, then $HM = \dot{\bigcup}_{p\in M} H_pM$ and J_M are smooth and define the unique CR manifold structure on M for which (M, HM, J_M) is a CR manifold and $\phi : M \to \mathfrak{M}$ a CR embedding.

The *characteristic bundle* H^0M of (M, HM, J) is defined to be the annihilator of HM in T^*M. It parametrizes the Levi form: recall that the *Levi form* of M at p is defined for $\xi \in H^0_pM$ and $X \in H_pM$ by

$$\mathcal{L}(\xi; X) = d\tilde\xi(X, JX) = \langle \xi, [J\tilde X, \tilde X]\rangle\,, \tag{2.3}$$

where $\tilde\xi \in \mathcal{C}^\infty(M, H^0M)$ and $\tilde X \in \mathcal{C}^\infty(M, HM)$ are smooth extensions of ξ and X. For each fixed ξ it is a Hermitian quadratic form for the complex structure J_p on H_pM.

The map $HM \ni X \to \frac{1}{2}(X - iJX) \in T^{1,0}M$ yields for each $p \in M$ an $\mathbb{R}$-linear isomorphism of H_pM with the complex linear space $T^{1,0}_pM$, in such a way that the antiinvolution J_p on H_pM becomes in $T^{1,0}_pM$ the multiplication by the imaginary unit i. In this way we associate to the Levi form $\mathcal{L}(\xi; \cdot)$ a unique Hermitian symmetric form $T^{1,0}_p \times T^{1,0}_p \ni (Z_1, Z_2) \to \mathcal{L}_\xi(Z_1, Z_2) \in \mathbb{C}$ such that $\mathcal{L}(\xi; X) = \frac{1}{4}\mathcal{L}_\xi(X - iJX, X - iJX)$ for all $X \in H_pM$.

In the next sections, to shorten notation, we shall write simply M, or $M^{n,k}$, for a CR manifold (M, HM, J) of type (n,k), as the CR structure will in general be clear from the context.

We recall that a CR manifold M is:

- of *finite kind* (or *finite type*) at $p \in M$ if the higher order commutators of $C^\infty(M, HM)$, evaluated at p, span T_pM;
- *strictly nondegenerate* (or *Levi nondegenerate*) at $p \in M$ if for each $\bar{Z} \in C^\infty(M, T^{0,1}M)$, with $\bar{Z}(p) \neq 0$, there exists $Z' \in C^\infty(M, T^{1,0}M)$ such that:
$$[Z', \bar{Z}](p) \notin T^{1,0}_pM + T^{0,1}_pM;$$
This is equivalent to the following: for every $Z \in T^{1,0}_pM \setminus \{0\}$ there exist $\xi \in H^0_pM$ and $Z' \in T^{1,0}_pM$ such that $\mathcal{L}_\xi(Z, Z') \neq 0$.
- *weakly nondegenerate* at $p \in M$ if for each $\bar{Z} \in C^\infty(M, T^{0,1}M)$, with $\bar{Z}(p) \neq 0$, there exist $m \in \mathbb{N}$ and $Z_1, \dots, Z_m \in C^\infty(M, T^{1,0}M)$ such that:

$$[Z_1, \dots, Z_m, \bar{Z}](p) = [Z_1, [Z_2, \dots, [Z_m, \bar{Z}] \dots]](p) \notin T^{1,0}_pM + T^{0,1}_pM.$$

3. The minimal orbit in a complex flag manifold

Throughout this paper, we shall consistently use the symbol $\hat{V}$ to indicate the *complexification* of a real vector space V.

A *complex flag manifold* is a coset space $\mathfrak{M} = \hat{\mathbf{G}}/\mathbf{Q}$, where $\hat{\mathbf{G}}$ is a connected complex semisimple Lie group and $\mathbf{Q}$ is parabolic in $\hat{\mathbf{G}}$. The manifold $\mathfrak{M}$ is a closed complex projective variety which only depends on the Lie algebras $\hat{\mathfrak{g}}$ of $\hat{\mathbf{G}}$ and $\mathfrak{q}$ of $\mathbf{Q}$: this is a consequence of the fact that the center of a connected and simply connected complex Lie group is contained in each of its parabolic subgroups.

A *real form* of $\hat{\mathbf{G}}$ is a real subgroup $\mathbf{G}$ of $\hat{\mathbf{G}}$ whose Lie algebra $\mathfrak{g}$ is a real form of $\hat{\mathfrak{g}}$ (i.e. $\hat{\mathfrak{g}} = \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{g}$). The real form $\mathfrak{g}$ is the set of fixed points of an anti-involution σ in $\hat{\mathfrak{g}}$: $\mathfrak{g} = \mathrm{Fix}_{\hat{\mathfrak{g}}}(\sigma) = \{X \in \hat{\mathfrak{g}} \,|\, \sigma(X) = X\}$.

A real form $\mathbf{G}$ acts on the complex flag manifold $\mathfrak{M}$ by left multiplication, and $\mathfrak{M}$ decomposes into a disjoint union of $\mathbf{G}$-orbits. In [41] it is shown that there are finitely many orbits, and a unique one which is closed (hence compact). This orbit M has minimal dimension and is connected. In particular, the connected component of the identity $\mathbf{G}^\circ$ of $\mathbf{G}$ is transitive on M. Thus, while studying M, we can as well assume that $\mathbf{G} = \mathbf{G}^\circ$ is connected.

Moreover, up to conjugation, we can arrange that the closed orbit is $M = \mathbf{G} \cdot o$, where $o = e\mathbf{Q}$. We shall denote by $\mathbf{G}_+ = \mathbf{G} \cap \mathbf{Q}$ the isotropy subgroup of $\mathbf{G}$ at o and by $\mathfrak{g}_+ = \mathfrak{g} \cap \mathfrak{q}$ its Lie algebra.

The closed orbit M has a CR structure induced by its embedding in the complex manifold $\mathfrak{M}$. This can also be described by using the canonical identifications: $T_o\mathfrak{M} \simeq \hat{\mathfrak{g}}/\mathfrak{q}$ and $T_oM \simeq \mathfrak{g}/\mathfrak{g}_+ \subset T_o\mathfrak{M}$. We have then:

$$H_oM \simeq \big(\mathfrak{g} \cap (\mathfrak{q} + \bar{\mathfrak{q}})\big)/\mathfrak{g}_+ \tag{3.1}$$

and

$$\text{(3.2)}\qquad \begin{cases} J_o : H_oM \to H_oM \quad \text{is defined by:} \\ J_o(X + \bar{X} + \mathfrak{g}_+) = iX - i\bar{X} + \mathfrak{g}_+ \quad \text{for } X \in \mathfrak{q}\,. \end{cases}$$

The bundle HM and the partial complex structure J are defined, at all points of M, in such a way that $\mathbf{G}$ acts on M as a group of CR automorphisms. By using the identification $T_{g\mathbf{Q}}\mathfrak{M} = \hat{\mathfrak{g}}/(\mathrm{Ad}(g)\mathfrak{q})$, we obtain:

$$\text{(3.3)}\qquad \begin{cases} T_{g\mathbf{G}_+}M \simeq \mathfrak{g}/(\mathrm{Ad}(g)\mathfrak{g}_+) \\ H_{g\mathbf{G}_+}M = \big(\mathfrak{g} \cap (\mathrm{Ad}(g)\mathfrak{q} + \mathrm{Ad}(g)\bar{\mathfrak{q}})\big)/\,\mathrm{Ad}(g)\mathfrak{g}_+ \\ J_{g\mathbf{G}_+}(X + \bar{X} + \mathrm{Ad}(g)\mathfrak{g}_+) = iX - i\bar{X} + \mathrm{Ad}(g)\mathfrak{g}_+ \quad \text{for } X \in \mathrm{Ad}(g)\mathfrak{q}. \end{cases}$$

Note that the embedding of M into $\mathfrak{M}$ is always generic, because $T_p\mathfrak{M} = T_pM + \overline{T_pM}$ at every $p \in M$.

4. Homogeneous CR manifolds and CR algebras

To a CR manifold M, which is homogeneous for the action of a real Lie group $\mathbf{G}$ of CR transformations, we associate a *CR algebra* $(\mathfrak{g}, \mathfrak{q})$. This is a pair consisting of the real Lie algebra $\mathfrak{g}$ of the group $\mathbf{G}$ and of a complex subalgebra $\mathfrak{q}$ of its complexification $\hat{\mathfrak{g}}$. This subalgebra $\mathfrak{q}$ is the inverse image of $T_o^{0,1}M$ by the complexification $\hat{\pi}_*$ of the differential $\pi_* : \mathfrak{g} \simeq T_e\mathbf{G} \to T_oM$ of the group action at e. Note that the fact that $\mathfrak{q}$ is a complex Lie subalgebra of $\hat{\mathfrak{g}}$ is a consequence of the formal integrability condition (2.2) for CR manifolds.

Let $(\mathfrak{g}, \mathfrak{q})$ be a CR algebra. The real Lie subalgebra $\mathfrak{g}_+ = \mathfrak{g} \cap \mathfrak{q}$ is called its *isotropy*. Let $\mathbf{G}$ be a connected and simply connected Lie group with Lie algebra $\mathfrak{g}$. Assume that the analytic subgroup $\mathbf{G}_+$ of $\mathbf{G}$ corresponding to the Lie subalgebra $\mathfrak{g}_+$ is closed in $\mathbf{G}$. Then the homogeneous space $M = \mathbf{G}/\mathbf{G}_+$ is a smooth paracompact manifold and has a unique CR structure such that:

- $T_o^{0,1}M = \hat{\pi}_*(\mathfrak{q})$
- $\mathbf{G}$ acts on M by CR diffeomorphisms.

We denote the CR manifold $\mathbf{G}/\mathbf{G}_+$ by $\tilde{M}(\mathfrak{g}, \mathfrak{q})$.

For general definitions and basic properties of CR algebras we refer the reader to [30]. In particular, we recall that a *morphism of CR algebras* $\phi : (\mathfrak{g}, \mathfrak{q}) \to (\mathfrak{g}', \mathfrak{q}')$ is a homomorphism $\phi : \mathfrak{g} \to \mathfrak{g}'$ of real Lie algebras whose complexification $\hat{\phi}$ satisfies $\hat{\phi}(\mathfrak{q}) \subset \mathfrak{q}'$; it is a *$CR$ submersion* if $\phi(\mathfrak{g}) + \mathfrak{g}'_+ = \mathfrak{g}'$ and $\hat{\phi}(\mathfrak{q}) + \mathfrak{q}' \cap \bar{\mathfrak{q}}' = \mathfrak{q}'$.

We say that the CR algebra $(\mathfrak{g}, \mathfrak{q})$ is:

- *effective* if there are no ideals of $\mathfrak{g}$ contained in $\mathfrak{g}_+$;
- *fundamental* if $\mathfrak{q} + \bar{\mathfrak{q}}$ generates $\hat{\mathfrak{g}}$;
- *ideal nondegenerate* if there is no ideal $\mathfrak{a}$ of $\mathfrak{g}$ with $\mathfrak{a} \subset \mathfrak{q} + \bar{\mathfrak{q}}$ and $\mathfrak{a} \not\subset \mathfrak{g}_+$;
- *weakly nondegenerate* if there are no complex subalgebras $\mathfrak{q}' \subset \hat{\mathfrak{g}}$ with $\mathfrak{q} \subsetneq \mathfrak{q}' \subset \mathfrak{q} + \bar{\mathfrak{q}}$;
- *strictly nondegenerate* if for every $Z \in \mathfrak{q} \setminus \bar{\mathfrak{q}}$ there exists $Z' \in \bar{\mathfrak{q}}$ such that $[Z, Z'] \notin \mathfrak{q} + \bar{\mathfrak{q}}$.

If $(\mathfrak{g}, \mathfrak{q})$ is the CR algebra associated to a $\mathbf{G}$-homogeneous CR manifold M, these notions express geometric properties of M (see [30]): effectiveness is equivalent to almost effectiveness (i.e. discreteness of the isotropy subgroup) of the $\mathbf{G}$ action; fundamental to finite kind; ideal nondegeneracy to holomorphic nondegeneracy (see [9]); weak and strict nondegeneracy to weak and strict nondegeneracy as defined at the end of §2.

When $(\mathfrak{g}, \mathfrak{q})$ is weakly degenerate, it was proved in [30] that there is a CR-fibration $\tilde{M}(\mathfrak{g}, \mathfrak{q}) \to M'$ of the corresponding homogeneous CR manifold $\tilde{M}(\mathfrak{g}, \mathfrak{q})$ on a CR manifold M' with the same CR codimension, having a non trivial complex fiber. For homogeneous simply connected CR manifolds, the condition of weak degeneracy of §3 is in fact necessary and sufficient for the existence of CR fibrations with non trivial complex fibers. Indeed, for general CR manifolds, the existence of a CR fibration with non trivial complex fibers implies weak degeneracy, as we have:

Proposition 4.1. *Let M and M' be CR manifolds. We assume that M' is locally embeddable and that there exists a CR fibration $M \xrightarrow{\pi} M'$ with totally complex fibers of positive dimension. Then M is weakly degenerate.*

Proof. Let f be any smooth CR function defined on a neighborhood U' of $p' \in M'$. Then $\pi^* f$ is a CR function in $U = \pi^{-1}(U')$, that is constant along the fibers of π. Then, if $L \in \mathcal{C}^\infty(M, T^{1,0}M)$ is tangent to the fibers of π in U, we obtain that $[\bar{Z}_1, \ldots, \bar{Z}_m, L](\pi^* f) = 0$ for every choice of $\bar{Z}_1, \ldots, \bar{Z}_m \in \mathcal{C}^\infty(M, T^{0,1}M)$. Assume by contradiction that M is weakly nondegenerate at some p with $\pi(p) = p'$. Then for some choice of $\bar{Z}_1, \ldots, \bar{Z}_m \in \mathcal{C}^\infty(M, T^{0,1}M)$ we would have $v_p = [\bar{Z}_1, \ldots, \bar{Z}_m, L] \notin T^{1,0}_p M \oplus T^{0,1}_p M$. Since the fibers of π are totally complex, $\pi_*(v_p) \neq 0$. By the assumption that M' is locally embeddable at p, the real parts of the (locally defined) CR functions give local coordinates in M' and therefore there is a CR function f defined on a neighborhood U' of p' with $v_p(\pi^* f) = \pi_*(v_p)(f) \neq 0$. This gives a contradiction, proving our statement. ∎

We can always reduce to the case of an almost effective action of $\mathbf{G}$: at the level of CR algebras, this corresponds to substituting to $(\mathfrak{g}, \mathfrak{q})$ its *effective quotient*, which is the CR algebra $(\mathfrak{g}/\mathfrak{a}, \mathfrak{q}/\hat{\mathfrak{a}})$, where $\mathfrak{a}$ is the maximal ideal of $\mathfrak{g}$ that is contained in $\mathfrak{g}_+$, and $\hat{\mathfrak{a}}$ its complexification in $\hat{\mathfrak{g}}$ (see [30, Lemma 4.7]).

5. Parabolic CR algebras

In the following, we shall restrict our consideration to the the case of *parabolic* CR algebras, i.e. those CR algebras $(\mathfrak{g}, \mathfrak{q})$ where $\mathfrak{g}$ is finite dimensional and $\mathfrak{q}$ is a parabolic subalgebra of $\hat{\mathfrak{g}}$. In this section we explain some of their simplest properties.

Proposition 5.1. *A parabolic CR algebra $(\mathfrak{g}, \mathfrak{q})$ is effective if and only if the following two conditions are satisfied:*

(*i*) *$\mathfrak{g}$ is semisimple,*

(*ii*) *no simple ideal of* $\hat{\mathfrak{g}}$ *is contained in* $\mathfrak{q} \cap \overline{\mathfrak{q}}$.

Proof. The statement follows by observing that: (a) for a parabolic $(\mathfrak{g}, \mathfrak{q})$ the radical $\mathfrak{r}$ of $\mathfrak{g}$ is contained in $\mathfrak{g}_+$; (b) if an ideal $\mathfrak{a}$ of $\hat{\mathfrak{g}}$ is contained in $\mathfrak{q} \cap \overline{\mathfrak{q}}$, then $\mathfrak{a} + \overline{\mathfrak{a}}$ is the complexification of an ideal $\mathfrak{b}$ of $\mathfrak{g}$ contained in $\mathfrak{g}_+$. ∎

To an effective parabolic CR algebra $(\mathfrak{g}, \mathfrak{q})$ we associate a CR manifold $M = M(\mathfrak{g}, \mathfrak{q})$, unique modulo isomorphisms, defined as the orbit $\mathbf{G} \cdot o$ in $\hat{\mathbf{G}}/\mathbf{Q}$, where:

- $\hat{\mathbf{G}}$ is a connected and simply connected Lie group with Lie algebra $\hat{\mathfrak{g}}$;
- $\mathbf{Q} = \mathbf{N}_{\hat{\mathbf{G}}}(\mathfrak{q})$ is the parabolic subgroup of $\hat{\mathbf{G}}$ with Lie algebra $\mathfrak{q}$;
- $\mathbf{G}$ is the analytic real subgroup of $\hat{\mathbf{G}}$ with Lie algebra $\mathfrak{g}$.

Note that for a parabolic $(\mathfrak{g}, \mathfrak{q})$ also $\tilde{M}(\mathfrak{g}, \mathfrak{q})$ is well defined and is the universal cover of $M(\mathfrak{g}, \mathfrak{q})$.

The following proposition reduces the study of effective parabolic CR algebras $(\mathfrak{g}, \mathfrak{q})$ to the case where $\mathfrak{g}$ is a *simple* real Lie algebra.

Proposition 5.2. *Let* $(\mathfrak{g}, \mathfrak{q})$ *be an effective parabolic* CR *algebra and let* $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_\ell$ *be the decomposition of* $\mathfrak{g}$ *into the direct sum of its simple ideals. Then:*

(*i*) $\mathfrak{q} = \mathfrak{q}_1 \oplus \cdots \oplus \mathfrak{q}_\ell$ *where* $\mathfrak{q}_j = \mathfrak{q} \cap \hat{\mathfrak{g}}_j$ *for* $j = 1, \ldots, \ell$;
(*ii*) *for each* $j = 1, \ldots, \ell$, $(\mathfrak{g}_j, \mathfrak{q}_j)$ *is an effective parabolic* CR *algebra;*
(*iii*) $(\mathfrak{g}, \mathfrak{q})$ *is ideal (resp. weakly, strictly) nondegenerate if and only if for each* $j = 1, \ldots, \ell$, *the* CR *algebra* $(\mathfrak{g}_j, \mathfrak{q}_j)$ *is ideal (resp. weakly, strictly) nondegenerate;*
(*iv*) $(\mathfrak{g}, \mathfrak{q})$ *is fundamental if and only if for each* $j = 1, \ldots, \ell$, *the* CR *algebra* $(\mathfrak{g}_j, \mathfrak{q}_j)$ *is fundamental.*

Proof. In fact $\hat{\mathfrak{g}} = \bigoplus_{j=1}^{\ell} \hat{\mathfrak{g}}_j$ is a decomposition of $\hat{\mathfrak{g}}$ into a direct sum of ideals. The decomposition (*i*) of $\mathfrak{q}$ follows then from the decomposition $\hat{\mathfrak{h}} = \bigoplus_{j=1}^{\ell} \left(\hat{\mathfrak{h}} \cap \hat{\mathfrak{g}}_j \right)$ of any Cartan subalgebra of $\hat{\mathfrak{g}}$ contained in $\mathfrak{q}$ (see [11, Ch.VII, §2, Prop.2]).

The proof of the other statements is straightforward. ∎

We note that, with the notation of Proposition 5.2,

$$M(\mathfrak{g}, \mathfrak{q}) \simeq M(\mathfrak{g}_1, \mathfrak{q}_1) \times \cdots \times M(\mathfrak{g}_\ell, \mathfrak{q}_\ell)$$
$$\tilde{M}(\mathfrak{g}, \mathfrak{q}) \simeq \tilde{M}(\mathfrak{g}_1, \mathfrak{q}_1) \times \cdots \times \tilde{M}(\mathfrak{g}_\ell, \mathfrak{q}_\ell)$$

where "$\simeq$" means isomorphism of CR manifolds.

When $\mathfrak{q}$ is parabolic in $\hat{\mathfrak{g}}$, its conjugate $\overline{\mathfrak{q}}$ with respect to the real form $\mathfrak{g}$ is also parabolic in $\hat{\mathfrak{g}}$. Therefore the intersection $\mathfrak{q} \cap \overline{\mathfrak{q}}$ contains a Cartan subalgebra $\hat{\mathfrak{h}}$ that is invariant under conjugation (see e.g. [11, Ch.VIII, §3, Prop.10]. We observe that $\mathfrak{g}_+$ contains an element A with $\mathrm{ad}_{\mathfrak{q} \cap \overline{\mathfrak{q}}}(A)$ semisimple and of maximal rank. The centralizer $\hat{\mathfrak{h}}$ of A in $\hat{\mathfrak{g}}$ is a Cartan subalgebra of $\hat{\mathfrak{g}}$ that is contained in $\mathfrak{q} \cap \overline{\mathfrak{q}}$ and is invariant under conjugation). The intersection $\mathfrak{h} = \hat{\mathfrak{h}} \cap \mathfrak{g}$ is a Cartan subalgebra of $\mathfrak{g}$. Thus we have:

Lemma 5.3. *Let $(\mathfrak{g},\mathfrak{q})$ be a parabolic CR algebra. Then $\mathfrak{g}_+ = \mathfrak{q}\cap\mathfrak{g}$ contains a Cartan subalgebra of $\mathfrak{g}$.* ∎

Moreover we have:

Proposition 5.4. *Let $(\mathfrak{g},\mathfrak{q})$ be an effective parabolic CR algebra. The set $\mathfrak{n}_+$ of the elements A of the radical $\mathfrak{r}(\mathfrak{g}_+)$ of $\mathfrak{g}_+$, for which $\mathrm{ad}_{\mathfrak{g}}(A):\mathfrak{g}\to\mathfrak{g}$ is nilpotent, is a nilpotent ideal of $\mathfrak{g}_+$ and there exists a reductive subalgebra $\mathfrak{w}$ of $\mathfrak{g}_+$ such that*

$$\mathfrak{g}_+ = \mathfrak{n}_+ \oplus \mathfrak{w}\,. \tag{5.1}$$

The reductive subalgebra $\mathfrak{w}$ is uniquely determined modulo the subgroup of inner automorphisms of $\mathfrak{g}_+$ generated by those of the form $\exp\big(\mathrm{ad}_{\mathfrak{g}_+}(X)\big)$ with $X\in\mathfrak{n}_+$.

Proof. Indeed $\mathfrak{q}$, being parabolic, contains the semisimple and nilpotent parts of its elements. If $X\in\mathfrak{q}$ belongs to the real form $\mathfrak{g}$, then also its semisimple and nilpotent parts belong to $\mathfrak{g}$. Therefore $\mathfrak{g}_+$ is splittable, i.e. contains the semisimple and nilpotent part of its elements and we can apply [11, Prop.7, §5, Ch.VII] to obtain our statement. ∎

Let $\mathfrak{z}$ denote the center of $\mathfrak{w}$ and let $\mathfrak{s}=[\mathfrak{w},\mathfrak{w}]$ be its semisimple ideal. Then

$$\mathfrak{w} \;=\; \mathfrak{z}\oplus\mathfrak{s}\,. \tag{5.2}$$

Thus, a Cartan subalgebra $\mathfrak{h}\subset\mathfrak{g}_+$ of $\mathfrak{g}$ can be taken as the direct sum

$$\mathfrak{h} \;=\; \mathfrak{z}\oplus\mathfrak{h}' \tag{5.3}$$

of the center $\mathfrak{z}$ of $\mathfrak{w}$ and a Cartan subalgebra $\mathfrak{h}'$ of $\mathfrak{s}$. Vice versa, every Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ contained in $\mathfrak{g}_+$ has the form (5.3) for some reductive subalgebra $\mathfrak{w}$ of $\mathfrak{g}_+$.

Choose a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, and denote by $\hat{\mathfrak{h}}$ its complexification, which is a Cartan subalgebra of $\hat{\mathfrak{g}}$. Fix a Cartan decomposition

$$\mathfrak{g} \;=\; \mathfrak{k}\oplus\mathfrak{p} \tag{5.4}$$

of $\mathfrak{g}$, where $\mathfrak{k}$ is a maximal compact subalgebra of $\mathfrak{g}$ and $\mathfrak{p}$ its orthogonal in $\mathfrak{g}$ with respect to the Killing form, such that:

$$\mathfrak{h} \;=\; \mathfrak{h}^+\oplus\mathfrak{h}^- \qquad \text{where} \quad \mathfrak{h}^+=\mathfrak{h}\cap\mathfrak{k} \quad\text{and}\quad \mathfrak{h}^-=\mathfrak{h}\cap\mathfrak{p}\,; \tag{5.5}$$

$\mathfrak{h}^+$ is the *toroidal* and $\mathfrak{h}^-$ the *vector part* of $\mathfrak{h}$.

Set $\mathfrak{h}_{\mathbb{R}} = (i\,\mathfrak{h}^+)\oplus\mathfrak{h}^-$. This is a real subalgebra of $\hat{\mathfrak{h}}$ and the Cartan subalgebra of a real split form of $\hat{\mathfrak{g}}$; we denote by $\mathfrak{h}^*_{\mathbb{R}}$ its dual. Let $\mathcal{R}=\mathcal{R}(\hat{\mathfrak{g}},\hat{\mathfrak{h}})\subset\mathfrak{h}^*_{\mathbb{R}}$ be the associated root system and denote by $\hat{\mathfrak{g}}^\alpha\subset\hat{\mathfrak{g}}$ the eigenspace corresponding to the root $\alpha\in\mathcal{R}$.

