Kazhdan and Haagerup Properties in algebraic groups over local fields

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Abstract. Given a Lie algebra \( \mathfrak{s} \), we call Lie \( \mathfrak{s} \)-algebra a Lie algebra endowed with a reductive action of \( \mathfrak{s} \). We characterize the minimal \( \mathfrak{s} \)-Lie algebras with a nontrivial action of \( \mathfrak{s} \), in terms of irreducible representations of \( \mathfrak{s} \) and invariant alternating forms.

As a first application, we show that if \( \mathfrak{g} \) is a Lie algebra over a field of characteristic zero whose amenable radical is not a direct factor, then \( \mathfrak{g} \) contains a subalgebra which is isomorphic to the semidirect product of \( \mathfrak{sl}_2 \) by either a nontrivial irreducible representation or a Heisenberg group (this was essentially due to Cowling, Dorofaeff, Seeger, and Wright). As a corollary, if \( G \) is an algebraic group over a local field \( K \) of characteristic zero, and if its amenable radical is not, up to isogeny, a direct factor, then \( G(K) \) has Property (T) relative to a noncompact subgroup. In particular, \( G(K) \) does not have Haagerup’s property. This extends a similar result of Cherix, Cowling and Valette for connected Lie groups, to which our method also applies.

We give some other applications. We provide a characterization of connected Lie groups all of whose countable subgroups have Haagerup’s property. We give an example of an arithmetic lattice in a connected Lie group which does not have Haagerup’s property, but has no infinite subgroup with relative Property (T). We also give a continuous family of pairwise non-isomorphic connected Lie groups with Property (T), with pairwise non-isomorphic (resp. isomorphic) Lie algebras.

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1. Introduction

In the sequel, all Lie algebras are finite-dimensional over a field of characteristic zero, denoted by \( K \), or \( K \) when it is a local field. If \( \mathfrak{g} \) is a Lie algebra, denote by \( \text{rad}(\mathfrak{g}) \) its radical and \( Z(\mathfrak{g}) \) its centre, \( D\mathfrak{g} \) its derived subalgebra, and \( \text{Der}(\mathfrak{g}) \) the Lie algebra of all derivations of \( \mathfrak{g} \). If \( \mathfrak{h}_1, \mathfrak{h}_2 \) are Lie subalgebras of \( \mathfrak{g} \), \( [\mathfrak{h}_1, \mathfrak{h}_2] \) denotes the Lie subalgebra generated by the brackets \( [h_1, h_2], (h_1, h_2) \in \mathfrak{h}_1 \times \mathfrak{h}_2 \).
Let \( g \) be a Lie algebra with radical \( \text{rad}(g) = r \) and semisimple Levi factor \( s \) (so that \( g \cong s \ltimes r \)). We focus here on aspects of \( g \) related to the action of \( s \). This suggests the following definitions.

If \( s \) is a Lie algebra, we define a \textit{Lie \( s \)-algebra} to be a Lie algebra \( n \) endowed with a morphism \( i : s \rightarrow \text{Der}(n) \), defining a \textit{completely reducible} action of \( s \) on \( n \). (This latter technical condition is empty if \( s \) is semisimple.)

A Lie \( s \)-algebra naturally embeds in the semidirect product \( s \ltimes n \), so that we write \( i(s)(n) = [s,n] \) for \( s \in s \), \( n \in n \).

By the \textit{trivial irreducible module} of \( s \) we mean a one-dimensional space vector space endowed with a trivial action of \( s \). We say that a module (over a Lie algebra or over a group) is \textit{full} if is is completely reducible and does not contain the trivial irreducible module.

**Definition 1.1.** Let \( s \) be a Lie algebra. We say that a Lie \( s \)-algebra \( n \) is \textit{minimal} if \([s,n] \neq 0\), and for every \( s \)-subalgebra \( n' \) of \( n \), either \( n' = n \) or \([s,n'] = 0\).

It is clear that a Lie \( s \)-algebra \( n \) satisfying \([s,n] \neq 0\) contains a minimal \( s \)-subalgebra. We begin by a characterization of minimal \( s \)-algebras:

**Theorem 1.2.** Let \( s \) be a Lie algebra. A solvable Lie \( s \)-algebra \( n \) is minimal if and only if it satisfies the following conditions 1), 2), 3), and 4):

1) \( n \) is 2-nilpotent (that is, \([n,Dn] = 0\)).
2) \([s,n] = n\).
3) \([s,Dn] = 0\).
4) \( n/Dn \) is irreducible as a \( s \)-module.

**Definition 1.3.** We call a solvable Lie \( s \)-algebra \( n \) \textit{almost minimal} if it satisfies conditions 1), 2), and 3) of Theorem 1.2.

This definition has the advantage to be invariant under field extensions. Note that an almost minimal solvable Lie \( s \)-algebra \( n \) satisfies Condition 4'): \( n/Dn \) is a full \( s \)-module.

The classification of (almost) minimal solvable Lie \( s \)-algebras can be deduced from the classification of irreducible \( s \)-modules. Let \( v \) be a full \( s \)-module (equivalently, an abelian Lie \( s \)-algebra satisfying \([s,v] = v\)). Recall that a bilinear form \( \varphi \) on \( v \) is called \( s \)-invariant if it satisfies \( \varphi([s,v],w) + \varphi(v,[s,w]) = 0 \) for all \( s \in s \), \( v,w \in v \). Let \( \text{Bil}_s(v) \) (resp. \( \text{Alt}_s(v) \)) denote the space of all \( s \)-invariant bilinear (resp. alternating bilinear) forms on \( v \).

**Definition 1.4.** We define the Lie \( s \)-algebra \( h(v) \) as follows: as a vector space, \( h(v) = v \oplus \text{Alt}_s(v)^* \); it is endowed with the following bracket:

\[
[(x,z),(x',z')] = (0,e_{x,x'}) \quad x,x' \in v \quad z,z' \in \text{Alt}_s(v)^*
\]

where \( e_{x,x'} \in \text{Alt}_s(v)^* \) is defined by \( e_{x,x'}(\varphi) = \varphi(x,x') \).

This is a 2-nilpotent Lie \( s \)-algebra under the action \([s,(x,z)] = ([s,x],0)\), which is almost minimal. Other almost minimal Lie \( s \)-algebras can be obtained by taking the quotient by a linear subspace of the centre. The following theorem states that this is the only way to construct almost minimal solvable Lie \( s \)-algebras.
Theorem 1.5. If \( n \) is an almost minimal solvable Lie \( s \)-algebra, then it is isomorphic (as a \( s \)-algebra) to \( h(v)/Z \), for some full \( s \)-module \( v \) and some subspace \( Z \) of \( \operatorname{Alt}_s(v)^* \). It is minimal if and only if \( v \) is irreducible.

Moreover, the almost minimal \( s \)-algebras \( h(v)/Z \) and \( h(v)/Z' \) are isomorphic if and only if \( Z' \) and \( Z \) are in the same orbit for the natural action of \( \operatorname{Aut}_s(v) \) on the Grassmannian of \( \operatorname{Alt}_s(v)^* \).

Remark 1.6. If \( s \) is semisimple, \( s \ltimes h(v) \) is the universal central extension of the perfect Lie algebra \( s \ltimes v \).

