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A New Formula for Discrete Series Characters on Split Groups

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Abstract. Let be G a connected semisimple real Lie group with a compact Cartan subgroup. Harish-Chandra proved formulas for discrete series characters which are explicit except for certain integer constants appearing in numerators. The main result of this paper is to work out a new formula for the constants when G is a split group.
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0. Introduction

Let G be a connected, semisimple Lie group. In Harish-Chandra's work on the Plancherel formula [7], the discrete series of irreducible unitary representations plays a crucial role. Their characters are actually analytic functions on the regular set of G, and it is a fundamental problem of harmonic analysis to find formulas for the characters as functions on the regular set. Harish-Chandra gave formulas which are completely explicit except for certain integer constants. Later on, Harish Chandra and Okamoto found the constants when G is a real rank one group, and Hecht did so for the holomorphic discrete series. For general G, Hirai has given explicit but very complicated formula. Lately, by means of the Schmid reduction [17], Herb [10] and Goresky, Kottwitz and MacPherson [4] have given explicit formulas for the constants. More precisely, let G' be the set of regular elements on G, and $H \subset G$ a Cartan subgroup. Fix $\widetilde{H}$ a connected component of $H \cap G'$. Then if $\Theta(\lambda)$ is the character of the discrete series representation with infinitesimal character λ, Harish-Chandra showed that

$$\Theta(\lambda)_{|\widetilde{H}} = \frac{\sum\limits_{w \in W(\mathfrak{g},\mathfrak{h})} n\left(w, \lambda, \widetilde{H}\right) \exp(w\lambda)}{\prod\limits_{\alpha \in \Phi(\mathfrak{g},\mathfrak{h}) : (\alpha,\lambda) > 0} \left(e^{\alpha/2} - e^{-\alpha/2}\right)} \ .$$

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[4] have given a geometric explicit formula for the constants $n\left(w,\lambda,\widetilde{H}\right)$. In this paper, whenever G is a split group and $\widetilde{H}$ is the identity component of a split Cartan subgroup, in Theorem 3.5 we work out a new formula for $n\left(w,\lambda,\widetilde{H}\right)$. Applications of this formula are found in the Ph. D. thesis of J. Bigeon [2].

This paper is organized as follows. In the first section we recall some notation. In the second section we find explicit candidate for the constants $n\left(w,\lambda,\widetilde{H}\right)$ and prove their main properties following similar ideas as in [4]. In the third section in Definition 19 and Theorem 3.5, we prove our formula for the constants $n\left(w,\lambda,\widetilde{H}\right)$ for the identity component of a split Cartan subgroup of a split group.

1. The function $\psi_\Phi(C,x,\tau)$

In this section we recall some notation from [4]. For this section, we fix a real vector space V, Φ a root system of V^* and $(\cdot,\cdot)$ an inner product in V^* invariant under the Weyl group $W(\Phi)$ of Φ. Unless we say otherwise, we assume that Φ spans V^*. We use the terminology in [4] and [3] on coroots, coweights and fundamental coweights. $(\cdot,\cdot)$ also induces an inner product on V and orthogonality (denoted by $\perp$) refers to these inner products.

Let C be a chamber of V^* relative to Φ, $\Delta_C(\Phi)=\{\alpha_1,...,\alpha_n\}$ (respectively Φ_C^+) the simple roots relative to C (respectively the positive roots relative to C). Hence the 1-dimensional faces of $\overline{C}$ are given by the fundamental weights $\{\omega_1,...,\omega_n\}$ associated to $\{\alpha_1,...,\alpha_n\}$.

Definition 1.1. If C is a closed convex polyhedral cone of V^*, we denote by C^* the subset $\{x\in V \mid \tau(x)\geq 0 \text{ for every } \tau\in C\}$.

Let $C^\vee=\bigcap_{i=1}^{n}\omega_i^{-1}(0,+\infty)$, hence $\overline{C^\vee}=\overline{C}^*$ is a simplicial closed cone in V. Unless we indicate the contrary, in this section all chambers in V^* are chambers in V^* relative to Φ and the chambers in V are given by the subsets $C^\vee$ of V, with C a chamber in V^*. The 1-dimensional faces of $\overline{C^\vee}$ are given by the fundamental coweights $\{H_1,...,H_n\}$. Since the map $C\longmapsto C^\vee$ is a bijection between the chambers in V^* and the chambers in V, for convenience we denote its inverse also by $C\longmapsto C^\vee$. Hence, the subsets $C^\vee$ of V, with C a chamber of V^* are exactly the chambers in V relative to $\Phi^\vee$. If $H\in V$, we set $\ker(H)=\{\varphi\in V^*\mid \varphi(H)=0\}$.

If $H\in V$, we write $\Phi_H=\{\alpha\in\Phi\mid\alpha(H)=0\}$, this is a root subsystem of Φ. If C is a chamber in V, let C^H denote the only chamber in V relative to $\Phi_H^\vee$ that contains C. If $H\in V$ and Φ^+ is a system of positive roots of Φ, we write Φ_H^+ for $\Phi^+\cap\Phi_H$.

For two chambers C_1 and C_2 in V, we denote by $l_\Phi(C_1,C_2)$ the number of root hyperplanes in V separating C_1 and C_2. Thus, if $w\in W(\Phi)$ is the only element in $W(\Phi)$ such that $wC_1=C_2$, then $l_{\Phi^\vee}(C_1,C_2)=l(w)$, where the length $l(w)$ is taken relative to Δ_{C_1}. Since $\ker(\alpha^\vee)$ separates C_1 and C_2 iff $\ker(\alpha)$ separates $C_1^\vee$ and $C_2^\vee$, we have

$$l_{\Phi^\vee}(C_1,C_2)=l_\Phi\left(C_1^\vee,C_2^\vee\right)\ .$$

We now recall the function $\psi_C(x,\tau)$ considered in [4] and some of its properties.

Let C be a closed convex polyhedral cone and let $\mathcal{F}_C$ be the set of faces of C.

Definition 1.2. If C is a closed convex polyhedral cone and F is a subset of C, we denote by $F^\perp := C^* \cap \{\tau \in V^* \mid \tau(F) = 0\}$.

If X is a subset of ..., let ξ_X denotes the characteristic function of X.

Definition 1.3. If C is a closed convex polyhedral cone of V, let $\psi_C : V \times V^* \to \mathbb{Z}$ be the map given by $\psi_C = \sum\limits_{F \in \mathcal{F}_C} (-1)^{\dim\langle F\rangle} \xi_{(F^\perp)^* \times F^*}$.

Thus we have $\psi_C(x,\tau) = 0$ if x does not lie in the linear subspace spanned by C and

$$\psi_{C^*}(\tau,x) = (-1)^{\dim V} \psi_C(x,\tau) \ . \tag{1}$$

If C is a subspace of V, we have

$$\psi_C(x,\tau) = \begin{cases} (-1)^{\dim V} & \text{if } \tau = 0 \\ 0 & \text{if } \tau \neq 0 \end{cases} . \tag{2}$$

Let C be a chamber in V relative to $\Phi^\vee$. If Φ generates V^*, let $n = \dim V$, $\{\alpha_1, ..., \alpha_n\} \subset V^*$ the simple roots of Φ relative to $C^\vee$ and let $\{H_1, ..., H_n\} \subset V$ be the dual basis of $\{\alpha_1, ..., \alpha_n\}$. For $x = \sum\limits_{i=1}^{n} a_i H_i$ and $\tau = \sum\limits_{i=1}^{n} b_i \alpha_i$, $I = \{1, ..., n\}$ we set

$$I_x = \{i \in I \mid a_i \geq 0\} \text{ and } I_\tau = \{i \in I \mid b_i \geq 0\} \ .$$

Since $\overline{C}$ is a simplicial closed cone, Lemma A.1 of [4], gives us

$$\psi_C(x,\tau) = \begin{cases} (-1)^{|I_\tau|} & \text{if } I_\tau = I - I_x \\ 0 & \text{otherwise} \end{cases} .$$

If V^* is not generated by Φ, let $V_L = \bigcap\limits_{\alpha \in \Phi} \ker(\alpha)$, $\pi : V \to (V/V_L)$ the canonical projection and let π' be the natural identification between the annihilator $(V_L)^*$ of V_L and $(V/V_L)^*$, then, by (A.2) in [4]

$$\psi_C(x,\tau) = \begin{cases} (-1)^{\dim V_L} \psi_{\pi(C)}(\pi(x), \pi'(\tau)) & \text{if } \tau \in (V_L)^* \\ 0 & \text{if } \tau \notin (V_L)^* \end{cases} . \tag{3}$$

Definition 1.4. If C is a chamber in V, $x \in V$ and $\tau \in V^*$, let $\psi_\Phi(C,x,\tau) = \sum\limits_{D} (-1)^{l_{\Phi^\vee}(C,D)} \psi_D(x,\tau)$, where D runs over all chambers in V relative to Φ.

If V^* is not generated by Φ, with the same notations as in (3) and since $(V_L)^*$ agrees with $\langle\Phi\rangle$, we have

$$\psi_\Phi(C,x,\tau) = \begin{cases} (-1)^{\dim V_L} \psi_{\pi^*(\Phi)}(\pi(C), \pi(x), \pi'(\tau)) & \text{if } \tau \in \langle\Phi\rangle \\ 0 & \text{if } \tau \notin \langle\Phi\rangle \end{cases} . \tag{4}$$

Lemma 1.5. *Fix H a coweight, $x \in V$ and let $Y = \ker(H)$ in V^*.*

i) For each chamber of V^*, $\psi_\Phi(C, x, \cdot) : V^* \to \mathbb{Z}$ is constant on the set of Φ-regular elements.
Assume that τ and τ' are Φ-regular elements of V^* lying in adjacent Φ-chambers, separated by the hyperplane Y and assume further that $\tau(H) > 0$ and $\tau'(H) < 0$. We denote by $\widetilde{x}$ the image of x in $V/\mathbb{R}H$ and by $\widetilde{\tau} \in Y$, the unique element lying on the line segment joining τ and τ'. Then

ii) $\psi_\Phi(C, x, \tau) - \psi_\Phi(C, x, \tau') = -2\psi_{\Phi_H}\left(C^H, \widetilde{x}, \widetilde{\tau}\right)$ if $\mathbb{R}H$ contains a coroot and 0 otherwise.

Proof. Lemma 1.2 in [4] and [5]. ■

If C is a chamber in V, we write by $\delta_C(\Phi) = \frac{1}{2}\sum_{\alpha\in\Phi_C^+}\alpha^\vee$ and $\Gamma(\Phi)$ (respectively $\Gamma_R(\Phi)$) for the lattice in V generated by the coweights (respectively coroots) of Φ. We denote by $\widehat{A_{sc}}(\Phi) = Hom_{\mathbb{Z}}(\Gamma(\Phi), \mathbb{C}^\times)$ and $\widehat{A_{ad}}(\Phi) = Hom_{\mathbb{Z}}(\Gamma_R(\Phi), \mathbb{C}^\times)$. The inclusion $\Gamma_R(\Phi) \subset \Gamma(\Phi)$ induces a surjection $\widehat{A_{sc}}(\Phi) \to \widehat{A_{ad}}(\Phi)$ whose kernel we denote by $Z^\vee(\Phi)$. Thus, we obtain the exact sequence

$$1 \to Z^\vee(\Phi) \to \widehat{A_{sc}}(\Phi) \to \widehat{A_{ad}}(\Phi) \to 1 \ .$$

We denote by $\langle\cdot,\cdot\rangle$ the natural $\mathbb{C}^\times$-valued pairing between $\Gamma(\Phi)$ and $\widehat{A_{sc}}(\Phi)$ or between $\Gamma_R(\Phi)$ and $\widehat{A_{ad}}(\Phi)$.
For $s \in \widehat{A_{sc}}(\Phi)$ such that $s^2 \in Z^\vee(\Phi)$, we define the subsystem of roots of Φ, $\Phi_s^\vee := \{\alpha^\vee \in \Phi^\vee \mid \langle\alpha^\vee, s\rangle = 1\}$ and $\Phi_s = \{\alpha \in \Phi \mid \alpha^\vee \in \Phi_s^\vee\}$.

