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Orbital Convolution Theory for Semi-direct Products

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Abstract. We extend previous results of the authors on orbital convolutions for compact groups, to compact times vector semidirect products. In particular, we define convolutions of noncompact coadjoint orbits and recover the character formulae and Plancherel formula of Lipsman.
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1. Introduction

Previous work of the authors ([5], [7], [6]), in the setting of compact groups, introduced the wrapping map Φ. This map associates, to each Ad-invariant distribution μ of compact support on the Lie algebra $\mathfrak{g}$, a central distribution $\Phi\mu$ on the Lie group G, via the formula, for $f \in C_c^\infty(G)$,

$$\langle \Phi\mu, f\rangle = \langle \mu, j \cdot f \circ \exp\rangle, \tag{1}$$

where j is the square root of the Jacobian of $\exp : \mathfrak{g} \to G$.

The remarkable thing about Φ is that it provides a convolution homomorphism between the Euclidean convolution structure on $\mathfrak{g}$ and the group convolution on G, that is

$$\Phi(\mu *_{\mathfrak{g}} \nu) = \Phi\mu *_G \Phi\nu. \tag{2}$$

This mapping is a global version of the Duflo isomorphism — there are no conditions on the supports of μ and ν (they need not, for example, lie in a fundamental domain). As pointed out in [5], we may interpret the dual of Φ, a map from the Gelfand space of $M_G(G)$ to that of $M_G(\mathfrak{g})$, in such a way as to obtain the Kirillov character formula for G.

In a recent paper [1], Andler, Sahi and Torossian have extended the Duflo isomorphism to arbitrary Lie groups. Their results give a version of equation (2) which holds for germs of hyperfunctions with support at the identity. In fact, equation (2) can be viewed as a statement that, for compact Lie groups, the

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results of [1] hold for invariant distributions of compact support, and hold globally in the sense that the restriction that the supports are compact is needed only in order to ensure that the convolutions exist. This observation allows one to develop calculational tools for invariant harmonic analysis based on convolutions of orbits and distributions in the Euclidean space $\mathfrak{g}$: see for example [6], where new character formulae and new approaches to the Plancherel formula are developed for compact Lie groups.

In this article, we extend these ideas to semi-direct product groups $G = V \rtimes K$, where V is a vector space and K a compact group. There are several significant differences between this case and the compact case previously treated — firstly, there is no identification between the adjoint and coadjoint pictures as the Killing form is indefinite, and secondly, perhaps more significantly, the fact that the orbits are no longer compact means that there are few Ad-invariant distributions of compact support — so the convolutions in formula (2) need careful interpretation.

Specifically, we shall:

(1) Give an explicit description of conjugacy classes in G, adjoint orbits in $\mathfrak{g}$ and coadjoint orbits in $\mathfrak{g}^*$, and find canonical G-invariant measures on these. (Sections 2 and 3).

(2) Show how the adjoint orbits exponentiate to conjugacy classes, calculate the function j, and calculate the Fourier transforms of adjoint orbits. (Section 5).

(3) Define classes of invariant distributions on G, $\mathfrak{g}$ and $\mathfrak{g}^*$, which include the canonical orbital measures, and which will play the role of the G-invariant distributions of compact support. These distributions are defined using the structure of the orbits — roughly speaking on $\mathfrak{g}$, they are K-invariant, have compact support in the $\mathfrak{k}$-direction, and consist of appropriate invariant means in the direction of the fibres. The spaces of such distributions are denoted $\mathcal{AP}'_G(G)$, $\mathcal{AP}'_G(\mathfrak{g})$, $\mathcal{AP}'_G(\mathfrak{g}^*)$. (Section 6)

(4) Show that the elements of $\mathcal{AP}'_G(\mathfrak{g})$ wrap to elements of $\mathcal{AP}'_G(G)$ and have Fourier transforms in $\mathcal{AP}'_G(\mathfrak{g}^*)$. (Section 6)

(5) Show that the spaces $\mathcal{AP}'_G(G)$, $\mathcal{AP}'_G(\mathfrak{g})$ and $\mathcal{AP}'_G(\mathfrak{g}^*)$ are closed under the operation of convolution. In order to do this, one needs to introduce spaces of test functions $\mathcal{AP}_G(G)$, $\mathcal{AP}_G(\mathfrak{g})$, $\mathcal{AP}_G(\mathfrak{g}^*)$ which have our distributions as duals, and define

$$\langle \mu * \nu, \psi \rangle = \langle \mu(X)\nu(Y), \psi(X+Y) \rangle.$$

The basic idea of the test space $\mathcal{AP}_G(\mathfrak{g})$ is to take functions which are C^∞ in the $\mathfrak{k}$-direction and almost periodic in the fibre direction. This idea is appropriately modified for G and $\mathfrak{g}^*$ (see Definitions 6.1 and 6.3).