The case of \( sl_2 \) is essential, and there is a simple description for it. Recall that if \( s = sl_2 \), then, up to isomorphism, there exists exactly one irreducible \( s \)-module \( v_n \) of dimension \( n \) for every \( n \geq 1 \). If \( n = 2m \) is even, it has a central extension by a one-dimensional subspace, giving a Heisenberg Lie algebra, on which \( sl_2 \) acts naturally (see 2. for details), denoted by \( h_{2m+1} \). Theorem 1.5 thus reduces as:

Proposition 1.7. Up to isomorphism, the minimal solvable Lie \( sl_2 \)-algebras are \( v_n \) and \( h_{2n-1} \) (\( n \geq 2 \)).

Let \( g \) be a Lie algebra, \( r \) its radical and \( s \) a semisimple factor. Write \( s = s_c \oplus s_{nc} \) by separating \( K \)-anisotropic and \( K \)-isotropic factors\(^1\). The ideal \( s_c \ltimes r \) is sometimes called the amenable radical of \( g \).

Definition 1.8. We call \( g \) M-decomposed if \( [s_{nc}, r] = 0 \). Equivalently, \( g \) is M-decomposed if the amenable radical is a direct factor of \( g \).

Proposition 1.9. Let \( g \) be a Lie algebra, and keep notation as above. Suppose that \( g \) is not M-decomposed. Then there exists a Lie subalgebra \( h \) of \( g \) which is isomorphic to \( sl_2 \ltimes v_n \) or \( sl_2 \ltimes h_{2n-1} \) for some \( n \geq 2 \).

This result is essentially due to Cowling, Dorofaeff, Seeger and Wright [6], where it is not explicitly stated, but it is actually proved in the proof of Proposition 8.2 there (under the assumption \( K = \mathbb{R} \), but their argument generalizes to any field of characteristic zero). This was a starting point for the present paper.

Let \( G \) be a locally compact, \( \sigma \)-compact group. Recall that \( G \) has the Haagerup Property if it has a metrically proper isometric action on a Hilbert space; in contrast, \( G \) has Kazhdan’s Property (T) if every isometric action of \( G \) on a Hilbert space has a fixed point. See 4. for a short reminder about Haagerup and Kazhdan Properties.

We provide corresponding statements for Proposition 1.9 in the realm of algebraic groups and connected Lie groups. As a consequence, we get the following theorem, which was the initial motivation for the results above. It was already proved, in a different way, for connected Lie groups by Cherix, Cowling and Valette [5, Chap. 4].

\(^1\) \( c \) and \( nc \) respectively stand for “non-compact” and “compact”; this is related to the fact that if \( S \) is a simple algebraic group defined over the local field \( K \), then its Lie algebra is \( K \)-isotropic if and only if \( S(K) \) is not compact.
Theorem 1.10. Let $G$ be either a connected Lie group, or $G = G(K)$, where $G$ is a linear algebraic group over the local field $K$ of characteristic zero. Let $\mathfrak{g}$ be its Lie algebra. The following are equivalent.

(i) $G$ has Haagerup’s property.

(ii) For every noncompact closed subgroup $H$ of $G$, $(G, H)$ does not have relative Property (T).

(iii) The following conditions are satisfied:

- $\mathfrak{g}$ is M-decomposed.
- All simple factors of $\mathfrak{g}$ have $K$-rank $\leq 1$.
- (in the case of Lie groups or when $K = \mathbb{R}$) No simple factor of $\mathfrak{g}$ is isomorphic to $\mathfrak{sp}(n, 1)$ ($n \geq 2$) or $\mathfrak{f}_4(-20)$.

(iv) $\mathfrak{g}$ contains no isomorphic copy of any one of the following Lie algebras

- $\mathfrak{sl}_2 \ltimes \mathfrak{v}_n$ or $\mathfrak{sl}_2 \ltimes \mathfrak{h}_{2n-1}$ for some $n \geq 2$,
- (in the case of Lie groups or when $K = \mathbb{R}$) $\mathfrak{sp}(2, 1)$.

Remark 1.11. The notion of M-decomposed (real) Lie algebras also appears in other contexts: heat kernel on Lie groups [12], Rapid Decay Property [7], weak amenability [6].

We derive some other results with the help of Theorem 1.5.

Proposition 1.12. There exists a continuous family $(\mathfrak{g}_t)$ of pairwise non-isomorphic real (or complex) Lie algebras satisfying the following properties:

(i) $\mathfrak{g}_t$ is perfect, and

(ii) the simply connected Lie group corresponding to $\mathfrak{g}_t$ has Property (T).

Note that Proposition 1.12 with only (i) may be of independent interest; we do not know if it had already been observed. On the other hand, it is well-known that there exist continuously many pairwise non-isomorphic complex $n$-dimensional nilpotent Lie algebras if $n \geq 7$.

Proposition 1.13. There exists a continuous family of pairwise non-isomorphic connected Lie groups with Property (T), and with isomorphic Lie algebras.

We also give the classification, when $K = \mathbb{R}$, of the minimal $\mathfrak{so}_3$-algebras (Proposition 2.3). We use it to prove (ii)$\Rightarrow$(i) in the following result (while the reverse implication is essentially due to Guentner, Higson and Weinberger [9, Theorem 5.1]).

Theorem 1.14. Let $G$ be a connected Lie group. Then the following are equivalent:

(i) $G$ is locally isomorphic to $\mathrm{SO}_3(\mathbb{R})^a \times \mathrm{SL}_2(\mathbb{R})^b \times \mathrm{SL}_2(\mathbb{C})^c \times \mathbb{R}$, for a solvable Lie group $R$ and integers $a, b, c$.

(ii) Every countable subgroup of $G$ has Haagerup’s property (when endowed with the discrete topology).
Remark 1.15.  Assertion (i) of Theorem 1.14 is equivalent to: (ii') The complexification $\mathfrak{g}_C$ of $\mathfrak{g}$ is an M-decomposed complex Lie algebra, and its semisimple part is isomorphic to $\mathfrak{s}_2(C)^n$ for some $n$.

For instance, $\text{SO}_3(\mathbb{R}) \ltimes \mathbb{R}^3$ has a countable subgroup which does not have Haagerup’s property. An explicit example is given by $\text{SO}_3(\mathbb{Z}[1/p]) \ltimes \mathbb{Z}[1/p]^3$. It can also be shown that this group has no infinite subgroup with relative Property (T). This answers an open question in [5, Section 7.1], and is in contrast with Theorem 1.10. This group is not finitely presented (this is a consequence of [1, Theorem 2.6.4]); we give a similar example in Remark 4.10 which is, in addition, finitely presented.

2. Lie algebras

2.1. Minimal subalgebras.

Proposition 2.1. Let $\mathfrak{n}$ be a solvable Lie $\mathfrak{s}$-algebra.

1) The Lie $\mathfrak{s}$-subalgebra $[\mathfrak{s}, \mathfrak{n}]$ is an ideal in $\mathfrak{n}$ (and also in $\mathfrak{s} \ltimes \mathfrak{n}$), and $[\mathfrak{s}, [\mathfrak{s}, \mathfrak{n}]] = [\mathfrak{s}, \mathfrak{n}]$.

2) If, moreover, $[\mathfrak{s}, D\mathfrak{n}] = 0$, then $[\mathfrak{s}, \mathfrak{n}]$ is an almost minimal Lie algebra (see Definition 1.3).