If C is a chamber in V we denote by C^s the unique chamber in V, relative to $\Phi_s^\vee$, which contains C.

Lemma 1.6. *Let $x \in V$ regular, $s \in \widehat{A_{sc}}(\Phi)$ and $\tau \in V^*$. For the next equality the sum runs over all chambers D in V relative to Φ.*

i) $\sum_D (-1)^{l_{\Phi^\vee}(C,D)}\langle\delta_D(\Phi) - \delta_C(\Phi), s\rangle\psi_D(x,\tau) = \psi_{\Phi_s}(C^s, x, \tau)$.

ii) If $rk(\Phi_s) < rk(\Phi)$, the left-hand side of the equality in i) vanishes for every $x \in V$ regular and $\tau \in V^*$, Φ-regular.

iii) If $s^2 \neq 1$ the left-hand side of the equality in i) vanishes for every $x \in V$ regular and $\tau \in V^*$, Φ-regular.

Proof. Assertions i) and iii) are Lemma 1.4 of [4]. A proof of ii) can be found within the proof of the Lemma 1.4 of [4]. ■

For the rest of this section we assume that $-1 \in W(\Phi)$.

If $\alpha \in \Phi$, let $\Phi_\alpha := \{\beta \in \Phi \mid (\beta, \alpha) = 0\}$. Since $-s_\alpha$ fix α, by a Lemma of Chevalley, $-s_\alpha \in W(\Phi_\alpha)$. Hence, $-1_{\Phi_\alpha} \in W(\Phi_\alpha)$. From the identity $-1_{\Phi_\alpha}.s_\alpha = -1_\Phi$ we conclude

$$rk(\Phi_\alpha) = rk(\Phi) - 1 \ . \tag{5}$$

Since, $\alpha^\vee \in \bigcap_{\beta\in\Phi_\alpha} \ker(\beta)$, (5) implies that $\alpha^\vee$ lies on a 1-dimensional face of a chamber C of V. Hence, there exists a coweight H_α such that $\alpha^\vee \in \mathbb{R}H_\alpha$. Thus $\Phi_\alpha = \Phi_{H_\alpha}$. Since $-1 \in W(\Phi_\alpha)$, every Φ-chamber of V^* is contained in a unique $\Phi_\alpha-$ chamber of V^*. If $\alpha \in \Phi$ and C is a chamber in V, we denote by C^α the unique chamber in V relative to $\Phi_\alpha^\vee$ which contains C.

Besides, on $Y = \ker(\alpha) \subset V$, in addition to the usual chambers relative to $\Phi_\alpha^\vee$, we consider the connected components of $Y - \bigcup_\beta \ker(\beta)$ where β runs over the roots that have a nonzero restriction to Y. These connected components are called chambers in Y relative to $\Phi^\vee$. Hence, if D is a fixed chamber in V, the map $C \longmapsto C_Y = \overset{\circ}{\widehat{Y \cap \overline{C}}}$ (interior relative to Y) is a bijection between the set of chambers in V relative to $\Phi_\alpha^\vee$ and lying on the same side of Y as D does, and the set of chambers in Y relative to $\Phi^\vee$.

Lemma 1.7. *For $\alpha \in \Phi$, C a fixed chamber in V so that $Y = \ker(\alpha)$ is a wall of C and x, x' regular elements in V, lying in adjacent chambers of V, separated by the hyperplane Y. Assume that x and C lie on the same side of Y. If $y \in Y$ is the unique element lying on the line segment joining x with x' and $D := C^\alpha \cap Y$, then,*

- *i) For each τ, $\psi_\Phi(C, \cdot, \tau) : V \to \mathbb{Z}$ is constant on every chamber in V.*
- *ii) For each x, $\psi_\Phi(C, x, \cdot) : V^* \to \mathbb{Z}$ is constant on every Φ-chamber.*
- *iii) y is regular in Y relative to Φ_α,*
- *iv) $\psi_\Phi(C, x, \tau) - \psi_\Phi(C, x', \tau) = 2\psi_{\Phi_\alpha}\left(D, y, \tau_{|Y}\right)$,*
- *v) If τ is Φ-regular, then, $\tau_{|Y}$ is Φ_α-regular.*

Proof. i), ii) follow from the definition of $\psi_\Phi(C, x, \tau)$ and iii), iv), v) are proved as in Lemma 2.1 of [4]. ∎

2. The function $b_{\Phi(\mathfrak{g},\mathfrak{t})}(\phi, C, x, v, t)$

In this section, we define and analyze the properties of the geometric function $b_{\Phi(\mathfrak{g},\mathfrak{t})}(\phi, C, x, v, t)$, a function that is similar to the one defined by [4] in Section 6.

Throughout this section, we fix a real semisimple Lie algebra $\mathfrak{g}_0$ and $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ a Cartan decomposition. We assume $rk(\mathfrak{g}_0) = rk(\mathfrak{k}_0)$ and fix $\mathfrak{t}_0$ a Cartan subalgebra of $\mathfrak{k}_0$, thus it is also a Cartan subalgebra of $\mathfrak{g}_0$. In addition we assume that $\mathfrak{g}_0$ is a split real form. Let $\Phi = \Phi(\mathfrak{g}, \mathfrak{t})$ (respectively $\Phi_K = \Phi_K(\mathfrak{g}, \mathfrak{t})$ and $\Phi_N = \Phi_N(\mathfrak{g}, \mathfrak{t})$) be the roots (respectively compact roots and noncompact roots) of $\mathfrak{g}$ relative to $\mathfrak{t}$.

Since $\mathfrak{g}_0$ has a compact Cartan subalgebra as well as a split Cartan subalgebra, we have that $-1 \in W(\Phi)$.

Let C be a chamber in $i\mathfrak{t}_0$ relative to $\Phi^\vee$, $\Phi^+_{C^\vee}$ the positive roots relative to $C^\vee$ and $\delta_C = \frac{1}{2}\sum_{\alpha\in\Phi^+_{C^\vee}} \alpha^\vee$. The chamber C determines a subsystem

$$\Phi_{2,C} = \{\alpha \in \Phi \mid \alpha(\delta_C) \in 2\mathbb{Z}\} .$$

We recall that if $u \in W(\Phi)$, we have

$$\Phi_{2,uC} = u\Phi_{2,C} \text{ and } W(\Phi_{2,uC}) = uW(\Phi_{2,C})\, u^{-1} \ . \tag{6}$$

We denote by $\widetilde{W}(\Phi_{2,C})$ the normalizer of $W(\Phi_{2,C})$ in $W(\Phi)$. Since for $\alpha \in \Phi_{2,C}$ and $w \in W(\Phi)$, $ws_\alpha w^{-1} = s_{w\alpha}$, we have

$$\widetilde{W}(\Phi_{2,C}) = \{w \in W(\Phi) \mid w\,(\Phi_{2,C}) = \Phi_{2,C}\} \ .$$

Applying Proposition 7.145 of [15] to $t \in W(\Phi)$ and $\frac{\delta_{tC}}{2}$, we obtain

$$\{w \in W(\Phi) \mid w\delta_{tC} - \delta_{tC} \in 2\Gamma_R\,(\Phi)\} = W(\Phi_{2,tC}) \tag{7}$$

and

$$\{w \in W(\Phi) \mid w\delta_{tC} - \delta_{tC} \in \Phi_{2,tC}\} = \widetilde{W}(\Phi_{2,tC}) \ . \tag{8}$$

Since $\mathfrak{g}_0$ is split real, by [1] Proposition 6.24, there exists a chamber C in $i\mathfrak{t}_0$ relative to $\Phi^\vee$ such that every simple root (relative to the order $\Phi^+_{C^\vee}$) is noncompact. For $\alpha \in \Delta_{C^\vee}(\Phi)$ a simple root relative to $C^\vee$, $\alpha\,(\delta_C) = 1$, hence, $\Phi_{2,C} = \Phi_K$. Therefore $W(\Phi_{2,C}) = W(\Phi_K)$ and $\widetilde{W}(\Phi_{2,C}) = \widetilde{W}(\Phi_K)$. Thus, C' is a chamber in $i\mathfrak{t}_0$ relative to $\Phi^\vee$ such that $\Phi_{2,C'} = \Phi_K$ iff $C' \in \widetilde{W}(\Phi_K)\, C$. Finally, if $x \in i\mathfrak{t}_0$ is regular relative to Φ, we denote by C_x (or D_x) the unique chamber in $i\mathfrak{t}_0$ relative to $\Phi^\vee$ that contains x.

Definition 2.1. Let C be a chamber in $i\mathfrak{t}_0$ relative to $\Phi^\vee$, $x \in i\mathfrak{t}_0$ regular relative to Φ, ϕ a Φ-regular element of $(i\mathfrak{t}_0)^*$ and $v, t \in W(\Phi)$, we denote by

$$b_\Phi\,(\phi, C, x, v, t) = \sum_{w \in vW\left(\Phi_{2,tC}\right)} (-1)^{l_{\Phi^\vee}(D_x, wC)}\, \psi_{wC}\,(x, v\phi) \ .$$

Our main goal will be to show in Theorem 3.5 that, for suitable ϕ, C, x, v and t, that $b_\Phi\,(\phi, C, x, v, t)$ are the constant $n\left(w, \lambda, \widetilde{H}\right)$ for the character of the discrete series on the identity component of a split Cartan subgroup. For this purpose we now show that the constants $b_\Phi\,(\phi, C, x, v, t)$ satisfy certain properties which characterize them in a unique way.

Proposition 2.2. *Let C be a chamber in $i\mathfrak{t}_0$ relative to $\Phi^\vee$, $x \in i\mathfrak{t}_0$ regular relative to Φ, ϕ a Φ-regular element of $(i\mathfrak{t}_0)^*$ and $v, t \in W(\Phi)$. Then*

i) For each ϕ, v, t and C, the map $b_\Phi\,(\phi, C, \cdot, v, t) : i\mathfrak{t}_0 \to \mathbb{Z}$ is constant on every chamber of $i\mathfrak{t}_0$ relative to $\Phi^\vee$.

ii) For each x, v, t and C, the map $b_\Phi\,(\cdot, C, x, v, t) : (i\mathfrak{t}_0)^* \to \mathbb{Z}$ is constant on the Φ-regular elements of every chamber in $(i\mathfrak{t}_0)^*$ relative to Φ.

iii) For every $u \in W(\Phi)$, $b_\Phi\,(\phi, C, ux, uv, t) = b_\Phi\,(\phi, C, x, v, t)$.

iv) For every $u \in \widetilde{W}(\Phi_{2,tC})$, $b_\Phi\,(\phi, uC, x, v, t) = b_\Phi\,(\phi, C, x, v, t)$.

v) $b_\Phi\,(\phi, C, x, v, t) = 0$ unless $v\phi$ is ≤ 0 in D_x.

vi) If $\Phi = \emptyset$ then $b_\Phi\,(\phi, C, x, e, e) = 1$.

vii) For every $u \in W(\Phi)$, $b_\Phi\,(u\phi, uC, x, vu^{-1}, utu^{-1}) = b_\Phi\,(\phi, C, x, v, t)$.

viii) For every $u \in \widetilde{W}(\Phi_{2,tC})$, $b_\Phi\,(\phi, uC, x, v, utu^{-1}) = b_\Phi\,(\phi, C, x, v, t)$.