(6) Prove the convolution formula $\Phi(\mu * \nu) = \Phi(\mu) * \Phi(\nu)$ for invariant elements $\mu, \nu \in \mathcal{AP}'_G(\mathfrak{g})$ for this notion of convolution (Theorem 6.1).

(7) Link this to the representation theory of G, and prove the Lipsman character formula (Sections 4,7).

Together, these steps constitute a demonstration that the "orbital convolutions" approach can work in the setting of non-compact groups. It will be apparent that we have made significant use of our detailed knowledge of the structure of the orbits in order to define the convolution algebras $\mathcal{AP}'_G$ over each of G, $\mathfrak{g}$ and $\mathfrak{g}^*$. It is for this technical reason that we cannot at present extend our results to other Lie groups, although there are some preliminary indications (see for example [3], [16], [15]) that the program might also work for nilpotent groups.

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2. The exponential map and the adjoint action

Let K be a compact connected Lie group with Lie algebra $\mathfrak{k}$ and suppose that K has a smooth linear action on a complex vector space V. We shall write the action as $v \mapsto kv$, $k \in K, v \in V$. We may then form the semi-direct product $G = V \rtimes K$. This is the product manifold $V \times K$ with multiplication

$$(v,k)(v',k') = (v + kv', kk'). \tag{3}$$

For simplicity [2], we shall assume throughout that K acts effectively on V, that is $K_0 = \{e\}$, where

$$K_0 = \{k \in K : kv = v \ \forall v \in V\}.$$

The semi-direct product is a Lie group with identity $(0,e)$ and inverse map $(v,k)^{-1} = (-k^{-1}v, k^{-1})$. Its Lie algebra may be identified as $\mathfrak{g} = V \oplus \mathfrak{k}$ with Lie bracket given, for $a, a_1 \in V$, $A, A_1 \in \mathfrak{k}$ by

$$\operatorname{ad}(a,A)(a_1,A_1) = [(a,A),(a_1,A_1)] = (A \cdot a_1 - A_1 \cdot a, [A,A_1]_{\mathfrak{k}}). \tag{4}$$

In this formula $a \mapsto A \cdot a$ denotes the differential of the action of K on V, given by

$$A \cdot a = \left.\frac{d}{dt}\right|_{t=0} (\exp_K tA)a,$$

where $A \in \mathfrak{k}$ and $a \in V$. Here, $\exp_K$ denotes the exponential map of K.

The exponential map $\exp_G : \mathfrak{g} \to G$ is given by

$$\exp_G(a,A) = \left(\left(\frac{I - e^{-A}}{A} \right) \cdot a, \, \exp_K A \right). \tag{5}$$

In this formula, the expression $\frac{I-e^{-A}}{A}$ denotes the linear operator on V defined by the formal power series $\sum_{n=0}^{\infty} (-1)^n \frac{A^n}{(n+1)!}$. (see [9])

[2] In fact, it is not difficult to see that this is no real restriction. K_0 is easily seen to be a closed normal subgroup of K, and the quotient action of K/K_0 on V is effective. The standard isomorphism theorems of group theory allow us to write

$$V \rtimes K \cong (V \rtimes K/K_0) \rtimes K_0$$

and one may easily combine the theorems which we will prove with those already known for the compact case to obtain the analagous results for $V \rtimes K$ with no restriction on the action.

One calculates easily that the conjugation action of G on itself is given by

$$(v,k)(v_1,k_1)(v,k)^{-1} = (v + kv_1 - kk_1k^{-1}v, kk_1k^{-1}). \tag{6}$$

This formula may be differentiated to see that the adjoint action of G on $\mathfrak{g}$ is

$$\mathrm{Ad}(v,k)(a,A) = (k \cdot a - (\mathrm{Ad}(k)A) \cdot v, \mathrm{Ad}(k)A). \tag{7}$$

By Weyl's unitary trick we may (and will) assume that V is equipped with a K-invariant inner product $(\ ,\ )$. We shall do this systematically throughout the paper.

Given $k \in K$, let

$$\begin{aligned} V_k &= \{v \in V\ : kv = v\} \quad \text{and} \\ V^k &= \{(I-k)\cdot v\ : v \in V\} \end{aligned} \tag{8}$$

so that we have the orthogonal decomposition

$$V = V_k \oplus V^k$$

Given $A \in \mathfrak{k}$, let

$$V_A = \{v \in V : A \cdot v = 0\}$$

and

$$V^A = A \cdot V.$$

This gives an orthogonal decomposition

$$V = V_A \ \oplus\ V^A. \tag{9}$$

We may now describe the geometrical structure of the conjugacy classes and the adjoint orbits as follows:

Lemma 2.1. *The conjugacy class of $(v_1,k_1) \in G$ is fibred over the K-orbit $K(v_1,k_1) = \{(k\cdot v_1, kk_1k^{-1})\}$ in G, the fibre at the point indexed by k being $V^{kk_1k^{-1}}$.* ∎