Proof. 1) Let $\mathfrak{v}$ be the subspace generated by the brackets $[\mathfrak{s}, \mathfrak{n}]$, $(\mathfrak{s}, \mathfrak{n}) \in \mathfrak{s} \times \mathfrak{n}$. Since the action of $\mathfrak{s}$ is completely reducible (see the definition of Lie $\mathfrak{s}$-algebra), it is immediate that $[\mathfrak{s}, \mathfrak{n}]$ and $[\mathfrak{s}, [\mathfrak{s}, \mathfrak{n}]]$ both coincide with the Lie subalgebra generated by $\mathfrak{v}$. Then, using Jacobi identity,

$$[\mathfrak{n}, [\mathfrak{s}, \mathfrak{n}]] = [\mathfrak{n}, [\mathfrak{s}, \mathfrak{n}]] \subseteq [\mathfrak{s}, [\mathfrak{n}, [\mathfrak{s}, \mathfrak{n}]]] + [[\mathfrak{s}, \mathfrak{n}], [\mathfrak{s}, \mathfrak{n}]] \subseteq [\mathfrak{s}, [\mathfrak{n}, \mathfrak{n}]] + [\mathfrak{s}, \mathfrak{n}] \subseteq [\mathfrak{s}, \mathfrak{n}].$$

2) Let $\mathfrak{z}$ be the linear subspace generated by the commutators $[\mathfrak{v}, \mathfrak{w}]$, $\mathfrak{v}, \mathfrak{w} \in \mathfrak{v}$. By Jacobi identity,

$$[\mathfrak{v}, \mathfrak{z}] = [[\mathfrak{v}, \mathfrak{v}], \mathfrak{z}] \subseteq [[\mathfrak{z}, \mathfrak{z}], \mathfrak{v}] + [\mathfrak{s}, [\mathfrak{v}, \mathfrak{z}]] \subseteq [[\mathfrak{s}, D\mathfrak{n}], \mathfrak{v}] + [\mathfrak{s}, D\mathfrak{n}] = 0.$$

Thus, the subspace $\mathfrak{n}' = \mathfrak{v} \oplus \mathfrak{z}$ is a 2-nilpotent Lie $\mathfrak{s}$-subalgebra of $\mathfrak{n}$. The Lie subalgebra $[\mathfrak{s}, \mathfrak{n}']$ contains $\mathfrak{v}$, hence also contains $\mathfrak{z}$, so $[\mathfrak{s}, \mathfrak{n}']$ is equal to $\mathfrak{n}'$. Thus Conditions 1) and 2) of Definition 1.3 are satisfied, while Condition 3) follows immediately from the hypothesis $[\mathfrak{s}, D\mathfrak{n}] = 0$.

Proof of Theorem 1.2. Suppose (ii). Condition 4 implies $\mathfrak{n} \neq 0$. Then Condition 2 implies $[\mathfrak{s}, \mathfrak{n}] = \mathfrak{n} \neq 0$. Let $\mathfrak{n}' \subseteq \mathfrak{n}$ be a $\mathfrak{s}$-subalgebra. Then, by irreducibility (Condition 4), either $D\mathfrak{n} + \mathfrak{n}' = D\mathfrak{n}$ or $D\mathfrak{n} + \mathfrak{n}' = \mathfrak{n}$. In the first case, $\mathfrak{n}'$ centralizes $\mathfrak{s}$. In the second case, $\mathfrak{n} = [\mathfrak{s}, \mathfrak{n}] = [\mathfrak{s}, \mathfrak{n}' + D\mathfrak{n}] = [\mathfrak{s}, \mathfrak{n}'] \subseteq \mathfrak{n}'$, using Conditions 1 and 2, and the fact that $\mathfrak{n}'$ is a $\mathfrak{s}$-subalgebra.

Conversely, suppose that (i) holds. Since $\mathfrak{n}$ is solvable, $D\mathfrak{n}$ is a proper $\mathfrak{s}$-subalgebra, so that, by minimality, $[\mathfrak{s}, D\mathfrak{n}] = 0$. By Proposition 2.1, $[\mathfrak{s}, \mathfrak{n}]$ is a nonzero almost minimal Lie $\mathfrak{s}$-subalgebra of $\mathfrak{n}$, hence satisfies 1), 2), 3). The minimality implies that 4) is also satisfied.
Proof of Proposition 1.9. Let \( s \) be an almost minimal solvable Lie \( s \)-algebra. Let \( v \) be the subspace generated by the brackets \([s, n]\), \((s, n) \in s \times n\). Since \( n \) is almost minimal, \( v \) is a complementary subspace of \( Dn \), and is a full \( s \)-module. If \( u \in Dn^\ast\), consider the alternating bilinear form \( \phi_u \) on \( v \) defined by \( \phi_u(x, y) = u([x, y]) \). This defines a mapping \( Dn^\ast \to \text{Alt}_s(v) \) which is immediately seen to be injective. By duality, this defines a surjective linear map \( \text{Alt}_s(v)^\ast \to Dn \), whose kernel we denote by \( Z \). It is immediate from the definition of \( \mathfrak{h}(v) \) that this map extends to a surjective morphism of Lie \( s \)-algebras \( \mathfrak{h}(v) \to n \) with kernel \( Z \). This proves that \( n \) is isomorphic to \( \mathfrak{h}(v)/Z \).

The second assertion is immediate.

The third assertion follows from the proof of the first one, where we made no choice. Namely, take an isomorphism \( \psi: \mathfrak{h}(v)/Z \to \mathfrak{h}(v)/Z' \). It gives by restriction an \( s \)-automorphism \( \varphi \) of \( v \), which induces a unique automorphism \( \hat{\varphi} \) of \( \mathfrak{h}(v) \). Let \( p \) and \( p' \) denote the natural projections in the following diagram of Lie \( s \)-algebras:

\[
\begin{array}{ccc}
\mathfrak{h}(v) & \xrightarrow{p} & \mathfrak{h}(v)/Z \\
\hat{\varphi} \downarrow & & \downarrow \psi \\
\mathfrak{h}(v) & \xrightarrow{p'} & \mathfrak{h}(v)/Z'
\end{array}
\]

This diagram is commutative: indeed, \( p' \circ \hat{\varphi} = \psi \circ p \) coincide in restriction to \( v \), and \( v \) generates \( \mathfrak{h}(v) \) as a Lie algebra. This implies \( Z = \text{Ker}(\psi \circ p) = \text{Ker}(p' \circ \hat{\varphi}) = \hat{\varphi}^{-1}(Z') \).

\[\square\]

2.2. The example \( \mathfrak{sl}_2 \).

If \( s = \mathfrak{sl}_2(K) \), then, up to isomorphism, there exists exactly one irreducible \( s \)-module \( v_n \) of dimension \( n \) for every \( n \geq 1 \).

Since \( v_n \) is absolutely irreducible for all \( n \), by Schur’s Lemma, \( \text{Bil}_s(v_n) \) is at most one dimensional for all \( n \). In fact, it is one-dimensional. Indeed, take the usual basis \((H, X, Y)\) of \( \mathfrak{sl}_2 \) satisfying \([H, X] = 2X\), \([H, Y] = -2Y\), \([X, Y] = H\), and take the basis \((e_0, \ldots, e_{n-1})\) of \( v_n \) so that \( H.e_i = (n - 1 - 2i)e_i \), \( X.e_i = (n - i)e_{i-1} \), and \( Y.e_i = (i + 1)e_{i+1} \), with the convention \( e_{-1} = e_n = 0 \). Then \( \text{Bil}_s(v_n) \) is generated by the form \( \varphi_n \) defined by

\[
\varphi_n(e_i, e_{n-1-i}) = (-1)^i \binom{i}{n-1}; \quad \varphi(e_i, e_j) = 0 \text{ if } i + j \neq n - 1.
\]

For odd \( n \), \( \varphi_n \) is symmetric so that \( \text{Alt}_s(v_n) = 0 \); for even \( n \), \( \varphi_n \) is symplectic and generates \( \text{Alt}_s(v_n) \). For even \( n \), denote by \( \mathfrak{h}_{n+1} \) the one-dimensional central extension \( \mathfrak{h}(v_n) \), well-known as the \((n + 1)\)-dimensional Heisenberg Lie algebra.