Proof. (i) This follows from the definition of b_Φ.

(iii) We have

$$\begin{aligned} b_\Phi(\phi, C, ux, uv, t) &= \sum_{w \in uvW(\Phi_{2,tC})} (-1)^{l_{\Phi^\vee}(D_{ux}, wC)} \psi_{wC}(ux, uv\phi) \\ &= \sum_{w \in vW(\Phi_{2,tC})} (-1)^{l_{\Phi^\vee}(uD_x, uwC)} \psi_{uwC}(ux, uv\phi) \\ &= b_\Phi(\phi, C, x, v, t) \ . \end{aligned}$$

(v) By Proposition A.5 of [4], $b_\Phi(\phi, C, x, v, t) = 0$ unless $(v\phi)(x) \leq 0$, and by i), $b_\Phi(\phi, C, x, v, t)$ depends only on the chamber D_x.

(vi) is a consequence of (2).

(vii) We have

$$\begin{aligned} b_\Phi(u\phi, uC, x, vu^{-1}, utu^{-1}) &= \sum_{w \in vW(\Phi_{2,tC})u^{-1}} (-1)^{l_{\Phi^\vee}(D_x, wuC)} \psi_{wuC}(x, v\phi) \\ &= b_\Phi(\phi, C, x, v, t) \ . \end{aligned}$$

(viii) If $u \in \widetilde{W}(\Phi_{2,tC})$, by (6, $W(\Phi_{2,utC}) = W(\Phi_{2,tC})$, hence viii) follows.

(ii) By the definition, the map $b_\Phi(\_, C, x, v, t) : (i\mathfrak{t}_0)^* \to \mathbb{Z}$ is constant in every Φ-chamber. By Lemma 1.6 i), for every $s \in \widehat{A_{sc}}(\Phi)$ such that $s^2 \in Z^\vee(\Phi)$, we have

$$\begin{aligned} &\sum_D (-1)^{l_{\Phi^\vee}(C,D)} \langle \delta_D - \delta_{tC}, s \rangle \psi_D(v^{-1}x, \phi) \\ &= (-1)^{l_{\Phi^\vee}(C,tC)} \psi_{\Phi_s}(\pi(tC), \pi(v^{-1}x), \pi'(\phi)) \ . \end{aligned} \tag{9}$$

where the sum runs over all the chambers D in $i\mathfrak{t}_0$ relative to $\Phi^\vee$. We will show that (9) is constant on the Φ-regular elements of every chamber of $(i\mathfrak{t}_0)^*$ relative to Φ. If $rk(\Phi_s) = rk(\Phi)$, by Lemma 1.5, (9) is constant on the Φ_s-regular elements of every chamber in $(i\mathfrak{t}_0)^*$ relative to $\Phi_s \subset \Phi$, hence it is constant on the Φ-regular elements of every chamber in $(i\mathfrak{t}_0)^*$ relative to Φ. If $rk(\Phi_s) < rk(\Phi)$ by Lemma 1.6 ii), (9) vanishes on all the Φ-regular elements in $(i\mathfrak{t}_0)^*$.

The map $s \longmapsto \langle \delta_D - \delta_{tC}, s \rangle$ is a character of the group

$$\left\{ s \in \widehat{A_{sc}}(\Phi) \mid s^2 \in Z^\vee(\Phi) \right\} \ .$$

Thus, if we denote by $N_\Phi = \left| \left\{ s \in \widehat{A_{sc}}(\Phi) \mid s^2 \in Z^\vee(\Phi) \right\} \right|$, we then have

$$\sum_{s \in \widehat{A_{sc}}(\Phi) | s^2 \in Z^\vee(\Phi)} \langle \delta_D - \delta_{tC}, s \rangle = \begin{cases} N_\Phi & \text{if } \delta_D - \delta_{tC} \in 2\Gamma_R(\Phi) \\ 0 & \text{if } \delta_D - \delta_{tC} \notin 2\Gamma_R(\Phi) \end{cases} \ .$$

That is, the sum vanishes unless the character is trivial.

Hence, summing (9) over $\left\{ s \in \widehat{A_{sc}}(\Phi) \mid s^2 \in Z^\vee(\Phi) \right\}$, we obtain

$$\begin{aligned} &N_\Phi \sum_{D | \delta_D - \delta_{tC} \in 2\Gamma_R(\Phi)} (-1)^{l_{\Phi^\vee}(C,D)} \psi_D(v^{-1}x, \phi) \\ &= \sum_{s \in \widehat{A_{sc}}(\Phi) | s^2 \in Z^\vee(\Phi)} \sum_D (-1)^{l_{\Phi^\vee}(C,D)} \langle \delta_D - \delta_{tC}, s \rangle \psi_D(v^{-1}x, \phi) \\ &= (-1)^{l_{\Phi^\vee}(C,tC)} \sum_{s \in \widehat{A_{sc}}(\Phi) | s^2 \in Z^\vee(\Phi)} \psi_{\Phi_s}(\pi(tC), \pi(v^{-1}x), \pi'(\phi)) \ . \end{aligned}$$

By Lemma 1.5, the last term is constant on the Φ-regular elements of any chamber in $(i\mathfrak{t}_0)^*$ relative to Φ. Now, we have

$$\begin{aligned}
&\sum_{D\mid\delta_D-\delta_{tC}\in 2\Gamma_R(\Phi)} (-1)^{l_{\Phi^\vee}(C,D)}\,\psi_D\left(v^{-1}x,\phi\right)\\
&=\sum_{w\in W(\Phi)\mid w\delta_{tC}-\delta_{tC}\in 2\Gamma_R(\Phi)} (-1)^{l_{\Phi^\vee}(C,wC)}\,\psi_{wC}\left(v^{-1}x,\phi\right)\ .
\end{aligned}$$

Thus (7) and iii) imply

$$\begin{aligned}
&\sum_{D\mid\delta_D-\delta_{tC}\in 2\Gamma_R(\Phi)} (-1)^{l_{\Phi^\vee}(C,D)}\,\psi_D\left(v^{-1}x,\phi\right)\\
&=\sum_{w\in W\left(\Phi_{2,tC}\right)} (-1)^{l_{\Phi^\vee}(C,wC)}\,\psi_{wC}\left(v^{-1}x,\phi\right)\\
&=(-1)^{l_{\Phi^\vee}\left(C,D_{v^{-1}x}\right)}\sum_{w\in W\left(\Phi_{2,tC}\right)} (-1)^{l_{\Phi^\vee}\left(D_{v^{-1}x},wC\right)}\,\psi_{wC}\left(v^{-1}x,\phi\right)\\
&=(-1)^{l_{\Phi^\vee}\left(C,D_{v^{-1}x}\right)}\,b_\Phi\left(\phi,C,v^{-1}x,e,t\right)\\
&=(-1)^{l_{\Phi^\vee}\left(C,D_{v^{-1}x}\right)}\,b_\Phi\left(\phi,C,x,v,t\right)\ .
\end{aligned}$$

Hence

$$\begin{aligned}
b_\Phi\left(\phi,C,x,v,t\right)\ &=\tfrac{1}{N_\Phi}(-1)^{l_{\Phi^\vee}\left(C,D_{v^{-1}x}\right)}(-1)^{l_{\Phi^\vee}(C,tC)}\\
&\sum_{s\in\widehat{A_{sc}}(\Phi)\mid s^2\in Z^\vee(\Phi)} \psi_{\Phi_s}\left(\pi\left(tC\right),\pi\left(v^{-1}x\right),\pi'\left(\phi\right)\right)\ .
\end{aligned}$$

Therefore, $b_\Phi\left(\phi,C,x,v,t\right)$ is constant on the Φ-regular elements of every chamber in $(i\mathfrak{t}_0)^*$ relative to Φ.

(iv) By Lemma 1.6 iii), if D runs over all chambers in $i\mathfrak{t}_0$ relative to $\Phi^\vee$, we have

$$\begin{aligned}
&\sum_{s\in\widehat{A_{sc}}(\Phi)\mid s^2=1}\sum_D (-1)^{l_{\Phi^\vee}(C,D)}\left\langle\delta_D-\delta_{tC},s\right\rangle\psi_D\left(v^{-1}x,\phi\right)\\
&=\sum_{s\in\widehat{A_{sc}}(\Phi)\mid s^2\in Z^\vee(\Phi)}\sum_D (-1)^{l_{\Phi^\vee}(C,D)}\left\langle\delta_D-\delta_{TC},s\right\rangle\psi_D\left(v^{-1}x,\phi\right)\ .
\end{aligned}\tag{10}$$

If $s\in\widehat{A_{sc}}\left(\Phi\right)$ is such that $s^2=1$, the map $s\longmapsto\left\langle\delta_D-\delta_{tC},s\right\rangle$ is a character of the group

$$\left\{s\in\widehat{A_{sc}}\left(\Phi\right)\mid s^2=1\right\}$$

and it is the trivial character iff $\delta_D-\delta_{tC}\in\Phi_{2,tC}$. Hence, if we denote by $Z_\Phi=\left|\left\{s\in\widehat{A_{sc}}\left(\Phi\right)\mid s^2=1\right\}\right|$ we have

$$\sum_{s\in\widehat{A_{sc}}(\Phi)\mid s^2=1}\left\langle\delta_D-\delta_{tC},s\right\rangle=\begin{cases} Z_\Phi & \text{if } \delta_D-\delta_{tC}\in\Gamma_R\left(\Phi_{2,tC}\right)\\ 0 & \text{if } \delta_D-\delta_{tC}\notin\Gamma_R\left(\Phi_{2,tC}\right)\end{cases}\ .$$

From (8) we obtain

$$\begin{aligned}
&\sum_{D\mid\delta_D-\delta_{tC}\in\Phi_{2,tC}} (-1)^{l_{\Phi^\vee}(C,D)}\,\psi_D\left(v^{-1}x,\phi\right)\\
&=\sum_{w\in W(\Phi)\mid w\delta_{tC}-\delta_{tC}\in\Phi_{2,tC}} (-1)^{l_{\Phi^\vee}(C,wC)}\,\psi_{wC}\left(v^{-1}x,\phi\right)\\
&=\sum_{w\in\widetilde{W}\left(\Phi_{2,tC}\right)} (-1)^{l_{\Phi^\vee}(C,wC)}\,\psi_{wC}\left(v^{-1}x,\phi\right)\ .
\end{aligned}$$