Lemma 2.2. *Let $(a,A) \in \mathfrak{g} = V \oplus \mathfrak{k}$. The adjoint orbit through (a,A) is fibred over the K-orbit in $\mathfrak{g}$ $\{(k\cdot a, \mathrm{Ad}(k)A) : k \in K\}$, the fibre at $(k\cdot a, \mathrm{Ad}(k)A)$ being $V^{\mathrm{Ad}(k)A}$.* ∎

It is generally true that the exponential map of a Lie group maps adjoint orbits to conjugacy classes. This may be seen directly in our situation from the following basic properties of $\exp = \exp_G$.

Proposition 2.3. (i) $\exp(\mathrm{Ad}(v,e)(a,A)) = (a - A\cdot v, \exp A)$

(ii) $\exp^{-1}\{e\} = \{(A\cdot v, A) : \exp_K A = e_K,\ v \in V\}$

(iii) *Let J_K be the Jacobian of* $\exp_K$ *and J_G the Jacobian of* $\exp_G$. *Then*

$$J_G(a,A) = \det\left(\left(\frac{1-e^{-A}}{A}\right)\Big|_V\right) J_K(A). \tag{10}$$

(Notice that the right-hand side is independent of a.)

Proof. (i) is a simple consequence of the formulae in Lemma 2.1.

For (ii), suppose that $\exp(a, A) = (0, e_K)$. Then $\exp_K A = e_K$ and $(\frac{1-e^{-A}}{A})\cdot a = 0$. Since $\exp_K A = e_K$, $1 - e^{-A} = 0$ and hence for such an A, $(\frac{1-e^{-A}}{A}) \cdot a = 0$ if and only if $a \in A \cdot V$.

To see (iii), write the formula for ad as

$$\mathrm{ad}(a, A)(a_1, A_1) = \left(\begin{array}{c|c} A & -(\ .\ ) \cdot a \\ \hline 0 & \mathrm{ad}\, A \end{array}\right) \begin{pmatrix} a_1 \\ A_1 \end{pmatrix},$$

where $(\ .\ ) \cdot a$ is the map $\mathfrak{k} \to V$ given by $A_1 \to A_1 \cdot a$

By Helgason [9] the differential of exp at 0 is given by

$$\exp_{*,0}(A, a) = \frac{1 - e^{-\mathrm{ad}(a,A)}}{\mathrm{ad}(a, A)} = \left(\begin{array}{c|c} \frac{1-e^{-A}}{A} & * \\ \hline 0 & \frac{1-e^{-\mathrm{ad}\, A}}{\mathrm{ad}\, A} \end{array}\right).$$

The formula given in (iii) follows. ■

Remark 2.4. If the point A lies in the exponential lattice of $\mathfrak{k}$, then under the exponential map, the affine subspace (V_A, A) is collapsed to the single point $(0, e_K)$; the Jacobian is zero on this subspace.

We now define canonical measures on conjugacy classes and adjoint orbits. For the compact group K this is done as follows (c.f. [5]). Let T be a maximal torus for K, and choose a set Φ^+ of positive roots for T. Let $\mathfrak{t}^+$ be the corresponding positive Weyl chamber in $\mathfrak{t}$. Let D be an Ad-invariant fundamental domain for G in $\mathfrak{g}$ and let $T^+ = \exp(D \cup \mathfrak{t}^+)$.

Each conjugacy class $\mathcal{C}$ intersects T in a Weyl group orbit; let $t_{\mathcal{C}}$ be the element of this orbit which lies in the closure of T^+ and $K_{\mathcal{C}}$ the stabilizer of $t_{\mathcal{C}}$ (generically, $K_{\mathcal{C}} = T$). The conjugacy class is isomorphic to $K/K_{\mathcal{C}}$ (via $gt_{\mathcal{C}}g^{-1} \mapsto gK_{\mathcal{C}}$) and thus carries a unique central measure $d\mu_{\mathcal{C}}$ which gives it total mass

$$\Bigg(\prod_{\substack{\alpha\in\Phi^+ \\ \alpha(\log\, t_{\mathcal{C}})\neq 0}} 2\sin\left(\frac{\alpha}{2}(\log\, t_{\mathcal{C}})\right)\Bigg)^2.$$

Similarly, each adjoint orbit $\mathcal{O}$ intersects $\mathfrak{t}$ in a Weyl orbit. Letting $H_{\mathcal{O}}$ be a point in this orbit in $\overline{\mathfrak{t}^+}$, we see that $\mathcal{O}$ carries a unique Ad-invariant measure $d\mu_{\mathcal{O}}$ so that it has total mass

$$\Bigg(\prod_{\substack{\alpha\in\Phi^+ \\ \alpha(H_{\mathcal{O}})\neq 0}} \alpha(H_{\mathcal{O}})\Bigg)^2.$$

Given [5] that the Jacobian of the exponential map of K is given by

$$J_K(H) = (j_K(H))^2,$$

where

$$j_K(H) = \prod_{\alpha\in\Phi^+} \frac{2\,\sin\frac{\alpha}{2}(H)}{\alpha(H)},$$

we see that $d\mu_{\mathcal{C}} = J(H)d\mu_{\mathcal{O}}$ whenever $\mathcal{C}$ is the image of $\mathcal{O}$, and $H \in \mathcal{O}$.