Proof of Proposition 1.9. Since \( s_{nc} \) is semisimple and isotropic, it is generated by its subalgebras \( K \)-isomorphic to \( \mathfrak{sl}_2 \). Since \([s_{nc}, t] \neq 0\), this implies that there exists some subalgebra \( s' \) of \( s_{nc} \) which is \( K \)-isomorphic to \( \mathfrak{sl}_2 \) and such that \([s', t] \neq 0\). Then the result is clear from Proposition 1.7. Notice that the proof gives the following slight refinement: \( \mathfrak{h} \) can be chosen so that \( \text{rad}(\mathfrak{h}) \subseteq \text{rad}(\mathfrak{g}) \). \[\square\]
2.3. The example $\mathfrak{so}_3$.

We now study a more specific example. Let us deal with the field $\mathbb{R}$ of real numbers, and with $\mathfrak{s} = \mathfrak{so}_3$.

Since the complexification of $\mathfrak{so}_3$ is isomorphic to $\mathfrak{sl}_2(\mathbb{C})$, the irreducible complex $\mathfrak{s}$-modules make up a family $(\mathfrak{u}_n)$ ($n \geq 1$); $\dim(\mathfrak{u}_n) = n$, which are the symmetric powers of the natural action of $\mathfrak{su}_2 = \mathfrak{so}_3$ on $\mathbb{C}^2$.

If $n = 2m + 1$ is odd, then this is the complexification of a real $\mathfrak{so}_3$-module $\mathfrak{d}^{R}_{2m+1}$ (of dimension $n$). If $n = 2m$ is even, $\mathfrak{d}_n$ is irreducible as a $4m$-dimensional real $\mathfrak{so}_3$-module, we call it $\mathfrak{u}_{4m}$.

These two families $(\mathfrak{d}^{R}_{2n+1})$ and $(\mathfrak{u}_{4n})$ make up all irreducible real $\mathfrak{so}_3$-modules.

Proposition 2.2. The irreducible real $\mathfrak{so}_3$-modules make up two families: a family $(\mathfrak{d}^{R}_{2n+1})$ of $(2n + 1)$-dimensional modules ($n \geq 0$), absolutely irreducible, and a family $(\mathfrak{u}_{4n})$ of $4n$-dimensional modules ($n \geq 1$), not absolutely irreducible, preserving a quaternionic structure. 

Since $(\mathfrak{d}^{R}_{2n+1})$ is absolutely irreducible, the space of invariant bilinear forms on $(\mathfrak{d}^{R}_{2n+1})$ is generated by a scalar product, so that $\text{Alt}_{\mathfrak{so}_3}(\mathfrak{d}^{R}_{2n+1}) = 0$.

On the other hand, $\text{Alt}_{\mathfrak{so}_3}(\mathfrak{u}_{4n})$ is three-dimensional, and is given by the imaginary part of an invariant quaternionic hermitian form.

In order to classify the minimal solvable $\mathfrak{so}_3$-algebras, we must determine the orbits of the natural action of $\text{Aut}_{\mathfrak{so}_3}(\mathfrak{u}_{4n})$ on $\text{Alt}_{\mathfrak{so}_3}(\mathfrak{u}_{4n})$. It is a standard fact that $\text{Aut}_{\mathfrak{so}_3}(\mathfrak{u}_{4n})$ is isomorphic to the group of nonzero quaternions, that $\text{Alt}_{\mathfrak{so}_3}(\mathfrak{u}_{4n})$ naturally identifies with the set of imaginary quaternions, and that the action of $\text{Aut}_{\mathfrak{so}_3}(\mathfrak{u}_{4n})$ on $\text{Alt}_{\mathfrak{so}_3}(\mathfrak{u}_{4n})$ is given by conjugation of quaternions. This implies that it acts transitively on each component of the Grassmannian.

For $i = 0, 1, 2, 3$, let $Z_i$ be a fixed $(3 - i)$-dimensional linear subspace of $\text{Alt}_{\mathfrak{s}}(\mathfrak{d}^*)$. Denote by $\mathfrak{h}_i^{0, 4n}$ the minimal Lie $\mathfrak{so}_3$-algebra $\mathfrak{h}(\mathfrak{u}_{4n})/Z_i$; of course, $\mathfrak{h}_i^{0, 4n} = \mathfrak{u}_{4n}$ and $\mathfrak{h}_i^{3, 4n} = \mathfrak{h}(\mathfrak{u}_{4n})$.

Proposition 2.3. Up to isomorphism, the minimal solvable Lie $\mathfrak{so}_3(\mathbb{R})$-algebras are $\mathfrak{d}^{R}_{2n+1}$ ($n \geq 1$) and $\mathfrak{h}_i^{0, 4n}$ ($n \geq 1$, $i = 0, 1, 2, 3$).

There is an analogous statement to Proposition 1.9.

Proposition 2.4. Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{R}$. Suppose that $[\mathfrak{s}_c, \mathfrak{r}] \neq 1$. Then $\mathfrak{g}$ has a Lie subalgebra which is isomorphic to either $\mathfrak{so}_3 \times \mathfrak{d}^{R}_{2n+1}$ or $\mathfrak{so}_3 \times \mathfrak{h}_i^{0, 4n}$ for some $i = 0, 1, 2, 3$ and some $n \geq 1$.

3. Corresponding results for algebraic groups and connected Lie groups

3.1. Minimal algebraic subgroups.

We now give the corresponding statements and results for algebraic groups.

Recall that the Lie algebra functor gives an equivalence of categories between the category of unipotent $K$-groups and the category of nilpotent Lie $K$-algebras. If $S$ is semisimple and simply connected with Lie algebra $\mathfrak{g}$, it induces an equivalence of categories between the category of unipotent $K$-groups and the category of nilpotent Lie $S$-algebras over $K$. If $S$ is not simply connected (in particular, if $S$ is not semisimple), this is no longer an essentially surjective functor, but it remains fully faithful.

A minimal (resp. almost minimal) solvable $S$-group $N$ is defined similarly as in the case of Lie algebras; it is automatically unipotent (since it satisfies $[S,N] = N$). Moreover, $N$ is a minimal (resp. almost minimal) solvable $K$-$S$-group if and only if its Lie algebra $\mathfrak{n}$ is a minimal (resp. almost minimal) solvable Lie $\mathfrak{s}$-algebra. Proposition 2.1 and Theorem 1.2 also immediately carry over into the context of algebraic groups.

If $S$ is reductive and $V$ is a $K$-$S$-module, we define the unipotent $K$-$S$-group $H(V)$ as follows: as a variety, $H(V) = V \oplus \text{Alt}_S(V)^*$; it is endowed with the following group law:

$$((x,z))(x',z') = (x+x', z+z' + e_{x,x'}) \quad x,x' \in V \quad z,z' \in \text{Alt}_S(V)^*$$

where $e_{x,x'} \in \text{Alt}_S(V)^*$ is defined by $e_{x,x'}(\varphi) = \varphi(x,x')$. This is a $K$-$S$-group under the action $s.(x,z) = (s.x,z)$. It is clear that its Lie algebra is isomorphic as a Lie $K$-$S$-algebra to $\mathfrak{h}(\mathfrak{v})$, where $V = \mathfrak{v}$ viewed as a $\mathfrak{s}$-module. Here is the analog of Theorem 1.5.