We shall denote by $\mathbf{W}_{\hat{\mathfrak{h}}}$ the Weyl group of $\mathcal{R}$: it is the group of isometries of $\mathfrak{h}^*_{\mathbb{R}}$ generated by the reflections s_α with respect to the elements $\alpha \in \mathcal{R}$. We recall that it is canonically identified to the quotient $\mathbf{N}_{\mathbf{Int}(\hat{\mathfrak{g}})}(\hat{\mathfrak{h}})/\mathbf{Z}_{\mathbf{Int}(\hat{\mathfrak{g}})}(\hat{\mathfrak{h}})$. This will be also called the *algebraic* Weyl group, to distinguish it from the *analytic* Weyl group $\mathbf{W}_{\mathfrak{h}}$, which is the image in $\mathbf{W}_{\hat{\mathfrak{h}}}$ of the composed homomorphism:

$$\mathbf{N}_{\mathbf{Int}(\mathfrak{g})}(\mathfrak{h})/\mathbf{Z}_{\mathbf{Int}(\mathfrak{g})}(\mathfrak{h}) \longrightarrow \mathbf{N}_{\mathbf{Int}(\hat{\mathfrak{g}})}(\hat{\mathfrak{h}})/\mathbf{Z}_{\mathbf{Int}(\hat{\mathfrak{g}})}(\hat{\mathfrak{h}}) \longrightarrow \mathbf{W}_{\hat{\mathfrak{h}}}\,.$$

We also consider the group $\mathbf{A}_{\hat{\mathfrak{h}}}$ of all the isometries of $\mathfrak{h}^*_{\mathbb{R}}$ that transform $\mathcal{R}$ into $\mathcal{R}$. The (algebraic) Weyl group $\mathbf{W}_{\hat{\mathfrak{h}}}$ is a normal subgroup of $\mathbf{A}_{\hat{\mathfrak{h}}}$. We have a natural isomorphism $\mathbf{N}_{\mathbf{Aut}(\hat{\mathfrak{g}})}(\hat{\mathfrak{h}})/\mathbf{Z}_{\mathbf{Aut}(\hat{\mathfrak{g}})}(\hat{\mathfrak{h}}) \xrightarrow{\sim} \mathbf{A}_{\hat{\mathfrak{h}}}$, yielding a commutative diagram:

$$\begin{array}{ccccccccc}
 & & & & \mathbf{1} & & \mathbf{1} & & \\
 & & & & \downarrow & & \downarrow & & \\
\mathbf{1} & \longrightarrow & \mathbf{Z}_{\mathbf{Int}(\hat{\mathfrak{g}})}(\hat{\mathfrak{h}}) & \longrightarrow & \mathbf{N}_{\mathbf{Int}(\hat{\mathfrak{g}})}(\hat{\mathfrak{h}}) & \longrightarrow & \mathbf{W}_{\hat{\mathfrak{h}}} & \longrightarrow & \mathbf{1} \\
 & & \| & & \downarrow & & \downarrow & & \\
\mathbf{1} & \longrightarrow & \mathbf{Z}_{\mathbf{Aut}(\hat{\mathfrak{g}})}(\hat{\mathfrak{h}}) & \longrightarrow & \mathbf{N}_{\mathbf{Aut}(\hat{\mathfrak{g}})}(\hat{\mathfrak{h}}) & \longrightarrow & \mathbf{A}_{\hat{\mathfrak{h}}} & \longrightarrow & \mathbf{1} \\
 & & & & & & \downarrow & & \\
 & & & & & & \mathbf{Aut}(\Delta) & & \\
 & & & & & & \downarrow & & \\
 & & & & & & \mathbf{1} & &
\end{array}$$

with exact rows and columns, in which we denoted by $\mathbf{Aut}(\Delta)$ the group of automorphisms of the Dynkin diagram Δ associated to $\mathcal{R}$.

We finally define the group $\mathbf{A}_{\mathfrak{h}}$ as the image in $\mathbf{A}_{\hat{\mathfrak{h}}}$ of $\mathbf{N}_{\mathbf{Aut}(\mathfrak{g})}(\mathfrak{h})$, identified to a subgroup of $\mathbf{N}_{\mathbf{Aut}(\hat{\mathfrak{g}})}(\hat{\mathfrak{h}})$, by the homomorphism described above.

Denote by $\mathfrak{C}(\mathcal{R})$ the set of Weyl chambers associated to the root system $\mathcal{R}$. Choose a Weyl chamber $C \subset \mathfrak{h}_{\mathbb{R}}$ and let $\prec$ be the corresponding partial order in $\mathfrak{h}^*_{\mathbb{R}}$, defined by

$$\alpha \prec \beta \quad \Longleftrightarrow \quad \alpha(H) < \beta(H) \qquad \forall H \in C\,. \tag{5.6}$$

Let $\mathcal{R}^+ = \mathcal{R}^+(C) = \{\alpha \in \mathcal{R} \,|\, \alpha \succ 0\}$ and $\mathcal{R}^- = \mathcal{R}^-(C) = \{\alpha \in \mathcal{R} \,|\, \alpha \prec 0\}$ be the set of positive and the set of negative roots with respect to C, respectively, and denote by $\mathcal{B} = \mathcal{B}(C)$ the set of simple roots in $\mathcal{R}^+$. If

$$\alpha = \textstyle\sum_i n_i\alpha_i, \quad \alpha_i \in \mathcal{B},\ n_i \in \mathbb{Z},$$

we set

$$\mathrm{supp}(\alpha) = \mathrm{supp}_C(\alpha) = \{\alpha_i \in \mathcal{B} \,|\, n_i \neq 0\}\,.$$

Let Φ be a subset of $\mathcal{B}$. The set $\check{\Phi}^r$ of those $\beta \in \mathcal{R}^-$ for which $\mathrm{supp}(\beta) \cap \Phi = \emptyset$ is a closed system of roots ($\beta_1, \beta_2 \in \check{\Phi}^r$ and $\beta_1 + \beta_2 \in \mathcal{R} \implies \beta_1 + \beta_2 \in \check{\Phi}^r$). Then

$$\mathfrak{q}_\Phi = \hat{\mathfrak{h}} \oplus \sum_{\beta \in \mathcal{R}^+} \hat{\mathfrak{g}}^\beta \oplus \sum_{\beta \in \check{\Phi}^r} \hat{\mathfrak{g}}^\beta \tag{5.7}$$

is a parabolic subalgebra of $\hat{\mathfrak{g}}$, and every parabolic subalgebra of $\hat{\mathfrak{g}}$ that contains $\mathfrak{h}$ can be described in this way, by a suitable choice of the Weyl chamber C and of the set of simple roots $\Phi \subset \mathcal{B}$.

Each set:

$$\begin{cases} \mathcal{Q} = \mathcal{Q}_\Phi = \mathcal{R}^+ \cup \check{\Phi}^r \\ \mathcal{Q}^r = \mathcal{Q}^r_\Phi = \{\alpha \in \mathcal{Q} | -\alpha \in \mathcal{Q}\} = \check{\Phi}^r \cup (-\check{\Phi}^r) \\ \mathcal{Q}^n = \mathcal{Q}^n_\Phi = \{\alpha \in \mathcal{Q} | -\alpha \notin \mathcal{Q}\} = \mathcal{R}^+ \setminus (-\check{\Phi}^r) \end{cases} \tag{5.8}$$

is a closed system of roots. We set:

$$\begin{cases} \mathfrak{q}^r = \mathfrak{q}^r_\Phi = \hat{\mathfrak{h}} \oplus \sum_{\alpha \in \mathcal{Q}^r} \hat{\mathfrak{g}}^\alpha \\ \mathfrak{q}^n = \mathfrak{q}^n_\Phi = \sum_{\alpha \in \mathcal{Q}^n} \hat{\mathfrak{g}}^\alpha \,, \end{cases} \tag{5.9}$$

to obtain the decomposition:

$$\mathfrak{q} = \mathfrak{q}^r \oplus \mathfrak{q}^n \,, \tag{5.10}$$

of $\mathfrak{q}$ into the direct sum of its nil radical $\mathfrak{q}^n$ and a reductive subalgebra $\mathfrak{q}^r$.

We say that a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is *adapted* to the parabolic effective CR algebra $(\mathfrak{g}, \mathfrak{q})$ if, in the decomposition (5.3), the Cartan subalgebra $\mathfrak{h}'$ of $\mathfrak{s}$ has *maximal vector part.*[1]

A real Lie subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ is *triangular* if all linear maps $\mathrm{ad}_\mathfrak{g}(X) \in \mathfrak{gl}_\mathbb{R}(\mathfrak{g})$ with $X \in \mathfrak{t}$ can be simultaneously represented by triangular matrices in a suitable basis of $\mathfrak{g}$. All maximal triangular subalgebras of $\mathfrak{g}$ are conjugate by an inner automorphism ([31], §5.4, or [38]). A real Lie subalgebra of $\mathfrak{g}$ containing a maximal triangular subalgebra of $\mathfrak{g}$ is called a *t-subalgebra.*

An effective parabolic CR algebra $(\mathfrak{g}, \mathfrak{q})$ will be called *minimal* if $\mathfrak{g}_+ = \mathfrak{q} \cap \mathfrak{g}$ is a t-subalgebra of $\mathfrak{g}$.

We observe that a maximal triangular subalgebra of $\mathfrak{g}$ contains a maximal Abelian subalgebra of semisimple elements having real eigenvalues. Hence:

Proposition 5.5. *An adapted Cartan subalgebra of an effective parabolic minimal CR algebra $(\mathfrak{g}, \mathfrak{q})$ has maximal vector part as a Cartan subalgebra of $\mathfrak{g}$.* ∎

[1] A Cartan subalgebra of a semisimple real Lie algebra $\mathfrak{s}$ with a maximal vector part is obtained in the following way: Assume that $\mathfrak{s} = \mathfrak{k} \oplus \mathfrak{p}$ is a Cartan decomposition of $\mathfrak{s}$. Take any maximal Abelian Lie subalgebra $\mathfrak{h}'_a$ of $\mathfrak{s}$ contained in $\mathfrak{p}$ and let $\mathfrak{h}'_t$ be the centralizer of $\mathfrak{h}'_a$ in $\mathfrak{k}$. Then $\mathfrak{h}' = \mathfrak{h}'_a \oplus \mathfrak{h}'_t$ is a Cartan subalgebra of $\mathfrak{s}$ with maximal vector part. All Cartan subalgebras of $\mathfrak{s}$ with maximal vector part are conjugate and can be obtained in this way from a suitable Cartan decomposition of $\mathfrak{s}$.

Theorem 5.6. *Let $\mathfrak{g}$ be a semisimple real Lie algebra and $\mathfrak{q}$ a parabolic subalgebra of its complexification $\hat{\mathfrak{g}}$. Then, up to CR isomorphisms, there is a unique parabolic minimal CR algebra $(\mathfrak{g}', \mathfrak{q}')$ with $\mathfrak{g}'$ isomorphic to $\mathfrak{g}$ and $\mathfrak{q}'$ isomorphic to $\mathfrak{q}$.*

Proof. Fix a maximal triangular subalgebra $\mathfrak{t}$ of $\mathfrak{g}$. Its complexification $\hat{\mathfrak{t}}$ is solvable and therefore is contained in a maximal solvable subalgebra, i.e. a Borel subalgebra, $\mathfrak{b}$ of $\hat{\mathfrak{g}}$. Modulo an inner automorphism of $\hat{\mathfrak{g}}$, we can assume that $\mathfrak{b} \subset \mathfrak{q}$. The CR algebra $(\mathfrak{g}, \mathfrak{q})$ is parabolic minimal.

Let $\mathfrak{q}$, $\mathfrak{q}'$ be parabolic subalgebras of $\hat{\mathfrak{g}}$ such that $\mathfrak{g}_+ = \mathfrak{q} \cap \mathfrak{g}$ and $\mathfrak{g}'_+ = \mathfrak{q}' \cap \mathfrak{g}$ are t-subalgebras of $\mathfrak{g}$. By an inner automorphism of $\mathfrak{g}$, we can assume that $\mathfrak{g}_+$ and $\mathfrak{g}'_+$ contain the same maximal triangular subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ and hence a same maximal Abelian subalgebra of $\mathfrak{g}$ of semisimple elements having real eigenvalues. Hence, using another inner automorphism of $\mathfrak{g}$, we can assume that $\mathfrak{q}$ and $\mathfrak{q}'$ contain the same maximal vectorial Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$.

The inner automorphism of $\hat{\mathfrak{g}}$ transforming $\mathfrak{q}$ into $\mathfrak{q}'$ can now be taken to be an element of the analytic Weyl group, leaving the Cartan subalgebra $\mathfrak{h}$ and hence $\mathfrak{g}$ invariant. It defines a CR isomorphism between $(\mathfrak{g}, \mathfrak{q})$ and $(\mathfrak{g}, \mathfrak{q}')$. ∎

We recall that a CR algebra $(\mathfrak{g}, \mathfrak{q})$ is *totally real* if $\mathfrak{q} = \overline{\mathfrak{q}}$, or, equivalently, if $\mathfrak{g}_+ = \mathfrak{q} \cap \mathfrak{g} = \mathcal{H}_+ = \mathfrak{g} \cap (\mathfrak{q} + \overline{\mathfrak{q}})$. This is equivalent to the fact that $M(\mathfrak{g}, \mathfrak{q})$ is totally real, i.e. a CR manifold with CR dimension 0. For a totally real effective parabolic CR algebra $(\mathfrak{g}, \mathfrak{q})$ the real subalgebra $\mathfrak{g}_+$ of $\mathfrak{g}$ is parabolic, hence a t-subalgebra of $\mathfrak{g}$. Thus we have:

Proposition 5.7. *A totally real effective parabolic CR algebra is minimal.* ∎

Effective parabolic minimal CR algebras correspond to minimal orbits. In fact we have:

Theorem 5.8. *The CR manifold $M(\mathfrak{g}, \mathfrak{q})$, associated to an effective parabolic subalgebra $(\mathfrak{g}, \mathfrak{q})$, is compact if and only if $(\mathfrak{g}, \mathfrak{q})$ is minimal.*

Proof. Indeed, since $\mathbf{G}$ is a linear group, a $\mathbf{G}$-homogeneous space $\mathbf{G}/\mathbf{G}_+$ is compact if and only if $\mathbf{G}_+$ contains a maximal connected triangular subgroup (see [32, II, Ch.5, §1.1]), i.e. if $\mathfrak{g}_+$ is a t-subalgebra of $\mathfrak{g}$. ∎

6. Parabolic minimal CR algebras and cross-marked Satake diagrams

Denote by $\sigma : \mathfrak{h}^*_{\mathbb{R}} \to \mathfrak{h}^*_{\mathbb{R}}$ the involution induced by the conjugation defined by the real form $\mathfrak{g}$ of $\hat{\mathfrak{g}}$. If ϑ is the complexification of the Cartan involution associated to the decomposition (5.4), then the conjugation equals $(-\vartheta)$ on $\mathfrak{h}_{\mathbb{R}}$, so that $\sigma = -(\vartheta|_{\mathfrak{h}_{\mathbb{R}}})^*$.

We note that $\sigma(\mathcal{R}) = \mathcal{R}$. A root $\alpha \in \mathcal{R}$ is called *real* if $\bar{\alpha} = \sigma(\alpha) = \alpha$, *imaginary* if $\bar{\alpha} = \sigma(\alpha) = -\alpha$. We shall denote by $\mathcal{R}_\bullet$ the set of *imaginary* roots in $\mathcal{R}$. We recall from [7]:

Proposition 6.1. *Let $\sigma : \mathfrak{h}^*_{\mathbb{R}} \to \mathfrak{h}^*_{\mathbb{R}}$ be the involution associated to the conjugation induced by the real form $\mathfrak{g}$ of $\hat{\mathfrak{g}}$. The real Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ has maximal vector part if and only if $\hat{\mathfrak{g}}^\alpha \subset \hat{\mathfrak{k}} = \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{k}$ for all $\alpha \in \mathcal{R}_\bullet$.*

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ with maximal vector part and $\mathcal{R} = \mathcal{R}(\hat{\mathfrak{g}}, \hat{\mathfrak{h}})$. Then there exists a Weyl chamber $C \in \mathfrak{C}(\mathcal{R})$ such that:

- *(i)* $\bar{\alpha} = \sigma(\alpha) \succ 0$ *for all* $\alpha \in \mathcal{R}^+(C) \setminus \mathcal{R}_\bullet$;
- *(ii)* *there are pairwise orthogonal roots* $\beta_1, \ldots, \beta_m \in \mathcal{R}_\bullet$ *such that* $s_{\beta_1} \circ \cdots \circ s_{\beta_m}$ *is the element* $w_{(C,\bar{C})}$ *of the Weyl group that transforms* C *into* $\bar{C}$*; in particular* $w_{(C,\bar{C})}$ *is an involution:* $w^2_{(C,\bar{C})} = \mathbf{1}$;
- *(iii)* *there is an involution* $\varepsilon_C \in \mathbf{A}_{\hat{\mathfrak{h}}}$, *such that* $\varepsilon_C(C) = C$, *that commutes with* σ *and with* $w_{(C,\bar{C})}$, *such that:*

$$\sigma = \varepsilon_C \circ w_{(C,\bar{C})} .$$

The Weyl chamber C is uniquely determined modulo the analytic Weyl group $\mathbf{W}_{\mathfrak{h}}$. ∎

A Weyl chamber C that satisfies conditions (i), (ii) and (iii) of Proposition 6.1 will be called *adapted to the conjugation* σ.

We shall denote by $\mathcal{B}_\bullet = \mathcal{B}_\bullet(C)$ the set $\mathcal{B}(C) \cap \mathcal{R}_\bullet$ of simple purely imaginary roots of $\mathcal{R}^+(C)$ and by $\Xi = \Xi(C)$ its complement in $\mathcal{B}$. The involution ε_C transforms Ξ into itself. Moreover, from Proposition 6.1 we obtain the conjugation formula: for every $\alpha \in \Xi$ there are integers $n_{\alpha,\beta} \geq 0$ such that

$$\bar{\alpha} = \varepsilon_C(\alpha) + \sum_{\beta \in \mathcal{B}_\bullet} n_{\alpha,\beta}\beta . \tag{6.1}$$

We associate to C the *Satake diagram* of $\mathfrak{g}$. It is obtained from the Dynkin diagram of $\hat{\mathfrak{g}}$ whose nodes correspond to the roots in $\mathcal{B}(C)$ by painting black those corresponding to imaginary roots and joining by a curved arrow those corresponding to distinct roots $\alpha_1, \alpha_2 \in \Xi$ with $\varepsilon_C(\alpha_1) = \alpha_2$.

We can associate to any automorphism $\eta \in \mathbf{A}_{\mathfrak{h}}$ the automorphism $\tilde{\eta} = \eta \circ w_{(\eta^{-1}(C),C)} \in \mathbf{A}_{\mathfrak{h}}$, that leaves C fixed. We observe that $\tilde{\eta}(\mathcal{B}(C)) = \mathcal{B}(C)$, and therefore $\tilde{\eta}$ defines a permutation of the nodes of the Satake diagram $\boldsymbol{\mathcal{S}}$. Thus the quotient group $\mathbf{A}_{\mathfrak{h}}/\mathbf{W}_{\mathfrak{h}}$ can be considered as the group $\mathbf{Aut}(\boldsymbol{\mathcal{S}})$ of *automorphisms of the Satake diagram* $\boldsymbol{\mathcal{S}}$.

Note that a Satake diagram is completely determined by the data of: (i) the underlying Dynkin diagram Δ, (ii) the color (white or black) of the nodes, (iii) the involution ε_C on the nodes of Δ.

The correspondence between real semisimple Lie algebras and their Satake diagrams is one to one.

We list in the appendix all the connected Satake diagrams of the non compact forms. We use the labels (A I, ..., G) devised by Cartan in his classification of symmetric spaces. We shall also consistently employ the indices attached to the simple roots in these diagrams throughout the paper.

We proved in Proposition 5.5 that, for a parabolic minimal CR algebra $(\mathfrak{g}, \mathfrak{q})$, the isotropy subalgebra $\mathfrak{g}_+$ contains a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ with maximal vector part. First we prove:

Proposition 6.2. *If $(\mathfrak{g},\mathfrak{q})$ is an effective parabolic minimal CR algebra and $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ with maximal vector part and contained in $\mathfrak{q}$, then there exists a σ-adapted Weyl chamber C for $(\hat{\mathfrak{g}},\hat{\mathfrak{h}})$ such that $\mathcal{R}^+(C)\subset\mathcal{Q}$.*

Proof. Modulo an inner automorphism of $\hat{\mathfrak{g}}$, we can assume that any given parabolic subalgebra $\mathfrak{q}$ of $\hat{\mathfrak{g}}$ contains a Borel subalgebra $\mathfrak{b}$ of the form

$$\mathfrak{b} = \hat{\mathfrak{h}} + \sum_{\alpha\in\mathcal{R}^+(C)} \mathfrak{g}^\alpha$$

for a Weyl chamber $C\in\mathfrak{C}(\mathcal{R})$ adapted to the conjugation σ defined by the real form $\mathfrak{g}$. Then $\mathfrak{b}\cap\mathfrak{g}$ is contained in $\mathfrak{g}_+$ and contains a maximal triangular subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ (see for instance [33, 4.4, 4.5] or [39, 1.1.3, 1.1.4]). The statement follows from the uniqueness stated in Theorem 5.6. ∎

Let $\boldsymbol{\mathcal{S}}$ be the Satake diagram of the semisimple real Lie algebra $\mathfrak{g}$. The nodes of $\boldsymbol{\mathcal{S}}$ correspond to the simple roots $\mathcal{B}(C)$ of a Weyl chamber $C\in\mathfrak{C}(\mathcal{R})$ adapted to the conjugation σ defined by $\mathfrak{g}$. Fix a subset Φ of $\mathcal{B}(C)$ and consider the diagram $(\boldsymbol{\mathcal{S}},\Phi)$ obtained from $\boldsymbol{\mathcal{S}}$ by adding a cross-mark on each node of $\boldsymbol{\mathcal{S}}$ corresponding to a root in Φ.

We associate to the pair $(\boldsymbol{\mathcal{S}},\Phi)$ the CR algebra $(\mathfrak{g},\mathfrak{q}_\Phi)$ with $\mathfrak{q}_\Phi$ defined by (5.7).

Two cross-marked Satake diagrams $(\boldsymbol{\mathcal{S}},\Phi)$ and $(\boldsymbol{\mathcal{S}},\Psi)$ are said to be *equivalent* if there exists an $\varepsilon\in\mathbf{Aut}(\boldsymbol{\mathcal{S}})$ such that $\Psi=\varepsilon(\Phi)$.

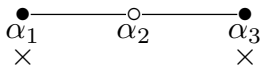
Proposition 6.2 and Theorem 5.6 yield:

Theorem 6.3. *The correspondence*

$$(\boldsymbol{\mathcal{S}},\Phi)\longleftrightarrow(\mathfrak{g},\mathfrak{q}_\Phi)$$

is bijective between cross-marked Satake diagrams (modulo automorphisms of cross-marked Satake diagrams) and minimal effective parabolic CR algebras (modulo CR isomorphisms).

Example 6.4. The diagram



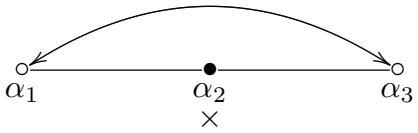
corresponds to

$$\mathfrak{g}=\mathfrak{sl}(2,\mathbb{H})\subset\mathfrak{sl}(4,\mathbb{C}),$$
$$\mathfrak{q}=\{Z\in\mathfrak{sl}(4,\mathbb{C})\,|\,Z(\langle e_1\rangle)\subset\langle e_1\rangle,\ Z(\langle e_1,e_2,e_3\rangle)\subset\langle e_1,e_2,e_3\rangle\},$$

where e_1,e_2,e_3,e_4 is the canonical basis of $\mathbb{C}^4$ with $e_1\mathbb{H}=\langle e_1,e_2\rangle$ and $e_3\mathbb{H}=\langle e_3,e_4\rangle$.

The associated minimal orbit is the CR manifold $M=M^{3,2}$ whose points are the pairs (ℓ_1,ℓ_3) consisting of a complex line ℓ_1 and a complex 3-plane ℓ_3 of $\mathbb{C}^4$ with $\ell_1\cdot\mathbb{H}\subset\ell_3$. It is strictly nondegenerate, of CR dimension 3 and CR codimension 2; all its nonzero Levi forms have one positive, one negative and one zero eigenvalues (see for instance [17]).

Example 6.5. The diagram:



corresponds to

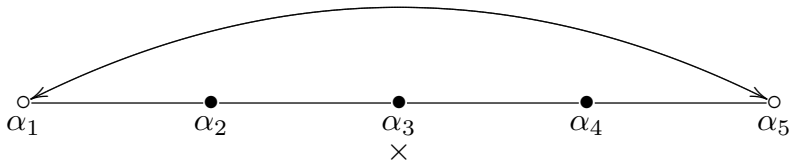
$$\mathfrak{g} = \mathfrak{su}(1,3) \subset \hat{\mathfrak{g}} = \mathfrak{sl}(4,\mathbb{C})$$
$$\mathfrak{q} = \{Z \in \mathfrak{sl}(4,\mathbb{C}) \,|\, Z(\langle e_1, e_2\rangle) \subset \langle e_1, e_2\rangle\}$$

where e_1, e_2, e_3, e_4 is a basis of $\mathbb{C}^4$ such that

$$\mathfrak{su}(1,3) = \left\{ Z \in \mathfrak{sl}(4,\mathbb{C}) \;\middle|\; \begin{pmatrix} 0&0&0&1\\0&1&0&0\\0&0&1&0\\1&0&0&0 \end{pmatrix} Z + Z^* \begin{pmatrix} 0&0&0&1\\0&1&0&0\\0&0&1&0\\1&0&0&0 \end{pmatrix} = 0 \right\}$$

The associated minimal orbit is a CR manifold $M = M^{3,1}$, of hypersurface type, with a Levi form having one positive, one negative and one zero eigenvalues, and is weakly nondegenerate but not strictly nondegenerate.