We now carry out an analogous procedure to define measures on conjugacy classes and coadjoint orbits for $V \rtimes K$.

We may decompose V into real weight spaces for T;

$$V = \oplus_{\lambda \in \Lambda} V_\lambda, \tag{11}$$

where if $v \in V_0$, we have $t \cdot v = v$, for all $t \in T$, and if $\lambda \neq 0$, V_λ is a $2m_\lambda$ dimensional space such that for $v \in V_\lambda$ $\;t \cdot v = \otimes^{m_\lambda} T_{\lambda(t)} v$. Here T_θ denotes the rotation whose matrix is $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$, and $m_\lambda \geq 1$ is the multiplicity of the weight λ. The direct sum is with respect to our fixed K-invariant inner product $(\ ,\ )$.

Each of the spaces V_λ, equipped with $(\ ,\ )$, is a Euclidean space. We choose (and fix once and for all) a Lebesgue measure dv_λ on V_λ. This choice leads to a measure $dv = \prod_\lambda dv_\lambda$ on V, and we may assume that Haar measure on $V \rtimes K$ is $dv\,dk$.

Notice that if $t \in T$, then in the notation of (8),

$$V_t = \oplus_{\{\lambda:\lambda(\log t)=0\}} V_\lambda \tag{12}$$

and

$$V^t = \oplus_{\{\lambda:\lambda(\log t)\neq 0\}} V_\lambda. \tag{13}$$

Consider a $V \rtimes K$ conjugacy class $\mathcal{C}$. By the formula for the conjugation and the remarks about the compact case, $\mathcal{C}$ contains a unique point of the form (v,t) where $t \in T^+$ and $v \in V_t$. The fibre at this point is V^t. Denote such a conjugacy class by $\mathcal{C}(v,t)$.

We may thus define a measure on $\mathcal{C}$ by (1) identifying the K orbit of (v,t) with the conjugacy class of t in K; we take the measure $d\mu_t$ on this orbit; and (2) defining a suitable measure on V^t which is then transferred to $V^{k\cdot t}$ by K-invariance. Such a measure is given by

$$\prod_{\{\lambda:\lambda(\log t)\neq 0\}} 4\sin^{2m_\lambda}\left(\frac{\lambda}{2}(\log t)\right) dv_\lambda. \tag{14}$$

The factors of $2\sin\left(\frac{\lambda}{2}(\log t)\right)$ come from the facts that $V^t = (1-t)V$, and that V_λ is an invariant space for $\otimes^{m_\lambda} I - T_\theta$ and $\det(I - T_\theta) = 4\sin^{2m_\lambda}\frac{\theta}{2}$.

Definition 2.5. For fixed (v,t) with $v \in V_t$, let

$$\kappa_1(t) = \prod_{\{\lambda:\lambda(\log t)\neq 0\}} 4\sin^{2m_\lambda}\left(\frac{\lambda(\log t)}{2}\right) \prod_{\{\alpha\in\Phi^+:\alpha(\log t)\neq 0\}} 4\sin^2\frac{\alpha(\log t)}{2}$$

Using the fact that $V^t = \oplus_{\{\lambda:\lambda(\log t)\neq 0\}} V_\lambda$, define the measure dv_1 on V^t by

$$dv_1 = \prod_{\{\lambda:\lambda(\log t)\neq 0\}} dv_\lambda$$

and take $d\dot{k}$ to be normalized measure on K/K_t.

Finally, define the measure $d\mu_{\mathcal{C}}$ on the conjugacy class $\mathcal{C} = \mathcal{C}(v,t) = \{(k\cdot(v+v_1), ktk^{-1}) : k \in K, v_1 \in V^t\}$ by $d\mu_{\mathcal{C}} = \kappa_1(t)dv_1 d\dot{k}$.