By ii) equality (10) becomes

$$Z_\Phi \sum_{w\in\widetilde{W}\left(\Phi_{2,tC}\right)} (-1)^{l_{\Phi^\vee}(C,wC)}\,\psi_{wC}\left(v^{-1}x,\phi\right) = \\ = N_\Phi\left(-1\right)^{l_{\Phi^\vee}\left(C,D_{v^{-1}x}\right)} b_\Phi\left(\phi,C,x,v,t\right)$$

hence

$$Z_\Phi \sum_{w\in\widetilde{W}\left(\Phi_{2,tC}\right)} (-1)^{l_{\Phi^\vee}\left(D_{v^{-1}x},wC\right)}\,\psi_{wC}\left(v^{-1}x,\phi\right) = N_\Phi b_\Phi\left(\phi,C,x,v,t\right)\ .$$

If $u\in\widetilde{W}(\Phi_{2,tC})$, by (6) we have $\widetilde{W}(\Phi_{2,tuC})=\widetilde{W}(\Phi_{2,tC})$, then

$$\begin{aligned} N_\Phi b_\Phi\left(\phi,uC,x,v,t\right) &= Z_\Phi \sum_{w\in\widetilde{W}\left(\Phi_{2,tuC}\right)} (-1)^{l_{\Phi^\vee}\left(D_{v^{-1}x},wuC\right)}\,\psi_{wuC}\left(v^{-1}x,\phi\right) \\ &= Z_\Phi \sum_{w\in\widetilde{W}\left(\Phi_{2,tC}\right)} (-1)^{l_{\Phi^\vee}\left(D_{v^{-1}x},wC\right)}\,\psi_{wC}\left(v^{-1}x,\phi\right) \\ &= N_\Phi b_\Phi\left(\phi,C,x,v,t\right) \end{aligned}$$

which concludes the proof of the Proposition. ■

For the rest of this section we fix $\alpha\in\Phi(\mathfrak{g},\mathfrak{t})$. Let be $Y_\alpha=\ker(\alpha)\subset\mathfrak{t}$, then by (5), $-1_{Y_\alpha}\in W(\Phi_\alpha(\mathfrak{g},\mathfrak{t}))$ and $\Phi_\alpha(\mathfrak{g},\mathfrak{t})$ spans $(Y_\alpha)^*$. We assume also that $\Phi_\alpha(\mathfrak{g},\mathfrak{t})$ satisfies the conditions in this section, namely $\Phi_\alpha(\mathfrak{g},\mathfrak{t})$ is the root system of a compact Cartan subalgebra of a split real Lie algebra. The algebra is the semisimple factor of the centralizer of α in $\mathfrak{g}_0$ and the compact Cartan subalgebra is Y_α. In Lemma 3.1 we will show how to chose such an α.

Given $y\in Y_\alpha$ regular relative to $\Phi_\alpha=\Phi_\alpha(\mathfrak{g},\mathfrak{t})$, we denote by $(C_{Y_\alpha})_y$ (or $(D_{Y_\alpha})_y$) the unique chamber in Y_α relative to $\Phi_\alpha^\vee$ where y lies.

Definition 2.3. Let C be a chamber in $i\mathfrak{t}_0$ relative to $\Phi^\vee$, let $y\in Y_\alpha$ be regular relative to Φ_α, let ϕ be a Φ-regular element in $(i\mathfrak{t}_0)^*$ and let $v,t\in W(\Phi)$. We define

$$b^\Phi_{\Phi_\alpha}\left(\phi,C,y,v,t\right) = \sum_{w\in W(\Phi_\alpha)\cap vW\left(\Phi_{2,tC}\right)} (-1)^{l_{\Phi_\alpha^\vee}\left((D_{Y_\alpha})_y,wC_{Y_\alpha}\right)}\,\psi_{wC_{Y_\alpha}}\left(y,(v\phi)_{|Y_\alpha}\right)\ .$$

Proposition 2.4. *For every $u\in W(\Phi)$ and notation as in Definition 2.3*

$$b^\Phi_{\Phi_\alpha}\left(u\phi,uC,y,vu^{-1},utu^{-1}\right) = b^\Phi_{\Phi_\alpha}\left(\phi,C,y,v,t\right)\ .$$

Proof. We have

$$\begin{aligned} & b^\Phi_{\Phi_\alpha}\left(u\phi,uC,y,vu^{-1},utu^{-1}\right) \\ &= \sum_{w\in W(\Phi_\alpha)\cap vu^{-1}W\left(\Phi_{2,utu^{-1}C}\right)} (-1)^{l_{\Phi^\vee}\left((D_{Y_\alpha})_y,wuC_{Y_\alpha}\right)}\,\psi_{wuC_{Y_\alpha}}\left(y,(vu^{-1}u\phi)_{|Y_\alpha}\right) \\ &= b^\Phi_{\Phi_\alpha}\left(\phi,C,y,v,t\right)\ . \end{aligned}$$

■

Proposition 2.5. *Let x and x' be regular elements in $i\mathfrak{t}_0$ relative to Φ lying in adjacent chambers of $i\mathfrak{t}_0$ relative to $\Phi^\vee$, and such that the chambers are separated by the hyperplane Y_α, fix a chamber C of $i\mathfrak{t}_0$ relative to $\Phi^\vee$, ϕ a Φ-regular element in $(i\mathfrak{t}_0)^*$, $v, t \in W(\Phi)$ and let $y \in Y_\alpha$ be the unique element lying on the line segment joining x with x'. Then*

i) y is regular relative to Φ_α .

ii) We have

$$b_\Phi(\phi, C, x, v, t) + b_\Phi(\phi, C, x', v, t) = b^\Phi_{\Phi_\alpha}(\phi, C, y, v, t) + b^\Phi_{\Phi_\alpha}(\phi, C, y, s_\alpha v, t) \ .$$

Proof. (i) It is a consequence of Lemma 1.7.

(ii) Because of Proposition 2.2, we can assume $x' = s_\alpha x$. Owing to Proposition 2.2 iii) and Proposition 2.4 it is enough to assume that Y_α is a wall of C and that x and C lie on the same side of Y_α. Hence

$$\begin{aligned} & b_\Phi(\phi, C, x, v, t) + b_\Phi(\phi, C, x, s_\alpha v, t) \\ = & \sum_{w \in vW(\Phi_{2,tC})} (-1)^{l_{\Phi^\vee}(D_x, wC)} \psi_{wC}(x, v\phi) \\ + & \sum_{w \in vW(\Phi_{2,tC})} (-1)^{l_{\Phi^\vee}(D_x, s_\alpha wC)} \psi_{s_\alpha wC}(x, s_\alpha v\phi) \\ = & \sum_{w \in vW(\Phi_{2,tC})} (-1)^{l_{\Phi^\vee}(D_x, wC)} \left[\psi_{wC}(x, v\phi) - \psi_{wC}(s_\alpha x, v\phi)\right] \ . \end{aligned}$$

We assume $\alpha(x) > 0$ (the case $\alpha(x) < 0$ is analogous). By Lemma A.2 of [4], we obtain

$$\begin{aligned} & b_\Phi(\phi, C, x, v, t) + b_\Phi(\phi, C, x, s_\alpha v, t) \\ = & \sum_{w \in vW(\Phi_{2,tC})} (-1)^{l_{\Phi^\vee}(D_x, wC)} \eta(w) \psi_{wC_{Y_\alpha}}\left(y, (v\phi)_{|Y_\alpha}\right) \ . \end{aligned}$$

Here

$$\eta(w) = \begin{cases} 1 & \text{if } Y_\alpha \text{ is a wall of } wC \text{ such that} \\ & x \text{ and } wC \text{ lie on the same side of } Y_\alpha \\ -1 & \text{if } Y_\alpha \text{ is a wall of } wC \text{ such that} \\ & x \text{ and } wC \text{ do not lie on the same side of } Y_\alpha \\ 0 & \text{if } Y_\alpha \text{ is not a wall of } wC \end{cases}$$

By Chevalley's Lemma, Y_α is a wall of wC iff $w \in W(\Phi_\alpha \cup \{\pm\alpha\})$. Since $\alpha \perp \Phi_\alpha$, $W(\Phi_\alpha \cup \{\pm\alpha\}) = W(\Phi_\alpha) \cup s_\alpha W(\Phi_\alpha)$. If $w \in W(\Phi_\alpha)$, x and wC lie on the same side of Y_α and x and $s_\alpha C$ do not lie on the same side of Y_α, we then have

$$\eta(w) = \begin{cases} 1 & \text{if } w \in W(\Phi_\alpha) \\ -1 & \text{if } w \in s_\alpha W(\Phi_\alpha) \\ 0 & \text{otherwise} \end{cases} \ .$$

Now because of Lemma 1.1 i) in [4] $l_{\Phi^\vee}(D_x, wC) = l_{\Phi^\vee_\alpha}\left(\left(D_{Y_\alpha}\right)_y, wC_{Y_\alpha}\right)$, hence

$$\begin{aligned}
& b_{\Phi}(\phi, C, x, v, t) + b_{\Phi}(\phi, C, x, s_{\alpha}v, t) \\
& = \sum_{w \in W(\Phi_{\alpha}) \cap vW\left(\Phi_{2,tC}\right)} (-1)^{l_{\Phi_{\alpha}^{\vee}}\left((D_{Y_{\alpha}})_y, wC_{Y_{\alpha}}\right)} \psi_{wC_{Y_{\alpha}}}\left(y, (v\phi)_{|Y_{\alpha}}\right) - \\
& - \sum_{w \in s_{\alpha}W(\Phi_{\alpha}) \cap vW\left(\Phi_{2,tC}\right)} (-1)^{l_{\Phi_{\alpha}^{\vee}}\left((D_{Y_{\alpha}})_y, wC_{Y_{\alpha}}\right)} \psi_{wC_{Y_{\alpha}}}\left(y, (v\phi)_{|Y_{\alpha}}\right) \ .
\end{aligned} \tag{11}$$

Since $(s_{\alpha}\phi)_{|Y_{\alpha}} = \phi_{|Y_{\alpha}}$ and for every $w \in W(\Phi_{\alpha})$, we have $(s_{\alpha}wC)_{Y_{\alpha}} = (wC)_{Y_{\alpha}}$ and thus (11) is equal to

$$b_{\Phi_{\alpha}}^{\Phi}(\phi, C, y, v, t) + b_{\Phi_{\alpha}}^{\Phi}(\phi, C, y, s_{\alpha}v, t) \ .$$

This proves Proposition 2.5. ∎

Proposition 2.6. *If C is a chamber in $i\mathfrak{t}_0$ relative to $\Phi^{\vee}$ such that Y_{α} is a wall of C we have*

i) For $\beta \in \Phi_{\alpha}$, then, $\beta \in (\Phi_{\alpha})_{2,C_{Y_{\alpha}}}$ iff $\beta \in \Phi_{2,C}$.

ii) $W(\Phi_{\alpha}) \cap W(\Phi_{2,C}) = W\left((\Phi_{\alpha})_{2,C_{Y_{\alpha}}}\right)$.