Example 6.6. The diagram:



corresponds to

$$\mathfrak{g} = \mathfrak{su}(1,5) \subset \hat{\mathfrak{g}} = \mathfrak{sl}(6,\mathbb{C})$$
$$\mathfrak{q} = \{Z \in \mathfrak{sl}(6,\mathbb{C}) \,|\, Z(\langle e_1, e_2, e_3\rangle) \subset \langle e_1, e_2, e_3\rangle\}$$

where $e_1, e_2, e_3, e_4, e_5, e_6$ is a basis of $\mathbb{C}^4$ such that

$$\mathfrak{su}(1,5) = \left\{ Z \in \mathfrak{sl}(4,\mathbb{C}) \;\middle|\; \begin{pmatrix} & & & 1\\ & I_3 & & \\ & & & \\ 1 & & & \end{pmatrix} Z + Z^* \begin{pmatrix} & & & 1\\ & I_3 & & \\ & & & \\ 1 & & & \end{pmatrix} = 0 \right\}$$

The associated minimal orbit is the CR manifold $M = M^{8,1}$, of hypersurface type, with a Levi form having two positive, two negative and four zero eigenvalues, and is weakly nondegenerate but not strictly nondegenerate.

Example 6.7. The two diagrams:

are isomorphic. Indeed the map $\digamma(\alpha_i) = \alpha_{4-i}$ for $i = 1, 2, 3$ defines an isomorphism of cross-marked Satake diagrams. The corresponding effective parabolic

minimal CR-algebras correspond to $\mathfrak{g} = \mathfrak{su}(1,3)$ and $\mathfrak{q} = \mathfrak{q}_{\{\alpha_1\}}$, $\mathfrak{q} = \mathfrak{q}_{\{\alpha_3\}}$, respectively. Let

$$K = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

and identify $\mathfrak{g}$ with the Lie algebra of 4×4 complex matrices with trace zero that satisfy $X^*K + KX = 0$. The CR isomorphism $(\mathfrak{g}, \mathfrak{q}_{\alpha_1}) \longrightarrow (\mathfrak{g}, \mathfrak{q}_{\alpha_3})$ is given by the map $\mathfrak{su}(1,3) \ni X \longrightarrow -{}^tX \in \mathfrak{su}(1,3)$.

7. $\mathfrak{g}$-equivariant fibrations

In this section we discuss morphisms of CR algebras of the special form $(\mathfrak{g}, \mathfrak{q}) \longrightarrow (\mathfrak{g}, \mathfrak{q}')$, for $\mathfrak{q} \subset \mathfrak{q}'$. They have been called in [30] $\mathfrak{g}$-*equivariant fibrations* and describe at the level of CR algebras the corresponding $\mathbf{G}$-equivariant smooth fibrations $M(\mathfrak{g}, \mathfrak{q}) \longrightarrow M(\mathfrak{g}, \mathfrak{q}')$. In this section we focus on the CR algebra aspects, preparing for the applications of the next sections.

We keep the notation of the previous sections. In particular, $\mathfrak{g}$ is a semisimple real Lie algebra, $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$ with maximal vector part, $\mathcal{R} = \mathcal{R}(\hat{\mathfrak{g}}, \hat{\mathfrak{h}})$, C a Weyl chamber adapted to the conjugation σ in $\mathfrak{h}^*_{\mathbb{R}}$ induced by the real form $\mathfrak{g}$ of $\hat{\mathfrak{g}}$, $\mathcal{B} = \mathcal{B}(C)$ is the set of simple roots in $\mathcal{R}^+ = \mathcal{R}^+(C)$.

Let $\Psi \subset \Phi \subset \mathcal{B}$. Then $\mathfrak{q}_\Phi \subset \mathfrak{q}_\Psi$ and the identity on $\mathfrak{g}$ defines a natural $\mathfrak{g}$-equivariant morphism of CR algebras (see [30]):

$$\pi : (\mathfrak{g}, \mathfrak{q}_\Phi) \longrightarrow (\mathfrak{g}, \mathfrak{q}_\Psi)\,. \tag{7.1}$$

Its fiber (see [30]) is:

$$(\mathfrak{g}', \mathfrak{q}'), \quad \text{where} \quad \begin{cases} \mathfrak{g}' = \mathfrak{g} \cap \mathfrak{q}_\Psi = \mathfrak{g} \cap \bar{\mathfrak{q}}_\Psi \\ \hat{\mathfrak{g}}' = \mathfrak{q}_\Psi \cap \bar{\mathfrak{q}}_\Psi \\ \mathfrak{q}' = \hat{\mathfrak{q}}_\Phi \cap \hat{\mathfrak{g}}' = \hat{\mathfrak{q}}_\Phi \cap \mathfrak{q}_\Psi \cap \bar{\mathfrak{q}}_\Psi = \mathfrak{q}_\Phi \cap \bar{\mathfrak{q}}_\Psi\,. \end{cases} \tag{7.2}$$

Denote by $\mathcal{R}'$ and $\mathcal{Q}'$ the sets of roots $\alpha \in \mathcal{R}$ for which $\hat{\mathfrak{g}}^\alpha$ is contained in $\hat{\mathfrak{g}}'$ and $\hat{\mathfrak{q}}'$, respectively:

$$\begin{cases} \mathcal{R}' = \mathcal{Q}_\Psi \cap \bar{\mathcal{Q}}_\Psi \\ \mathcal{Q}' = \mathcal{Q}_\Phi \cap \bar{\mathcal{Q}}_\Psi\,, \end{cases} \tag{7.3}$$

define:

$$\begin{cases} \mathcal{R}'' = \mathcal{R}' \cap (-\mathcal{R}') \;=\; \mathcal{Q}^r_\Psi \cap \bar{\mathcal{Q}}^r_\Psi \\ \mathcal{Q}'' = \mathcal{Q}' \cap \mathcal{R}'' \\ \mathcal{A} = \mathcal{R}' \setminus \mathcal{R}'' \;=\; \big(\mathcal{Q}^n_\Psi \cap \bar{\mathcal{Q}}_\Psi\big) \cup \big(\bar{\mathcal{Q}}^n_\Psi \cap \mathcal{Q}_\Psi\big) \end{cases} \tag{7.4}$$

and set:

$$\begin{cases} \hat{\mathfrak{g}}'' = \hat{\mathfrak{h}} \oplus \bigoplus_{\alpha \in \mathcal{R}''} \hat{\mathfrak{g}}^\alpha \\ \mathfrak{q}'' = \mathfrak{q}' \cap \hat{\mathfrak{g}}'' \\ \hat{\mathfrak{a}} = \bigoplus_{\alpha \in \mathcal{A}} \hat{\mathfrak{g}}^\alpha\,. \end{cases} \tag{7.5}$$

Then $\mathcal{R}''$ is σ-invariant, $\hat{\mathfrak{g}}'' = \mathfrak{q}^r_\Psi \cap \bar{\mathfrak{q}}^r_\Psi$ is reductive, $\mathfrak{q}''$ is parabolic in $\hat{\mathfrak{g}}''$ and $\hat{\mathfrak{a}} = (\mathfrak{q}^n_\Psi \cap \bar{\mathfrak{q}}_\Psi) + (\mathfrak{q}_\Psi \cap \bar{\mathfrak{q}}^n_\Psi)$ is an ideal in $\hat{\mathfrak{g}}'$, which is invariant with respect to the conjugation defined by the real form $\mathfrak{g}$.

Lemma 7.1. $\hat{\mathfrak{a}} \subset \mathfrak{q}_\Phi$.

Proof. We first show that $\mathcal{Q}^n_\Psi \cap \bar{\mathcal{Q}}_\Psi \cap \mathcal{R}_\bullet = \emptyset$. Assume by contradiction that there is $\alpha \in \mathcal{Q}^n_\Psi \cap \bar{\mathcal{Q}}_\Psi \cap \mathcal{R}_\bullet$. ¿From $\alpha \in \mathcal{Q}^n_\Psi$ we obtain that $\bar{\alpha} = -\alpha \in \bar{\mathcal{Q}}^n_\Psi$, that is $\alpha \notin \bar{\mathcal{Q}}_\Psi$, which gives a contradiction.

Since $\mathcal{Q}^n_\Psi$ is contained in $\mathcal{R}^+$ and $\mathcal{Q}^n_\Psi \cap \bar{\mathcal{Q}}_\Psi$ does not contain imaginary roots, also its conjugate $\overline{\mathcal{Q}^n_\Psi \cap \bar{\mathcal{Q}}_\Psi} = \bar{\mathcal{Q}}^n_\Psi \cap \mathcal{Q}_\Psi$ is contained in $\mathcal{R}^+$. Hence $\mathcal{A} \subset \mathcal{R}^+ \subset \mathcal{Q}_\Phi$. ∎

Lemma 7.2. *$\mathcal{B}'' = \mathcal{B} \cap \mathcal{R}''$ is a basis of $\mathcal{R}''$.*

Proof. Indeed, assume that $\alpha \in \mathcal{R}''$ is the sum of two positive roots: $\alpha = \beta + \gamma$ with $\beta, \gamma \in \mathcal{R}^+$. Then $\alpha \in \mathcal{Q}^r_\Psi$ implies that also $\beta, \gamma \in \mathcal{Q}^r_\Psi$. If $\beta, \gamma \notin \mathcal{R}_\bullet$, then by the same argument applied to $\bar{\alpha} = \bar{\beta} + \bar{\gamma} \in \mathcal{Q}^r_\Psi$ we obtain that β, γ also belong to $\bar{\mathcal{Q}}^r_\Psi$ and hence to $\mathcal{R}''$.

Consider now the case where, for instance, $\beta \in \mathcal{R}_\bullet$. Then $\bar{\beta} = -\beta \in \mathcal{Q}^r_\Psi$ implies that $\beta \in \mathcal{R}''$ and therefore $\gamma = \alpha - \beta \in \mathcal{R}''$, showing that also in this case α is not simple in $[\mathcal{R}'']^+ = \mathcal{R}^+ \cap \mathcal{R}''$. This shows that $\mathcal{B}''$ is exactly the set of simple roots in $[\mathcal{R}'']^+$, and thus a basis of $\mathcal{R}''$. ∎

We have obtained:

Proposition 7.3. *The CR algebra $(\mathfrak{g}'', \mathfrak{q}'')$ is parabolic minimal. Its cross-marked Satake diagram $(\boldsymbol{\mathcal{S}}'', \Phi'')$ is the subdiagram of $(\boldsymbol{\mathcal{S}}, \Phi)$ consisting of the simple roots α such that:*

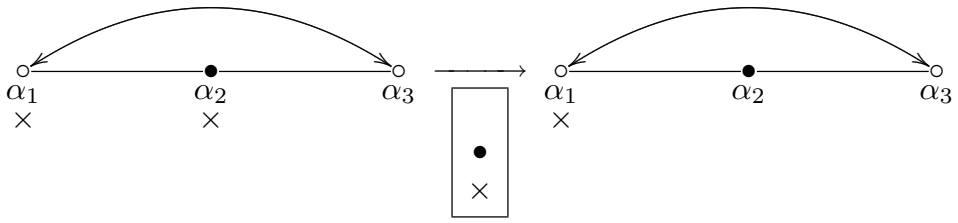
either (i) $\alpha \in \mathcal{R}_\bullet \setminus \Psi$, or (ii) $\alpha \notin \mathcal{R}_\bullet$ and $(\{\alpha\} \cup \mathrm{supp}(\bar{\alpha})) \cap \Psi = \emptyset$.

The cross-marks are left on the nodes corresponding to roots in $\Phi \cap \mathcal{B}''$. ∎

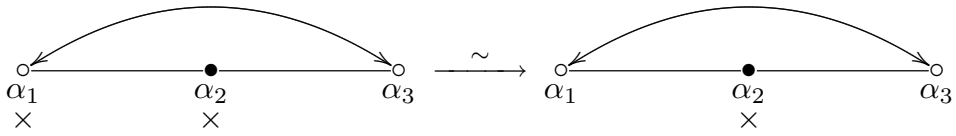
We say that a Satake diagram is *σ-connected* if either it is connected or consists of two connected components, joined by curved arrows.

Theorem 7.4. *Let (7.1) be a $\mathfrak{g}$-equivariant fibration. Then the effective quotient of its fiber is the parabolic minimal CR algebra whose cross-marked Satake diagram consists of the union of all σ-connected components of the diagram $\boldsymbol{\mathcal{S}}''$ described in Proposition 7.3, containing at least one cross-marked node.* ∎

Example 7.5. Let $\mathfrak{g} = \mathfrak{su}(1,3)$ and let $\Phi = \{\alpha_1, \alpha_2\}$, $\Psi = \{\alpha_1\}$. Then the cross-marked Satake diagrams corresponding to the CR algebra $(\mathfrak{g}, \mathfrak{q}_\Phi)$, the base $(\mathfrak{g}, \mathfrak{q}_\Psi)$ and the corresponding effective fiber are given by:



In the case $\Psi = \{\alpha_2\}$ we have instead:



The fiber is trivial and the map is a CR morphism, but not a CR isomorphism. The corresponding map $M(\mathfrak{g}, \mathfrak{q}_\Phi) \to M(\mathfrak{g}, \mathfrak{q}_\Psi)$ is an analytic diffeomorphism and a CR map, but not a CR diffeomorphism.

A $\mathfrak{g}$-equivariant morphism of CR algebras (7.1) is a *CR-fibration* if the quotient map

$$\mathfrak{q}_\Phi / (\mathfrak{q}_\Phi \cap \bar{\mathfrak{q}}_\Phi) \to \mathfrak{q}_\Psi / (\mathfrak{q}_\Psi \cap \bar{\mathfrak{q}}_\Psi) \tag{7.6}$$

is onto. Set $M_\Phi = M(\mathfrak{g}, \mathfrak{q}_\Phi)$, $M_\Psi = M(\mathfrak{g}, \mathfrak{q}_\Psi)$, and $F = M(\mathfrak{g}'', \mathfrak{q}'')$. The condition that (7.1) is a *CR-fibration* is equivalent to the fact that every point of M_Φ has an open neighborhood which is CR diffeomorphic to the product of an open submanifold of M_Ψ and F.

We have the criterion:

Proposition 7.6. *The following conditions are equivalent:*

(*i*) (7.1) *is a CR-fibration;*
(*ii*) $\mathcal{Q}^r_\Psi \setminus \mathcal{Q}_\Phi \subset \bar{\mathcal{Q}}_\Psi$;
(*iii*) $\mathcal{Q}^r_\Psi \cap \mathcal{Q}^n_\Phi \subset \bar{\mathcal{Q}}^r_\Psi$.

Proof. First we prove the equivalence (*i*) $\Leftrightarrow$ (*ii*). A necessary and sufficient condition in order that (7.1) be a CR-fibration is that the sum of the CR dimensions of $(\mathfrak{g}, \mathfrak{q}_\Psi)$ and of the fiber $(\mathfrak{g}', \mathfrak{q}')$ equals the CR-dimension of the total space $(\mathfrak{g}, \mathfrak{q}_\Phi)$:

$$\begin{aligned}\dim_{\mathbb{C}} \mathfrak{q}_\Phi - \dim_{\mathbb{C}} \mathfrak{q}_\Phi \cap \bar{\mathfrak{q}}_\Phi = \dim_{\mathbb{C}} \mathfrak{q}_\Psi - \dim_{\mathbb{C}} \mathfrak{q}_\Psi \cap \bar{\mathfrak{q}}_\Psi \\ + \dim_{\mathbb{C}} \mathfrak{q}_\Phi \cap \bar{\mathfrak{q}}_\Psi - \dim_{\mathbb{C}} \mathfrak{q}_\Phi \cap \bar{\mathfrak{q}}_\Phi \,.\end{aligned}$$

Since all subspaces considered in this formula contain $\hat{\mathfrak{h}}$, this is equivalent to:

$$(*) \qquad |\mathcal{Q}_\Phi| = |\mathcal{Q}_\Psi| - |\mathcal{Q}_\Psi \cap \bar{\mathcal{Q}}_\Psi| + |\mathcal{Q}_\Phi \cap \bar{\mathcal{Q}}_\Psi| = |\mathcal{Q}_\Psi \setminus \bar{\mathcal{Q}}_\Psi| + |\mathcal{Q}_\Phi \cap \bar{\mathcal{Q}}_\Psi|\,,$$

(where we used $|A|$ for the number of elements of the finite set A). Since $\mathcal{Q}_\Phi \subset \mathcal{Q}_\Psi$, we always have:

$$\mathcal{Q}_\Phi \subset \big(\mathcal{Q}_\Psi \setminus \bar{\mathcal{Q}}_\Psi\big) \cup \big(\mathcal{Q}_\Phi \cap \bar{\mathcal{Q}}_\Psi\big)\,.$$

The two sets on the right hand side are disjoint. Hence $(*)$ is equivalent to:

$$\mathcal{Q}_\Psi \setminus \bar{\mathcal{Q}}_\Psi \subset \mathcal{Q}_\Phi.$$

As $\mathcal{Q}^n_\Psi \subset \mathcal{R}^+ \subset \mathcal{Q}_\Phi$, this is equivalent to

$$\mathcal{Q}^r_\Psi \setminus \mathcal{Q}_\Phi \subset \bar{\mathcal{Q}}_\Psi.$$

Next we prove that (*ii*) $\Rightarrow$ (*iii*). We distinguish several cases.

If $\alpha \in \mathcal{Q}^r_\Psi \cap \mathcal{R}_\bullet$, then $\bar{\alpha} = -\alpha \in \mathcal{Q}^r_\Psi$, that is $\alpha \in \bar{\mathcal{Q}}^r_\Psi$.

If $\alpha \in \mathcal{Q}^r_\Psi \cap \mathcal{Q}^n_\Phi$ and $\alpha \notin \mathcal{R}_\bullet$, then $\bar{\alpha} \succ 0$, hence $\alpha \in \bar{\mathcal{Q}}_\Psi$. On the other hand $-\alpha \in \mathcal{Q}^r_\Psi \setminus \mathcal{Q}_\Phi$ and, by (*ii*), $-\alpha \in \bar{\mathcal{Q}}_\Psi$, thus $\alpha \in \bar{\mathcal{Q}}^r_\Psi$.

Finally we prove that (*iii*) $\Rightarrow$ (*ii*). Let $\alpha \in \mathcal{Q}^r_\Psi \setminus \mathcal{Q}_\Phi$. Then $-\alpha \in \mathcal{Q}^r_\Psi \cap \mathcal{Q}^n_\Phi$, and (*iii*) implies that $-\alpha \in \bar{\mathcal{Q}}^r_\Psi$, which is equivalent to $\alpha \in \bar{\mathcal{Q}}^r_\Psi$. ∎

In particular, we obtain:

Proposition 7.7. *If $\bar{\mathcal{Q}}_\Psi = \mathcal{Q}_\Psi$, then (7.1) is a CR-fibration.*

Proof. Indeed condition (iii) of Proposition 7.6 is trivially satisfied if $\bar{\mathcal{Q}}_\Psi = \mathcal{Q}_\Psi$. ∎

We recall (see [30]) that a CR algebra $(\mathfrak{g}, \mathfrak{q})$ is *totally complex* if $\mathfrak{q} + \bar{\mathfrak{q}} = \hat{\mathfrak{g}}$. This condition is equivalent to $\mathfrak{g} + \mathfrak{q} = \hat{\mathfrak{g}}$ and to the fact that every homogeneous CR manifold M with associated CR algebra $(\mathfrak{g}, \mathfrak{q})$ is actually a complex manifold.

Proposition 7.8. *If $\bar{\mathcal{Q}}_\Psi \cup \mathcal{Q}_\Psi = \bar{\mathcal{Q}}_\Phi \cup \mathcal{Q}_\Phi$, then (7.1) is a CR-fibration with a totally complex fiber.*

Proof. Indeed we obtain: $\mathcal{Q}_\Psi \setminus \mathcal{Q}_\Phi \subset \bar{\mathcal{Q}}_\Phi \subset \bar{\mathcal{Q}}_\Psi$, and hence (ii) of Proposition 7.6 follows because $\mathcal{Q}^r_\Psi \supset \mathcal{Q}^r_\Phi$, $\mathcal{Q}^n_\Psi \subset \mathcal{Q}^n_\Phi$.

To show that the fiber is totally complex, we need to verify that $\mathfrak{q}_\Psi \cap \bar{\mathfrak{q}}_\Psi = \mathfrak{q}_\Phi \cap \bar{\mathfrak{q}}_\Psi + \mathfrak{q}_\Psi \cap \bar{\mathfrak{q}}_\Phi$. This is obvious because $\mathfrak{q}_\Phi \subset \mathfrak{q}_\Psi \subset \mathfrak{q}_\Phi + \bar{\mathfrak{q}}_\Phi$. ∎

Our next aim is to characterize $\mathfrak{g}$-equivariant CR fibrations in terms of cross marked Satake diagrams. For this we introduce some notation.

The *component* $\check{\Psi}(\alpha)$ of a root $\alpha \in \mathcal{B}(C)$ is the set of roots $\beta \in \mathcal{B}(C)$ belonging to the connected component of the node corresponding to α in the graph obtained from $\boldsymbol{\mathcal{S}}$ by deleting those nodes that correspond to roots in $\Psi \setminus \{\alpha\}$ and the lines and arrows issuing from them.

Given a subset $\mathcal{E}$ of $\mathcal{B}(C)$, its *exterior boundary* $\partial_e\mathcal{E}$ in $\boldsymbol{\mathcal{S}}$ is the set of roots α in $\mathcal{B}(C) \setminus \mathcal{E}$ such that, for some $\beta \in \mathcal{E}$, $\alpha + \beta \in \mathcal{R}$.

It will be convenient in the following to identify the nodes of $\boldsymbol{\mathcal{S}}$ with the corresponding roots in $\mathcal{B}(C)$. In particular, for a connected subset $\mathcal{E}$ of a Satake diagram $\boldsymbol{\mathcal{S}}$, we set $\delta(\mathcal{E}) = \sum_{\alpha\in\mathcal{E}} \alpha \in \mathcal{R}$.

We recall the notation $\Xi = \mathcal{B}(C) \setminus \mathcal{R}_\bullet$ for the set of non imaginary simple roots.

Lemma 7.9. *If $\alpha \in \mathcal{R} \setminus \mathcal{R}_\bullet$, then*

$$\operatorname{supp}(\bar{\alpha}) \supset \big(\partial_e(\operatorname{supp}(\alpha)) \cap \mathcal{R}_\bullet\big) \cup \varepsilon_C\big(\operatorname{supp}(\alpha) \setminus \mathcal{R}_\bullet\big).$$

Proof. By inspecting the conjugation diagrams in [7], we find that, if $\alpha \in \Xi$:

$$\operatorname{supp}(\bar{\alpha}) = \big(\check{\Xi}(\alpha) \setminus \{\alpha\}\big) \cup \{\varepsilon_C(\alpha)\}. \tag{7.7}$$

If $\alpha = \sum k_i\alpha_i \in \mathcal{R} \setminus \mathcal{R}_\bullet$, then

$$\operatorname{supp}(\bar{\alpha}) \supset \Big(\bigcup_{\substack{k_i>0\\ \alpha_i\in\Xi}} \operatorname{supp}(\bar{\alpha}_i)\Big) \setminus \big(\operatorname{supp}(\alpha) \cap \mathcal{R}_\bullet\big),$$

in particular $\operatorname{supp}(\bar{\alpha})$ contains $\varepsilon_C\big(\operatorname{supp}(\alpha) \setminus \mathcal{R}_\bullet\big)$.

If $\beta \in \partial_e\big(\operatorname{supp}(\alpha)\big) \cap \mathcal{R}_\bullet$, then, since $\operatorname{supp}(\alpha) \not\subset \mathcal{R}_\bullet$, there exists $\alpha_i \in \operatorname{supp}(\alpha) \cap \Xi$ such that $\beta \in \check{\Xi}(\alpha_i)$. This implies that $\operatorname{supp}(\bar{\alpha}) \ni \beta$. ∎

Theorem 7.10. *A necessary and sufficient condition for* (7.1) *to be a CR* $\mathfrak{g}$*-equivariant fibration is that for every* $\alpha \in \Phi \setminus \Psi$ *either one of the following conditions hold:*

(*i*) $\check{\Psi}(\alpha) \subset \mathcal{R}_\bullet$;
(*ii*) $\check{\Psi}(\alpha) \not\subset \mathcal{R}_\bullet$, $\varepsilon_C(\check{\Psi}(\alpha) \setminus \mathcal{R}_\bullet) \cap \Psi = \emptyset$, *and* $\partial_e \check{\Psi}(\alpha) \cap \mathcal{R}_\bullet = \emptyset$.

Proof. Condition (*ii*) in Proposition 7.6 is equivalent to the assertion that, for every root β:

$$\left.\begin{array}{l} \mathrm{supp}(\beta) \cap \Psi = \emptyset \\ \mathrm{supp}(\beta) \cap \Phi \neq \emptyset \end{array}\right\} \implies \mathrm{supp}(\bar{\beta}) \cap \Psi = \emptyset. \tag{7.8}$$

Fix $\alpha \in \Phi \setminus \Psi$ and let $\beta = \delta(\check{\Psi}(\alpha))$. Then, according to Lemma 7.9, either $\beta \in \mathcal{R}_\bullet$ or $\mathrm{supp}(\bar{\beta}) \supset \varepsilon_C\big(\check{\Psi}(\alpha) \setminus \mathcal{R}_\bullet\big) \cup \big(\partial_e \check{\Psi}(\alpha) \setminus \mathcal{R}_\bullet\big)$, showing that either (*i*) or (*ii*) must be valid.