We have shown:

Proposition 2.6. *The measure $d\mu_{\mathcal{C}}$ is an invariant measure on $\mathcal{C}(v,t)$.*

Following the same path as in Proposition 2.6, we may define an associated measure on the adjoint orbits. Each adjoint orbit $\mathcal{O}$ contains a unique point of the form (a_H, H) where $H \in \mathfrak{t}^+$ and $a_H \in V_H$. The fibre at this point is

$$V^H = \oplus_{\{\lambda:\lambda(H)\neq 0\}} V_\lambda,$$

and V_λ is a $2m_\lambda$-dimensional eigenspace for the $\mathfrak{t}$ action with eigenvalue $\lambda(H)$, so we may define a measure on V^H by $da_1 = \Pi_{\{\lambda:\lambda(H)\neq 0\}}\, dv_\lambda$, and a constant

$$\kappa_2(H) = \Pi_{\lambda:\lambda(H)\neq 0}\, \lambda(H)^{2m_\lambda}\, \Pi_{\{\alpha\in\Phi^+:\alpha(H)\neq 0\}}\, \alpha(H)^2. \tag{15}$$

This leads to:

Proposition 2.7. *An invariant measure on the adjoint orbit $\{(k\cdot(a_H + a_1),\, \mathrm{Ad}(k)H) : k \in K,\, a_1 \in V^H\}$, where $a_H \in V_H$, is given by*

$$d\mu_{\mathcal{O}}(a_1, \dot{k}) = \kappa_2(H)\, da_1\, d\dot{k}, \tag{16}$$

where $d\dot{k}$ is the normalized measure on K/K_H and a is written in the $\oplus_{\{\lambda:\lambda(H)\neq 0\}} V_\lambda$ coordinates as (a_λ).

Proposition 2.8. *Suppose that the exponential map takes $\mathcal{O}$ to C. Then*

$$d\mu_C = J\, d\mu_{\mathcal{O}}.$$

Proof. This follows exactly as in the compact case once we write out the expression for $J(H) = \det\left(\frac{1-e^{-H}}{H}\big|_V\right) J_K(H)$ for $H \in \mathfrak{t}$, using the basis $V = \oplus V_\lambda$. With this basis, the first determinant becomes

$$\prod_{\lambda\in\Lambda} \left(\frac{1 - e^{-i\lambda(H)}}{\lambda(H)}\right)^{m_\lambda} = \prod_{\lambda\in\Lambda} \left(\frac{2\sin\frac{\lambda(H)}{2}}{\lambda(H)}\right)^{m_\lambda}.$$

Now using the result for K and comparing the formulae given above, the result follows. ■

Note that neither adjoint orbits nor conjugacy classes are symplectic manifolds (in general), so we cannot expect a canonical Liouville measure. Nevertheless, the above choices are natural and compatible.

3. The coadjoint action

The coadjoint orbits may be described similarly to the adjoint orbits — although unlike the compact case, there is not a one-to-one G-correspondence between $\mathfrak{g}$ and $\mathfrak{g}^*$: the Killing form is degenerate.

We may identify the dual of $\mathfrak{g}$ as $V^* \oplus \mathfrak{k}^*$. The coadjoint action is defined by $\langle \mathrm{Ad}^*(g)\beta, X\rangle = \langle \beta, \mathrm{Ad}(g^{-1})X\rangle$ for $\beta \in \mathfrak{g}^*$ and $X \in \mathfrak{g}$, and may be conveniently written as

$$\mathrm{Ad}^*(v,k)(\varphi, f) = (k\cdot\varphi,\, k\cdot f + v\times(k\cdot\varphi)), \tag{17}$$

where $v \times \varphi \in \mathfrak{k}^*$ is defined by

$$(v \times \varphi)(A) = (\varphi, A \cdot v) \qquad \text{for } A \in \mathfrak{k}.$$

This notation, generalizing the traditional crossed product of vector calculus, is due to Rawnsley [13], see also [12].

Note that by our assumption that the action of K is effective on V, we see that

$$\mathfrak{k}^* = \{v \times \varphi : v \in V, \varphi \in V^*\}.$$

The following Lemma, which gives the structure of the coadjoint orbits, is now obvious.

Lemma 3.1. *The coadjoint orbit through $(\varphi, f) \in V^* \times \mathfrak{k}^*$ is given by*

$$\{k \cdot (\varphi, f) + v \times (k \cdot \varphi) : v \in V, k \in K\}.$$

This is fibred over the K orbit $K \cdot (\varphi, f)$ with fibre at the point $(k \cdot \varphi,\ \mathrm{Ad}^*(k)f)$ being $V \times k \cdot \varphi$.)