Proof. We assume $\Phi_{2,C} \neq \emptyset$ (otherwise the assertion is trivial).

i) We decompose $\beta(\delta_C)$ as the sum

$$\beta\left(\delta_{C_{Y_{\alpha}}}(\Phi_{\alpha})\right) + \sum_{\gamma \in \Phi_{C^{\vee}}^{+} - \Phi_{\alpha} | \beta(\gamma^{\vee}) = 0} \beta(\gamma^{\vee}) + \sum_{\gamma \in \Phi_{C^{\vee}}^{+} - \Phi_{\alpha} | \beta(\gamma^{\vee}) \neq 0} \beta(\gamma^{\vee}) \ .$$

Notice that every summand is an integer. In order to show that the parity of $\beta(\delta_C)$ and $\beta\left(\delta_{C_{Y_{\alpha}}}(\Phi_{\alpha})\right)$ agree, it is enough to verify that $\sum_{\gamma \in \Phi_{C^{\vee}}^{+} - \Phi_{\alpha} | \beta(\gamma^{\vee}) \neq 0} \beta(\gamma^{\vee})$ is even.

Let $\Delta_{C^{\vee}} = \{\alpha_1, ..., \alpha_n\}$ and let $\{\omega_1, ..., \omega_n\}$ be the associated fundamental weights. Since Y_{α} is a wall of C, we may order $\Delta_{C^{\vee}}$ such that $\omega_2^{\vee}, ..., \omega_n^{\vee} \in Y_{\alpha}$, that is $(\alpha, \omega_i) = 0$ for every i, $2 \leq i \leq n$.

We prove that for $\gamma \in \Phi_{C^{\vee}}^{+} - \Phi_{\alpha}$ and $\beta(\gamma^{\vee}) \neq 0$ then

a) $s_{\alpha}(\gamma) \neq \gamma$

b) $s_{\alpha}(\gamma) \in \Phi_{C^{\vee}}^{+} - \Phi_{\alpha}$

c) $\beta(\gamma^{\vee}) = \beta\left((s_{\alpha}(\gamma))^{\vee}\right)$.

Coupling γ with $s_{\alpha}(\gamma)$ in $\sum_{\alpha \in \Phi_{C^{\vee}}^{+} - \Phi_{\alpha} | \beta(\gamma^{\vee}) \neq 0} \beta(\gamma^{\vee})$, a), b) and c) show that this is an even number.

We now verify a), b) and c).

a) $(s_{\alpha}(\gamma), \alpha) = (\gamma, s_{\alpha}(\alpha)) = -(\gamma, \alpha)$. Now $\gamma \notin \Phi_{\alpha}$, hence $(\gamma, \alpha) \neq 0$. Thus if $s_{\alpha}(\gamma) = \gamma$, we would have $(\gamma, \alpha) = 0$.

c) $\beta\left((s_{\alpha}(\gamma))^{\vee}\right) = \frac{2}{(s_{\alpha}(\gamma), s_{\alpha}(\gamma))}(s_{\alpha}(\gamma), \beta) = \frac{2}{(\gamma,\gamma)}(\gamma, s_{\alpha}(\beta)) = \frac{2}{(\gamma,\gamma)}(\gamma, \beta)$. Hence $\beta\left((s_{\alpha}(\gamma))^{\vee}\right) = \beta(\gamma^{\vee})$.

b) $(s_\alpha(\gamma),\alpha) = -(\gamma,\alpha) \neq 0$. Thus $s_\alpha(\gamma) \notin \Phi_\alpha$. Since $\gamma \in \Phi^+_{C^\vee}$, for every i, $2 \le i \le n$, $(s_\alpha(\gamma),\omega_i) = (\gamma, s_\alpha(\omega_i)) = (\gamma,\omega_i) \ge 0$.

We return to the proof of i). If $(\gamma,\omega_i) = 0$ for every i, $2 \le i \le n$ we would have $\gamma = \pm\alpha$, a contradiction, because $\beta \in \Phi_\alpha$. Hence, there exists i, $2 \le i \le n$, such that $(s_\alpha(\gamma),\omega_i) = (\gamma,\omega_i) > 0$, thus $s_\alpha(\gamma) \in \Phi^+_{C^\vee}$, and i) follows.

We now show ii). By i) is enough to prove that $W(\Phi_\alpha) \cap W(\Phi_{2,C}) \subset W\Big((\Phi_\alpha)_{2,C_{Y_\alpha}}\Big)$.

Let $w \in (W(\Phi_\alpha) \cap W(\Phi_{2,C})) - W\Big((\Phi_\alpha)_{2,C_{Y_\alpha}}\Big)$ of minimum length (minimum relative to $(\Phi_\alpha)^+_{(C_{Y_\alpha})^\vee} = \Phi^+_{C^\vee} \cap \Phi_\alpha$). Thus, its length $l(w)$ is positive and there exists $\beta \in \Delta_{(C_{Y_\alpha})^\vee}$ such that $l(s_\beta w) < l(w)$. Also $s_\beta w \in W\Big((\Phi_\alpha)_{2,C_{Y_\alpha}}\Big) \subset W(\Phi_{2,C})$. Then

$$s_\beta = s_\beta w w^{-1} \in W(\Phi_{2,C}) \subset \widetilde{W}(\Phi_{2,C}) = \{w \in W(\Phi) \mid w(\Phi_{2,C}) = \Phi_{2,C}\}$$

That is, $s_\beta(\Phi_{2,C}) = \Phi_{2,C}$. Since

$$\beta(\delta_C) = \sum_{\gamma \in \Phi^+_{C^\vee} | \beta(\gamma^\vee)=0} \beta(\gamma^\vee) + \sum_{\gamma \in \Phi^+_{C^\vee} | \beta(\gamma^\vee)\neq 0} \beta(\gamma^\vee)$$

in order to show that $\beta(\delta_C)$ it is even is enough to prove that $\sum_{\gamma \in \Phi^+_{C^\vee} | \beta(\gamma^\vee)\neq 0} \beta(\gamma^\vee)$ is even.

We prove that for $\gamma \in \Phi^+_{C^\vee}$, $\gamma \neq \beta$ and $\beta(\gamma^\vee) \neq 0$, then,

d) $\beta\big((s_\beta(\gamma))^\vee\big) = -\beta(\gamma^\vee)$ and hence $s_\beta(\gamma) \neq \gamma$ and $\beta\big((s_\beta(\gamma))^\vee\big) \neq 0$

e) $s_\beta(\gamma) \in \Phi^+_{C^\vee}$.

We now show d) and e).

d) $\beta\big((s_\beta(\gamma))^\vee\big) = \frac{2}{(s_\beta(\gamma),s_\beta(\gamma))}(\beta, s_\beta(\gamma)) = \frac{2}{(\gamma,\gamma)}(s_\beta(\beta),\gamma) = -\frac{2}{(\gamma,\gamma)}(\beta,\gamma)$. Hence $\beta\big((s_\beta(\gamma))^\vee\big) = -\beta(\gamma^\vee)$

e) Since β is simple and $\beta \neq \gamma$ we have $s_\beta(\gamma) \in \Phi^+_{C^\vee}$.

Coupling γ with $s_\beta(\gamma)$, d) and e) shows that $\sum_{\gamma \in \Phi^+_{C^\vee} | \beta(\gamma^\vee)\neq 0} \beta(\gamma^\vee) = \beta(\beta^\vee) = 2$

We now conclude the proof of ii). Hence, $\beta(\delta_C)$ is even and thus $\beta \in \Phi_{2,C} \cap \Phi_\alpha$. By i) $s_\beta \in W\Big((\Phi_\alpha)_{2,C_{Y_\alpha}}\Big)$. Thus $w = s_\beta(s_\beta w) \in W\Big((\Phi_\alpha)_{2,C_{Y_\alpha}}\Big)$, a contradiction, which concludes the proof of the Proposition. ∎

Proposition 2.7. *Let C be a chamber in $i\mathfrak{t}_0$ relative to $\Phi^\vee$ such that Y_α is a wall of C, $y \in Y_\alpha$ is regular relative to Φ_α, $\phi \in (i\mathfrak{t}_0)^*$ is Φ-regular, $v \in W(\Phi)$ and $t \in \widetilde{W}(\Phi_{2,C})$. Then*

$$b^{\Phi}_{\Phi_\alpha}(\phi, C, y, v, t) = \begin{cases} b_{\Phi_\alpha}\Big((w_C\phi)_{|Y_\alpha}, C_{Y_\alpha}, y, w_\alpha, e\Big) & \text{if } v = w_\alpha w_C \text{ with} \\ & w_\alpha \in W(\Phi_\alpha) \text{ and} \\ & w_C \in W(\Phi_{2,C}) \\ 0 & \text{if } v \notin W(\Phi_\alpha)W(\Phi_{2,C}) \end{cases}.$$

Proof. Since $t \in \widetilde{W}(\Phi_{2,C})$, by Proposition 2.2 viii)

$$b^{\Phi}_{\Phi_\alpha}(\phi, C, y, v, t) = b^{\Phi}_{\Phi_\alpha}(\phi, C, y, v, e) \ .$$

In the statement of the Proposition 2.7, the decomposition $v = w_\alpha w_C$ with $w_\alpha \in W(\Phi_\alpha)$ and $w_C \in W(\Phi_{2,C})$ is defined up to an element in $W(\Phi_\alpha) \cap W(\Phi_{2,C})$, because if $v = w'_\alpha w'_C$ with $w_\alpha \in W(\Phi_\alpha)$ and $w_C \in W(\Phi_{2,C})$ is another decomposition, then $u = w'^{-1}_\alpha w_\alpha = w'_C w_C^{-1} \in W(\Phi_\alpha) \cap W(\Phi_{2,C})$. Since Y_α is a wall of C, by Proposition 2.6, $W(\Phi_\alpha) \cap W(\Phi_{2,C}) = W\Big((\Phi_\alpha)_{2,C_{Y_\alpha}}\Big)$ and by Proposition 2.2 iv) and vii)

$$\begin{aligned} b_{\Phi_\alpha}\Big((u^{-1}w_C\phi)_{|Y_\alpha}, C_{Y_\alpha}, y, w_\alpha u, e\Big) &= b_{\Phi_\alpha}\Big((w_C\phi)_{|Y_\alpha}, uC_{Y_\alpha}, y, w_\alpha, u\Big) \\ &= b_{\Phi_\alpha}\Big((w_C\phi)_{|Y_\alpha}, C_{Y_\alpha}, y, w_\alpha, e\Big) \ . \end{aligned}$$

Hence $b_{\Phi_\alpha}\Big((w_C\phi)_{|Y_\alpha}, C_{Y_\alpha}, y, w_\alpha, e\Big)$ is well defined.

We now prove the proposed equality. If $v \notin W(\Phi_\alpha)\, W(\Phi_{2,C})$, both members vanish.

If $v = w_\alpha w_C$ with $w_\alpha \in W(\Phi_\alpha)$ and $w_C \in W(\Phi_{2,C})$, in Definition 2.3 the sum runs over all

$$w \in W(\Phi_\alpha) \cap w_\alpha W(\Phi_{2,C}) = w_\alpha\left(W(\Phi_\alpha) \cap W(\Phi_{2,C})\right) = w_\alpha W\Big((\Phi_\alpha)_{2,C_{Y_\alpha}}\Big) \ .$$

Hence

$$\begin{aligned} b^{\Phi}_{\Phi_\alpha}(\phi, C, y, v, t) &= b^{\Phi}_{\Phi_\alpha}(\phi, C, y, v, e) \\ &= \sum_{w \in w_\alpha W\left((\Phi_\alpha)_{2,C_{Y_\alpha}}\right)} (-1)^{l_{\Phi^\vee_\alpha}\left((D_{Y_\alpha})_y, wC_{Y_\alpha}\right)} \psi_{wC_{Y_\alpha}}\Big(y, (v\phi)_{|Y_\alpha}\Big) \\ &= b_{\Phi_\alpha}\Big((w_C\phi)_{|Y_\alpha}, C_{Y_\alpha}, y, w_\alpha, e\Big) \end{aligned}$$

and this completes the proof of this Proposition. ∎

In the next section we will consider C a chamber in $i\mathfrak{t}_0$ relative to $\Phi^\vee$ such that for every $\alpha \in \Delta_{C^\vee}$, Φ_α meets the conditions of this section, this is why we need to formulate an analogous statement of Proposition 2.7 valid in this case.