**Theorem 3.1.** If $N$ is an almost minimal solvable $K$-$S$-group, then it is isomorphic (as a $K$-$S$-group) to $H(V)/Z$, for some full $K$-$S$-module $V$ and some $K$-subspace $Z$ of $\text{Alt}_S(V)^*$. It is minimal if and only if $V$ is irreducible.

Moreover, the almost minimal $K$-$S$-groups $H(V)/Z$ and $H(V)/Z'$ are isomorphic if and only if $Z'$ and $Z$ are in the same orbit for the natural action of $\text{Aut}_S(V)$ on the Grassmannian of $\text{Alt}_S(V)^*$.

### 3.2. The example $\text{SL}_2$.  

The simply connected $K$-group with Lie algebra $\mathfrak{sl}_2$ is $\text{SL}_2$. Denote by $V_n$ and $H_{2n-1}$ the $\text{SL}_2$-groups corresponding to $\mathfrak{v}_n$ and $\mathfrak{h}_{2n-1}$. These are the solvable minimal $\text{SL}_2$-groups over $K$. The only non-simply connected $K$-group with Lie algebra $\mathfrak{sl}_2$ is the adjoint group $\text{PGL}_2$; thus the minimal solvable $\text{PGL}_2$-groups over $K$ are $V_{2n-1}$ for $n \geq 2$.

**Remark 3.2.** It is convenient, in algebraic groups, to deal with the unipotent radical rather than with the radical. It is straightforward to see that a reductive subgroup $S$ of a linear algebraic group centralizes the radical if and only if it centralizes the unipotent radical. Indeed, suppose $[S,R_u] = 1$. We always have $[S,R/R_u] = 1$ since $R/R_u$ is central in $G^0/R_u$ and $S$ is connected ($G^0$ denoting the unit component of $G$). This easily implies that $S$ acts trivially.

Write, for $s \in S$ and $r \in R$, $s.r = ru(s,r)$, where $u(s,r) \in R_u$ and $u(s,r) = 1$ if $r \in R_u$. Then, for all $s,t \in S$ and $r \in R$ $st.r = sru(t,r) = (s.r)(s.u(t,r)) = ru(s,r)u(t,r)$, so that $u(st,r) = u(s,r)u(t,r)$. This implies that if $s \in D^nS = S$, then $u(s,r) \in D^nR_u$. Taking $n$ sufficiently large, we obtain $u(s,r) = 1$ for all $s \in S$ and $r \in R$, that is, $[S,R] = 1$. 

\[\text{Note: }\]
Let $G$ be a linear algebraic group over the field $K$ of characteristic zero, $R$ its radical, $S$ a Levi factor, decomposed as $S_{nc}S_c$ by separating $K$-isotropic and $K$-anisotropic factors.

**Proposition 3.3.** Suppose that $[S_{nc}, R] \neq 1$. Then $G$ has a $K$-subgroup which is $K$-isomorphic to either $\text{SL}_2 \ltimes V_n$, $\text{PGL}_2 \ltimes V_{2n-1}$, or $\text{SL}_2 \ltimes H_{2n-1}$ for some $n \geq 2$.

Let us mention the translation into the context of connected Lie groups, which is immediate from the Lie algebraic version.

**Proposition 3.4.** Let $G$ be a real Lie group. Suppose that $[S_{nc}, R] \neq 1$. Then there exists a Lie subgroup $H$ of $G$ which is locally isomorphic to $\text{SL}_2(R) \ltimes V_n(R)$ or $\text{SL}_2(R) \ltimes H_{2n-1}(R)$ for some $n \geq 2$.

**Remark 3.5.**
1) An analogous result holds with complex Lie groups.
2) The Lie subgroup $H$ is not necessarily closed; this is due to the fact that $\text{SL}_2(R)$ and $H_{2n-1}(R)$ have noncompact centre. For instance, take an element $z$ of the centre of $H$ that generates an infinite discrete subgroup, and take the image of $H$ in the quotient of $H \times R/Z$ by $(z, \alpha)$, where $\alpha$ is irrational.
3) It can be easily be shown that, if the Lie group $G$ is linear, then the subgroup $H$ is necessarily closed. In a few words, this is because the derived subgroup of the radical is unipotent, hence simply connected, and the centre of the semisimple part is finite.

### 3.3. The example $\text{SO}_3$.

We go on with the notation of Proposition 2.2.

In the context of algebraic $R$-groups as in the context of connected Lie groups, the simply connected group corresponding to $\mathfrak{so}_3(R)$ is $\text{SU}(2)$. The only non-simply connected corresponding group is $\text{SO}_3(R)$.

The irreducible $\text{SU}(2)$-modules corresponding to $\mathfrak{d}^R_{2n+1}$ and $\mathfrak{u}_{4n}$ are denoted by $D^R_{2n+1}$ and $U_n$. Among those, only $D^R_{2n+1}$ provide $\text{SO}_3(R)$-modules.

Denote by $HU_{4n}^i$ the unipotent $R$-group corresponding to $\mathfrak{hu}_{4n}^i$, $i = 0, 1, 2, 3$.

**Remark 3.6.** It can be shown that the maximal unipotent subgroups of $\text{Sp}(n, 1)$ are isomorphic to $HU_{4n}^3$.

**Proposition 3.7.** Up to isomorphism, the minimal solvable $\text{SO}_3(R)$-algebras are $D^R_{2n+1}$ for $n \geq 1$; the other minimal solvable Lie $\text{SU}(2)$-algebras are $HU_{4n}^i$, for $n \geq 1$, $i = 0, 1, 2, 3$.

**Proposition 3.8.** Let $G$ be a linear algebraic $R$-group. Suppose that $[S_c, R] \neq 1$. Then $G$ has a $R$-subgroup which is $R$-isomorphic to either $\text{SU}(2) \ltimes D^R_{2n+1}$, $\text{SO}_3(R) \ltimes D^R_{2n+1}$, or $\text{SU}(2) \ltimes HU_{4n}^i$ for some $i = 0, 1, 2, 3$ and some $n \geq 1$.

Let $G$ be a real Lie group. Suppose that $[S_c, R] \neq 1$. Then $G$ has a Lie subgroup which is locally isomorphic to either $\text{SU}(2) \ltimes D^R_{2n+1}$ or $\text{SU}(2) \ltimes HU_{4n}^i$ for some $i = 0, 1, 2, 3$ and some $n \geq 1$. 


4. Application to Haagerup and Kazhdan Properties

4.1. Reminder.

Recall [5, Chap. 1] that a locally compact, \( \sigma \)-compact group \( G \) has the **Haagerup Property** if there exists a metrically proper, isometric action of \( G \) on some affine Hilbert space.

If \( H \) is a subgroup of \( G \), the pair \( (G, H) \) has **Kazhdan Property (T)**, or that \( H \) has Kazhdan’s Property (T) relatively to \( G \), if every isometric action of \( G \) on any affine Hilbert space has a fixed point in restriction to \( H \). In the case when \( H = G \), \( G \) is said to have Property (T) (see [10] or [2]).

As an immediate consequence of these definitions, if \( (G, H) \) has Property (T) and \( H \) is not relatively compact in \( G \), then \( G \) does not have the Haagerup Property; this is a frequent obstruction to Haagerup Property, although it is not the only one (see Remark 4.10).

The class of groups with the Haagerup Property generalizes the class of amenable groups as a strong negation of Kazhdan’s Property (T). For other motivations of the Haagerup Property and equivalent definitions, see [5].

In the following lemma, we summarize the hereditary properties of the Haagerup and Kazhdan Properties that we will use in the sequel.