Let $\mathfrak{k}_\varphi$ denote the stabilizer of φ in $\mathfrak{k}$, $\mathfrak{k}_\varphi = \{A \in \mathfrak{k} : A \cdot \varphi = 0\}$. Use the K-invariant form on $\mathfrak{k}$ to form $\mathfrak{k}^\perp$. Similarly, use the form to define an Ad^*-invariant form on $\mathfrak{k}^*$ and notice that $\mathfrak{k}_\varphi^* = \{f \in \mathfrak{k}^* : f(A) = 0 \ \text{ if } \ A \perp \mathfrak{k}_\varphi\}$. Furthermore, $\mathfrak{k}_\varphi^\perp = V \times \varphi$. To see this, notice that for $v \in V$ and $A \in \mathfrak{k}_\varphi$, $(v \times \varphi)(A) = \varphi(A \cdot v) = (A \cdot \varphi, v) = 0$. Since $v \times \varphi = 0$ if and only if $v \perp \mathfrak{k} \cdot \varphi$, we see that $\dim(\mathrm{V} \times \varphi) = \dim(\mathfrak{k} \cdot \varphi)$, which is the dimension of $(\mathfrak{k}_\varphi^\perp)$. Thus the two spaces are equal. From this it follows that there is a canonical point on each fibre. Specifically:

Lemma 3.2. *On any fibre there is a point (φ, f) with $f \in \mathfrak{k}_\varphi^*$.*

Proof. Let (φ, f) be any point on the orbit, and write $f = f_1 + f_2$ with $f_1 \in \mathfrak{k}_\varphi^*$ and $f_2 \in k_\varphi^{*\perp}$. Choose $v \in V$ so that $v \times \varphi = f_2$; we have

$$\mathrm{Ad}^*(-v, e)(\varphi, f) = (\varphi, f_1). \tag{18}$$

■

We shall sometimes denote the choice of such a pair by writing $(\varphi, f) \in V^* \times \mathfrak{k}_\varphi^*$.

Corollary 3.3. *If $(\varphi, f) \in V^* \times \mathfrak{k}_\varphi^*$, the stabilizer of (φ, f) (under the coadjoint action) is $V_\varphi \rtimes (K_\varphi)_f$, where $V_\varphi = \{v \in V : v \times \varphi = 0\}$ and $(K_\varphi)_f$ is the stabilizer of f in K_φ.*

We may now describe the canonical Liouville measure on the coadjoint orbit through $(\varphi, f) \in V^* \times \mathfrak{k}_\varphi^*$. Let K_φ be the stabilizer of φ in K. The Lie algebra of the connected component of K_φ is $\mathfrak{k}_\varphi$. We let $f \in \mathfrak{k}_\varphi^*$. The stabilizer of f in K_φ is denoted $K_{\varphi,f}$ and its Lie algebra, $\mathfrak{k}_{\varphi,f}$ is generically a Cartan subalgebra in $\mathfrak{k}_\varphi$.

Clearly, the K-orbit of (φ, f) may be canonically identified with $K/K_{\varphi,f}$, and this coset space carries a unique normalized K-invariant measure.

Now K_φ need not be connected, but its connected component $(K_\varphi)_e$ is a compact connected Lie group whose Lie algebra is $\mathfrak{k}_\varphi$. Thus, since $f \in \mathfrak{k}_\varphi^*$, $((K_\varphi)_e)_f$

is also connected — this is a fundamental property of the Adjoint representation. Now $((K_\varphi)_f)_e$ is also a connected Lie group, which contains $((K_\varphi)_e)_f$ (since the latter is connected). Since these two connected Lie groups have the same Lie algebra, they are equal.

Now $K_\varphi/(K_\varphi)_e$ is a compact discrete, hence finite, group. Let us denote it by F_φ; we may then realize K_φ as $F_\varphi \ltimes (K_\varphi)_e$. It follows that

$$(K_\varphi)_f = \{(\ell, k) : \ell \in F_\varphi \text{ and } k \in (K_\varphi)_e \text{ and } \ell k f = f\}.$$

In a similar manner, $(K_\varphi)_f/((K_\varphi)_f)_e$ is a finite group and by the comments of the previous paragraph, it may be realized as a subgroup $F_{\varphi,f}$ of F_φ.

The canonical 2-form on the orbit of (φ, f) may be calculated as follows. Firstly, notice that for (a, A) and $(a_1, A_1) \in V \oplus \mathfrak{k}$, we have

$$\begin{aligned} ((\varphi, f), [(a, A), (a_1, A_1)]) &= \varphi(A \cdot a_1) - \varphi(A_1 \cdot a) + (f, [A, A_1]) \\ &= a_1 \times \varphi(A) - a \times \varphi(A_1) + w_f([A, A_1]). \end{aligned} \tag{19}$$

Decomposing $A = B + C$ and $A_1 = B_1 + C_1$, where $B, B_1 \in \mathfrak{k}_\varphi$ and $C, C_1 \in \mathfrak{k}_\varphi^\perp$, we get

$$\omega_{(\varphi,f)}((a, A), (a_1, A_1)) = (a_1 \times \varphi)(C) - (a \times \varphi)(C_1) + \omega_f(C, C_1). \tag{20}$$

The last term passes to the canonical symplectic 2-form on the $(K_\varphi)_e$ orbit through f, and its wedge product with itself $\frac{1}{2}(\dim\mathfrak{k}_\varphi - \dim(\mathfrak{k}_\varphi)_f)$ times is nothing but the Liouville orbital measure of $d\mu_{\mathcal{O}f}$.