We show first that $W((\Phi_\alpha)_K) \subset W(\Phi_\alpha) \cap W(\Phi_K) \subset \widetilde{W}((\Phi_\alpha)_K)$. We only need to show the last inclusion. For this we fix Φ^+ a system of positive roots in Φ. Let be $w \in W(\Phi_\alpha) \cap W(\Phi_K)$ and $\beta \in (\Phi_\alpha)_K$. Since $w\beta \in (\Phi_\alpha)_K$ and $w\rho\left((\Phi^+_\alpha)_K\right) - \rho\left((\Phi^+_\alpha)_K\right) \in \Gamma_R\left((\Phi^+_\alpha)_K\right)$, we have $w \in \widetilde{W}((\Phi_\alpha)_K)$.

The next Proposition is the analogue of Proposition 2.7 in the case that $W((\Phi_\alpha)_K) = W(\Phi_\alpha) \cap W(\Phi_K)$.

Proposition 2.8. *Let C be a chamber in $i\mathfrak{t}_0$ relative to $\Phi^\vee$ such that Y_α is a wall of C, $y \in Y_\alpha$ regular relative to Φ_α, $\phi \in (i\mathfrak{t}_0)^*$, Φ-regular, $v \in W(\Phi)$ and $t \in W(\Phi)$ such that $W(\Phi_{2,tC}) = W(\Phi_K)$.*
Then, if $W((\Phi_\alpha)_K) = W(\Phi_\alpha) \cap W(\Phi_K)$ and $v_1 \in W(\Phi_\alpha)$ is such that $W\Big((\Phi_\alpha)_{2,v_1C_{Y_\alpha}}\Big) = W((\Phi_\alpha)_K)$, we have

$$b^{\Phi}_{\Phi_\alpha}(\phi, C, y, v, t) = \begin{cases} b_{\Phi_\alpha}\Big((w_K\phi)_{|Y_\alpha}, C_{Y_\alpha}, y, w_\alpha, v_1\Big) & \text{if } v = w_\alpha w_K \text{ with } \\ & w_\alpha \in W(\Phi_\alpha) \text{ and } \\ & w_K \in W(\Phi_K) \\ 0 & \text{if } v \notin W(\Phi_\alpha)\, W(\Phi_K) \end{cases} \ .$$

Proof. As in Proposition 2.9 one shows that $b_{\Phi_\alpha}\Big((w_K\phi)_{|Y_\alpha}, C_{Y_\alpha}, y, w_\alpha, v_1\Big)$ is independent of the decomposition $v = w_\alpha w_K$ with $w_\alpha \in W(\Phi_\alpha)$ and $w_K \in W(\Phi_K)$ chosen because if $v = w'_\alpha w'_K$ with $w'_\alpha \in W(\Phi_\alpha)$ and $w'_K \in W(\Phi_K)$ is another decomposition, then $u = w'^{-1}_\alpha w_\alpha = w'_K w_K^{-1} \in W(\Phi_\alpha) \cap W(\Phi_K)$.

Since $W(\Phi_\alpha) \cap W(\Phi_K) \subset \widetilde{W}((\Phi_\alpha)_K)$, by Proposition 2.2 iv) and vii)

$$\begin{aligned} b_{\Phi_\alpha}\Big((u^{-1}w_K\phi)_{|Y_\alpha}, C_{Y_\alpha}, y, w_\alpha u, v_1\Big) &= b_{\Phi_\alpha}\Big((w_K\phi)_{|Y_\alpha}, uC_{Y_\alpha}, y, w_\alpha, uv_1\Big) \\ &= b_{\Phi_\alpha}\Big((w_K\phi)_{|Y_\alpha}, C_{Y_\alpha}, y, w_\alpha, v_1\Big) \end{aligned}$$

Thus $b_{\Phi_\alpha}\Big((w_K\phi)_{|Y_\alpha}, C_{Y_\alpha}, y, w_\alpha, v_1\Big)$ is well defined. By Proposition 2.2 viii), $b_{\Phi_\alpha}\Big((w_K\phi)_{|Y_\alpha}, C_{Y_\alpha}, y, w_\alpha, v_1\Big)$ does not depend on $v_1 \in W(\Phi_\alpha)$ whenever $W\Big((\Phi_\alpha)_{2,v_1C_{Y_\alpha}}\Big) = W((\Phi_\alpha)_K)$.

We now prove the asserted equality. If $v \notin W(\Phi_\alpha)\, W(\Phi_K)$ both members vanish.

If $v = w_\alpha w_K$ with $w_\alpha \in W(\Phi_\alpha)$ and $w_K \in W(\Phi_K)$, in Definition 2.3 the sum runs over all

$$w \in W(\Phi_\alpha) \cap w_\alpha W(\Phi_K) = w_\alpha \left(W(\Phi_\alpha) \cap W(\Phi_K)\right) = w_\alpha W((\Phi_\alpha)_K) \ .$$

Hence

$$\begin{aligned} & b^{\Phi}_{\Phi_\alpha}(\phi, C, y, v, t) \\ &= \sum_{w \in w_\alpha W\left((\Phi_\alpha)_{2,v_1C_{Y_\alpha}}\right)} (-1)^{l_{\Phi_\alpha^\vee}\left((D_{Y_\alpha})_y, wC_{Y_\alpha}\right)}\, \psi_{wC_{Y_\alpha}}\Big(y, (v\phi)_{|Y_\alpha}\Big) \\ &= b_{\Phi_\alpha}\Big((w_K\phi)_{|Y_\alpha}, C_{Y_\alpha}, y, w_\alpha, v_1\Big) \end{aligned}$$

and this completes the proof of the Proposition. ■

3. The character constants

Let G be a connected semisimple Lie group. For every closed subgroup H of G, we denote the real Lie algebra by $\mathfrak{h}_0$ and its complexification by $\mathfrak{h}$. We will assume that G is acceptable and has a complexification $G_{\mathbb{C}}$. Let K be a maximal compact subgroup of G, $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ the corresponding Cartan decomposition and θ the involution associated, both in $\mathfrak{g}$ and in G. From now on we assume $rk\,(G) = rk\,(K)$ and we fix a compact Cartan subgroup $T \subset K$.

By [7] the discrete series characters are completely explicit except for certain integer constants. Furthermore, in order to compute the constants, by the reductions in [6] and [HS] we only need to consider the case in which G is split and to determine the discrete series characters on the identity component of a split θ-stable Cartan subgroup.

From now on, we also assume that G has a split Cartan subgroup H. We denote by H_e the identity connected component of H. Then as in [13], formula 13.30 for a Cayley transform $\widetilde{c}$, for every chamber C in $(i\mathfrak{t}_0)^*$ relative to $\Phi = \Phi(\mathfrak{g}, \mathfrak{t})$, $\Phi^+ = \Phi^+(\mathfrak{g}, \mathfrak{t})$ system of positive roots on Φ, for a connected component H_0 of $H_e \cap G'$ and $\lambda \in (i\mathfrak{t}_0)^*$ regular we have

$$\Theta\left(\Phi^+(\mathfrak{g}, \mathfrak{t}), \lambda\right)_{|H_0} \circ \widetilde{c}^{-1} = \frac{\sum\limits_{w \in W(\mathfrak{g},\mathfrak{t})} (\det w)\, c(w, C, \Phi^+) \exp(w\lambda)}{\prod\limits_{\alpha \in \Phi^+(\mathfrak{g},\mathfrak{t})} (e^{\alpha/2} - e^{-\alpha/2})} .$$

The purpose of this section is to provide a formula for the constants $c(w, C, \Phi^+)$. We recall that the constants $c(w, C, \Phi^+)$ depend only on data involving the Lie algebra $\mathfrak{g}$. Moreover, because of [10] Lemma 2.1, the constants do not depend on the choice of the split Cartan subgroup.

As in [10], we can remove the assumption that G is acceptable and that G has finite center. In the first case we have to consider the two-fold covering of G which is acceptable and in the last case we have the techniques in [11] for relative discrete series.

We recall, [10], that $\alpha \in \Phi$ is a good root iff Φ_α admits a strongly orthogonal spanning set of noncompact roots, and Φ^+ is a good order iff every simple root is a good root.

Lemma 3.1. *Let $\mathfrak{g}_0$ be a split semisimple real Lie algebra who has a compact Cartan subalgebra, then Φ has a good order. More precisely if $\mathfrak{g}$ is of type A_1, B_{2n+1}, D_{2n}, E_7, E_8 or G_2 there is a good order such that all simple roots are noncompact. If $\mathfrak{g}$ is of type B_{2n}, C_n or F_4, there is a good order such that every long simple roots is noncompact and all the short simple roots are compact.*

Proof. Lemma 3.1 of [10]. ∎

As before, C is a chamber in $(i\mathfrak{t}_0)^*$ relative to Φ, let $\alpha \in \Delta_C$ and $i : \ker\alpha \to \mathfrak{t}$ the inclusion. We denote by i^* the natural map

$$i^* : (i\mathfrak{t}_0)^* \to (i\mathfrak{t}_0 \cap \ker\alpha)^*$$

and by C_α the chamber in $(i\mathfrak{t}_0 \cap \ker\alpha)^*$ such that $i^*(C) \subset C_\alpha$.

We extend our definition of $c(w, C_\alpha, \Phi^+_\alpha)$ to every $w \in W(\Phi)$ by setting

$$c(w, C_\alpha, \Phi^+_\alpha) = \\ = \begin{cases} c(w_\alpha, w_K C_\alpha, \Phi^+_\alpha) & \text{if } w = w_\alpha w_K,\ w_\alpha \in W(\Phi_\alpha),\ w_K \in W(\Phi_K) \\ 0 & \text{if } w \notin W(\Phi_\alpha)\, W(\Phi_K) \end{cases} .$$

By ii) of the following Theorem, $c(w, C_\alpha, \Phi^+_\alpha)$ is well defined, that is, it is independent of the decomposition $w = w_\alpha w_K$ with $w_\alpha \in W(\Phi_\alpha)$, $w_K \in W(\Phi_K)$.

Theorem 3.2. *The constants $c(w, C, \Phi^+)$ satisfy the following properties*

i) $c(uw, C, u\Phi^+) = c(w, C, \Phi^+)$, *for every* $u \in W(\Phi)$.

ii) $c(wv, v^{-1}C, \Phi^+) = c(w, C, \Phi^+)$, *for every* $v \in W(\Phi_K)$.

iii) If there exists $\tau \in C$ such that $(w\tau)(H_i) > 0$ for some fundamental coweight H_i relative to Φ^+, then $c(w, C, \Phi^+) = 0$.

iv) $c(e, C, \emptyset) = 1$.

v) For every simple root α in Φ^+ and α good root, then $c(w, C, \Phi^+) + c(s_\alpha w, C, \Phi^+) = c(w, C_\alpha, \Phi^+_\alpha) + c(s_\alpha w, C_\alpha, \Phi^+_\alpha)$.