Fix again $\alpha \in \Phi \setminus \Psi$ and let $\alpha_j \in \check{\Psi}(\alpha)$. If $\alpha_j \in \mathcal{R}_\bullet$ then $\bar{\alpha}_j = -\alpha_j$ and $\mathrm{supp}(\bar{\alpha}_j) \cap \Psi = \emptyset$. If $\alpha_j \notin \mathcal{R}_\bullet$, formula (7.7) implies that either $\mathrm{supp}(\bar{\alpha}_j) \subset \check{\Psi}(\alpha)$ or $\bar{\alpha}_j = \varepsilon_C(\alpha_j)$. In both cases $\mathrm{supp}(\bar{\alpha}_j) \cap \Psi = \emptyset$. For a generic $\beta \in \mathcal{R} \setminus \mathcal{R}_\bullet$ such that $\mathrm{supp}(\beta) \subset \check{\Psi}(\alpha)$ we have that:

$$\mathrm{supp}(\bar{\beta}) \subset \bigcup_{\alpha_j \in \mathrm{supp}(\beta)} \mathrm{supp}(\bar{\alpha}_j),$$

hence $\mathrm{supp}(\bar{\beta}) \cap \Psi = \emptyset$. ∎

8. A rigidity theorem for minimal orbits

In this section we will discuss some topological properties of homogeneous CR manifolds, having an associated CR algebra which is parabolic minimal. We generalize here some results proved in [26], [29] in the case of parabolic minimal CR algebras corresponding to *semisimple Levi-Tanaka algebras.*

Let $(\mathfrak{g}, \mathfrak{q}_\Phi)$ be an effective parabolic minimal CR algebra, with associated cross-marked Satake diagram $(\boldsymbol{\mathcal{S}}, \Phi)$. We say that $(\mathfrak{g}, \mathfrak{q}_\Phi)$ has property (F) if Φ does not contain any real root.

Let (5.4) be a Cartan decomposition of $\mathfrak{g}$ with (5.5). Denote by $\mathfrak{k}^{(1)} = [\mathfrak{k}, \mathfrak{k}]$ the maximal semisimple ideal of $\mathfrak{k}$ and set $\mathfrak{k}_+ = \mathfrak{k} \cap \mathfrak{g}_+ = \mathfrak{k} \cap \mathfrak{q}_\Phi$. We have:

Proposition 8.1. *Assume that* $(\mathfrak{g}, \mathfrak{q}_\Phi)$ *has property* (F). *Then:*

$$\mathfrak{k} = \mathfrak{k}^{(1)} + \mathfrak{k}_+ . \tag{8.1}$$

Proof. Let $\hat{\mathfrak{k}}$ be the complexification of $\mathfrak{k}$. Since $\hat{\mathfrak{k}}_+ = \mathbb{C} \otimes \mathfrak{k}_+ = \hat{\mathfrak{k}} \cap \mathfrak{q}_\Phi \cap \bar{\mathfrak{q}}_\Phi$, our contention (8.1) is equivalent to:

$$\hat{\mathfrak{k}} = \hat{\mathfrak{k}}^{(1)} + \hat{\mathfrak{k}}_+ . \tag{8.2}$$

For $\alpha \notin \mathcal{R}_\bullet$, set $\hat{\mathfrak{k}}^\alpha = (\mathfrak{g}^\alpha \oplus \mathfrak{g}^{-\bar{\alpha}}) \cap \hat{\mathfrak{k}}$. This is a 1-dimensional subspace of $\hat{\mathfrak{g}}$, and we obtain the direct sum decomposition:

$$\hat{\mathfrak{k}} = \hat{\mathfrak{h}}^+ \oplus \sum_{\alpha \in \mathcal{R}_\bullet} \hat{\mathfrak{g}}^\alpha \oplus \sum_{\substack{\alpha \succ 0 \\ \bar{\alpha} \succ 0}} \hat{\mathfrak{k}}^\alpha . \tag{8.3}$$

First we note that $\hat{\mathfrak{h}}^+ \subset \mathfrak{q}_\Phi \cap \bar{\mathfrak{q}}_\Phi$. Next we observe that, if α is not real, then $\alpha(\mathfrak{h}^+) \neq \{0\}$ and hence: $\mathfrak{g}^\alpha = [\hat{\mathfrak{h}}^+, \mathfrak{g}^\alpha] \subset \hat{\mathfrak{k}}^{(1)}$ if $\alpha \in \mathcal{R}_\bullet$, and $\hat{\mathfrak{k}}^\alpha = [\hat{\mathfrak{h}}^+, \hat{\mathfrak{k}}^\alpha] \subset \hat{\mathfrak{k}}^{(1)}$ if $\alpha \succ 0$, $\bar{\alpha} \neq \pm\alpha$.

By property (F), for a simple real α we have $\alpha \in \mathcal{Q}^r \cap \bar{\mathcal{Q}}^r$ and hence $\hat{\mathfrak{k}}^\alpha \subset \hat{\mathfrak{k}}_+$.

To complete the proof, we argue by contradiction. If (8.2) is not valid, there exists some real root $\alpha \succ 0$, minimal with respect to "$\prec$", with $\hat{\mathfrak{k}}^\alpha \not\subset \hat{\mathfrak{k}}^{(1)} \cup \hat{\mathfrak{k}}_+$. This α is not simple, and hence $\alpha = \beta + \gamma$, with $\beta, \gamma \succ 0$ and we can assume that β is not imaginary. We consider the different cases.

If $\gamma \in \mathcal{R}_\bullet$, then $\hat{\mathfrak{k}}^\alpha = [\mathfrak{g}^\gamma, \hat{\mathfrak{k}}^\beta] \subset \hat{\mathfrak{k}}^{(1)}$.

We note that, for $\beta, \gamma \notin \mathcal{R}_\bullet$, the commutator $[\hat{\mathfrak{k}}^\beta, \hat{\mathfrak{k}}^\gamma]$ is contained in a sum of eigenspaces $\hat{\mathfrak{g}}^\eta$ where η is a root in $\{\beta+\gamma, -\bar{\beta}-\bar{\gamma}, \beta-\bar{\gamma}, \gamma-\bar{\beta}\}$. Hence:

If $\gamma \notin \mathcal{R}_\bullet$ and $\beta - \bar{\gamma} \notin \mathcal{R}$, then $\hat{\mathfrak{k}}^\alpha = [\hat{\mathfrak{k}}^\beta, \hat{\mathfrak{k}}^\gamma] \subset \hat{\mathfrak{k}}^{(1)}$.

If $\gamma \notin \mathcal{R}_\bullet$ and $\beta - \bar{\gamma}$ is a positive root, we have that $0 \prec \beta - \bar{\gamma} \prec \beta \prec \alpha$. Thus, by the assumption that α is minimal, $\hat{\mathfrak{k}}^{\beta-\bar{\gamma}} \subset \hat{\mathfrak{k}}^{(1)} + \hat{\mathfrak{k}}_+$ and $\hat{\mathfrak{k}}^\alpha \subset [\hat{\mathfrak{k}}^\beta, \hat{\mathfrak{k}}^\gamma] + \hat{\mathfrak{k}}^{\beta-\bar{\gamma}} \subset \hat{\mathfrak{k}}^{(1)} + \hat{\mathfrak{k}}_+$.

Analogously, if $\gamma \notin \mathcal{R}_\bullet$ and $\gamma - \bar{\beta}$ is a positive root, then $0 \prec \gamma - \bar{\beta} \prec \gamma \prec \alpha$; by the assumption that α is minimal, $\hat{\mathfrak{k}}^{\gamma-\bar{\beta}} \subset \hat{\mathfrak{k}}^{(1)} + \hat{\mathfrak{k}}_+$ and $\hat{\mathfrak{k}}^\alpha \subset [\hat{\mathfrak{k}}^\beta, \hat{\mathfrak{k}}^\gamma] + \hat{\mathfrak{k}}^{\gamma-\bar{\beta}} \subset \hat{\mathfrak{k}}^{(1)} + \hat{\mathfrak{k}}_+$. ∎

As a corollary, we obtain:

Theorem 8.2. *Let M be a connected homogeneous CR manifold whose associated CR algebra is a parabolic minimal $(\mathfrak{g}, \mathfrak{q}_\Phi)$ that satisfies property (F). Then M is compact and has a finite fundamental group.*

Proof. Let $\mathbf{G}$ denote the semisimple group with Lie algebra $\mathfrak{g}$ that acts transitively on M. Let (5.4) be a Cartan decomposition of $\mathfrak{g}$ and let $\mathbf{K}^{(1)}$ be the analytic subgroup generated by $\hat{\mathfrak{k}}^{(1)}$. Since $\hat{\mathfrak{k}}^{(1)}$ is semisimple and compact, the group $\mathbf{K}^{(1)}$ is semisimple and compact. By Proposition 8.1, $\mathbf{K}^{(1)}$ is transitive on M, because for each $p \in M$ the orbit $\mathbf{K}^{(1)} \cdot p$ is open and closed in M, and hence fore coincides with M, which, in particular, is compact. The universal covering $\tilde{\mathbf{K}}^{(1)}$ of $\mathbf{K}^{(1)}$ is compact. If $\tilde{\mathbf{K}}^{(1)}_+$ is the analytic subgroup of $\tilde{\mathbf{K}}^{(1)}$ generated by $\hat{\mathfrak{k}}^{(1)}_+ = \hat{\mathfrak{k}}^{(1)} \cap \mathfrak{q}_\Phi$, then $\tilde{M} = \tilde{\mathbf{K}}^{(1)} / \tilde{\mathbf{K}}^{(1)}_+$ is simply connected and is the universal covering of M. Therefore, having a compact universal covering, M has a finite fundamental group. ∎

Example 8.3. Fix a positive integer p, and let $\mathfrak{g} \simeq \mathfrak{su}(p,p)$ be the set of $(2p) \times (2p)$ complex matrices Z with 0 trace that satisfy:

$$Z^* K + K Z = 0 \text{ , where } K = \begin{pmatrix} I_p & \\ & -I_p \end{pmatrix} .$$

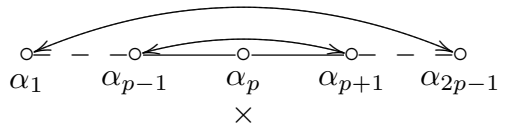
Let $e_1, \dots, e_{2p}$ be the canonical basis of $\mathbb{C}^{2p}$ and let $\mathfrak{q} \subset \hat{\mathfrak{g}} \simeq \mathfrak{sl}(2p, \mathbb{C})$ be the set of $(2p) \times (2p)$ matrices in $\mathfrak{sl}(2p, \mathbb{C})$ such that

$$Z(\langle e_1 + e_{p+1}, \dots, e_p + e_{2p} \rangle) \subset \langle e_1 + e_{p+1}, \dots, e_p + e_{2p} \rangle .$$

Then $(\mathfrak{g}, \mathfrak{q})$ is parabolic minimal. The corresponding CR manifold $M = M(\mathfrak{g}, \mathfrak{q})$ is the Grassmannian of p-planes ℓ_p in $\mathbb{C}^{2p}$ which are totally isotropic for K (i.e. $v^* K v = 0$ for all $v \in \ell_p$). We have

$$M \simeq \left\{ \ell_p = \{(v, u(v)) \in \mathbb{C}^{2p} \,|\, v \in \mathbb{C}^p\} \,|\, u \in \mathbf{U}(p) \right\} \simeq \mathbf{U}(p)$$

where $\mathbf{U}(p)$ is the group of unitary $p \times p$ matrices, i.e. $\mathbf{U}(p) = \{u \in \mathbf{GL}(p, \mathbb{C}) \mid u^* u = \mathbf{1}\}$. Then $\pi_1(M) \simeq \mathbb{Z}$ is infinite. In this case the cross-marked Satake diagram is:



and property (F) is not valid, since $\Phi = \{\alpha_p\}$ consists of a real root.

Proposition 8.4. *Let $(\mathfrak{g}, \mathfrak{q}_\Phi)$ be an effective parabolic minimal CR algebra. Then there exists a unique minimal (with respect to inclusion) parabolic subalgebra $\mathfrak{q}$ of $\hat{\mathfrak{g}}$ that is contained in $\mathfrak{q}_\Phi$ and satisfies $\mathfrak{q}_\Phi \cap \bar{\mathfrak{q}}_\Phi = \mathfrak{q} \cap \bar{\mathfrak{q}}$.*

The parabolic CR algebra $(\mathfrak{g}, \mathfrak{q})$ is minimal and we have:

- *(i)* $\mathfrak{q} = \mathfrak{q}_\Psi$ *for a* $\Psi \supset \Phi$ *;*
- *(ii)* $\mathfrak{q}_\Psi + \bar{\mathfrak{q}}_\Psi$ *is a Lie subalgebra of* $\hat{\mathfrak{g}}$ *.*

Moreover, if property (F) is valid for $(\mathfrak{g}, \mathfrak{q}_\Phi)$, then it is also valid for $(\mathfrak{g}, \mathfrak{q}_\Psi)$.

Proof. Let $A \in \mathfrak{h}_\mathbb{R}$ be such that:

$$\mathcal{Q}_\Phi = \{\alpha \in \mathcal{R} \,|\, \alpha(A) \geq 0\}$$

(for this characterization of the parabolic set of roots see for instance [39]). Set $A = A_- + i A_+$, with $A_- \in \mathfrak{h}^-$ and $A_+ \in \mathfrak{h}^+$. Fix a real positive ε sufficiently small, so that $|\alpha(i A_+)| < \varepsilon^{-1} |\alpha(A_-)|$ whenever $\alpha(A_-) \neq 0$, and set $B = A_- + i \varepsilon A_+$. Then $B \in \mathfrak{h}_\mathbb{R}$. We observe that

$$\mathcal{Q} = \{\alpha \in \mathcal{R} \,|\, \alpha(B) \geq 0\}$$

is the the set of roots of a parabolic minimal corresponding to some $\Psi \subset \mathcal{B}$. Indeed $\alpha(A_-) > 0$ implies that $\alpha(B) > 0$ and $\alpha(B) = \varepsilon \alpha(A)$ when $\alpha(A_-) = 0$. This shows in particular that $\mathcal{R}^+ \subset \mathcal{Q}_\Psi$, and hence $\mathcal{Q} = \mathcal{Q}_\Psi$ for some subset Ψ of simple roots of $\mathcal{R}^+$. This observation also yields (i), while $\mathfrak{q}_\Psi + \bar{\mathfrak{q}}_\Psi$ is the parabolic subalgebra corresponding to the set

$$\mathcal{Q}' = \{\alpha \in \mathcal{R} \,|\, \alpha(A_-) \geq 0\} .$$

For a real root α, we have $\alpha(A) = \alpha(B)$, and hence the two parabolic sets $\mathcal{Q}_\Phi$ and $\mathcal{Q}_\Psi$ contain the same real roots. This implies that they either both have or both do not have property (F).

Let us prove that $\Phi \subset \Psi$. Let $\alpha \in \mathcal{Q}^n_\Phi \subset \mathcal{R}^+$. If $\alpha(A_-) = 0$, then $\alpha(B) = \varepsilon \alpha(A) > 0$ and $\alpha \in \mathcal{Q}^n_\Psi$. If $\alpha(A_-) \neq 0$, then $\alpha \in \mathcal{R}^+ \setminus \mathcal{R}_\bullet$, and hence $\alpha, \bar{\alpha} \in \mathcal{R}^+ \subset \mathcal{Q}_\Phi$ implies that $\alpha(A_-) \geq |\alpha(iA_+)|$. Thus $\alpha(B) > 0$ and again $\alpha \in \mathcal{Q}^n_\Psi$. Since $\mathcal{Q}^n_\Phi \subset \mathcal{Q}^n_\Psi$, we have $\Phi \subset \Psi$.

Finally we show that $\mathfrak{q} = \mathfrak{q}_\Psi$ satisfies the minimality condition. To this aim, we will show that every parabolic Lie subalgebra $\mathfrak{q}'$ of $\hat{\mathfrak{g}}$ with $\mathfrak{q}' \subset \mathfrak{q}_\Phi$ and $\mathfrak{q}' \cap \bar{\mathfrak{q}}' = \mathfrak{q}_\Phi \cap \bar{\mathfrak{q}}_\Phi$ contains $\mathfrak{q}_\Psi$. Since $\mathfrak{q}'$ contains $\hat{\mathfrak{h}}$, we have $\mathfrak{q}' = \hat{\mathfrak{h}} \oplus \sum_{\alpha \in \mathcal{Q}'} \hat{\mathfrak{g}}^\alpha$ for a parabolic subset $\mathcal{Q}' \subset \mathcal{R}$.

We claim that $\mathcal{R}^+ \subset \mathcal{Q}'$. Indeed, for $\alpha \in \mathcal{R}^+ \setminus \mathcal{R}_\bullet$, we have $\alpha \in \mathcal{Q}_\Phi \cap \bar{\mathcal{Q}}_\Phi = \mathcal{Q}' \cap \bar{\mathcal{Q}}' \subset \mathcal{Q}'$. If $\alpha \in \mathcal{R}^+ \cap \mathcal{R}_\bullet$, then either $\alpha \in \mathcal{Q}_\Phi^r$ and again $\alpha \in \mathcal{Q}_\Phi \cap \bar{\mathcal{Q}}_\Phi = \mathcal{Q}' \cap \bar{\mathcal{Q}}' \subset \mathcal{Q}'$, or $\alpha \in \mathcal{Q}_\Phi^n$ and $-\alpha \notin \mathcal{Q}_\Phi \supset \mathcal{Q}'$ implies $\alpha \in \mathcal{Q}'$, because $\mathcal{Q}'$ is parabolic. This implies that $(\mathfrak{g}, \mathfrak{q}')$ is parabolic minimal and $\mathfrak{q}' = \mathfrak{q}_{\Psi'}$ for some $\Psi' \supset \Phi$.

To prove minimality, we need to show that $\Psi' \subset \Psi$, i.e. that $\alpha(B) > 0$ for all $\alpha \in \Psi'$. First we observe that $\Psi' \cap \mathcal{R}_\bullet = \Psi \cap \mathcal{R}_\bullet = \Phi \cap \mathcal{R}_\bullet$, as we showed above that ${\mathcal{Q}'}^n \cap \mathcal{R}_\bullet = \mathcal{Q}_\Phi^n \cap \mathcal{R}_\bullet$ for all parabolic sets $\mathcal{Q}'$ with $\mathcal{Q}' \cap \bar{\mathcal{Q}}' = \mathcal{Q}_\Phi \cap \bar{\mathcal{Q}}_\Phi$.

For $\alpha \in \Psi' \setminus \mathcal{R}_\bullet$, either $\alpha \in \Phi \subset \Psi$, or $\alpha(A) = 0$ and $\bar{\alpha}(A) > 0$. This implies that $\alpha(A_-) > 0$ and hence $\alpha(B) > 0$.

This completes the proof. ∎

Corollary 8.5. *Let $\Phi \subset \Psi \subset \mathcal{B}$ be as in the statement of the previous proposition and let $\mathbf{G}$ be any connected Lie group with Lie algebra $\mathfrak{g}$. Denote by $\widetilde{\mathrm{Ad}}$ the complexification of the adjoint action of $\mathbf{G}$ in $\mathfrak{g}$ and by*

$$\mathbf{N}_{\mathbf{G}}(\mathfrak{a}) = \{g \in \mathbf{G} \,|\, \widetilde{\mathrm{Ad}}(g)(\mathfrak{a}) \subset \mathfrak{a}\}$$

the normalizer of a subspace $\mathfrak{a}$ of $\hat{\mathfrak{g}}$ in $\mathbf{G}$. Then

$$\mathbf{N}_{\mathbf{G}}(\mathfrak{q}_\Psi) = \mathbf{N}_{\mathbf{G}}(\mathfrak{q}_\Phi)\,. \tag{8.4}$$

Proof. Indeed, if $g \in \mathbf{N}_{\mathbf{G}}(\mathfrak{q}_\Phi)$, then $\widetilde{\mathrm{Ad}}(g)(\mathfrak{q}_\Psi)$ is still a parabolic subalgebra of $\hat{\mathfrak{g}}$, minimal among those that are contained in $\mathfrak{q}_\Phi$ and satisfy $\mathfrak{q} \cap \bar{\mathfrak{q}} = \mathfrak{q}_\Phi \cap \bar{\mathfrak{q}}_\Phi$. Hence, by the uniqueness stated in Proposition 8.4, it coincides with $\mathfrak{q}_\Psi$ and therefore $g \in \mathbf{N}_{\mathbf{G}}(\mathfrak{q}_\Psi)$. Thus we proved the inclusion $\mathbf{N}_{\mathbf{G}}(\mathfrak{q}_\Phi) \subset \mathbf{N}_{\mathbf{G}}(\mathfrak{q}_\Psi)$.

Note that $\widetilde{\mathrm{Ad}}$ can be considered as a homomorphism $\mathbf{G} \to \mathbf{Int}(\hat{\mathfrak{g}})$, and $\mathbf{N}_{\mathbf{G}}(\mathfrak{a}) = \widetilde{\mathrm{Ad}}^{-1}(\mathbf{N}_{\mathbf{Int}(\hat{\mathfrak{g}})}(\mathfrak{a})$ for every subspace $\mathfrak{a}$ of $\hat{\mathfrak{g}}$. Then the opposite inclusion $\mathbf{N}_{\mathbf{G}}(\mathfrak{q}_\Psi) \subset \mathbf{N}_{\mathbf{G}}(\mathfrak{q}_\Phi)$ follows from the fact that any parabolic subgroup of a connected complex Lie group is the normalizer of its Lie algebra. ∎

Theorem 8.6. *Let $(\mathfrak{g}, \mathfrak{q}_\Phi)$ be an effective parabolic minimal CR algebra, satisfying condition (F). Let $\hat{\mathbf{G}}$ be any connected algebraic complex semisimple Lie group whose Lie algebra is the complexification $\hat{\mathfrak{g}}$ of $\mathfrak{g}$ and let $\mathbf{Q}$ be its parabolic subgroup with Lie algebra $\mathfrak{q}$. Denote by $\mathbf{G}$ the analytic subgroup of $\hat{\mathbf{G}}$ with Lie algebra $\mathfrak{g}$. Then:*

- (i) *$\mathbf{G}_+ = \mathbf{G} \cap \mathbf{Q}$ is connected;*
- (ii) *$M = \mathbf{G}/\mathbf{G}_+$ is compact and simply connected.*

Proof. First we consider the case where $(\mathfrak{g}, \mathfrak{q}_\Phi)$ is a totally real parabolic minimal CR algebra satisfying condition (F). In this case $\mathfrak{g}_+$ is a parabolic real Lie subalgebra of $\mathfrak{g}$. Denote by $\mathbf{N}$ the normalizer of $\mathfrak{g}_+$ in $\mathbf{G}$. Denote

by Σ the real root system of $\mathfrak{g}$ with respect to $\mathfrak{h}^-$ and let $\pi : \mathcal{R} \to \Sigma$ denote the natural projection. By [40] we know that the fundamental group of $\mathbf{G}/\mathbf{N}$ is a quotient group of the free group generated by the elements of $\pi(\Phi)$ having multiplicity 1. But condition (F) implies that no root in $\pi(\Phi)$ has multiplicity 1, and hence $\pi_1(\mathbf{G}/\mathbf{N}) = \mathbf{1}$. Thus $\mathbf{G}/\mathbf{N}$ is simply connected. Since $\mathbf{N} \supset \mathbf{G}_+$, we deduce that $\mathbf{G}_+ = \mathbf{N}$ is connected. By Theorem 8.2, M is also compact. Thus the proof is complete in this case.

Consider now a general effective parabolic minimal $(\mathfrak{g}, \mathfrak{q}_\Phi)$. Let Ψ be the subset of $\mathcal{B}$ found in Proposition 8.4. Then we consider the parabolic subalgebra $\mathfrak{q}_\Pi = \mathfrak{q}_\Psi + \bar{\mathfrak{q}}_\Psi$, with $\Pi \subset \Psi$ and we apply Proposition 7.8 to the $\mathfrak{g}$-equivariant fibration $(\mathfrak{g}, \mathfrak{q}_\Psi) \to (\mathfrak{g}, \mathfrak{q}_\Pi)$. The fiber is a parabolic minimal totally complex CR algebra $(\mathfrak{g}', \mathfrak{q}')$. Thus, with $\mathbf{G}_\Psi$ equal to $\mathbf{N}_\mathbf{G}(\mathfrak{q}_\Psi)$, and $\mathbf{G}_\Pi$ equal to $\mathbf{N}_\mathbf{G}(\mathfrak{q}_\Pi) = \mathbf{Q}_\Pi \cap \mathbf{G}$, we obtain a $\mathbf{G}$-equivariant fibration $\tilde{M} = \mathbf{G}/\mathbf{G}_\Psi \to \mathbf{G}/\mathbf{G}_\Pi$ whose fiber is a complex flag manifold. Since both the base space $\mathbf{G}/\mathbf{G}_\Pi$, by the first part of this proof, and the fiber are simply connected, also the total space $\tilde{M}$ is simply connected.

By Corollary 8.5, we have $\mathbf{N}_\mathbf{G}(\mathfrak{q}_\Psi) = \mathbf{N}_\mathbf{G}(\mathfrak{q}_\Phi) = \mathbf{G} \cap \mathbf{Q}_\Phi$, and therefore M is diffeomorphic to $\tilde{M}$ and therefore simply connected. By Theorem 8.2, M is also compact. ∎

Theorem 8.6 and Theorem 8.2 yield a rigidity theorem for CR manifolds:

Corollary 8.7. *Let $\mathbf{G}$ be a semisimple real Lie group and M a connected $\mathbf{G}$-homogeneous CR manifold. If the associated CR algebra $(\mathfrak{g}, \mathfrak{q})$ is parabolic minimal and has property (F), then M is simply connected and CR-diffeomorphic to $M(\mathfrak{g}, \mathfrak{q})$.* ∎

9. Fundamental CR algebras

We give a criterion to read off the property of being fundamental from the cross-marked Satake diagram:

Theorem 9.1. *An effective parabolic minimal CR algebra $(\mathfrak{g}, \mathfrak{q}_\Phi)$ is fundamental if and only if its corresponding cross-marked Satake diagram $(\boldsymbol{\mathcal{S}}, \Phi)$ has the property:*

$$\alpha \in \Phi \setminus \mathcal{R}_\bullet \Longrightarrow \varepsilon_C(\alpha) \notin \Phi\,. \tag{9.1}$$

Here ε_C is the involution in $\mathcal{B}(C)$ defined in Proposition 6.1.