The first two terms of this expression can be understood as $\psi \wedge \psi$, where ψ is the 1-form on $\mathfrak{k} \cdot \varphi \times \mathfrak{k}_\varphi^\perp$ given by

$$\psi(a, C) = (a \times \varphi)(C). \tag{21}$$

To evaluate the wedge product of this 1-form with itself $(\dim \mathfrak{k}.\varphi) + \dim \mathfrak{k}_\varphi^\perp = 2\dim \mathfrak{k}_\varphi^\perp$ times, note that the tangent space of the orbit $K\varphi$ at φ may be canonically identified as $\mathfrak{k} \cdot \varphi$ (a subspace of V). As observed above, $\mathfrak{k}_\varphi^\perp$ (a subspace of $\mathfrak{k}$) has the same dimension, and indeed, the pairing $\mathfrak{k}_\varphi^\perp \times \mathfrak{k}.\varphi \to \mathbb{C}$ given by $(C, a) = a \times \varphi(C)$ identifies $\mathfrak{k}_\varphi^\perp$ as $(K \cdot \varphi)^*$. The symplectic form

$$\langle (C, a), (C_1, a_1) \rangle = (a_1 \times \varphi)(C) - (a \times \varphi)(C_1) \tag{22}$$

then identifies $\psi \wedge \psi$ as the canonical symplectic form on the cotangent bundle.

The associated measure on $\mathfrak{k} \cdot \varphi$ may be described in coordinates as follows. By Lemma 3.1, we may assume without loss of generality that $C \in \mathfrak{t}$. Then if $a = \sum a_\lambda$, we have $(a \times \varphi)(C) = \varphi(C.a) = \sum_{\lambda \in \Lambda} \lambda(C)\varphi(a_\lambda)$.

Define the constant

$$\kappa_3(C, a) = \Pi_{\lambda \in \Lambda}\, \lambda(C)\varphi(a_\lambda). \tag{23}$$

Thus we have proved

Proposition 3.4. *In the coordinates introduced above, the Liouville measure on the coadjoint orbit through $(\varphi, f) \in V^* \times \mathfrak{k}_\varphi^*$ is*

$$\kappa_3(C, a)\, da\, dC\, d\mu_f(B). \tag{24}$$

4. The representation theory of G

The description of the irreducible representations of $V \rtimes K$ is standard, using the Mackey machine. The relationship between the Mackey induced representation and the Kirillov orbit method is explained in [13]. We give a brief outline of how the correspondence works in the notation above.

According to Mackey, the irreducible representations of $V \rtimes K$ are in one-one correspondence with pairs (φ, η), where $\varphi \in V^*$ and $\eta \in \widehat{K_\varphi}$. Elements φ of the same K-orbit in V^* and equivalent irreducible representations η of K_φ give equivalent representations of $V \rtimes K$. The corresponding irreducible representation of $V \rtimes K$ is the induced representation

$$\varrho_{(\varphi,\eta)} = e^{i\varphi} \otimes \eta \uparrow_{V \rtimes K_\varphi}^{V \rtimes K}.$$

The orbit method description of $\varrho_{(\varphi,\eta)}$ is achieved as follows. Note that $\eta \in \widehat{K_\varphi}$ and $K_\varphi = (K_\varphi)_e \rtimes F_\varphi$, we may choose $\eta_0 \in \widehat{(K_\varphi)_e}$ and $\phi \in \widehat{F_{\varphi_{\eta_0}}}$ so that η is induced from $\tilde{\eta}(k, f) = \eta_0(k)\phi(f)$.

To relate this to the method of orbits approach to the representations of solvable Lie groups detailed by Auslander and Kostant [2], we remark that $(K_\varphi)_e$ has Lie algebra $\mathfrak{k}_\varphi$ and so η_0 has highest weight, say $f \in \mathfrak{t}_\varphi^*$, where $\mathfrak{t}_\varphi$ is a Cartan subalgebra of $\mathfrak{k}_\varphi^{\mathbb{C}}$. Letting Φ_φ^+ be a choice of positive roots of $(\mathfrak{k}_\varphi, \mathfrak{t}_\varphi)$, we have

$$(\mathfrak{k}_\varphi)_f = \mathfrak{t}_\varphi \oplus \sum_{\{\alpha \in \Phi_\varphi : <f,\alpha>=0\}} \mathfrak{k}_\varphi^{(\alpha)}$$

and the associated polarisation of $\mathfrak{k}_\varphi$ at f is

$$\mathfrak{h}_{\varphi,f} = \mathfrak{t}^{\mathbb{C}} \oplus \sum_{\{\alpha \in \Phi_\varphi : <f,\alpha> \geq 0\}} \mathfrak{k}_\varphi^{(\alpha)}.$$

We now describe polarisations at (φ, f) for $\mathfrak{g}$. We saw in Corollary 3.3, that the stabilizer of (φ, f) is $V_\varphi \rtimes (K_\varphi)_f$. Thus at the Lie algebra level, the polarisation must contain $V_\varphi^{\mathbb{C}} \oplus \mathfrak{h}_{\varphi,f}$, if it is to be compatible with the choice of polarisation for f outlined above.