**Lemma 4.1.** The Haagerup Property for locally compact, \( \sigma \)-compact groups is closed under taking (H1) closed subgroups, (H2) finite direct products, (H3) direct limits [5, Proposition 6.1.1], (H4) extensions with amenable quotient [5, Example 6.1.6], and (H5) is inherited from lattices [5, Proposition 6.1.5].

Relative Property (T) is inherited by dense images: if \( (G, H) \) has Property (T) and \( f : G \to K \) is a continuous morphism, then \( (K, f(H)) \) has Property (T).

4.2. Continuous families of Lie groups with Property (T).

**Proof of Proposition 1.12.** We must construct a continuous family of connected Lie groups with Property (T) and with perfect and pairwise non-isomorphic Lie algebras.

Consider \( \mathfrak{s} = \mathfrak{sp}_{2n}(\mathbb{R}) \) \( (n \geq 2) \). Let \( \mathfrak{v}_i, i = 1, 2, 3, 4 \) be four non-trivial absolutely irreducible, \( \mathfrak{s} \)-modules which are pairwise non-isomorphic and all preserve a symplectic form\(^3\). Then \( \mathfrak{v} = \bigoplus_{i=1}^{4} \mathfrak{v}_i \) is a full \( \mathfrak{s} \)-module and \( \text{Aut}_s(\mathfrak{v}) = \prod_{i=1}^{4} \text{Aut}_s(\mathfrak{v}_i) \simeq (\mathbb{R}^4)^4 \). In particular, \( \text{Alt}_s(\mathfrak{v})^* \simeq \mathbb{R}^4 \) and \( \text{Aut}_s(\mathfrak{v}) \) acts diagonally on it. The action on the 2-Grassmannian, which is 4-dimensional, is trivial on the scalars, so that its orbits are at most 3-dimensional. So there exists a continuous family \( (P_t) \) of 2-planes in \( \text{Alt}_s(\mathfrak{v})^* \) which are in pairwise distinct orbits for the action of \( \text{Aut}_s(\mathfrak{v}) \). By Theorem 3.1, the Lie \( \mathfrak{s} \)-algebras \( \mathfrak{h}(\mathfrak{v})/P_t \) are pairwise non-isomorphic, and so the Lie algebras \( \mathfrak{s} \ltimes \mathfrak{h}(\mathfrak{v})/P_t \) are pairwise non-isomorphic. The Lie algebras \( \mathfrak{g}_t \) are perfect, and the corresponding Lie groups \( G_t \) have Property (T): this immediately follows from Wang’s classification [13, Theorem 1.9].

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\(^3\)There exist infinitely many such modules, which can be obtained by taking large irreducible components of the odd tensor powers of the standard \( 2n \)-dimensional \( \mathfrak{s} \)-module.
Remark 4.2. These examples have 2-nilpotent radical. This is, in a certain sense, optimal, since there exist only countably many isomorphism classes of Lie algebras over $\mathbb{R}$ with abelian radical, and only a finite number for each dimension.

Proof of Proposition 1.13. We must construct a continuous family of locally isomorphic, pairwise non-isomorphic connected Lie groups with Property (T). The proof is actually similar to that of Proposition 1.12. Use the same construction, but, instead of taking the quotient $H_t$, take the quotient $H_t$ by its biggest compact normal subgroup $P_t/\Gamma_t$, we obtain $G_t$. Accordingly, the groups $H_t$ are pairwise non-isomorphic.

4.3. Characterization of groups with the Haagerup Property.

Proposition 4.3. Let $K$ be a local field of characteristic zero and $n \geq 1$. Then the pairs $(\text{SL}_2(K) \ltimes V_n(K), V_n(K))$, $(\text{PGL}_2(K) \ltimes V_n(K), V_n(K))$, $(\text{SL}_2(K) \ltimes H_n(K), H_n(K))$, $(\text{SL}_2(R) \ltimes V_n(R), V_n(R))$, and $(\text{SL}_2(R) \ltimes H_n(R), H_n(R))$ have Property (T).

Proof. The first (and the fourth) case is well-known; it follows, for instance, from Furstenberg’s theory [8] of invariant probabilities on projective spaces, which implies that $\text{SL}_2(K)$ does not preserve any probability on $V_n(K)$ (more precisely, on its Pontryagin dual) other than the Dirac measure at zero. See, for instance, the proof [10, Chap. 2, Proposition 2]. The second case is an immediate consequence of the first. For the third (resp. fifth) case, we invoke [5, Proposition 4.1.4], with $S = \text{SL}_2(K)$, $N = H_n(K)$, even if the hypotheses are slightly different (unless $K = R$ or $C$): the only modification is that, since here $[N,S]$ is not necessarily connected, we must show that its image in the unitary group $U_n$ is connected so as to justify Lie’s Theorem. Otherwise, it would have a nontrivial finite quotient. This is a contradiction, since $[N,S]$ is generated by divisible elements; this is clear, since, as the group of $K$-points of an unipotent group, it has a well-defined logarithm.

Corollary 4.4. Let $G$ be either a connected Lie group, or $G = G(K)$, where $G$ is a linear algebraic group over the local field $K$ of characteristic zero. Suppose that the Lie algebra $g$ of $G$ contains a subalgebra $h$ isomorphic to either $\mathfrak{sl}_2 \ltimes \mathfrak{v}_n$ or $\mathfrak{sl}_2 \ltimes \mathfrak{h}_{2n-1}$ for some $n \geq 2$. Then $G$ has a noncompact closed subgroup with relative Property (T). In particular, $G$ does not have Haagerup’s property.

Proof. Let us begin by the case of algebraic groups. By [3, Chap. II, Corollary 7.9], since $h$ is perfect, it is the Lie algebra of a closed $K$-subgroup $H$ of $G$. Since $H$ must be $K$-isomorphic to either $\text{SL}_2 \ltimes V_m$, $\text{PGL}_2 \ltimes V_{2m-1}$, or $\text{SL}_2 \ltimes H_{2m-1}$ for some $m \geq 2$, Proposition 4.3 implies that $G(K)$ has a noncompact closed subgroup with relative Property (T).

In the case of Lie groups, we obtain a Lie subgroup which is the image of an immersion $i$ of $\text{SL}_2(R) \ltimes N$, where $N$ is either $V_n(R)$ or $H_{2n-1}(R)$, for some $n \geq 2$, into $G$. By Proposition 4.3, $(G, \overline{i}(N))$ has Property (T). We claim that $i(N)$ is not compact. Suppose the contrary. Then it is solvable and connected, hence it is a torus. It is normal in the closure $H$ of $i(G)$. Since the automorphism group of a torus is totally disconnected, the action by conjugation of $H$ on $\overline{i}(N)$ is trivial; that is, $i(N)$ is central in $H$. This is a contradiction.
Proof of Theorem 1.10. As we already noticed in the reminder, (i) ⇒ (ii) is immediate from the definition. We are going to prove (ii) ⇒ (iv) ⇒ (iii) ⇒ (i).

For the implication (iii) ⇒ (i), in the algebraic case, $G$ is isomorphic, up to a finite kernel, to $S_{nc}(K) \times \text{Mr}(K)$, where Mr denotes the amenable radical of $G$. The group $\text{Mr}(K)$ is amenable, hence has Haagerup's property. The group $S_{nc}(K)$ also has Haagerup's property: if $K$ is Archimedean, it maps, with finite kernel, onto a product of groups isomorphic to $\text{PSO}_0(n, 1)$ or $\text{PSU}(n, 1)$ ($n \geq 2$), and these groups have Haagerup’s property, by a result of Faraut and Harzallah, see [2, Chap. 2]. If $K$ is non-Archimedean, then $S_{nc}(K)$ acts properly on a product of trees (one for each simple factor) [4], and this also implies that it has Haagerup’s property [2, Chap. 2].