Proof. It is a consequence of [6] and [10] Lemmas 2.2, 2.3, 2.4, 2.6, 2.9. ■

Proposition 3.3. *The constants $c\,(w, C, \Phi^+)$ are uniquely determinate by properties i), ii), iii), iv) and v) of Theorem 3.2.*

Proof. Lemma 3.2 of [10]. ■

Now we will define a candidate for $c\,(w, C, \Phi^+)$.

Definition 3.4. Given $w \in W(\Phi)$, let C_B be a chamber in $(i\mathfrak{t}_0)^*$ relative to Φ such that $\Phi^+_{C_B}$ is a good order, $t \in W(\Phi)$ such that $\Phi_{2,tC_B} = \Phi_K$, C a chamber in $(i\mathfrak{t}_0)^*$ relative to Φ, and Φ^+ an order in Φ. Then, for $x_{\Phi^+} \in i\mathfrak{t}_0$, $\Phi^\vee$-regular such that $x_{\Phi^+} \in C^\vee_{\Phi^+}$, $\phi_C \in (i\mathfrak{t}_0)^*$ Φ-regular, and $\phi_C \in C$, we set

$$a\left(w, C, \Phi^+\right) = b_\Phi\left(\phi_C, C_B, x_{\Phi^+}, w, t\right) \ .$$

By Proposition 2.2 i), ii), iv) and viii) $a\,(w, C, \Phi^+)$ does not depend neither on the choice of the $\Phi^\vee$-regular element $x_{\Phi^+} \in C^\vee_{\Phi^+}$ nor on the choice of the Φ-regular element $\phi_C \in C$. Moreover, it does not depend neither on the chamber C_B chosen such that $\Phi^+_{C_B}$ is a good order, nor on $t \in W(\Phi)$ chosen such that $\Phi_{2,tC_B} = \Phi_K$.

Theorem 3.5. *For every $w \in W(\Phi)$, C chamber in $(i\mathfrak{t}_0)^*$ relative to Φ and Φ^+ a system of positive roots in Φ, one has that $c\,(w, C, \Phi^+) = a\,(w, C, \Phi^+)$.*

Proof. By Proposition 3.3 it is enough to show that the constants $a\,(w, C, \Phi^+)$ satisfy properties i), ii), iii), iv) and v) of Theorem 3.2.

(i) Since $x_{v\Phi^+} = vx_{\Phi^+}$, for every $v \in W(\Phi)$, by Proposition 2.2 iii) we have $a\,(uw, C, u\Phi^+) = a\,(w, C, \Phi^+)$ for every $u \in W(\Phi)$.

(ii) Since $\Phi_{2,tC_B} = \Phi_K$, by Proposition 2.2 iv), vii) and viii) we have $a\,(wv, v^{-1}C, \Phi^+) = a\,(w, C, \Phi^+)$ for every $v \in W(\Phi_K)$.

(iii) If there exists $\tau \in C$ such that $(w\tau)\,(H_i) > 0$ for some fundamental coweight H_i relative to Φ^+, there is no loss of generality in assuming that τ is Φ-regular. Then, $w\tau$ is not ≤ 0 in C_{Φ^+}, and because of Proposition 2.2 v), then $b_\Phi\,(\tau, C_B, x_{\Phi^+}, w, t) = a\,(w, C, \Phi^+) = 0$.

(iv) $a\,(e, C, \emptyset) = 1$ by Proposition 2.2 vi).

(v) By ii), $c\,(w, C_\alpha, \Phi^+_\alpha)$ is well defined, that is, it is independent of the decomposition $v = w_\alpha w_K$ with $w_\alpha \in W(\Phi_\alpha)$ and $w_K \in W(\Phi_K)$ taken. If α is a good root of Φ, by Proposition 2.2 i) and vii), Proposition 2.4 and Proposition 2.5 we have

$$\begin{aligned}
&a\left(w, C, \Phi^+_B\right) + a\left(s_\alpha w, C, \Phi^+_B\right) = \\
&= b_\Phi\left(\phi_C, C_B, x_{\Phi^+_B}, w, t\right) + b_\Phi\left(\phi_C, C_B, x_{\Phi^+_B}, s_\alpha w, t\right) \\
&= b_\Phi\left(\phi_C, C_B, x_{\Phi^+_B}, w, t\right) + b_\Phi\left(\phi_C, C_B, s_\alpha x_{\Phi^+_B}, w, t\right) \\
&= b^\Phi_{\Phi_\alpha}\left(\phi_C, C_B, y, w, t\right) + b^\Phi_{\Phi_\alpha}\left(\phi_C, C_B, y, s_\alpha w, t\right) \ .
\end{aligned}$$

In order to conclude that this is equal to $a\left(w, C_\alpha, \left(\Phi^+_B\right)_\alpha\right) + a\left(s_\alpha w, C_\alpha, \left(\Phi^+_B\right)_\alpha\right)$, we analyze each series of simple groups.

For $\mathfrak{g}$ of type B_{2n+1}, D_{2n}, E_7, E_8 or G_2, the choice of good ordering implies: $W(\Phi_{2,C_B}) = W(\Phi_K)$, $\alpha \in \Delta_B$ and Y_α is a wall of $C_B^\vee$. By Proposition 2.7, for $w \notin W(\Phi_\alpha)\, W(\Phi_K)$ we have

$$b^{\Phi}_{\Phi_\alpha}(\phi_C, C_B, y, u, t) = a\left(w, C_\alpha, \Phi_B^+\right) = 0 \ .$$

If $w = w_\alpha w_K$, $w_\alpha \in W(\Phi_\alpha)$, $w_K \in W(\Phi_K)$, by Proposition 2.7 we have

$$\begin{aligned} b^{\Phi}_{\Phi_\alpha}(\phi_C, C_B, y, u, t) &= b_{\Phi_\alpha}\Big((w_K\phi_C)_{|Y_\alpha}, (C_B)_{Y_\alpha}, y, w_\alpha, v_1 \Big) \\ &= a\left(w_\alpha, (w_K C)_\alpha, \Phi_B^+\right) \\ &= a\left(w, C_\alpha, \Phi_B^+\right) \ . \end{aligned}$$

Hence we have

$$a\left(w, C, \Phi_B^+\right) + a\left(s_\alpha w, C, \Phi_B^+\right) = a\left(w, C_\alpha, \left(\Phi_B^+\right)_\alpha\right) + a\left(s_\alpha w, C_\alpha, \left(\Phi_B^+\right)_\alpha\right)$$

and since, by Lemma A.2 of [4],$(\phi_C)_{|Y_\alpha}$ is Φ_α-regular, the equality follows for $\mathfrak{g}$ of type B_{2n+1}, D_{2n}, E_7, E_8 or G_2.

If $\mathfrak{g}$ is of type A_1, $\Phi = \{\pm\beta\}$,and by [13], 13.35

$$c(w, C, \{\beta\}) = \begin{cases} 1 & \text{if } (wC, \beta) < 0 \\ 0 & \text{otherwise} \end{cases} \ .$$

A direct computation shows that $a(w, C, \{\beta\}) = c(w, C, \{\beta\})$. By iv), $c(e, C, \emptyset) = a(e, C, \emptyset)$ hence the equality follows for $\mathfrak{g}$ of type A_1.

The case when $\mathfrak{g}$ is of type B_{2n}, C_n or F_4, is analogous to the previous case but we need to appeal to Proposition 2.8 instead of Proposition 2.7.

In order to verify the hypothesis of Proposition 2.8 we proceed case by case to show $W((\Phi_\alpha)_K) = W(\Phi_\alpha) \cap W(\Phi_K)$ for every simple root α, for the good order Φ_B^+. Since in Φ_B^+ all the long simple roots are noncompact and all the short simple roots are compact, if α, β are simple roots for Φ_B^+, of the same length, there is $w \in W(\Phi_K)$ so that $w\alpha = \beta$. Indeed, by inspection in B_{2n}, C_n or F_4 we are reduced to an argument in $su(2,1)$.

(a) $\mathfrak{g}$ of type B_{2n}, $n > 1$ (that is $\mathfrak{g}_0 = so(2n+1, 2n)$). As usual

$$\begin{aligned} &\Phi = \{\pm e_i \pm e_j\}_{1 \le i < j \le 2n} \cup \{\pm e_i\}_{1 \le i \le 2n} \\ &\Phi^+ = \{e_i \pm e_j\}_{1 \le i < j \le 2n} \cup \{e_i\}_{1 \le i \le 2n} \\ &\Delta = \{e_i - e_{i+1}\}_{1 \le i < 2n} \cup \{e_{2n}\} \ . \end{aligned}$$

Since Φ^+ is a good order, $\alpha_i = e_i - e_{i+1}$ is noncompact for $1 \le i < 2n$ and $\alpha_{2n} = e_{2n}$ is compact. Thus

$$\Phi_K = \{\pm e_{2i} \pm e_{2j}\}_{1 \le i < j \le n} \cup \{e_{2i}\}_{1 \le i \le n} \cup \{\pm e_{2i-1} \pm e_{2j-1}\}_{1 \le i < j \le n} \ .$$

Hence Φ_K is of type $B_n \times D_n$ and $W(\Phi_K)$ agrees with the direct product $W_{n,e} \times W_n$, where $W_{n,e}$ is the group of all permutations and even sign changes of the set $\{e_1, e_3, ..., e_{2n-1}\}$ and W_n is the group of all permutations and sign changes of the set $\{e_2, e_4, ..., e_{2n}\}$.

Since all the long simple roots are conjugate by $W(\Phi_K)$ we only need to consider Φ_{α_1} and $\Phi_{\alpha_{2n}}$ Now

$$\Phi_{\alpha_1} = \{\pm(e_1 + e_2)\} \cup \{\pm e_i \pm e_j\}_{3 \le i < j \le 2n} \cup \{\pm e_i\}_{3 \le i \le 2n} \ .$$

Hence Φ_{α_1} is of type $B_{2n-2} \times A_1$ and $W(\Phi_{\alpha_1})$ is the direct product $W_{2n-2} \times W_1$, where W_{2n-2} is the group of all permutations and sign changes on the set $\{e_3, ..., e_{2n}\}$ and $W_1 = \{e, s_{e_1+e_2}\}$. On the other hand $W((\Phi_{\alpha_1})_K)$ agrees with the direct product $W_{n,e} \times W_{n-1}$, where $W_{n,e}$ is the group of all permutations and even sign changes on the set $\{e_3, ..., e_{2n-1}\}$ and W_{n-1} is the group of all permutations and sign changes on the set $\{e_4, ..., e_{2n}\}$. Thus $W((\Phi_{\alpha_1})_K) = W(\Phi_{\alpha_1}) \cap W(\Phi_K)$.

$$\Phi_{\alpha_{2n}} = \{\pm e_i \pm e_j\}_{1 \leq i < j \leq 2n-1} \cup \{\pm e_i\}_{1 \leq i \leq 2n-1} \ .$$

Hence, $W(\Phi_{\alpha_{2n}})$ is the group of all permutations and sign changes on the set $\{e_1, ..., e_{2n-1}\}$. On the other hand, $W((\Phi_{\alpha_{2n}})_K)$ is the direct product $W_{n,e} \times W_{n-1}$, where $W_{n,e}$ is the group of all permutations and even sign changes on the set $\{e_1, e_3, ..., e_{2n-1}\}$ and W_{n-1} is the group of all permutations and sign changes on the set $\{e_2, e_4, ..., e_{2n-2}\}$. Thus $W((\Phi_{\alpha_{2n}})_K) = W(\Phi_{\alpha_{2n}}) \cap W(\Phi_K)$.