Proof. Assume that α_1 and $\alpha_2 = \varepsilon_C(\alpha_1)$ both belong to Φ, and let $\Psi = \{\alpha_1, \alpha_2\}$. Then $\Psi \subset \Phi$ and hence $\mathfrak{q}_\Phi \subset \mathfrak{q}_\Psi$. To show that $(\mathfrak{g}, \mathfrak{q}_\Phi)$ is not fundamental, it is sufficient to check that $\mathfrak{q}_\Psi = \bar{\mathfrak{q}}_\Psi$. To this aim it suffices to verify that $\mathcal{Q}^n_\Psi = \overline{\mathcal{Q}}^n_\Psi$. Let $\mathcal{B}(C) = \{\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_\ell\}$. Every root $\alpha \in \mathcal{Q}^n_\Psi$ can be written in the form $\alpha = \sum_{i=1}^{\ell} k_i \alpha_i$ with $k_1 + k_2 > 0$. Since C is adapted to the conjugation σ, using (6.1) we obtain:

$$\bar\alpha = \textstyle\sum_{i=1}^{\ell} k_i \varepsilon_C(\alpha_i) + \sum_{\beta\in\mathcal{B}_\bullet(C)} k_{\alpha,\beta}\beta = \sum_{i=1}^{\ell} k'_i \alpha_i\,,$$

with $k'_1 + k'_2 = k_2 + k_1 > 0$, showing that also $\bar\alpha \in \mathcal{Q}^n_\Psi$. This shows that the condition is necessary.

Assume vice versa that there exists a proper parabolic subalgebra $\mathfrak{q}'$ of $\hat{\mathfrak{g}}$ with $\mathfrak{q}_\Phi \subset \mathfrak{q}' = \overline{\mathfrak{q}}\,'$. Then $\mathfrak{q}' = \mathfrak{q}_\Psi$ for some $\Psi \subset \Phi$, $\Psi \neq \emptyset$. Since $\overline{\mathcal{Q}}^n_\Psi = \mathcal{Q}^n_\Psi \subset \mathcal{R}^+(C)$, we have $\Psi \cap \mathcal{R}_\bullet = \emptyset$. Hence, again by (6.1), we obtain that $\varepsilon_C(\alpha) \in \Psi$ for all $\alpha \in \Psi$. ∎

Corollary 9.2. *Fundamental effective parabolic minimal CR algebras have the (F) property.* ∎

¿From Theorems 9.1, 7.4 and Proposition 7.7 we obtain:

Theorem 9.3. *Let $(\mathfrak{g}, \mathfrak{q}_\Phi)$ be an effective parabolic minimal CR algebra and let $(\boldsymbol{\mathcal{S}}, \Phi)$ be its corresponding cross-marked Satake diagram. Let*

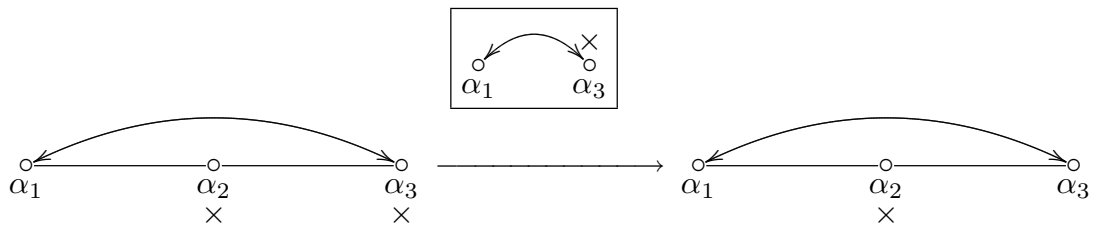
$$\Psi = \{\alpha \in \Phi \setminus \mathcal{R}_\bullet \,|\, \varepsilon_C(\alpha) \in \Phi\}\,.$$

Then

- *(i) The diagram $\boldsymbol{\mathcal{S}}'$ obtained from $\boldsymbol{\mathcal{S}}$ by erasing all the nodes corresponding to the roots in Ψ and the lines and arrows issued from them is still a Satake diagram, corresponding to a semisimple real Lie algebra $\mathfrak{g}'$.*
- *(ii) $(\mathfrak{g}, \mathfrak{q}_\Psi)$ is a totally real effective parabolic minimal CR algebra.*
- *(iii) The natural map $(\mathfrak{g}, \mathfrak{q}_\Phi) \longrightarrow (\mathfrak{g}, \mathfrak{q}_\Psi)$, defined by the inclusion $\mathfrak{q}_\Phi \subset \mathfrak{q}_\Psi$, is a $\mathfrak{g}$-equivariant CR fibration. The effective quotient of its fiber is the fundamental parabolic minimal CR algebra $(\mathfrak{g}'', \mathfrak{q}_{\Phi'})$, associated to the cross-marked Satake diagram $(\boldsymbol{\mathcal{S}}'', \Phi')$, where $\Phi' = \Phi \setminus \Psi$ and $\boldsymbol{\mathcal{S}}''$ is the union of the σ-connected components of $\boldsymbol{\mathcal{S}}'$ that contain some root of Φ'.* ∎

We call the map in *(iii)* the *fundamental reduction* of $(\mathfrak{g}, \mathfrak{q}_\Phi)$ and the totally real CR algebra $(\mathfrak{g}, \mathfrak{q}_\Psi)$ its *base*.

Example 9.4. Let $\mathfrak{g} \simeq \mathfrak{su}(2,2)$ and let $\Phi = \{\alpha_2, \alpha_3\}$ (we refer to the diagram below). We have $\varepsilon_C(\alpha_i) = \alpha_{4-i}$ for $i = 1, 2, 3$ and hence $\Psi = \{\alpha \in \Phi \,|\, \varepsilon_C(\alpha) \in \Phi\} = \{\alpha_2\}$. In particular $(\mathfrak{g}, \mathfrak{q}_{\{\alpha_2,\alpha_3\}})$ is not fundamental. We obtain by Theorem 9.3 a $\mathfrak{g}$-equivariant CR fibration $(\mathfrak{g}, \mathfrak{q}_{\{\alpha_2,\alpha_3\}}) \longrightarrow (\mathfrak{g}, \mathfrak{q}_{\{\alpha_2\}})$ with fundamental fiber $(\mathfrak{g}', \mathfrak{q}'_{\{\alpha_3\}})$, with $\mathfrak{g}' \simeq \mathfrak{sl}(2, \mathbb{C})$.



Corollary 9.5. *Let $\mathbf{G}$ be a semisimple Lie group and M a $\mathbf{G}$-homogeneous CR manifold. Assume that the CR algebra $(\mathfrak{g}, \mathfrak{q}_\Phi)$ associated to M is parabolic minimal. Let $(\mathfrak{g}, \mathfrak{q}_\Psi)$ be the base of its fundamental reduction. Then there exists a (totally real) $\mathbf{G}$-homogeneous CR manifold N, with associated CR algebra $(\mathfrak{g}, \mathfrak{q}_\Psi)$, and a $\mathbf{G}$-equivariant submersion $\omega : M \longrightarrow N$ such that the induced map $\omega_* : \pi_1(M) \longrightarrow \pi_1(N)$ is an isomorphism.*

Proof. Let o be a point of M and let $\mathbf{G}_+$ the stabilizer of o in $\mathbf{G}$. Let $\mathbf{H}$ be the analytic subgroup of $\mathbf{G}$ generated by $\mathfrak{g} \cap \mathfrak{q}_\Psi$. Then $\mathbf{H}$ contains $\mathbf{G}_+^\circ$. We claim that $\mathbf{H} \cdot \mathbf{G}_+ = \mathbf{G}'_+$ is a Lie subgroup of $\mathbf{G}$. Indeed, for all $g \in \mathbf{G}_+$, we have $\mathrm{Ad}(g)(\mathfrak{q}_\Phi) = \mathfrak{q}_\Phi$. Since g is real, we also have $\mathrm{Ad}(g)(\bar{\mathfrak{q}}_\Phi) = \bar{\mathfrak{q}}_\Phi$ and therefore $\mathrm{Ad}(g)(\mathfrak{q}_\Psi) = \mathfrak{q}_\Psi$ because $\mathfrak{q}_\Psi$ is generated by $\mathfrak{q}_\Phi + \bar{\mathfrak{q}}_\Phi$. This implies that $\mathrm{ad}(g)(\mathbf{H}) = \mathbf{H}$ for all $g \in \mathbf{G}_+$, and hence $\mathbf{G}'_+$ is a subgroup of $\mathbf{G}$. It is a Lie subgroup because its Lie algebra is real parabolic. Then $N = \mathbf{G}/\mathbf{G}'_+$ is a $\mathbf{G}$-homogeneous manifold. By the inclusion $\mathbf{G}_+ \subset \mathbf{G}'_+$ we obtain a $\mathbf{G}$-equivariant submersion $\omega : M \longrightarrow N$. By construction the fiber is connected. It has a natural structure of CR manifold, associated to a fundamental CR algebra $(\mathfrak{g}'', \mathfrak{q}_{\Phi'})$, as in Theorem 9.3, which is parabolic and minimal. By Corollary 8.7 the fiber is simply connected. Hence $\omega_* : \pi_1(M) \longrightarrow \pi_1(N)$ is an isomorphism. ∎

10. Totally real and totally complex CR algebras

From the discussion in the previous section we obtain the criterion:

Theorem 10.1. *A simple effective parabolic minimal CR algebra $(\mathfrak{g}, \mathfrak{q}_\Phi)$, with corresponding cross-marked Satake diagram $(\boldsymbol{\mathcal{S}}, \Phi)$, is totally real if and only if the following conditions hold true:*

(*i*) $\Phi \cap \mathcal{R}_\bullet = \emptyset$;
(*ii*) $\varepsilon_C(\Phi) = \Phi$. ∎

Theorem 10.2. *A simple effective parabolic minimal CR algebra $(\mathfrak{g}, \mathfrak{q}_\Phi)$ with associated cross-marked Satake diagram $(\boldsymbol{\mathcal{S}}, \Phi)$ is totally complex if and only if either:*

(*i*) $\mathfrak{g}$ *is compact, or*
(*ii*) $\mathfrak{g}$ *is of the complex type and all cross-marked nodes are in the same connected component of $\boldsymbol{\mathcal{S}}$, or*
(*iii*) $(\boldsymbol{\mathcal{S}}, \Phi)$ *is one of the following:*

(A II) $\begin{cases} \Phi = \{\alpha_1\} \\ \Phi = \{\alpha_\ell\} \end{cases}$

(D II) $\begin{cases} \Phi = \{\alpha_\ell\} \\ \Phi = \{\alpha_{\ell-1}\} \end{cases}$

Proof. The CR algebra $(\mathfrak{g}, \mathfrak{q})$ is totally complex if and only if $\mathfrak{g} + \mathfrak{q} = \hat{\mathfrak{g}}$. This is equivalent to the fact that the standard CR manifold $\mathbf{G} \cdot o$ is open in the complex flag manifold $\hat{\mathbf{G}}/\mathbf{Q}$. Since it is also closed, it follows that $\mathbf{G}$ is transitive on $\hat{\mathbf{G}}/\mathbf{Q}$. The result then follows from [42, Corollary 1.7]. ∎

This yields also a characterization of ideal nondegenerate parabolic minimal CR algebras. Indeed, for general parabolic CR algebras, we have:

Theorem 10.3. *Let $(\mathfrak{g}, \mathfrak{q})$ be a simple effective parabolic CR algebra. Then $(\mathfrak{g}, \mathfrak{q})$ is either totally complex or ideal nondegenerate.*

Proof. We recall (see [30]) that an effective CR algebra $(\mathfrak{g}, \mathfrak{q})$ is ideal nondegenerate if $\mathcal{H}_+ = (\mathfrak{q} + \bar{\mathfrak{q}}) \cap \mathfrak{g}$ does not contain a non zero ideal of $\mathfrak{g}$. When $\mathfrak{g}$ is simple, this is equivalent to the fact that $\mathcal{H}_+ \neq \mathfrak{g}$, i.e. that $(\mathfrak{g}, \mathfrak{q})$ is not totally complex. ∎

11. Weak nondegeneracy

In this section we characterize those parabolic minimal $(\mathfrak{g}, \mathfrak{q}_\Phi)$ that are weakly nondegenerate. We recall from Proposition 4.1 that this means that there is no nontrivial complex CR fibration $M(\mathfrak{g}, \mathfrak{q}_\Phi) \to N$ with totally complex fibers. In turns this is equivalent to the fact that $M(\mathfrak{g}, \mathfrak{q}_\Phi)$ is not, locally, CR equivalent to the product of a CR manifold with the same CR codimension and of a complex manifold of positive dimension.

From Proposition 7.8 we obtain:

Lemma 11.1. *A fundamental effective parabolic minimal CR algebra $(\mathfrak{g}, \mathfrak{q}_\Phi)$ is weakly degenerate if and only if there is $\Psi \subsetneq \Phi$ such that the $\mathfrak{g}$-equivariant fibration $(\mathfrak{g}, \mathfrak{q}_\Phi) \to (\mathfrak{g}, \mathfrak{q}_\Psi)$ is a CR fibration with totally complex fiber.* ∎

Lemma 11.2. *Let $(\mathfrak{g}, \mathfrak{q}_\Phi)$ be a minimal fundamental effective parabolic CR algebra. A necessary and sufficient condition in order that $(\mathfrak{g}, \mathfrak{q}_\Phi)$ be weakly degenerate is that there exists $\Psi \subset \Phi$ satisfying conditions in Theorem 7.10 and such that $\mathfrak{q}_\Psi \subset \mathfrak{q}_\Phi + \bar{\mathfrak{q}}_\Phi$.* ∎

We now give a characterization of the pairs (Φ, Ψ) for which (7.1) is a CR fibration with totally complex fiber in terms of properties of the roots α in $\Phi \setminus \Psi$.

Lemma 11.3. *Let $(\mathfrak{g}, \mathfrak{q}_\Phi)$ be a minimal fundamental effective parabolic CR algebra, with $\mathfrak{g}$ of the real type (i.e. $\hat{\mathfrak{g}}$ is also simple). Let $\emptyset \neq \Psi \subset \Phi$ and assume that (7.1) is a CR fibration. Then for each $\alpha \in \Phi \setminus \Psi$ we have the following possibilities:*

(*i*) $\check{\Psi}(\alpha) \subset \mathcal{R}_\bullet$;
(*ii*) (a) $\check{\Psi}(\alpha) \cap \mathcal{R}_\bullet = \emptyset$ *and* (b) $\big(\check{\Psi}(\alpha)\big) \cap \varepsilon_C\big(\check{\Psi}(\alpha)\big) = \emptyset$;
(*iii*) (a) $\emptyset \neq \check{\Psi}(\alpha) \cap \mathcal{R}_\bullet \neq \check{\Psi}(\alpha)$ *and* (a) $\varepsilon_C\big(\check{\Psi}(\alpha) \setminus \mathcal{R}_\bullet\big) = \check{\Psi}(\alpha) \setminus \mathcal{R}_\bullet$.

Proof. Fix $\alpha \in \Phi \setminus \Psi$ with $\check\Psi(\alpha) \not\subset \mathcal{R}_\bullet$ and let $\delta = \delta\big(\check\Psi(\alpha)\big)$.

If $\beta \in \check\Psi(\alpha) \setminus \mathcal{R}_\bullet$ and $\varepsilon_C(\beta) \in \check\Psi(\alpha)$, then $\mathrm{supp}(\bar\delta) \ni \varepsilon_C(\beta)$. Since it is connected and does not meet Ψ, we obtain $\mathrm{supp}(\bar\delta) \subset \check\Psi(\alpha)$. This implies that $\check\Psi(\alpha) \setminus \mathcal{R}_\bullet = \varepsilon_C\big(\check\Psi(\alpha) \setminus \mathcal{R}_\bullet\big)$. In this way we have shown that either $\varepsilon_C\big(\check\Psi(\alpha) \setminus \mathcal{R}_\bullet\big) \cap \check\Psi(\alpha) \setminus \mathcal{R}_\bullet = \emptyset$ or $\varepsilon_C\big(\check\Psi(\alpha) \setminus \mathcal{R}_\bullet\big) = \check\Psi(\alpha) \setminus \mathcal{R}_\bullet$

If $\check\Psi(\alpha) \cap \mathcal{R}_\bullet$ is not empty, then there exists $\beta \in \check\Psi(\alpha) \setminus \mathcal{R}_\bullet$ such that $\partial_e\{\beta\} \cap \check\Psi(\alpha) \cap \mathcal{R}_\bullet \neq \emptyset$. Hence $\mathrm{supp}(\bar\beta) \cap \check\Psi(\alpha) \neq \emptyset$ and, by the same argument as above, $\varepsilon_C(\beta) \in \check\Psi(\alpha)$ and we get (*iii*.b).

Finally we consider the case where $\check\Psi(\alpha) \cap \mathcal{R}_\bullet = \emptyset$. The boundary $\partial_e\big(\check\Psi(\alpha)\big)$ is not empty, thus it contains a root $\beta \in \Psi$ and $\beta \notin \mathcal{R}_\bullet$ because of Theorem 7.10. The fact that $\mathfrak{g}, \mathfrak{q}_\Psi$ is fundamental implies that $\varepsilon_C(\beta) \notin \Psi$. In particular $\varepsilon_C(\beta) \notin \partial_e\big(\check\Psi(\alpha)\big)$. Applying again Theorem 7.10 we have $\varepsilon_C\big(\check\Psi(\alpha) \setminus \mathcal{R}_\bullet\big) \cap \Psi = \emptyset$, hence $\varepsilon_C(\beta) \notin \check\Psi(\alpha)$. Since $\varepsilon_C(\beta) \in \partial_e\big(\mathrm{supp}(\bar\delta)\big)$ and $\mathrm{supp}(\bar\delta) \cap \mathcal{R}_\bullet \not\subset \check\Psi(\alpha)$, it follows that $\mathrm{supp}(\bar\delta) \cap \check\Psi(\alpha) = \emptyset$, thus $\check\Psi(\alpha) \cap \varepsilon_C\big(\check\Psi(\alpha)\big) = \emptyset$. ∎

Lemma 11.4. *With the same hypotheses of Lemma 11.3, the effective quotient of the fiber of the $\mathfrak{g}$-equivariant CR fibration $(\mathfrak{g}, \mathfrak{q}_\Phi) \longrightarrow (\mathfrak{g}, \mathfrak{q}_\Psi)$ has cross-marked Satake diagram*

$$\boldsymbol{\mathcal{S}}' = \bigcup_{\alpha \in \Phi \setminus \Psi} \check\Psi(\alpha) \cup \varepsilon_C\big(\check\Psi(\alpha) \setminus \mathcal{R}_\bullet\big)$$

and $\Phi' = \Phi \cap \boldsymbol{\mathcal{S}}'$.

In particular it is totally complex if and only if for each $\alpha \in \Phi \setminus \Psi$, condition (i) or (ii) of Lemma 11.3 holds.

Proof. The effective quotient is described in [30] and at the end of §4. From Theorem 7.4 we know that $\boldsymbol{\mathcal{S}}' \subset \bigcup_{\alpha \in \Phi \setminus \Psi} \check\Psi(\alpha) \cup \varepsilon_C\big(\check\Psi(\alpha) \setminus \mathcal{R}_\bullet\big)$. Equality then follows from the observation that if $\beta \in \check\Psi(\alpha) \setminus \mathcal{R}_\bullet$ then $\mathrm{supp}(\bar\beta) \cap \Psi = \emptyset$.

To prove the second statement, we can assume that there exists exactly one root $\alpha \in \Phi \setminus \Psi$. In cases (*i*) and (*ii*) of Lemma 11.3 the cross-marked Satake diagram of the fiber is of the types described in Theorem 10.2 (*i*), (*ii*) and is totally complex. If we are in case (*iii*) of Lemma 11.3, then $\check\Psi(\alpha) \cap \mathcal{R}_\bullet \neq \emptyset$, and the fiber is totally complex if and only if $\big(\check\Psi(\alpha), \Phi \cap \check\Psi(\alpha)\big)$ is one of the diagrams in Theorem 10.2 (*iii*).

Since $\partial_e\big(\check\Psi(\alpha)\big) \cap \mathcal{R}_\bullet = \emptyset$ and $\varepsilon_C\big(\partial_e\check\Psi(\alpha)\big) \cap \big(\check\Psi(\alpha) \cup \partial_e\check\Psi(\alpha)\big) = \emptyset$, We have that ε_C is not the identity, hence $\boldsymbol{\mathcal{S}}$ must be of type A III, A IV, D Ib, D IIIb, E II or E III. We exclude types A III, A IV, D Ib and E II because they do not contain subdiagrams of type A II or D II, so we are left with types D IIIb and E III.

Type D IIIb must be excluded because in this case we have $\alpha = \alpha_1$ or $\alpha_{\ell-2}$, $\check\Psi(\alpha) = \{\alpha_1, \dots, \alpha_{\ell-2}\}$ and $\partial_e\big(\check\Psi(\alpha)\big) = \{\alpha_{\ell-1}, \alpha_\ell\} = \varepsilon_C\big(\partial_e\check\Psi(\alpha)\big)$.

Similarly type E III must be excluded because we have $\alpha = \alpha_3$ or α_5, $\check\Psi(\alpha) = \{\alpha_2, \alpha_3, \alpha_4, \alpha_5\}$ and $\partial_e\big(\check\Psi(\alpha)\big) = \{\alpha_1, \alpha_6\} = \varepsilon_C\big(\partial_e\check\Psi(\alpha)\big)$. ∎

Theorem 11.5. *Let $(\mathfrak{g}, \mathfrak{q}_\Phi)$ be a simple fundamental effective parabolic minimal CR algebra and assume that it is not totally complex. Let Π be the set of simple roots α in Φ that satisfy either one of:*

(i) $\check{\Phi}(\alpha) \subset \mathcal{R}_\bullet$;

(ii) $\big(\check{\Phi}(\alpha) \cup \partial_e \check{\Phi}(\alpha)\big) \cap \mathcal{R}_\bullet = \emptyset$ *and* $\varepsilon_C\big(\check{\Phi}(\alpha)\big) \cap \Phi = \emptyset$.

Then $(\mathfrak{g}, \mathfrak{q}_\Phi)$ is weakly nondegenerate if and only if $\Pi = \emptyset$.

Set $\Psi = \Phi \setminus \Pi$. Then $(\mathfrak{g}, \mathfrak{q}_\Phi) \to (\mathfrak{g}, \mathfrak{q}_\Psi)$ is a $\mathfrak{g}$-equivariant CR fibration with totally complex fiber and fundamental weakly nondegenerate base.

Proof. Fix $\alpha \in \Phi \setminus \Psi$. Then conditions (i) and (ii) are necessary and sufficient for $(\mathfrak{g}, \mathfrak{q}_\Phi) \to (\mathfrak{g}, \mathfrak{q}_{\Phi\setminus\{\alpha\}})$ to be a $\mathfrak{g}$-equivariant CR fibration with totally complex fiber. This observation, Lemma 11.4 and Lemma 11.2 yield our first statement.

To prove the last part of the Theorem, we make the following

Claim. Let $\alpha, \beta \in \Phi$ with $\alpha \in \Pi$. Then β satisfies either (i) or (ii) for Φ if and only if β satisfies (i) or (ii) for $\Phi' = \Phi \setminus \{\alpha\}$.

Assuming that this claim is true, we conclude as follows. If $\Pi = \{\beta_1, \dots, \beta_k\}$, we have $\mathfrak{g}$-equivariant CR fibrations with totally complex fibers:

$$(\mathfrak{g}, \mathfrak{q}_\Phi) \to (\mathfrak{g}, \mathfrak{q}_{\Phi\setminus\{\beta_1\}}) \to (\mathfrak{g}, \mathfrak{q}_{\Phi\setminus\{\beta_1,\beta_2\}}) \to \cdots \to (\mathfrak{g}, \mathfrak{q}_{\Phi\setminus\Pi}).$$

Their composition is still a $\mathfrak{g}$-equivariant CR fibration with totally complex fiber, and the base $(\mathfrak{g}, \mathfrak{q}_\Psi)$ is weakly nondegenerate.

Now we prove the claim. If $\beta \notin \partial_e\big(\check{\Phi}(\alpha)\big) \cup \partial_e \varepsilon_C\big(\check{\Phi}(\alpha)\big)$, then $\check{\Phi}(\beta) = \check{\Phi}'(\beta)$, $\varepsilon_C\big(\check{\Phi}(\beta)\big) = \varepsilon_C\big(\check{\Phi}'(\beta)\big)$, and there is nothing to prove.

Assume $\beta \in \partial_e\big(\check{\Phi}(\alpha)\big)$; then $\check{\Phi}'(\beta) = \check{\Phi}(\beta) \cup \check{\Phi}(\alpha)$. If $\check{\Phi}(\alpha) \subset \mathcal{R}_\bullet$, then $\check{\Phi}(\beta) \subset \mathcal{R}_\bullet$ if and only if $\check{\Phi}'(\beta) \subset \mathcal{R}_\bullet$.

If $\check{\Phi}'(\beta) \cap \mathcal{R}_\bullet = \emptyset$, we need to prove that, if β satisfies (i) or (ii), then $\check{\Phi}(\alpha) \cap \varepsilon_C\big(\check{\Phi}(\beta)\big) = \emptyset$. This is true because otherwise $\varepsilon_C(\beta) \in \partial_e\big(\check{\Phi}(\alpha)\big)$, and this yields a contradiction because we assumed that $(\mathfrak{g}, \mathfrak{q}_\Phi)$ is fundamental.

Finally if $\beta \in \partial_e \varepsilon_C\big(\check{\Phi}(\alpha)\big)$ then $\beta \notin \mathcal{R}_\bullet$ and $\varepsilon_C(\beta) \in \partial_e\big(\check{\Phi}(\alpha)\big)$, again contradicting the assumption that $(\mathfrak{g}, \mathfrak{q}_\Phi)$ is fundamental. ∎

12. Strict nondegeneracy

In this section we give necessary and sufficient conditions for a weakly nondegenerate CR algebra to be strictly nondegenerate. We recall from the introduction that the CR geometry of strict nondegenerate homogeneous CR manifolds can be related to the *standard* models and investigated by using the *Levi-Tanaka algebras* (cf. [30], [35], [36]). Therefore, by classifying the weakly degenerate minimal orbits that do not have the strict nondegeneracy property, we single out a class of homogeneous CR manifolds with a highly non trivial CR structure that cannot be discussed by using the standard Levi-Tanaka models. This also explains the need to introduce CR algebras, as a generalization of the Levi-Tanaka algebras, in [30].