The same argument also works for connected Lie groups when the semisimple part has finite centre; in particular, this is fulfilled for linear Lie groups and their finite coverings. The case when the semisimple part has infinite centre is considerably more involved, see [5, Chap. 4].

(ii) ⇒ (iv) Suppose that (iv) is not satisfied. If $g$ contains a copy of $\mathfrak{sl}_2 \ltimes \mathfrak{h}_2$, then, by Corollary 4.4, $G$ does not satisfy (ii). If $K = \mathbb{R}$, we consider $G$ as a Lie group with finitely many components. By a standard argument, since $\text{Sp}(2, 1)$ is simply connected with finite centre (of order 2), an embedding of $\mathfrak{sp}(2, 1)$ into $g$ corresponds to a closed embedding of $\text{Sp}(2, 1)$ or $\text{PSp}(2, 1)$ into $G$. Since $\text{Sp}(2, 1)$ has Property (T) [2, Chap. 3], this contradicts (ii).

(iv) ⇒ (iii) If $g$ is not M-decomposed, then, by Proposition 1.9, it contains a copy of $\mathfrak{sl}_2 \ltimes \mathfrak{v}_n$ or $\mathfrak{sl}_2 \ltimes \mathfrak{h}_{2n-1}$ for some $n \geq 2$.

If $g$ has a simple factor $s$, then $s$ embeds in $g$ through a Levi factor. If $s$ has $K$-rank $\geq 2$, then it contains a subalgebra isomorphic to either $\mathfrak{sl}_3$ or $\mathfrak{sp}_4$ [11, Chap I, (1.6.2)], and such a subalgebra contains a subalgebra isomorphic to $\mathfrak{sl}_2 \ltimes \mathfrak{v}_2$ (resp. $\mathfrak{sl}_2 \ltimes \mathfrak{v}_3$) [2, 1.4 and 1.5].

Finally, if $K = \mathbb{R}$ and $s$ is isomorphic to either $\mathfrak{sp}(n, 1)$ for some $n \geq 2$ or $\mathfrak{f}_{4(-20)}$, then it contains a copy of $\mathfrak{sp}(2, 1)$.

\begin{remark}
Conversely, $\mathfrak{sp}(n, 1)$ and $\mathfrak{f}_{4(-20)}$ do not contain any subalgebra isomorphic to $\mathfrak{sl}_2 \ltimes \mathfrak{v}_n$ or $\mathfrak{sl}_2 \ltimes \mathfrak{h}_{2n-1}$ for any $n \geq 2$; this can be shown using results of [6] about weak amenability.
\end{remark}

4.4. Subgroups of Lie groups.

Let us exhibit some subgroups in the groups above.

First observation. Let $G$ denote $\text{SL}_2 \ltimes V_n$, $\text{PGL}_2 \ltimes V_{2n-1}$, or $\text{SL}_2 \ltimes H_{2n-1}$ for some $n \geq 2$, and $R$ its radical. Then, for every field $K$ of characteristic zero, $G(K)$ contains $G(\mathbb{Z})$ as a subgroup. On the other hand, the pair $(G(\mathbb{Z}), R(\mathbb{Z}))$ has Property (T), this is because $G(\mathbb{Z})$ is a lattice in $G(\mathbb{R})$.

Second observation. Now, let $G$ denote $\text{SU}(2) \ltimes D^R_{2n+1}$, $\text{SO}_3(\mathbb{R}) \ltimes D^R_{2n+1}$, or $\text{SU}(2) \ltimes \text{HU}^R_{3n}$ for some $i = 0, 1, 2, 3$. These groups all have a $\mathbb{Q}$-form: this is obvious at least for all but $\text{SU}(2) \ltimes \text{HU}^R_{3n}$ for $i = 1, 2$; for these two, this is because the subspace $Z_i$ can be chosen rational in the definition of $\text{HU}^R_{3n}$.

Let $R$ be the radical of $G$ and $S$ a Levi factor defined over $\mathbb{Q}$. Let $F$ be a number field of degree three over $\mathbb{Q}$, not totally real. Let $\mathcal{O}$ be its ring of
integers. Then $G(O)$ embeds diagonally as an irreducible lattice in $G(R) \times G(C)$. Its projection $\Gamma$ in $G(R)$ does not have Haagerup’s property, since otherwise $G(C)$ would also have Haagerup’s property (by (H5) in Lemma 4.1), and this is excluded since it does not satisfy $[S_{nc}, R] = 1$, by Theorem 1.10 (noting that the anisotropic Levi factor becomes isotropic after complexification).

**Proposition 4.6.** Let $G$ be a real Lie group, $R$ its radical, $S$ a semisimple factor. Suppose that $[S, R] \neq 1$. Then $G$ has a countable subgroup without Haagerup’s property.

**Proof.** First case: $[S_{nc}, R] \neq 1$. Then, by Proposition 3.4, $G$ has a Lie subgroup $H$ isomorphic to a quotient of $\tilde{H} = \frac{\text{SL}_2(R) \times R(R)}{\text{SL}_2(R) \times R(R)}$ by a discrete central subgroup, where $R = V_n$ or $H_{2n-1}$, for some $n \geq 2$. Denote by $\tilde{H}(Z)$ the inverse image of $\text{SL}_2(Z) \times R(Z)$ in $\tilde{H}$. By the observation above, $(H(Z), R(Z))$ has Property (T), so that its image in $H$, which we denote by $H(Z)$, satisfies $(H(Z), R_G(Z))$ has Property (T), where $R_G(Z)$ means the image of $R(Z)$ in $G$. Observe that $R_G(Z)$ is infinite: if $R = V_n$, this is $V_n(Z)$; if $R = H_{2n-1}$, this is a quotient of $H_{2n-1}(Z)$ by some central subgroup. Accordingly, $H(Z)$ does not have Haagerup’s property.

Second case: $[S_{c}, R] \neq 1$. By Proposition 3.8, $G$ has a Lie subgroup $H$ isomorphic to a central quotient of $\text{SU}(2)(R) \times R$, where $R = D^{R}_{2n+1}$ or $H_{4n}^U$, for some $n \geq 1$ and $i = 0, 1, 2, 3$.

First suppose that the radical of $H$ is simply connected. Then, by the second observation above, $H$ has a subgroup without the Haagerup property.

Now, let us deal with the case when $H = \tilde{H}/Z$, where $Z$ is a discrete central subgroup. Then $\tilde{H}$ has a subgroup $\Gamma$ as above which does not have Haagerup’s property. Let $W$ denote the centre of $\tilde{H}$. The kernel of the projection of $\Gamma$ to $H$ is given by $\Gamma \cap Z$. We use the following trick: we apply an automorphism $\alpha$ of $\tilde{H}$ such that $\alpha(\Gamma) \cap Z$ is finite. It follows that the image of $\alpha(\Gamma)$ in $H$ does not have Haagerup’s property.

This allows to suppose that $\Gamma \cap Z$ is finite, so that the image of $\Gamma$ in $H$ does not have Haagerup’s property. Let us construct such an automorphism.

Observe that the representations of $\text{SU}(2)$ can be extended to the direct product $R^* \times \text{SU}(2)$ by making $R^*$ act by scalar multiplication. This action lifts to an action of $R^* \times \text{SU}(2)$ on $H_{4n}^U$, where the scalar $a$ acts on the derived subgroup of $H_{4n}^U$ by multiplication by $a^2$.