(b) $\mathfrak{g}$ of type B_2 (that is $\mathfrak{g}_0 = so(3,2)$).

$$\begin{aligned} &\Phi = \{\pm e_1 \pm e_2\} \cup \{\pm e_1, \pm e_2\} \\ &\Phi^+ = \{e_1 \pm e_2\} \cup \{e_1, e_2\} \\ &\Delta = \{e_1 - e_2\} \cup \{e_2\} \ . \end{aligned}$$

Since Φ^+ is a good order on Φ, $\alpha_1 = e_1 - e_2$ is noncompact and $\alpha_2 = e_2$ is compact, thus

$$\Phi_K = \{e_2\} ,$$

hence, $W(\Phi_K) = \{e, s_{e_2}\}$. Now

$$\Phi_{\alpha_1} = \{\pm (e_1 + e_2)\} ,$$

hence, $W(\Phi_{\alpha_1}) = \{e, s_{e_1+e_2}\}$. On the other hand $W((\Phi_{\alpha_1})_K) = \{e\}$, thus $W((\Phi_{\alpha_1})_K) = W(\Phi_{\alpha_1}) \cap W(\Phi_K)$ and

$$\Phi_{\alpha_2} = \{\pm e_1\} \ .$$

Hence, $W(\Phi_{\alpha_2}) = \{e, s_{e_1}\}$. On the other hand, $W((\Phi_{\alpha_2})_K) = \{e\}$, thus $W((\Phi_{\alpha_2})_K) = W(\Phi_{\alpha_2}) \cap W(\Phi_K)$.

(c) $\mathfrak{g}$ of type C_n (that is $\mathfrak{g}_0 = sp(n, \mathbb{R})$).

$$\begin{aligned} &\Phi = \{\pm e_i \pm e_j\}_{1 \leq i < j \leq n} \cup \{\pm 2e_i\}_{1 \leq i \leq n} \\ &\Phi^+ = \{e_i \pm e_j\}_{1 \leq i < j \leq n} \cup \{2e_i\}_{1 \leq i \leq n} \\ &\Delta = \{e_i - e_{i+1}\}_{1 \leq i < n} \cup \{2e_n\} \ . \end{aligned}$$

Since Φ^+ is a good order on Φ, $\alpha_i = e_i - e_{i+1}$ is compact for $1 \leq i < n$ and $\alpha_n = 2e_n$ is noncompact. Thus

$$\Phi_K = \{\pm (e_i - e_j)\}_{1 \leq i < j \leq n} \ .$$

Hence, $W(\Phi_K)$ is the group of all permutations of the set $\{e_1, ..., e_n\}$.

Since all the short simple roots are conjugate by $W(\Phi_K)$ we only need to consider Φ_{α_1} and $\Phi_{\alpha_{2n}}$. Now

$$\Phi_{\alpha_1} = \{\pm (e_1 + e_2)\} \cup \{\pm e_i \pm e_j\}_{3 \leq i < j \leq 2n} \cup \{\pm 2e_i\}_{3 \leq i \leq 2n} \ .$$

Hence Φ_{α_1} is of type $C_{n-2}\times A_1$ and $W(\Phi_{\alpha_1}) = W_{n-2}\times W_1$, where W_{n-2} is the group of all permutations and sign changes on the set $\{e_3,...,e_n\}$ and $W_1 = \{e, s_{e_1+e_2}\}$. On the other hand, $W((\Phi_{\alpha_1})_K)$ is the group of all permutations of the set $\{e_3,...,e_n\}$. Thus $W((\Phi_{\alpha_1})_K) = W(\Phi_{\alpha_1})\cap W(\Phi_K)$. Also

$$\Phi_{\alpha_n} = \{\pm e_i \pm e_j\}_{1\le i<j\le n-1} \cup \{\pm 2e_i\}_{1\le i\le n-1} \ .$$

Hence, $W(\Phi_{\alpha_n})$ is the group of all permutations and sign changes on the set $\{e_1,...,e_{n-1}\}$. On the other hand $W((\Phi_{\alpha_n})_K)$ is the group of all permutations on the set $\{e_1,...,e_{n-1}\}$. Thus $W((\Phi_{\alpha_n})_K) = W(\Phi_{\alpha_n})\cap W(\Phi_K)$.

(d) $\mathfrak{g}$ of type F_4 ($\mathfrak{g}_0 = F_I$ as in [14] Appendix C).

$$\begin{aligned}
&\Phi = \{\pm e_i \pm e_j\}_{1\le i<j\le 4} \cup \{\pm e_i\}_{1\le i\le 4} \cup \left\{\tfrac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4)\right\}\\
&\Phi^+ = \{e_i \pm e_j\}_{1\le i<j\le 4} \cup \{e_i\}_{1\le i\le 4} \cup \left\{\tfrac{1}{2}(e_1 \pm e_2 \pm e_3 \pm e_4)\right\}\\
&\Delta = \{\alpha_1,\alpha_2,\alpha_3,\alpha_4\} \text{ where}\\
&\qquad \begin{matrix} \alpha_1 = \tfrac{1}{2}(e_1-e_2-e_3-e_4) & \alpha_2 = e_4 \\ \alpha_3 = e_3-e_4 & \alpha_4 = e_2 - e_3 \end{matrix} \ .
\end{aligned}$$

Since Φ^+ is a good order on Φ, α_3, α_4 are noncompact and α_1, α_2 are compact, thus

$$\begin{aligned}
\Phi_K &= \{\alpha_1,\alpha_2,\alpha_1+\alpha_2,\alpha_3+\alpha_4,\alpha_2+\alpha_3+\alpha_4,2\alpha_2+\alpha_3+\alpha_4\}\\
&\cup\{\alpha_1+\alpha_2+\alpha_3+\alpha_4,\alpha_1+2\alpha_2+\alpha_3+\alpha_4\}\\
&\cup\{2\alpha_1+2\alpha_2+\alpha_3+\alpha_4,2\alpha_1+4\alpha_2+3\alpha_3+\alpha_4\} \ .
\end{aligned}$$

Hence Φ_K is of type $C_3\times A_1$ with simple roots

$$\begin{matrix} \beta_1 = \tfrac{1}{2}(e_1-e_2-e_3-e_4) & \beta_2 = e_4 \\ \beta_3 = e_2-e_4 & \beta_4 = e_1+e_3 \end{matrix} \ .$$

Thus $W(\Phi_K) = W_3\times W_1$, and if

$$v_1 = e_1-e_3,\quad v_2 = e_2+e_4,\quad v_3 = e_2-e_4,\quad v_4 = e_1+e_3$$

$W_1 = \{e, s_{v_4}\}$ and W_3 is the group of all permutations and sign changes on the set $\{v_1,v_2,v_3\}$.

Since all the short simple roots are conjugate by $W(\Phi_K)$ and all the long simple roots are conjugate by $W(\Phi_K)$ we only need to consider Φ_{α_2} and Φ_{α_3}. Now

$$\begin{aligned}
\Phi^+_{\alpha_2} &= \{e_i \pm e_j\}_{1\le i<j\le 3} \cup \{e_i\}_{1\le i\le 3}\\
\Delta_{\alpha_2} &= \{e_1-e_2, e_2-e_3, e_3\} \ .
\end{aligned}$$

Hence, Φ_{α_2} is of type B_3 and view on the canonical basis of $\mathbb{R}^4$, $W(\Phi_{\alpha_2})$ is the group of all permutations and sign changes on the set $\{e_1,e_2,e_3\}$ which fix $\alpha_2 = e_4$. Let us denote by (13) the permutation between e_1 and e_3 which fix e_2 and e_4, and for $j=1,2,3$ let $(-j)$ denote the reflection $e_j\to -e_j$ on e_j. Since $W(\Phi_K)(e_1+e_3) = \pm(e_1+e_3)$ we have that $W(\Phi_{\alpha_1})\cap W(\Phi_K)$ is included in the subgroup $\{e,(13)\}\times\{e,(-2)\}\times\{e,(-1)(-3)\}$. On the other hand

$$\left(\Phi^+_{\alpha_2}\right)_K = \{e_1+e_3, e_1-e_3, e_2\} \ .$$

Hence $(\Phi_{\alpha_2})_K$ is of type $A_1\times A_1\times A_1$ and $W((\Phi_{\alpha_2})_K) = \{e, s_{e_1+e_3}\}\times\{e, s_{e_1-e_3}\}\times \{e, s_{e_2}\}$. Thus $W((\Phi_{\alpha_2})_K) = W(\Phi_{\alpha_2})\cap W(\Phi_K)$.

$$\begin{aligned}
\Phi^+_{\alpha_3} &= \{e_1\pm e_2\}\cup\{e_i\}_{1\le i\le 2}\cup\{e_3+e_4\}\cup\left\{\tfrac{1}{2}(e_1\pm e_2\pm(e_3+e_4))\right\}\\
\Delta_{\alpha_3} &= \left\{e_2, \tfrac{1}{2}(e_1-e_2-e_3-e_4), e_3+e_4\right\} \ .
\end{aligned}$$

Hence Φ_{α_3} is of type C_3 and if

$$w_1 = e_1 + e_2, \quad w_2 = e_1 - e_2, \quad w_3 = e_3 + e_4 \ ,$$

Using the ordered basis $\{w_1, w_2, w_3, \alpha_3\}$ of $\mathbb{R}^4$, $W(\Phi_{\alpha_3})$ is the group of all permutations and sign changes of the set $\{w_1, w_2, w_3\}$ and fix α_3. Since $W(\Phi_K)(e_1+e_3) = \pm(e_1+e_3)$ and $e_1+e_3 = \frac{1}{2}(w_1+w_2+w_3+\alpha_4)$, $W(\Phi_{\alpha_1}) \cap W(\Phi_K)$ is included in the subgroup of all permutations of the set $\{w_1, w_2, w_3\}$. On the other hand

$$\left(\Phi^+_{\alpha_3}\right)_K = \left\{e_2, \frac{1}{2}(e_1+e_2-e_3-e_4), \frac{1}{2}(e_1-e_2-e_3-e_4)\right\} \ .$$

Hence $(\Phi_{\alpha_3})_K$ is of type A_2 and

$$\begin{aligned} W((\Phi_{\alpha_3})_K) &= \left\{e, s_{e_2}, s_{\frac{1}{2}(e_1+e_2-e_3-e_4)}, s_{\frac{1}{2}(e_1-e_2-e_3-e_4)}\right\} \\ &\cup \left\{s_{e_2}s_{\frac{1}{2}(e_1+e_2-e_3-e_4)}, s_{e_2}s_{\frac{1}{2}(e_1-e_2-e_3-e_4)}\right\} \ . \end{aligned}$$

Thus $W((\Phi_{\alpha_3})_K) = W(\Phi_{\alpha_3}) \cap W(\Phi_K)$, which concludes the proof of Theorem 3.5. ∎

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