First we reformulate weak and strict nondegeneracy in terms of the root system:

Lemma 12.1. *A fundamental effective parabolic minimal CR algebra* $(\mathfrak{g}, \mathfrak{q})$ *is weakly nondegenerate if and only if for every root* $\alpha \in \bar{\mathcal{Q}} \setminus \mathcal{Q}$ *there exist a sequence* $(\beta_i \in \mathcal{Q})_{1 \leq i \leq n}$ *such that*

$$\alpha_j = \alpha + \sum_{i \leq j} \beta_i \in \mathcal{R} \quad \forall j = 1, \dots, n\,, \qquad \alpha_n \notin \mathcal{Q} \cup \bar{\mathcal{Q}}\,. \tag{12.1}$$

Proof. The statement is an easy consequence of [30, Theorem 6.2]. ∎

Likewise we have:

Lemma 12.2. *A fundamental effective parabolic minimal CR algebra* $(\mathfrak{g}, \mathfrak{q})$ *is strictly nondegenerate if and only if for every root* $\alpha \in \bar{\mathcal{Q}} \setminus \mathcal{Q}$ *there exists* $\beta \in \mathcal{Q}$ *such that* $\alpha + \beta \in \mathcal{R}$ *and* $\alpha + \beta \notin \mathcal{Q} \cup \bar{\mathcal{Q}}$. ∎

Next we prove that it suffices to check this condition on purely imaginary roots:

Proposition 12.3. *A necessary and sufficient condition for a fundamental effective weakly nondegenerate parabolic minimal CR algebra* $(\mathfrak{g}, \mathfrak{q})$ *to be strictly nondegenerate is that for every root* $\alpha \in \mathcal{R}_\bullet \cap \bar{\mathcal{Q}} \setminus \mathcal{Q}$ *there exists* $\beta \in \mathcal{Q}$ *such that* $\alpha + \beta \in \mathcal{R}$ *and* $\alpha + \beta \notin \mathcal{Q} \cup \bar{\mathcal{Q}}$.

Proof. The condition is obviously necessary.

To prove sufficiency, consider a root $\alpha \in \bar{\mathcal{Q}} \setminus \mathcal{Q}$, $\alpha \notin \mathcal{R}_\bullet$; since $\alpha \prec 0$, we have $-\alpha \in \mathcal{R}^+ \setminus \mathcal{R}_\bullet$. This implies that $-\bar{\alpha} \in \mathcal{R}^+ \subset \mathcal{Q}$. Then $-\alpha \in \bar{\mathcal{Q}}$ and $\alpha \in \bar{\mathcal{Q}}^r$. By the assumption that $(\mathfrak{g}, \mathfrak{q})$ is weakly nondegenerate, using Lemma 12.1 we can find a sequence of roots (β_i) satisfying (12.1). Take the sequence $(\beta_i)_{1 \leq i \leq n}$ of minimal length; we claim that for every permutation τ of the indices, the sequence $(\beta_{\tau(i)})_{1 \leq i \leq n}$ still satisfies (12.1).

Indeed, fix a Chevalley basis $\{X_\alpha\}_{\alpha \in \mathcal{R}}$. Then, for every transposition $(i, i+1)$:

$$\begin{aligned} &\mathfrak{q} + \bar{\mathfrak{q}} \not\ni [X_{\beta_n}, \dots, X_{\beta_{i+1}}, X_{\beta_i}, \dots, X_{\beta_1}, X_\alpha] = \\ &\quad = [X_{\beta_n}, \dots, X_{\beta_i}, X_{\beta_{i+1}}, \dots, X_{\beta_1}, X_\alpha] + [X_{\beta_n}, \dots, [X_{\beta_{i+1}}, X_{\beta_i}], \dots, X_{\beta_1}, X_\alpha]. \end{aligned}$$

The last addendum in the right hand side belongs to $\mathfrak{q} + \bar{\mathfrak{q}}$ by our assumption that $(\beta_i)_{1 \leq i \leq n}$ has minimal length. Thus

$$[X_{\beta_n}, \dots, X_{\beta_i}, X_{\beta_{i+1}}, \dots, X_{\beta_1}, X_\alpha] \in \hat{\mathfrak{g}} \setminus (\mathfrak{q} + \bar{\mathfrak{q}})\,.$$

In particular $\alpha + \beta_i \in \mathcal{R}$ for every i. At least one of the β_i's, say β_{i_0}, does not belong to $\bar{\mathcal{Q}}$, so $\alpha + \beta_{i_0} \notin \bar{\mathcal{Q}}$. Indeed, since $\alpha \in \bar{\mathcal{Q}}^r$, if $\alpha + \beta_i \in \bar{\mathcal{Q}}$, then also $\beta_i = (\alpha + \beta_i) + (-\alpha) \in \bar{\mathcal{Q}}$. By a permutation, we can take $\beta_{i_0} = \beta_n$. Then we claim that $\alpha + \beta_n \notin \mathcal{Q} \cup \bar{\mathcal{Q}}$. Indeed we already choose β_n so that $\alpha + \beta_n \notin \bar{\mathcal{Q}}$. If $\alpha + \beta_n \in \mathcal{Q}$, we have $[X_{\beta_1}, \dots, X_{\beta_{n-1}}, X_{\beta_n}, X_\alpha] = [X_{\beta_1}, \dots, X_{\beta_{n-1}}, [X_{\beta_n}, X_\alpha]] \in \mathfrak{q}$, because $X_{\beta_i} \in \mathfrak{q}$ for every $i = 1, \dots, n$, and hence $\alpha_n \in \mathcal{Q}$, contradicting (12.1). ∎

Theorem 12.4. *Let $(\mathfrak{g}, \mathfrak{q}_\Phi)$ be an effective parabolic minimal CR algebra, with $\mathfrak{g}$ simple. If $(\mathfrak{g}, \mathfrak{q}_\Phi)$ is weakly nondegenerate, but is not strictly nondegenerate, then Φ is contained in a connected component of $\mathcal{B} \cap \mathcal{R}_\bullet$.*

The strictly nondegenerate $(\mathfrak{g}, \mathfrak{q}_\Phi)$ with $\mathfrak{g}$ simple and $\Phi \subset \mathcal{R}_\bullet$ are those listed below:

(B Ib / B II) $\quad \Phi = \{\alpha_{p+1}\}$

(C IIa / IIb) $\quad \Phi = \{\alpha_{2i-1}\},\ 1 \le i \le p$

(D Ia) $\quad \Phi = \{\alpha_{p+1}\}$

(D II) $\quad \Phi = \{\alpha_2\}$

(E III) $\quad \Phi = \begin{cases} \{\alpha_4\} \\ \{\alpha_3, \alpha_4\} \\ \{\alpha_4, \alpha_5\} \\ \{\alpha_3, \alpha_4, \alpha_5\} \end{cases}$

$\begin{pmatrix} \text{E IV} \\ \text{E VII} \\ \text{E IX} \end{pmatrix} \quad \Phi = \begin{cases} \{\alpha_3\} \\ \{\alpha_5\} \\ \{\alpha_3, \alpha_4\} \\ \{\alpha_3, \alpha_5\} \\ \{\alpha_4, \alpha_5\} \\ \{\alpha_2, \alpha_3\} \\ \{\alpha_2, \alpha_5\} \end{cases}$

(F II) $\quad \Phi = \begin{cases} \{\alpha_2\} \\ \{\alpha_4\} \end{cases}$

Proof. We prove the first statement. The proof of the second will be omitted, as it requires a straightforward case by case analysis, chasing over the different Satake diagrams.

Suppose that $(\mathfrak{g}, \mathfrak{q}_\Phi)$ is weakly, but not strictly, nondegenerate. Then there is some root $\alpha \in \bar{\mathcal{Q}}_\Phi \setminus \mathcal{Q}_\Phi$, $\alpha \prec 0$, such that $\alpha + \beta \in \mathcal{Q}_\Phi \cup \bar{\mathcal{Q}}_\Phi$ for all $\beta \in \mathcal{Q}_\Phi$ for which $\alpha + \beta \in \mathcal{R}$. By Proposition 12.3 we can take $\alpha \in \mathcal{R}_\bullet$. Let $\mathcal{B}'$ be the connected component of $\mathrm{supp}(\alpha)$ in $\mathcal{B} \cap \mathcal{R}_\bullet$. Since $\alpha \notin \mathcal{Q}_\Phi$, we have $\mathcal{B}' \cap \Phi \neq \emptyset$.

Since we assumed that $(\mathfrak{g}, \mathfrak{q}_\Phi)$ is weakly nondegenerate, for each $\gamma \in \Phi$ the set $\check{\Phi}(\gamma)$ is not contained in $\mathcal{R}_\bullet$. As $\mathrm{supp}(\alpha) \cap \Phi \neq \emptyset$, this implies that there is some $\beta \in \mathcal{Q}_\Phi$, with $\beta \prec 0$, such that $\beta \notin \mathcal{R}_\bullet$ and $\alpha + \beta \in \mathcal{R}$. Since $\beta \in \mathcal{Q}^r_\Phi$ and $-\alpha \in \mathcal{Q}^n_\Phi$, we obtain that $\alpha + \beta \notin \mathcal{Q}_\Phi$. If $\mathcal{B}' \cap \Phi$ contains some α_i which does not belong to $\mathrm{supp}(\alpha)$, this α_i would belong to $\mathrm{supp}(\overline{\alpha + \beta})$. Indeed $\alpha + \beta \notin \mathcal{R}_\bullet$, hence $\mathrm{supp}(\overline{\alpha + \beta})$ contains all simple imaginary roots γ that are

not in $\mathrm{supp}(\alpha+\beta)$ and such that $\partial_e\check{\Xi}(\gamma)\cap\mathrm{supp}(\alpha+\beta)\neq\emptyset$. This shows that $\mathcal{B}'\cap\Phi=\mathrm{supp}(\alpha)\cap\Phi$.

Let $\mathcal{A}=(\mathcal{R}_\bullet\cap[\Phi\setminus\mathcal{B}'])\cup\varepsilon_C(\Phi\setminus\mathcal{R}_\bullet)$. We want to show that $\mathcal{A}=\emptyset$.

Assume by contradiction that $\mathcal{A}$ is not empty. Then there exists a segment S in $\mathcal{B}\setminus\Phi$ joining $\mathcal{A}$ to $\mathrm{supp}(\alpha)$, i.e. such that $\partial_e S\cap\mathcal{A}\neq\emptyset$, $\partial_e S\cap\mathrm{supp}(\alpha)\neq\emptyset$. By taking S of minimal lenght, we can also assume that $S\cap(\mathcal{A}\cup\mathrm{supp}(\alpha))=\emptyset$.

Let $\beta=-\delta(S)$. Then $\beta\prec 0$, $\beta\in\mathcal{Q}^r_\Phi$ and $\beta\notin\mathcal{R}_\bullet$, so that $\alpha+\beta\in\mathcal{R}\setminus\mathcal{Q}_\Phi$.

If there is some α_i in $\partial_e S\cap\mathcal{A}\cap\mathcal{R}_\bullet\neq\emptyset$, then $\alpha+\beta\in\mathcal{R}$, $\mathrm{supp}(\overline{\alpha+\beta})\ni\alpha_i$, and $\alpha+\beta\notin\bar{\mathcal{Q}}_\Phi$, contradicting our assumption.

If $\partial_e S\cap\mathcal{A}\cap\mathcal{R}_\bullet=\emptyset$, there is α_i in $\Phi\setminus\mathcal{R}_\bullet$ with $\varepsilon_C(\alpha_i)\in\partial_e S\cap\mathcal{A}$. Set $\beta'=\beta-\varepsilon_C(\alpha_i)$. Then $\beta'\in\mathcal{Q}_\Phi$, and $\alpha+\beta'\in\mathcal{R}\setminus\left(\mathcal{Q}_\Phi\cup\bar{\mathcal{Q}}_\Phi\right)$, yielding a contradiction; this shows that $\mathcal{A}$ is empty, completing the proof of our first claim. ∎

13. Essential pseudoconcavity for minimal orbits

Let (M,HM,J) be a CR manifold of finite kind (cf. §2). We say that (M,HM,J) is *essentially pseudoconcave* (see [17]) if it is possible to define a Hermitian symmetric smooth scalar product h on the fibers of HM such that for each $\xi\in H^0M$ the Levi form $\mathcal{L}_\xi$ has zero trace with respect to h. For a homogeneous CR manifold, this last condition is equivalent to the fact that for each $\xi\in H^0M$ the Levi form $\mathcal{L}_\xi$ is either 0 or has at least one positive and one negative eigenvalue.

The CR functions defined on essentially pseudoconcave CR manifolds enjoy some nice properties, like local smoothness and the local maximum modulus principle; CR sections of CR complex line bundles have the weak unique continuation property (see [17], [18]). When M is compact and essentially pseudoconcave, global CR functions are constant and CR-meromorphic functions form a field of finite transcendence degree (see [19]).

In this section we classify the essentially pseudoconcave minimal orbits of complex flag manifolds.

We keep the notation of the previous sections. In particular, $(\mathfrak{g},\mathfrak{q}_\Phi)$ is an effective parabolic minimal CR algebra, with associated cross-marked Satake diagram $(\boldsymbol{\mathcal{S}},\Phi)$. Moreover, we introduce a Chevalley system for $(\hat{\mathfrak{g}},\hat{\mathfrak{h}})$, i.e. a family $(Z_\alpha)_{\alpha\in\mathcal{R}}$ with the properties ([11, Ch.VIII,§2]):

- (i) $Z_\alpha\in\hat{\mathfrak{g}}^\alpha$ for all $\alpha\in\mathcal{R}$;
- (ii) $[Z_\alpha,Z_{-\alpha}]=-H_\alpha$, where H_α is the unique element of $[\hat{\mathfrak{g}}^\alpha,\hat{\mathfrak{g}}^{-\alpha}]$ such that $\alpha(H_\alpha)=2$;
- (iii) the $\mathbb{C}$-linear map that transforms each $H\in\hat{\mathfrak{h}}$ into $-H$ and Z_α into $Z_{-\alpha}$ for every $\alpha\in\mathcal{R}$ is an automorphism of the complex Lie algebra $\hat{\mathfrak{g}}$.

In particular, $(Z_\alpha)_{\alpha\in\mathcal{R}}\cup(H_\alpha)_{\alpha\in\mathcal{B}}$ is a basis of $\hat{\mathfrak{g}}$ as a $\mathbb{C}$-linear space. We denote by $(\xi^\alpha)_{\alpha\in\mathcal{R}}\cup(\omega^\alpha)_{\alpha\in\mathcal{B}}$ the corresponding dual basis in $\hat{\mathfrak{g}}^*$.

Let $\mathfrak{M}$ be the complex flag manifold $\hat{\mathbf{G}}/\mathbf{Q}$ and M the minimal orbit $\mathbf{G}/\mathbf{G}_+$ of $\mathbf{G}$ in $\mathfrak{M}$. As usual, $o \simeq e \cdot \mathbf{G}_+ \simeq e \cdot \mathbf{Q}$ is the base point. We note that

$$T^{1,0}_o\mathfrak{M} \simeq \hat{\mathfrak{g}}/\mathfrak{q} \simeq \langle Z_\alpha \mid -\alpha \in \mathcal{Q}^n \rangle_{\mathbb{C}} \, .$$

Therefore a Hermitian metric in $\mathfrak{M}$ is expressed at the point o by:

$$\tilde{h}_o \, = \sum_{\alpha,\beta \in \mathcal{Q}^n} c_{\alpha,\bar{\beta}} \xi^{-\alpha} \otimes \bar{\xi}^{-\beta} \, .$$

where $(c_{\alpha,\bar{\beta}})$ is Hermitian symmetric and positive definite. For the minimal orbit we have:

$$T^{1,0}_o M \simeq \bar{\mathfrak{q}}/(\mathfrak{q} \cap \bar{\mathfrak{q}}) \simeq \left\langle Z_\alpha \mid -\alpha \in \mathcal{Q}^n \, , \; \alpha \in \bar{\mathcal{Q}} \right\rangle_{\mathbb{C}} \, .$$

Thus a Hermitian metric h in $T^{1,0}M$ can be represented at o by:

$$h_o \, = \sum_{\alpha,\beta \in \mathcal{Q}^n \setminus \bar{\mathcal{Q}}^n} c_{\alpha,\bar{\beta}} \xi^{-\alpha} \otimes \bar{\xi}^{-\beta} \, .$$

where $(c_{\alpha,\bar{\beta}})$ is again Hermitian symmetric and positive definite.

The subspace $\hat{\mathfrak{t}} = \sum_{\alpha \in \mathcal{Q}^n \cap \bar{\mathcal{Q}}^n} \hat{\mathfrak{g}}^{-\alpha}$ is a nilpotent Lie subalgebra of $\hat{\mathfrak{g}}$, which is the complexification of a real subalgebra $\mathfrak{t} = \hat{\mathfrak{t}} \cap \mathfrak{g}$ of $\mathfrak{g}$. It can be identified to the quotient T_oM/H_oM and hence its dual space $\mathfrak{t}^*$ to the stalk H^0_oM of the characteristic bundle of M at o.

From this discussion we obtain the criterion:

Proposition 13.1. *A necessary and sufficient condition for M to be essentially pseudoconcave is that there exists a positive definite Hermitian symmetric matrix $(c_{\alpha,\bar{\beta}})_{\alpha,\beta \in \mathcal{Q}^n \setminus \bar{\mathcal{Q}}^n}$ such that*

$$\sum_{\substack{\alpha,\beta \in \mathcal{Q}^n \setminus \bar{\mathcal{Q}}^n \\ \alpha + \bar{\beta} = \gamma}} c_{\alpha,\bar{\beta}} [Z_\alpha, \bar{Z}_\beta] = 0 \qquad \forall \gamma \in \mathcal{Q}^n \cap \bar{\mathcal{Q}}^n \, . \tag{13.1}$$

Proof. Indeed (13.1) ie equivalent to the formula we obtain by changing α, β, γ into $-\alpha, -\beta, -\gamma$. ∎

Denote by $\check{\mathfrak{T}}^{1,0}$ the $\mathbb{C}$-linear subspace of $\hat{\mathfrak{g}}$ with basis $(Z_\alpha)_{\alpha \in \mathcal{Q}^n \setminus \bar{\mathcal{Q}}^n}$. To each $\gamma \in \mathcal{Q}^n \cap \bar{\mathcal{Q}}^n$ we associate a complex-valued form of type $(1,1)$ in $\check{\mathfrak{T}}^{1,0}$:

$$\mathbf{L}_\gamma : \check{\mathfrak{T}}^{1,0} \times \check{\mathfrak{T}}^{1,0} \ni (Z, W) \to \mathbf{L}_\gamma(Z, W) = (1/i) \kappa_{\hat{\mathfrak{g}}}(Z_{-\gamma}, [Z, \bar{W}]) \in \mathbb{C} \, , \tag{13.2}$$

where $\kappa_{\hat{\mathfrak{g}}}$ is the Killing form in $\hat{\mathfrak{g}}$. When $\gamma = \bar{\gamma}$ is real, we take $Z_{-\gamma}$ in $\mathfrak{g}$, to obtain a Hermitian symmetric $\mathbf{L}_\gamma$.

We obviously have:

Lemma 13.2. *The following are equivalent:*

- *(i) $M = M(\mathfrak{g}, \mathfrak{q})$ is essentially pseudoconcave;*
- *(ii) There exists a Hermitian symmetric positive definite form $\mathbf{h}$ in $\check{\mathfrak{T}}^{1,0}$ such that all $\mathbf{L}_\gamma$, for $\gamma \in \mathcal{Q}^n \cap \bar{\mathcal{Q}}^n$ have zero trace with respect to $\mathbf{h}$;*
- *(iii) For each $\gamma \in \mathcal{Q}^n \cap \bar{\mathcal{Q}}^n$ the Hermitian quadratic forms in $\check{\mathfrak{T}}^{1,0}$:*

$$\check{\mathfrak{T}}^{1,0} \ni Z \to \Re \mathbf{L}_\gamma(Z, \bar{Z}) \in \mathbb{R} \quad \text{and} \quad \check{\mathfrak{T}}^{1,0} \ni Z \to \Im \mathbf{L}_\gamma(Z, \bar{Z}) \in \mathbb{R} \tag{13.3}$$

are either 0 or have at least one positive and one negative eigenvalue.

Proof. The equivalence was proved in [17]. ∎

Proposition 13.3. *Let $(\mathfrak{g},\mathfrak{q})$ be an effective parabolic minimal fundamental CR algebra. A necessary and sufficient condition for $M = M(\mathfrak{g},\mathfrak{q})$ to be essentially pseudoconcave is that for all real roots $\gamma \in \mathcal{Q}^n \cap \bar{\mathcal{Q}}^n$ the Hermitian symmetric form $\mathbf{L}_\gamma$ is either zero or has at least one positive and one negative eigenvalue.*

Proof. The condition is obviously necessary. We prove sufficiency. Let Γ be a subset of $\mathcal{Q}^n \cap \bar{\mathcal{Q}}^n$ and let $\mathcal{H}(\Gamma)$ the $\mathbb{R}$-linear space consisting of the Hermitian symmetric parts of all linear combinations $\sum_{\gamma\in\Gamma} a_\gamma \mathbf{L}_\gamma$ with $a_\gamma \in \mathbb{C}$. When $\gamma \in \mathcal{Q}^n \cap \bar{\mathcal{Q}}^n$ is not real, the Hermitian symmetric part h of $a\mathbf{L}_\gamma$, for $a \in \mathbb{C}$, satisfies $h(Z_\alpha, \bar{Z}_\alpha) = 0$ for all $\alpha \in \mathcal{Q}^n \setminus \bar{\mathcal{Q}}^n$. More generally, if Γ_0 is the set of all $\gamma \in \mathcal{Q}^n \cap \bar{\mathcal{Q}}^n$ for which $\sum_{\alpha\in\mathcal{Q}^n\setminus\bar{\mathcal{Q}}^n} \mathbf{L}_\gamma(Z_\alpha, \bar{Z}_\alpha) = 0$, then the matrices $(h(Z_\alpha, \bar{Z}_\beta))_{\alpha,\beta\in\mathcal{Q}^n\setminus\bar{\mathcal{Q}}^n}$ corresponding to $h \in \mathcal{H}(\Gamma_0)$ have zero trace and thus every $h \in \mathcal{H}(\Gamma_0)$ that is $\neq 0$ has at least one positive and one negative eigenvalue.

Choose Γ as a maximal subset of $\mathcal{Q}^n \cap \bar{\mathcal{Q}}^n$ that contains Γ_0 and has the property that all non zero $h \in \mathcal{H}(\Gamma)$ have at least one positive and one negative eigenvalue.

If $\Gamma = \mathcal{Q}^n \cap \bar{\mathcal{Q}}^n$, then $M(\mathfrak{g},\mathfrak{q})$ is essentially pseudoconcave. Assume by contradiction that there is $\gamma \in \mathcal{Q}^n \cap \bar{\mathcal{Q}}^n \setminus \Gamma$.

Then γ is real, $\mathbf{L}_\gamma$ is Hermitian symmetric and $\mathcal{H}(\Gamma\cup\{\gamma\}) = \mathcal{H}(\Gamma) + \mathbb{R}\cdot \mathbf{L}_\gamma$. Moreover, there is at least one root $\alpha_0 \in \mathcal{Q}^n \setminus \bar{\mathcal{Q}}^n$ such that $\gamma = \alpha_0 + \bar{\alpha}_0$. Assume that there is another root $\alpha_1 \in \mathcal{Q}^n \setminus \bar{\mathcal{Q}}^n$ with $\alpha_1 + \bar{\alpha}_1 = \gamma$ and $\mathbf{L}_\gamma(Z_{\alpha_0}, \bar{Z}_{\alpha_0})\cdot\mathbf{L}_\gamma(Z_{\alpha_1}, \bar{Z}_{\alpha_1}) < 0$. If $h \in \mathcal{H}(\Gamma)$, then $h(Z_{\alpha_0}, \bar{Z}_{\alpha_0}) = h(Z_{\alpha_1}, \bar{Z}_{\alpha_1}) = 0$. Then the matrix associated in the basis (Z_α) to a linear combinations $h + c\mathbf{L}_\gamma$ with $c \in \mathbb{R}$, $c \neq 0$, has two entries of opposite sign on the main diagonal and therefore at least one negative and one positive eigenvalue. This would contradict the maximality of Γ. Hence we must assume that all terms $\mathbf{L}_\gamma(Z_\alpha, \bar{Z}_\alpha)$ have the same sign.

By the assumption that $\mathbf{L}_\gamma$ has at least one positive and one negative eigenvalue, we deduce that there are roots $\beta_1, \beta_2 \in \mathcal{Q}^n \setminus \bar{\mathcal{Q}}^n$ such that $\beta_2 \neq \bar{\beta}_1$ and $\beta_1 + \bar{\beta}_2 = \bar{\beta}_1 + \beta_2 = \gamma$, so that $\mathbf{L}_\gamma(Z_{\beta_1}, \bar{Z}_{\beta_2}) \neq 0$. If $h(Z_{\beta_2}, \bar{Z}_{\beta_2}) = 0$ for all $h \in \mathcal{H}(\Gamma)$, then the matrix corresponding to $h + c\mathbf{L}_\gamma$, for $h \in \mathcal{H}(\Gamma)$, $c \in \mathbb{R}$, $c \neq 0$ in the basis (Z_α) contains a principal 2×2 minor matrix, corresponding to β_1, β_2, of the form

$$\begin{pmatrix} a & \lambda \\ \bar{\lambda} & 0 \end{pmatrix} \quad \text{with} \quad a \in \mathbb{R} \quad \text{and} \quad \lambda \in \mathbb{C},\ \lambda \neq 0\,.$$

Thus it would have at least one positive and one negative eigenvalue, contradicting the choice of Γ.