Now, working in the unit component of the centre $W$ of $\tilde{H}$, which we treat as a vector space, we can take $a$ so that $a^2 \cdot (\Gamma \cap W)$ avoids $Z \cap W$ (a clearly exists, since $\Gamma$ and $Z$ are countable).

**Definition 4.7.** Let $G$ be a locally compact group. We say that $G$ has Haagerup’s property if every $\sigma$-compact open subgroup of $G$ does.

**Remark 4.8.** In view of (H3) of Lemma 4.1, this is equivalent to: every compactly generated, open subgroup of $G$ has Haagerup’s property, and also equivalent to the existence of a $C_0$-representation with almost invariant vectors [5, Chap. 1].
In particular, $G$ having Haagerup’s property and $(G, H)$ having Property (T) still imply $H$ relatively compact.

All properties of the class of groups with Haagerup’s property claimed in Lemma 4.1 also clearly remain true for general locally compact groups.

If $G$ is a topological group, denote by $G_d$ the group $G$ endowed with the discrete topology.

**Proof of Theorem 1.14.** We remind that we must prove, for a connected Lie group $G$, the equivalence between

(i) $G$ is locally isomorphic to $\text{SO}_3(\mathbb{R})^a \times \text{SL}_2(\mathbb{R})^b \times \text{SL}_2(\mathbb{C})^c \times R$, with $R$ solvable and integers $a, b, c$, and

(ii) $G_d$ has Haagerup’s property.

The implication $(i) \Rightarrow (ii)$ is, essentially, a deep and recent result of Guentner, Higson, and Weinberger [9, Theorem 5.1], which implies that $\text{PSL}_2(\mathbb{C})_d$ has Haagerup’s property. Let $G$ be as in (i), and $S$ its semisimple factor. Then $G/S$ is solvable, so that, by (H4) of Lemma 4.1, we can reduce to the case when $G = S$. Now, let $Z$ be the centre of the semisimple group $G$, and embed $G_d$ in $(G/Z)_d \times G$, where $G_d$ means $G$ endowed with the discrete topology. This is a discrete embedding. Since $G$ has Haagerup’s property, this reduces the problem to the case when $G$ has trivial centre. So, we are reduced to the cases of $\text{SO}_3(\mathbb{R})$, $\text{PSL}_2(\mathbb{R})$, and $\text{PSL}_2(\mathbb{C})$. The two first groups are contained in the third, so that the result follows from the Guentner-Higson-Weinberger Theorem.

Conversely, suppose that $G$ does not satisfy (i). If $[S, R] \neq 1$, then, by Proposition 4.6, $G_d$ does not have Haagerup’s property. Otherwise, observe that the simple factors allowed in (i) are exactly those of geometric rank one (viewing $\text{SL}_2(\mathbb{C})$ as a complex Lie group). Hence, $S$ has a factor $W$ which is not of geometric rank one. Then the result is provided by Lemma 4.9 below.

**Lemma 4.9.** Let $S$ be a simple Lie group which is not locally isomorphic to $\text{SO}_3(\mathbb{R})$, $\text{SL}_2(\mathbb{R})$ or $\text{SL}_2(\mathbb{C})$. Then $S_d$ does not have Haagerup’s property.

**Proof.** Let $Z$ be the centre of $S$, so that $S/Z \simeq G(\mathbb{R})$ for some $\mathbb{R}$-algebraic group $G$. By assumption, $G(\mathbb{C})$ has factors of higher rank, hence does not have Haagerup’s property. Let $F$ be a number field of degree three over $\mathbb{Q}$, not totally real. Let $O$ be its ring of integers. Then $G(O)$ embeds diagonally as an irreducible lattice in $G(\mathbb{R}) \times G(\mathbb{C})$, and is isomorphic to its projection in $G(\mathbb{R})$. Let $\Gamma$ be the inverse image in $S \times G(\mathbb{C})$ of $G(O)$. Then $\Gamma$ is a lattice in $S \times G(\mathbb{C})$. Hence, by [5, Proposition 6.1.5], $\Gamma$ does not have Haagerup’s property. Note that the projection $\Gamma'$ of $\Gamma$ into $S$ has finite kernel, contained in the centre of $G(\mathbb{C})$. So $\Gamma'$ neither has Haagerup’s property, and is a subgroup of $S$.

**Remark 4.10.** In contrast with Theorem 1.10, Theorem 1.14 is no longer true if we replace the statement “$G_d$ has Haagerup’s property” by “$G_d$ has no infinite subgroup with relative Property (T)”. Indeed, let $G = K \ltimes V$, where $K$ is locally isomorphic to $\text{SO}_3(\mathbb{R})^n$ and $V$ is a vector space on which $K$ acts nontrivially. Suppose that $(G_d, H)$ has Property (T) for some subgroup $H$. Then $(G_d/V, H/(H \cap V))$ has Property (T). In view of the Guentner-Higson-Weinberger
Theorem (see the proof of Theorem 1.14), $H/(H \cap V)$ is finite. On the other hand, since $G$ has Haagerup’s property, $H \cap V$ must be relatively compact, and this implies that $H \cap V = 1$. Thus, $H$ is finite.

Motivated by this example, it is easy to exhibit finitely generated groups without the Haagerup Property and do not have infinite subgroups with relative Property (T). For instance, let $n \geq 3$, and $q$ be the quadratic form $\sqrt{2}x_0^2 + x_1^2 + x_2^2 + \cdots + x_{n-1}^2$. Let $G(R) = \text{SO}(q)(R) \rtimes \mathbb{R}^n$ and write, for any commutative $\mathbb{Z}(\sqrt{2})$-algebra $R$, $H(R) = \text{SO}(q)(R)$. Then $G = G(\mathbb{Z}[\sqrt{2}])$ is such an example. The fact that $\Gamma$ has no infinite subgroup $\Lambda$ with relative Property (T) can be seen without making use of the Guentner-Higson-Weinberger Theorem: first observe that $H(\mathbb{Z}[\sqrt{2}])$ is a cocompact lattice in $\text{SO}(n-1,1)$, hence has Haagerup’s property. So the projection of $\Lambda$ in $H(\mathbb{Z}[\sqrt{2}])$ is finite. So, upon passing to a finite index subgroup, we can suppose that $\Lambda$ is contained in the subgroup $\mathbb{Z}[\sqrt{2}]^n$ of $\Gamma = \text{SO}(q)(\mathbb{Z}[\sqrt{2}]) \rtimes \mathbb{Z}[\sqrt{2}]^n$. But then the closure $L$ of $\Lambda$ in the subgroup $\mathbb{R}^n$ of the amenable group $G(\mathbb{R}) = \text{SO}(q)(\mathbb{R}) \rtimes \mathbb{R}^n$ is not compact, and $(G(\mathbb{R}), L)$ has Property (T). This is a contradiction.

On the other hand, $\Gamma$ does not have Haagerup’s property, since it is a lattice in $G(\mathbb{R}) \rtimes G^\sigma(\mathbb{R})$ (use (H5) of Lemma 4.1), where $\sigma$ is the nontrivial automorphism of $\mathbb{Q}(\sqrt{2})$, and $G^\sigma(\mathbb{R}) \simeq \text{SO}(n-1,1) \rtimes \mathbb{R}^n$ does not have Haagerup’s property, by Theorem 1.10. Note that $\Gamma$, as a cocompact lattice in a connected Lie group, is finitely presented.

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