Therefore, if $\Gamma \neq \mathcal{Q}^n_\Phi \cap \bar{\mathcal{Q}}^n_\Phi$, we have:

(i) there exists $\alpha_0 \in \mathcal{Q}^n \setminus \bar{\mathcal{Q}}^n$ such that $\alpha_0 + \bar{\alpha}_0 = \gamma \in \mathcal{Q}^n \cap \bar{\mathcal{Q}}^n$;

(ii) there exists $\alpha_1, \alpha_2 \in \mathcal{Q}^n \setminus \bar{\mathcal{Q}}^n$ with $\alpha_2 \neq \alpha_1$, $\alpha_2 \neq \bar{\alpha}_1$ and $\alpha_1 + \bar{\alpha}_2 = \gamma$;

(*iii*) for all $\alpha, \beta \in \mathcal{Q}^n \setminus \bar{\mathcal{Q}}^n$ with $\alpha \neq \beta$, $\beta \neq \bar{\alpha}$ and $\alpha + \bar{\beta} = \gamma$, we have $\alpha + \bar{\alpha} \in \mathcal{Q}^n \cap \bar{\mathcal{Q}}^n$ and $\beta + \bar{\beta} \in \mathcal{Q}^n \cap \bar{\mathcal{Q}}^n$.

The roots $\alpha_0, \bar{\alpha}_0, \alpha_1, \bar{\alpha}_1, \alpha_2, \bar{\alpha}_2$ generate a root system $\mathcal{R}'$ in their span in $\mathfrak{h}^*_{\mathbb{R}}$, that is closed under conjugation. Since we have the relations $\alpha_0 + \bar{\alpha}_0 = \alpha_1 + \bar{\alpha}_2 = \alpha_2 + \bar{\alpha}_0$, the span of $\mathcal{R}'$ has dimension ≤ 4. Moreover, $\alpha_0 + \bar{\alpha}_0$, $\alpha_1 + \bar{\alpha}_1$ and $\alpha_2 + \bar{\alpha}_2$ must be three distinct roots in $\mathcal{R}'$. Indeed, set $\alpha_1 + \bar{\alpha}_1 = \gamma_1$, $\alpha_2 + \bar{\alpha}_2 = \gamma_2$. By assumption $\gamma_1 \neq \gamma \neq \gamma_2$. Moreover we obtain $\alpha_1 - \alpha_2 = \gamma_1 - \gamma = \gamma - \gamma_2$, i.e. $\gamma_1 + \gamma_2 = 2\gamma$, which implies that $\gamma_1 \neq \gamma_2$ when $\gamma_1 \neq \gamma \neq \gamma_2$.

Thus the dimension of the span of $\mathcal{R}'$ is ≤ 4. An inspection of the Satake diagrams corresponding to bases of at most 4 simple roots shows that no such root system contains 3 distinct positive real roots that are sum of a root and its conjugate. Denote by Ω the set of positive real roots γ that are of the form $\gamma = \alpha + \bar{\alpha}$ with $\alpha \in \mathcal{R}$. To verify our claim, we only need to consider the diagrams with $\ell = 3, 4$ and $\Omega \neq \emptyset$:

(A IIIa, IIIb)
$$\begin{array}{ll} \mathfrak{su}(1,3): & \Omega = \{\alpha_1 + \alpha_2 + \alpha_3\} \\ \mathfrak{su}(2,2): & \Omega = \{\alpha_1 + \alpha_2 + \alpha_3\} \\ \mathfrak{su}(1,4): & \Omega = \{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4\} \\ \mathfrak{su}(2,3): & \Omega = \{\alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4\} \end{array}$$

(C IIa)
$$\begin{array}{ll} \mathfrak{sp}(1,2): & \Omega = \{\alpha_1 + 2\alpha_2 + \alpha_3\} \\ \mathfrak{sp}(1,3): & \Omega = \{\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4\} \end{array}$$

Thus we obtained a contradiction, proving our statement. ∎

Theorem 13.4. *Let $(\mathfrak{g}, \mathfrak{q}_\Phi)$ be a simple effective and fundamental parabolic minimal CR algebra. Then $M(\mathfrak{g}, \mathfrak{q}_\Phi)$ is always essentially pseudoconcave if $\mathfrak{g}$ is either of the complex type, or compact, or of real type* A II, A IIIb, B, C IIb, D I, D II, D IIIa, E II, E IV, E VI, E VII, E IX. *In the remaining cases $M(\mathfrak{g}, \mathfrak{q}_\Phi)$ is essentially pseudoconcave if and only if we have one of the following:*

(A IIIa-IV)
$$\begin{cases} \Phi \subset \mathcal{R}_\bullet \\ \Phi \subset \{\alpha_i \,|\, i < p\} \cup \{\alpha_i \,|\, i > q\} \end{cases}$$

(C IIa)
$$\begin{cases} \Phi \subset \{\alpha_{2h-1} \,|\, 1 \leq h \leq p\} \\ \Phi \subset \{\alpha_i \,|\, i > 2p\} \end{cases}$$

(D IIIb)
$$\Phi \cap \{\alpha_{\ell-1}, \alpha_\ell\} = \emptyset$$

(E III)
$$\begin{cases} \{\alpha_4\} \subset \Phi \subset \mathcal{R}_\bullet \\ \Phi = \{\alpha_3, \alpha_5\} \end{cases}$$

(F II)
$$\Phi \subset \{\alpha_1, \alpha_2\}$$

[See the table of Satake diagrams for the types and the references to the roots in the statement.]

Proof. We exclude in the statement the split forms, because in these cases $(\mathfrak{g}, \mathfrak{q})$ is not fundamental. When $\mathfrak{g}$ is compact, $(\mathfrak{g}, \mathfrak{q})$ is totally complex and thus essentially pseudoconcave, since the condition on the Levi form is trivially fulfilled.

For $\mathfrak{g}$ of the complex types or of the real types A II, A IIIb, B, C IIb, D I, D II, D IIIa, E II, E IV, E VI, E VII, E IX the statement follows from the fact that $\mathcal{Q}^n \cap \bar{\mathcal{Q}}^n$ cannot possibly contain a root of the form $\alpha + \bar{\alpha}$ with $\alpha \in \mathcal{Q}^n \setminus \bar{\mathcal{Q}}^n$.

To discuss the remaining cases, we shall use the following:

Lemma 13.5. *Let $\mathfrak{g}$ be a semisimple real Lie algebra, with a Cartan decomposition, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, and $\mathfrak{h}$ a Cartan subalgebra which is invariant with respect to the corresponding Cartan involution ϑ and with maximal vector part. Denote by σ the conjugation of $\hat{\mathfrak{g}}$ with respect to the real form $\mathfrak{g}$ and let $\tau = \sigma \circ \vartheta$ the conjugation with respect the compact form $\mathfrak{k} \oplus i\,\mathfrak{p}$ of $\hat{\mathfrak{g}}$. Set $\mathcal{R} = \mathcal{R}(\hat{\mathfrak{g}}, \hat{\mathfrak{h}})$. Then there exists a Chevalley system $\{X_\alpha\}_{\alpha \in \mathcal{R}}$ with $X_\alpha \in \hat{\mathfrak{g}}^\alpha$ such that:*

$$\begin{cases} [X_\alpha, X_{-\alpha}] = -H_\alpha \quad \forall \alpha \in \mathcal{R} \\ [H_\alpha, X_\beta] = \beta(H_\alpha) X_\beta \\ [X_\alpha, X_\beta] = N_{\alpha,\beta} X_{\alpha+\beta} \\ \tau(X_\alpha) = \sigma(X_\alpha) = \bar{X}_\alpha = X_{-\alpha} \quad \forall \alpha \in \mathcal{R}_\bullet \end{cases}$$

where the H_α and the coefficients $N_{\alpha,\beta}$ satisfy:

$$\begin{cases} \beta(H_\alpha) = q - p \\ N_{\alpha,\beta} = \pm(q+1) \\ N_{\alpha,\beta} \cdot N_{-\alpha,\alpha+\beta} = -p(q+1) \\ \text{if } \beta - q\alpha, \ldots, \beta + p\alpha \text{ is the } \alpha\text{-string through } \beta. \end{cases}$$

Proof. (*of Lemma 13.5*) For the proof of this lemma we refer the reader to [11, Ch.VIII], or [15, Ch.III]. ∎

Lemma 13.6. *With the notation of Lemma 13.5: let $\alpha, \beta \in \mathcal{R}$, with $\alpha \in \mathcal{R}_\bullet$, and $\alpha + \beta \in \mathcal{R}$, $\alpha - \beta \notin \mathcal{R}$, $\beta + \bar{\beta} \in \mathcal{R}$. Let*

$$\beta, \ldots, \beta + p\alpha \qquad \text{and} \qquad \beta + \bar{\beta} - q'\alpha, \ldots, \beta + \bar{\beta} + p'\alpha$$

be the α-strings through β and $\beta + \bar{\beta}$, respectively. Then we have:

$$[X_{\alpha+\beta}, \bar{X}_{\alpha+\beta}] = \left[[X_\alpha, X_\beta], \overline{[X_\alpha, X_\beta]}\right] = (p - p'(1+q'))\,[X_\beta, \bar{X}_\beta]\,. \tag{13.4}$$

Proof. (*of Lemma 13.6*) We observe that $[X_\alpha, X_\beta] = \pm X_{\alpha+\beta}$, because $\beta - \alpha \notin \mathcal{R}$. We have:

$$\begin{aligned} [X_{\alpha+\beta}, \bar{X}_{\alpha+\beta}] &= \left[[X_\alpha, X_\beta], \overline{[X_\alpha, X_\beta]}\right] = \left[[X_\alpha, X_\beta], [X_{-\alpha}, \bar{X}_\beta]\right] \\ &= [[[X_\alpha, X_\beta], X_{-\alpha}], \bar{X}_\beta] + [X_{-\alpha}, [[X_\alpha, X_\beta], \bar{X}_\beta]] \\ &= [[[X_\alpha, X_{-\alpha}], X_\beta], \bar{X}_\beta] + [X_{-\alpha}, [X_\alpha, [X_\beta, \bar{X}_\beta]]] \\ &= \left(-\beta(H_\alpha) + N_{\alpha,\beta+\bar{\beta}} N_{-\alpha,\beta+\bar{\beta}+\alpha}\right)\,[X_\beta, \bar{X}_\beta]\,, \end{aligned}$$

which, by Lemma 13.5, yields (13.4). ∎

Continuation of the Proof of Theorem 13.4.

We proceed by a case by case analysis of the simple real Lie algebras containing real roots γ of the form $\gamma = \alpha + \bar{\alpha}$.

A IIIa − IV The positive real roots that are of the form $\alpha+\bar{\alpha}$ for some $\alpha \in \mathcal{R}$ are:

$$\gamma_h = \textstyle\sum_{j=h}^{p+q-h} \alpha_j \quad \text{for} \quad h = 1, \dots, p.$$

(*i*) Assume that $\Phi \subset \mathcal{R}_\bullet$. All γ_h's belong to $\mathcal{Q}^n_\Phi \cap \bar{\mathcal{Q}}^n_\Phi$ and are sums $\alpha + \bar{\alpha}$ with $\alpha \in \mathcal{Q}^n_\Phi \setminus \bar{\mathcal{Q}}^n_\Phi$. To prove that $\mathbf{L}_{\gamma_h}$ has at least one positive and one negative eigenvalue, we consider the roots $\beta = \sum_{j=h}^{q-1} \alpha_j$ and $\delta = \sum_{j=p+1}^{p+q-h}$. They both belong to $\mathcal{Q}^n_\Phi \setminus \bar{\mathcal{Q}}^n_\Phi$ and $\beta + \bar{\beta} = \delta + \bar{\delta} = \gamma_h$. We have $\delta = \bar{\beta} + \eta$ with $\eta \in \mathcal{R}_\bullet$ and $\bar{\beta} - \eta \notin \mathcal{R}$. Since $\gamma_h \pm \eta \notin \mathcal{R}$, by Lemma 13.6 we obtain:

$$[X_\delta, \bar{X}_\delta] = [[X_{\bar{\beta}}, X_\eta], \overline{[X_{\bar{\beta}}, X_\eta]}] = -\,[X_\beta, \bar{X}_\beta]\,.$$

(*ii*) Assume that $\Phi \cap (\mathcal{R}_\bullet \cup \{\alpha_p, \alpha_q\}) = \emptyset$ and let $\Phi = \{\alpha_{j_1}, \dots, \alpha_{j_r}, \alpha_{h_1}, \dots, \alpha_{h_s}\}$ with $1 \le j_1 < \cdots < j_r < p < q < h_1 < \cdots < h_s \le \ell = p + q - 1$. We can assume that $r \ge 1$ and, if $s \ge 1$, that $p - j_r < h_1 - q$. Let $h'_1 = p+q-h_1$ if $s \ge 1$, and $h'_1 = 0$ otherwise. The real roots in $\mathcal{Q}^n_\Phi \cap \bar{\mathcal{Q}}^n_\Phi$ are the γ_k's with $1 \le k \le j_r$. All $\mathbf{L}_{\gamma_k}$'s with $k \le h'_1$ are 0. To show that the $\mathbf{L}_{\gamma_k}$'s with $h'_1 < k \le j_r$ have at least one positive and one negative eigenvalues, we consider $\alpha = \sum_{i=k}^{j_r} \alpha_i$ and $\beta = \sum_{i=k}^{\ell - j_r} \alpha_i$. They both belong to $\mathcal{Q}^n_\Phi \setminus \bar{\mathcal{Q}}^n_\Phi$, are distinct, and $\alpha + \bar{\beta} = \gamma_k$.

(*iii*) When $\Phi \cap \{\alpha_p, \alpha_q\} \neq \emptyset$, we can assume, modulo a CR isomorphism, that $\alpha_p \in \Phi$. Then $\gamma_p \in \mathcal{Q}^n_\Phi \cap \bar{\mathcal{Q}}^n_\Phi$ and all pairs (α, β) of roots in $\mathcal{Q}^n_\Phi \setminus \bar{\mathcal{Q}}^n_\Phi$ with $\alpha + \bar{\beta} = \gamma_p$ are of the form (β_k, β_k) with $\beta_k = \sum_{i=p}^{k} \alpha_i$ for some $p \le k < q$. By Lemma 13.6, we have

$$[X_{\beta_k}, \bar{X}_{\beta_k}] = [X_{\alpha_p}, \bar{X}_{\alpha_p}]\,,$$

and hence the corresponding $\mathbf{L}_{\gamma_p}$ is $\neq 0$ and semi-definite.

(*iv*) Assume that $\Phi \cap \mathcal{R}_\bullet \neq \emptyset$ and $\Phi \not\subset \mathcal{R}_\bullet$. We can assume, modulo a CR isomorphism, that there is $\alpha_j \in \Phi$ with $j \le p$ and that $\alpha_i \notin \Phi$ if either $j < i \le p$, or $q \le i \le p + q - j$. Let r be the largest integer $< q$ such that $\alpha_r \in \Phi$. We observe that $\gamma_j \in \mathcal{Q}^n_\Phi \cap \bar{\mathcal{Q}}^n_\Phi$ and that all pairs (α, β) of roots in $\mathcal{Q}^n_\Phi \setminus \bar{\mathcal{Q}}^n_\Phi$ with $\alpha + \bar{\beta} = \gamma_j$ are of the form (β_k, β_k) with $\beta_k = \sum_{i=j}^{k} \alpha_i$ for some $r \le k < q$. As in the previous case, for all $p \le k < q$:

$$[X_{\beta_k}, \bar{X}_{\beta_k}] = [X_{\beta_p}, \bar{X}_{\beta_p}]\,,$$

and hence $\mathbf{L}_{\gamma_j}$ is $\neq 0$ and semi-definite.

C IIa The positive real roots that can be written a sum $\alpha + \bar{\alpha}$ with $\alpha \in \mathcal{R}$ are:

$$\gamma_h = \alpha_{2h-1} + \alpha_\ell + 2\sum_{i=2h}^{\ell-1} \alpha_i \quad \text{for} \quad h = 1, \dots, p.$$

(*i*) Assume that $\Phi = \{\alpha_{2h_1-1}, \dots, \alpha_{2h_r-1}\}$ with $1 \le h_1 < \cdots < h_r \le p$. The roots in $\mathcal{Q}^n_\Phi \cap \bar{\mathcal{Q}}^n_\Phi$ that are of the form $\alpha + \bar\alpha$ are the γ_h with $1 \le h \le h_r$. The root γ_{h_r} is the only one that can be written as $\alpha + \bar\alpha$ with $\alpha \in \mathcal{Q}^n_\Phi \setminus \bar{\mathcal{Q}}^n_\Phi$. But this root can also be written as $\alpha + \bar\beta$ with $\alpha = \alpha_{2h_r-1} + \gamma_{h_r}$ and $\beta = \alpha_{2h_r-1}$, and therefore $\mathbf{L}_{\gamma_{h_r}}$ has at least one positive and one negative eigenvalue.

(*ii*) Assume that $\Phi = \{\alpha_{k_1}, \dots, \alpha_{k_r}\}$ with $2p < k_1 < \cdots < k_r \le \ell$. Then all γ_h belong to $\mathcal{Q}^n_\Phi \cap \bar{\mathcal{Q}}^n_\Phi$. Fix $1 \le h \le p$, and consider the roots $\beta = \sum_{i=2h}^{2p} \alpha_i + \alpha_\ell + 2\sum_{i=2p+1}^{\ell-1} \alpha_i$ and $\alpha = \alpha_{2h-1}$. Then $\beta, \alpha+\beta \in \mathcal{Q}^n_\Phi \setminus \bar{\mathcal{Q}}^n_\Phi$ and $\beta + \bar\beta = (\alpha+\beta) + \overline{(\alpha+\beta)} = \gamma_h$. By Lemma 13.6 we have:

$$[X_{\alpha+\beta}, \bar X_{\alpha+\beta}] = [[X_\alpha, X_\beta], [X_{-\alpha}, \bar X_\beta]] = -[X_\beta, \bar X_\beta]\,,$$

showing that $\mathbf{L}_{\gamma_h}$ has at least one positive and one negative eigenvalue.

(*iii*) Assume that $\Phi \supset \{\alpha_{2h-1}, \alpha_k\}$ with $1 \le h \le p$ and $k > 2p$. We can take h to be the largest integer $\le p$ with $\alpha_{2h-1} \in \Phi$ and k to be the smallest integer $> 2p$ with $\alpha_k \in \Phi$. Then $\gamma_h \in \mathcal{Q}^n_\Phi \cap \bar{\mathcal{Q}}^n_\Phi$. The set of pairs (α, β) of elements of $\mathcal{Q}^n_\Phi \setminus \bar{\mathcal{Q}}^n_\Phi$ with $\alpha + \bar\beta = \gamma_h$ consists of the pairs (β_r, β_r), where:

$$\beta_r = \sum_{i=2h-1}^{r} \alpha_i + \alpha_\ell + 2\sum_{i=r}^{\ell-1} \alpha_i$$

for $r = 2p+1, \dots, k$. We observe that $\beta_r = \beta_{r+1} + \alpha_r$, and that $\gamma_h \pm \alpha_r \notin \mathcal{R}$. Hence by Lemma 13.6 we have:

$$[X_{\beta_r}, \bar X_{\beta_r}] = [[X_{\alpha_r}, X_{\beta_{r+1}}], [X_{-\alpha_r}, \bar X_{\beta_{r+1}}]] = [X_{\beta_{r+1}}, \bar X_{\beta_{r+1}}]\,,$$

for all $r = 2p+1, \dots, k-1$. Hence $\mathbf{L}_{\gamma_h}$ is $\neq 0$ and semi-definite.

D IIIb The positive real roots that can be written as $\alpha + \bar\alpha$ with $\alpha \in \mathcal{R}$ are:

$$\gamma_h = \alpha_{2h-1} + \alpha_{\ell-1} + \alpha_\ell + 2\sum_{i=1}^{\ell-2} \alpha_i\,, \quad h = 1, \dots, p, \quad \text{for} \quad p = \tfrac{\ell-1}{2}\,.$$

(*i*) Assume that $\Phi \cap \{\alpha_{\ell-1}, \alpha_\ell\} \neq \emptyset$. Then $\gamma_p \in \mathcal{Q}^n_\Phi \cap \bar{\mathcal{Q}}^n_\Phi$, and the same discussion of case A IV shows that $\mathbf{L}_{\gamma_p}$ is $\neq 0$ and semi-definite.

(*ii*) Assume that $\Phi = \{\alpha_{2h_1-1}, \dots, \alpha_{2h_r-1}\}$ with $1 \le h_1 < \cdots < h_r \le p$. Then $\gamma_1, \dots, \gamma_{h_r} \in \mathcal{Q}^n_\Phi \cap \bar{\mathcal{Q}}^n_\Phi$, but only γ_{h_r} can be represented as a sum $\alpha + \bar\alpha$ with $\alpha \in \mathcal{Q}^n_\Phi \setminus \bar{\mathcal{Q}}^n_\Phi$. If $h_r = p$, we reduce to the case of A IV. Assume that $h_r < p$. Then we consider the two distinct roots:

$$\beta = \alpha_{2h_r-1} + \alpha_{2h_r} \qquad \text{and} \qquad \delta = \beta + \gamma_{h_{r+1}}\,.$$

They both belong to $\mathcal{Q}^n_\Phi \setminus \bar{\mathcal{Q}}^n_\Phi$ and $\beta + \bar\delta = \gamma_{h_r}$, showing that $\mathbf{L}_{\gamma_{h_r}}$ has at least one positive and one negative eigenvalue.

E III Set $\gamma_1 = \alpha_1 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6$, $\gamma_2 = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$. These are the real positive roots in $\mathcal{R}$ that can be written as a sum $\alpha + \bar\alpha$ for a root $\alpha \in \mathcal{R}$. Note that γ_1, γ_2 both belong to $\mathcal{Q}^n_\Phi \cap \bar{\mathcal{Q}}^n_\Phi$ for every choice of Φ. The discussion of the signature of $\mathbf{L}_{\gamma_1}$ reduces to the one we did for A IV.

(*i*) Assume that $\Phi \cap \{\alpha_1, \alpha_6\} \neq \emptyset$. In this case the discussion for A IV shows that $\mathbf{L}_{\gamma_1}$ is $\neq 0$ and semi-definite.

(*ii*) Assume that $\Phi = \{\alpha_3\}$ (the case $\Phi = \{\alpha_5\}$ is analogous). Then the set of pairs (α, β) of roots of $\mathcal{Q}^n_\Phi \setminus \bar{\mathcal{Q}}^n_\Phi$ such that $\alpha + \bar\beta = \gamma_2$ contains only

the pair (α,α) with $\alpha=\alpha_1+\alpha_2+2\alpha_3+2\alpha_4+\alpha_5$. Hence $\mathbf{L}_{\gamma_2}$ has rank 1 and is $\neq 0$ and semi-definite.

(iii) Assume that either $\alpha_4\in\Phi\subset\mathcal{R}_\bullet$, or $\Phi=\{\alpha_3,\alpha_5\}$. Then the set of pairs (α,β) of roots of $\mathcal{Q}^n_\Phi\setminus\bar{\mathcal{Q}}^n_\Phi$ such that $\alpha+\bar\beta=\gamma_2$ is empty, so that $\mathbf{L}_{\gamma_2}=0$. The discussion for A IV shows that in this case $\mathbf{L}_{\gamma_1}$ has one positive and one negative eigenvalue.

F II The real root $\gamma=\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4$ is the only positive root which can be written in the form $\alpha+\bar\alpha$ for some $\alpha\in\mathcal{R}$. It belongs to $\mathcal{Q}^n_\Phi\cap\bar{\mathcal{Q}}^n_\Phi$ for every choice of Φ.

(i) Assume that $\alpha_3\in\Phi$. Then (α_4,α_4) is the only pair (α,β) of roots in $\mathcal{Q}^n_\Phi\setminus\bar{\mathcal{Q}}^n_\Phi$ with $\alpha+\bar\beta=\gamma$. Thus $\mathbf{L}_\gamma$ has rank 1 and hencefore is $\neq 0$ and semi-definite.

(ii) Assume that $\Phi\subset\{\alpha_1,\alpha_2\}$. Set $\beta=\alpha_1+2\alpha_2+2\alpha_3+\alpha_4=\bar\alpha_4-\alpha_3$ and $\alpha=\alpha_3$. Then β and $\beta+\alpha$ both belong to $\mathcal{Q}^n_\Phi\setminus\bar{\mathcal{Q}}^n_\Phi$. With the notation of Lemma 13.6, we have $p=1$, $p'=1$, $q'=1$. Thus:

$$[X_{\alpha+\beta},\bar X_{\alpha+\beta}]=[[X_\alpha,X_\beta],[X_{-\alpha},\bar X_\beta]]=-[X_\beta,\bar X_\beta]\,,$$

showing that $\mathbf{L}_\gamma$ has at least one positive and one negative eigenvalue. ∎

14. *CR* functions on minimal orbits

Let M be a CR manifold and let $\mathcal{O}_M(M)$ be the space of smooth CR functions on M (see §2). We say that M is *locally CR separable* if the functions of $\mathcal{O}_M(M)$ locally separate points, and CR *separable* if the functions in $\mathcal{O}_M(M)$ separate points.

In this section we discuss CR separability for the minimal orbit $M=M(\mathfrak{g},\mathfrak{q})$ associated to the parabolic minimal CR algebra $(\mathfrak{g},\mathfrak{q})$.

When $\mathfrak{g}=\mathfrak{g}_1\oplus\mathfrak{g}_2$ is the product of two semisimple ideals, we set $\mathfrak{q}_i=\mathfrak{q}\cap\hat{\mathfrak{g}}_i$, for $i=1,2$. By Proposition 5.2 each $\mathfrak{q}_i$ is parabolic in $\hat{\mathfrak{g}}_i$, and the $(\mathfrak{g}_i,\mathfrak{q}_i)$'s are parabolic minimal. By the remarks following Proposition 5.2, we have $M\simeq M_1\times M_2$ and therefore $\mathcal{O}_{M_1}(M_1)\otimes\mathcal{O}_{M_2}(M_2)$ is dense in $\mathcal{O}_M(M)$.

When M is totally real, all smooth functions in M are CR and the CR separability is trivial. Thus the CR separability of a general $M(\mathfrak{g},\mathfrak{q})$ reduces to that of the fibers of its fundamental reduction (cf. Theorem 9.3 and its Corollary).

Indeed, let $M\xrightarrow{\rho}N$ be the fundamental reduction. We have $\mathcal{O}_N(N)=\mathcal{C}^\infty(N,\mathbb{C})$. Thus the CR functions certainly separate points on distinct fibers. Furthermore, let f be a CR function defined on a fiber $\rho^{-1}(x_0)$ $(x_0\in N)$. We choose a CR trivialization $U\times\rho^{-1}(x_0)\ni(x,y)\longrightarrow\phi(x,y)\in\rho^{-1}(U)$, where U is an open neighborhood of x_0 in N and ϕ is a CR diffeomorphism. Then f extends to a CR function F in $\rho^{-1}(U)$, by $F(\phi(x,y))=f(y)$. Take a cut-off function $\chi\in\mathcal{C}^\infty(U,\mathbb{C})$, with compact support in U and equal to 1 in x_0. Then $\tilde f(z)=\chi(\rho(z))\cdot F(z)$ for $z\in\rho^{-1}(U)$, extended by $\tilde f=0$ outside $\rho^{-1}(U)$, is a CR function in M that extends f. This shows that M is (locally) CR separable if and only if the fiber $\rho^{-1}(x_0)$ is (locally) CR separable.

In this way we can restrain our discussion of CR separability to the case where $(\mathfrak{g}, \mathfrak{q})$ is simple, effective and fundamental.

First we indicate how CR separability can be read off the cross-marked Satake diagram. We have:

Theorem 14.1. *Let $(\mathfrak{g}, \mathfrak{q}_\Phi)$ be a simple fundamental effective parabolic minimal CR algebra and $M = M(\mathfrak{g}, \mathfrak{q}_\Phi)$ its associated CR manifold. Then M is locally CR separable if and only if its cross-marked Satake diagram is one of the following:*

$$\begin{array}{cc} (\mathrm{A\,IIIa-IV}) & \Phi \subset \{\alpha_p, \alpha_q\} \\ (\mathrm{D\,IIIb}) & \Phi \subset \{\alpha_{\ell-1}, \alpha_\ell\} \\ (\mathrm{E\,III}) & \Phi \subset \{\alpha_1, \alpha_6\}\,. \end{array}$$

In these cases M is also CR separable by real analytic CR functions.

Proof. We prove that if there is a simple root α satisfying either:

(*i*) $\alpha \in \Phi \cap \mathcal{R}_\bullet$, or

(*ii*) $\alpha \in \Phi \setminus \mathcal{R}_\bullet$, $\bar\alpha \in \mathcal{B}$ and $\alpha + \bar\alpha \notin \mathcal{R}$,

then M is not CR separable. Inspection of the Satake diagrams then shows that the only possibilities left are those listed above. Finally in the examples below we show that in those cases M has a CR embedding into an affine complex space E, hence is separable by analytic CR functions (the restrictions to M of the global holomorphic functions in E).

Let α be a simple root, satisfying either (*i*) or (*ii*). Then $\alpha \pm \bar\alpha \notin \mathcal{R}$. Denote by $\hat{\mathfrak{a}}$ the subalgebra generated by $\hat{\mathfrak{g}}^\alpha + \hat{\mathfrak{g}}^{-\alpha} + \hat{\mathfrak{g}}^{\bar\alpha} + \hat{\mathfrak{g}}^{-\bar\alpha}$. It is semisimple, $\mathfrak{q} \cap \hat{\mathfrak{a}}$ is parabolic in $\hat{\mathfrak{a}}$ and $(\mathfrak{a}, \mathfrak{q} \cap \hat{\mathfrak{a}})$ is totally complex.

Let $\hat{\mathfrak{b}} = \hat{\mathfrak{a}} \cap \hat{\mathfrak{k}}$ (in case (*i*) we have $\hat{\mathfrak{b}} = \hat{\mathfrak{a}}$). Then $\mathfrak{b}$ is compact semisimple and $(\mathfrak{b}, \mathfrak{k} \cap \hat{\mathfrak{b}})$ is totally complex. If $\mathbf{B} \subset \mathbf{G}$ is the analytic subgroup with Lie algebra $\mathfrak{b}$ then $\mathbf{B} \cdot o$ is a compact complex submanifold of M of positive dimension. All smooth CR functions on M restrict to holomorphic functions in $\mathbf{B} \cdot o$, that are constant in $\mathbf{B} \cdot o$ by Liouville's Theorem, and therefore M is not CR separable. ∎

Example 14.2. Fix positive integers $p < q$ and let $n = p + q$. We identify the simple real Lie algebra $\mathfrak{g} \simeq \mathfrak{su}(p, q)$ with the set of $(n \times n)$ complex matrices Z with zero trace that satisfy:

$$Z^* K \,+\, K\, Z \,=\, 0\,, \text{ where } K = \begin{pmatrix} I_p & \\ & -I_q \end{pmatrix}\,.$$

Let $e_1, \ldots, e_n$ be the canonical basis of $\mathbb{C}^n$ and let $\mathfrak{q}_{\alpha_\mathfrak{p}} \subset \hat{\mathfrak{g}} \simeq \mathfrak{sl}(n, \mathbb{C})$ be the set of $(n \times n)$ matrices in $\mathfrak{sl}(n, \mathbb{C})$ such that

$$Z(\langle e_1 + e_{p+1}\,, \ldots\,, e_p + e_{2p}\rangle) \subset \langle e_1 + e_{p+1}\,, \ldots\,, e_p + e_{2p}\rangle\,.$$

Then $(\mathfrak{g}, \mathfrak{q}_{\alpha_\mathfrak{p}})$ is parabolic minimal.

The corresponding CR manifold $M = M(\mathfrak{g}, \mathfrak{q}_{\alpha_\mathfrak{p}})$ is the Grassmannian of p-planes ℓ_p in $\mathbb{C}^n$ which are totally isotropic for K (i.e. $v^* K v = 0$ for all $v \in \ell_p$). We have

$$M \simeq \left\{\ell_p = \{(v, u(v)) \in \mathbb{C}^n \,|\, v \in \mathbb{C}^p\} \,\middle|\, u \in \mathbf{U}(\mathbb{C}^p, \mathbb{C}^q)\right\} \simeq \mathbf{U}(\mathbb{C}^p, \mathbb{C}^q)$$

where $\mathbf{U}(\mathbb{C}^p, \mathbb{C}^q) = \{u \in \mathcal{M}_{q\times p}(\mathbb{C}) \,|\, u^* u = I_p\}$ is the set of unitary $q \times p$ matrices.

Give $\mathbf{U}(\mathbb{C}^p, \mathbb{C}^q)$ the CR structure induced by the standard embedding in $\mathcal{M}_{q\times p}(\mathbb{C})$. The compact subgroup $\mathbf{K}^{(1)} \simeq \mathbf{SU}(p)\times\mathbf{SU}(q)$ of matrices of $\mathbf{SU}(p,q)$ of the form $\begin{pmatrix} A_p & 0 \\ 0 & B_q \end{pmatrix}$ acts transitively by CR automorphisms on $\mathbf{U}(\mathbb{C}^p, \mathbb{C}^q)$, the action being given by: $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \cdot u = BuA^{-1}$.

The associated CR algebra is $(\mathfrak{k}^{(1)}, \mathfrak{q}')$ where $\mathfrak{k}^{(1)} \simeq \mathfrak{su}(p) \oplus \mathfrak{su}(q)$ and $\mathfrak{q}'$ is the set of matrices in $\mathfrak{sl}(p) \oplus \mathfrak{sl}(q)$ of the form $\begin{pmatrix} A_p & 0 & 0 \\ 0 & A_p & D \\ 0 & 0 & C_{q-p} \end{pmatrix}$.

The group $\mathbf{K}^{(1)}$ acts transitively on M by Theorem 8.2, and the associated CR algebra is $(\mathfrak{k}^{(1)}, \hat{\mathfrak{k}}^{(1)} \cap \mathfrak{q}) = (\mathfrak{k}^{(1)}, \mathfrak{q}')$. Thus the diffeomorphism $M \simeq \mathbf{U}(\mathbb{C}^p, \mathbb{C}^q)$ is in fact a CR isomorphism.

In this way we obtain the embedding $M \hookrightarrow \mathcal{M}_{q\times p}(\mathbb{C}) \simeq \mathbb{C}^{qp}$. This is a CR embedding into a Stein manifold, and therefore M is separable, since $\mathcal{O}_M(M)$ contains the restrictions to M of all the holomorphic functions on $\mathcal{M}_{q\times p}(\mathbb{C}) \simeq \mathbb{C}^{qp}$.

Example 14.3. Fix a positive integer p and let $n = 2p+1$. We identify the simple real Lie algebra $\mathfrak{g} = \mathfrak{so}^*(2n)$ with the set of $(2n \times 2n)$ complex matrices Z with zero trace that satisfy:

$$\begin{cases} ZJ = J\bar{Z}\,, \\ {}^tZK + KZ = 0\,, \end{cases}$$

where:

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \quad \text{and} \quad K = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}.$$

Let $\mathfrak{q}_{\alpha_1}$ be the parabolic subalgebra of matrices in $\mathfrak{g}$ that stabilize the subspace

$$V_n = \langle e_1 + e_{n+2p}, \ldots, e_p + e_{n+p+1}, e_{p+1} - e_{n+p}, \ldots, e_{2p} - e_{n+1}, e_{2p+1}\rangle.$$

Then $(\mathfrak{g}, \mathfrak{q}_{\alpha_1})$ is parabolic minimal.

The maximal compact subgroup $\mathbf{K} \simeq \mathbf{U}(n)$ of $\mathbf{G}$ of matrices of the form: $\begin{pmatrix} A_n & 0 \\ 0 & {}^tA_n^{-1} \end{pmatrix}$, $A_n \in \mathbf{U}(n)$, acts transitively by CR isomorphisms on $M(\mathfrak{g}, \mathfrak{q}_{\alpha_1})$. The associated CR algebra is $(\mathfrak{k}, \mathfrak{q}')$ where $\mathfrak{k} \simeq \mathfrak{u}(n)$ and $\mathfrak{q}' = \hat{\mathfrak{k}} \cap \mathfrak{q}_{\alpha_1}$. This is the subalgebra of matrices in $\mathfrak{so}(2n, \mathbb{C})$ of the form $\begin{pmatrix} A_n & 0 \\ 0 & -{}^tA_n \end{pmatrix}$ where $A_n \in \mathfrak{gl}(n, \mathbb{C})$ is of the form $\begin{pmatrix} B_p & C_p & v_p \\ D_p & -{}^tB_p & w_p \\ 0 & 0 & is \end{pmatrix}$ with $B_p = {}^tB_p$, $D_p = {}^tD_p$.

We let $\mathbf{K}$ act on $\mathfrak{so}(n, \mathbb{C})$ by: $k \cdot X = A\,X\,{}^tA$ if $k = \begin{pmatrix} A_n & 0 \\ 0 & {}^tA_n^{-1} \end{pmatrix}$. Let N be the $\mathbf{K}$-orbit of $o = \begin{pmatrix} 0 & -I_p & 0 \\ I_p & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. The associated CR algebra is $(\mathfrak{k}, \mathfrak{q}')$ and the isotropy is connected and contains a generator of $\pi_1\big(\mathbf{U}(n)\big)$. Thus M is CR isomorphic to N.

The CR manifold N is embedded in the Stein manifold $\mathfrak{so}(n, \mathbb{C}) \simeq \mathbb{C}^{n(n-1)/2}$, hence it follows that M is separable by the restrictions to M of the holomorphic functions in $\mathfrak{so}(n, \mathbb{C}) \simeq \mathbb{C}^{n(n-1)/2}$.

Example 14.4. Let D be the exceptional bounded symmetric domain of type V. Its Shilov boundary S is a real flag manifold (see [13, Part III,Ch.IV§2.8]) for

the group E III and is compact, hence it is a minimal orbit $M(\mathfrak{g},\mathfrak{q})$ where $\mathfrak{g}$ is of type E III. Furthermore it has CR dimension 8 and CR codimension 8 (see [24, p. 180]), hence $\mathfrak{q}=\mathfrak{q}_{\alpha_1}$ or $\mathfrak{q}=\mathfrak{q}_{\alpha_6}$. Thus $M(\mathfrak{g},\mathfrak{q}_{\alpha_1})\simeq S$ is an embedded CR submanifold of $\mathbb{C}^{16}$ and it is separable, since $\mathcal{O}_M(M)$ contains the restrictions of the holomorphic functions in $\mathbb{C}^{16}$.

A similar argument could have been applied also to discuss the two previous examples 14.2 and 14.3. Indeed the three classes of minimal orbits of examples 14.2, 14.3 and 14.4 are exactly the Shilov boundaries of the bounded symmetric domains that are not of tube type (and that are not totally real; see also [15, Ch.X, Ex. D.1] and [20]).

Identify $\hat{\mathfrak{g}}$ with a complex Lie algebra of left invariant complex valued vector fields in $\mathbf{G}$. Given a $\mathbb{C}$ linear subspace $\mathfrak{a}$ of $\hat{\mathfrak{g}}$, denote by $\mathcal{O}_{\mathbf{G},\mathfrak{a}}$ the sheaf of smooth complex valued functions f on $\mathbf{G}$ such that $L(f)=0$ for all $L\in\mathfrak{a}$.

Let $\pi:\mathbf{G}\longrightarrow M$ be the principal fibration, $\mathcal{F}_M(\mathbf{G})=\pi^*\mathcal{O}_M(M)$ and $\mathcal{T}_M$ the sheaf of local smooth complex valued vector fields L on $\mathbf{G}$ such that $L(f)=0$ for all $f\in\mathcal{F}_M(\mathbf{G})$. Let $\mathfrak{q}'=\{X\in\hat{\mathfrak{g}}|X_e\in(\mathcal{T}_M)_e\}$. We note that, since $\mathcal{F}_M(\mathbf{G})$ is invariant for the left action of $\mathbf{G}$ on functions, the elements of $\mathfrak{q}'$ define left invariant global sections of $\mathcal{T}_M$.

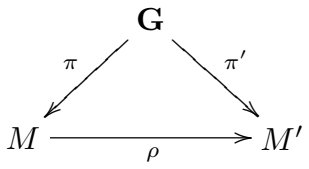
Lemma 14.5. *$\mathfrak{q}'$ is a parabolic subalgebra of $\hat{\mathfrak{g}}$ and $\mathfrak{q}\subset\mathfrak{q}'$.*

Proof. The sheaf $\mathcal{T}_M$ is invariant for the left action of $\mathbf{G}$, hence it is generated at every point by the global left invariant complex vector fields that belong to $\mathfrak{q}'$. Since $\mathcal{T}_M$ is involutive, $\mathfrak{q}'$ is a subalgebra. Clearly it contains $\mathfrak{q}$, and thus is parabolic. ∎

Lemma 14.6. *We have: $\mathcal{F}_M(\mathbf{G})=\mathcal{O}_{\mathbf{G},\mathfrak{q}'}(\mathbf{G})=\mathcal{O}_{\mathbf{G},\mathfrak{q}}(\mathbf{G})$.*

Proof. The inclusions $\mathcal{F}_M(\mathbf{G})\subset\mathcal{O}_{\mathbf{G},\mathfrak{q}'}(\mathbf{G})\subset\mathcal{O}_{\mathbf{G},\mathfrak{q}}(\mathbf{G})$ follow from the definition of $\mathfrak{q}'$ and Lemma 14.5. To complete the proof we need to show that $\mathcal{O}_{\mathbf{G},\mathfrak{q}}(\mathbf{G})\subset\mathcal{F}_M(\mathbf{G})$. An $f\in\mathcal{O}_{\mathbf{G},\mathfrak{q}}(\mathbf{G})$ is constant on the analytic subgroup of $\mathbf{G}$ that has Lie algebra $\mathfrak{g}\cap\mathfrak{q}$, i.e. on $\mathbf{G}_+$, (recall that $\mathbf{G}_+$ is connected). By left invariance, f is constant on the left cosets $g\mathbf{G}_+$ and hence is the pullback of a function $\tilde{f}$ defined in M. Furthermore $\tilde{f}$ is CR on M because $T^{0,1}_oM=\pi_*(\mathfrak{q})$ (here $\hat{\mathfrak{g}}$ is identified with the complexification of $T_e\mathbf{G}$) and therefore $L(\tilde{f})=0$ for all $L\in T^{0,1}M$ by left invariance. ∎

Let $M'=M(\mathfrak{g},\mathfrak{q}')$, $\pi':\mathbf{G}\longrightarrow M'$ and $\rho=\pi'\circ\pi^{-1}:M\longrightarrow M'$ the natural projection. We have the commutative diagram:



Lemma 14.7. *M' is CR separable.*

Proof. Assume that M' is not locally CR separable. Then there exists a tangent vector $X\in\mathfrak{g}$ at $e\in\mathbf{G}$ such that $\pi'_*(X)\neq 0$ and $\pi'_*(X)(f)=0$ for

every CR function f on M'. Then $X \in \mathfrak{q}'$, hence $X \in \mathfrak{g}'_+$ and $\pi'_*(X) = 0$, yielding a contradiction.

By Theorem 14.1 the CR manifold M', being locally CR separable, is also CR separable. ∎

Lemma 14.8. *$\mathcal{F}_M(\mathbf{G}) = \mathcal{F}_{M'}(\mathbf{G})$. Hence: $\mathcal{O}_M(M) = \rho^*\big(O_{M'}(M')\big)$.*

Proof. This follows by applying Lemma 14.6 to M'. ∎

Lemma 14.9. *The complex Lie subalgebra $\mathfrak{q}'$ is minimal in the set of (not necessarily proper) parabolic subalgebras of $\hat{\mathfrak{g}}$, containing $\mathfrak{q}$, such that $M(\mathfrak{g}, \mathfrak{q})$ is CR separable.*

Proof. Suppose that $\mathfrak{q}''$ is a parabolic subalgebra of $\hat{\mathfrak{g}}$ with $\mathfrak{q} \subset \mathfrak{q}'' \subset \mathfrak{q}'$ and such that $M'' = M(\mathfrak{g}, \mathfrak{q}'')$ is CR separable. Then $\mathcal{F}_{M''}(\mathbf{G}) = \mathcal{F}_{M'}(\mathbf{G})$ by Lemma 14.6. This implies that $\mathfrak{q}'' \cap \bar{\mathfrak{q}}'' = \mathfrak{q}' \cap \bar{\mathfrak{q}}'$, yielding $M'' = M'$ and thus $\mathfrak{q}'' = \mathfrak{q}'$. ∎

The discussion above leads to the following:

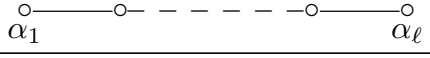
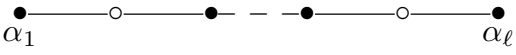
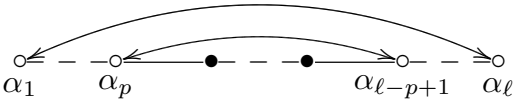
Theorem 14.10. *Let $(\mathfrak{g}, \mathfrak{q}_\Phi)$ be a simple effective fundamental parabolic minimal CR algebra and $M = M(\mathfrak{g}, \mathfrak{q}_\Phi)$. Then there exists $\Psi \subset \Phi$ and a $\mathbf{G}$-equivariant fibration $\rho : M \to M_s = M(\mathfrak{g}, \mathfrak{q}_\Psi)$ such that M_s is CR separable and $\mathcal{O}_M(M) = \rho^*\big(O_{M_s}(M_s)\big)$.*

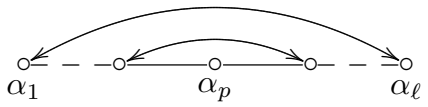
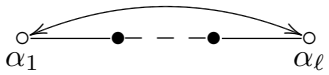
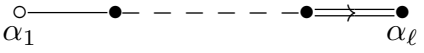
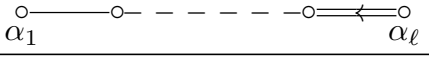
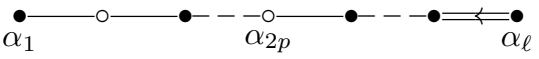
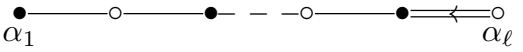
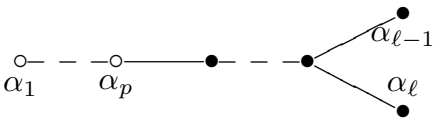
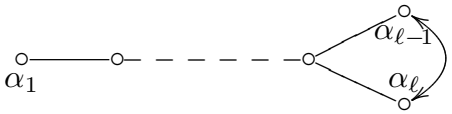
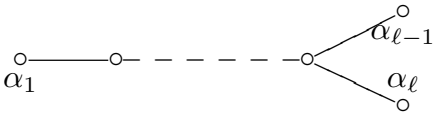
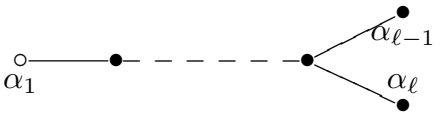
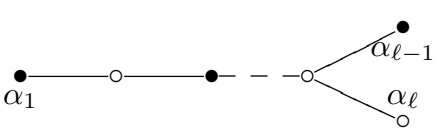
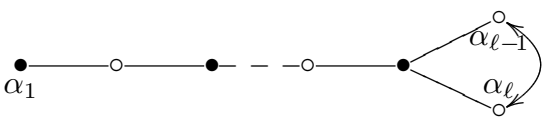
Furthermore $\Psi = \Phi \cap \Sigma$, where Σ is defined according to the type of $\mathfrak{g}$:

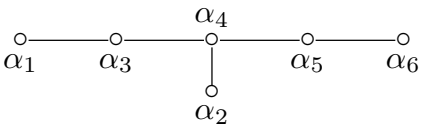
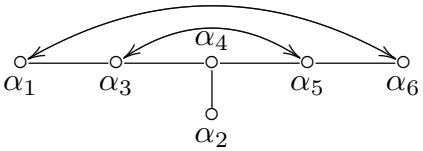
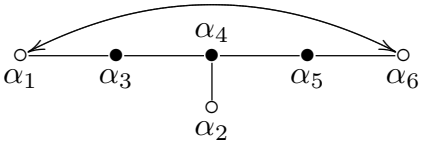
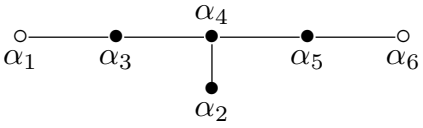
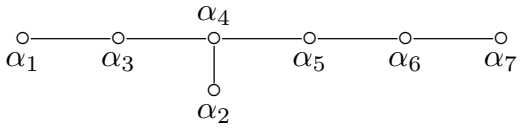
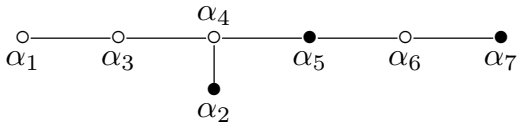
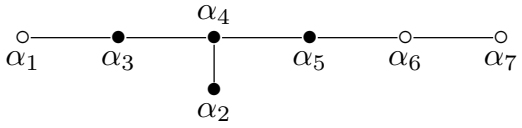
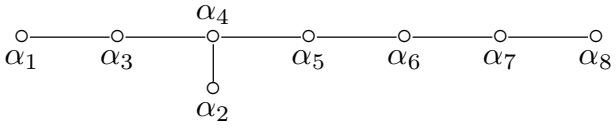
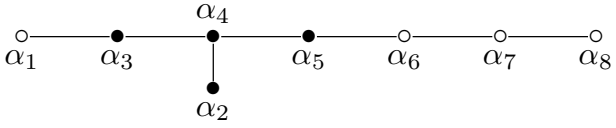
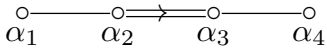
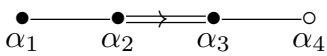
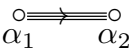
Type A IIIa – IV : $\Sigma = \{\alpha_p, \alpha_q\}$;
Type D IIIb : $\Sigma = \{\alpha_{\ell-1}, \alpha_\ell\}$;
Type E III : $\Sigma = \{\alpha_1, \alpha_6\}$;
All other types : $\Sigma = \emptyset$.

The space $\mathcal{O}_M(M)$ is one dimensional when $\Psi = \emptyset$, infinite dimensional when $\Psi \neq \emptyset$. ∎

Appendix: Table of noncompact real forms and Satake diagrams

Name	$\mathfrak{g}$	Satake diagram
A I	$\mathfrak{sl}(\ell+1, \mathbb{R})$	
A II	$\mathfrak{sl}(p, \mathbb{H})$ $2p-1=\ell$	
A IIIa	$\mathfrak{su}(p,q)$ $p+q=\ell+1$ $2 \le p \le \ell/2$	

Name	$\mathfrak{g}$	Satake diagram
A IIIb	$\mathfrak{su}(p,p)$ $1 \le p = (\ell+1)/2$	
A IV	$\mathfrak{su}(1,\ell)$	
B I	$\mathfrak{so}(p,2\ell+1-p)$ $2 \le p \le \ell$	
B II	$\mathfrak{so}(1,2\ell)$	
C I	$\mathfrak{sp}(2\ell,\mathbb{R})$	
C IIa	$\mathfrak{sp}(p,\ell-p)$ $2p < \ell$	
C IIb	$\mathfrak{sp}(p,p)$ $2p = \ell$	
D Ia	$\mathfrak{so}(p,2\ell-p)$ $2 \le p \le \ell-2$	
D Ib	$\mathfrak{so}(\ell-1,\ell+1)$	
D Ic	$\mathfrak{so}(2\ell,\mathbb{R})$	
D II	$\mathfrak{so}(1,2\ell-1)$	
D IIIa	$\mathfrak{so}*(2\ell)$ $\ell = 2p$	
D IIIb	$\mathfrak{so}*(2\ell)$ $\ell = 2p+1$	

Name	$\mathfrak{g}$	Satake diagram
E I		
E II		
E III		
E IV		
E V		
E VI		
E VII		
E VIII		
E IX		
F I		
F II		
G		

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