In section 3. we wrote $V = V_\varphi \oplus \mathfrak{k} \cdot \varphi$ and $\mathfrak{k} = \mathfrak{k}_\varphi \oplus \mathfrak{k}_\varphi^\perp$. So far, we have chosen a polarising space for $V_\varphi \oplus \mathfrak{k}_\varphi$. We now must add a polarising space for $\mathfrak{k} \cdot \varphi \oplus \mathfrak{k}_\varphi^\perp$.

Writing the $\mathfrak{k}$ coordinate as $A = (B, C)$ with respect to the first decomposition and the V coordinates as $a = (w, u)$ with respect to the second, we have the pairing:

$$\psi(u, C) = (u \times \varphi)(C)$$

between $u \in \mathfrak{k} \cdot \varphi$ and $C \in \mathfrak{k}_\varphi^\perp$. We take a maximal isotropic subspace of $\left(\mathfrak{k} \cdot \varphi \oplus \mathfrak{k}_\varphi^\perp\right)^{\mathbb{C}}$ relative to the symplectic form $\psi \wedge \psi$ given in equation (22)

By using the K-invariant inner product in V, we can identify $\mathfrak{k} \cdot \varphi$ with a subspace of V. In section 3. we wrote $V = V_\varphi \oplus \mathfrak{k} \cdot \varphi$ and $\mathfrak{k} = \mathfrak{k}_\varphi \oplus \mathfrak{k}_\varphi^\perp$. So far, we have chosen a polarising space for $V_\varphi \oplus \mathfrak{k}_\varphi$. We now must add a polarising space for $\mathfrak{k} \cdot \varphi \oplus \mathfrak{k}_\varphi^\perp$.

An obvious choice of maximal isotropic space $\mathfrak{h}_\varphi^1$ above is to choose $\mathfrak{h}_\varphi^1 = (\mathfrak{k} \cdot \varphi)^{\mathbb{C}}$. In this case we get $\mathfrak{h} = V \oplus \mathfrak{h}_{\varphi,f}$ and, if η denotes the representation of $\mathfrak{k}_\varphi$

associated to $\mathfrak{h}_{\varphi,f}$, then the associated representation of $V \rtimes K$ may be calculated as acting by the left regular representation in

$$\begin{aligned}
\mathcal{H}_{\varphi,f} &= \{\phi \in C^\infty(V \rtimes K) : X\phi = 2\pi i(\varphi, f)(X)\phi \qquad \forall\ X \in \mathfrak{h}\} \\
&= \big\{\phi \in C^\infty(V \rtimes K) : \phi(v,k) = e^{2\pi i\varphi(v)}\phi(0,k) \\
&\qquad\qquad \text{and} \quad X\phi(0,k) = 2\pi i f(X)\phi(0,k) \quad X \in \mathfrak{h}_{\varphi,f}\} \\
&= \big\{\phi \in C^\infty(V \rtimes K, \mathcal{H}_\eta) : \phi(v,k_1) = e^{2\pi i\varphi(v)}\eta(k_1)\phi(0,k) \quad \forall\ k \in K_\varphi\big\}
\end{aligned}$$

This space may be put into 1-1 correspondence with a subspace of $L^2(K/K_\varphi)$, via $\mathcal{F}(\phi)(k) = \phi(0,k)$. Hence it inherits a norm. The completion of $\mathcal{H}_{\varphi,f}$ under this norm is exactly the Mackey induced representation $e^{i\varphi} \otimes \eta \uparrow_{V\times K_\varphi}^{V\times K}$.

5. Fourier transforms

Given the pairing $\beta(X)$ between $\mathfrak{g}^*$ and $\mathfrak{g}$, we can define the Fourier transform of a function f on $\mathfrak{g}$ to be the function $\widehat{f}$ on $\mathfrak{g}^*$ defined by

$$\widehat{f}(\beta) = \int f(X)e^{i\beta(X)}dX \tag{25}$$

(provided the integral is defined) and similarly, if ψ is a function on $\mathfrak{g}^*$, its inverse Fourier transform is the function on $\mathfrak{g}$ defined by

$$\widehat{\psi}(X) = \int \psi(\beta)e^{-i\beta(X)}d\beta. \tag{26}$$

These definitions may be extended in the usual way to certain classes of distributions on $\mathfrak{g}$ and $\mathfrak{g}^*$. (We are so far being imprecise about what conditions are necessary on the distributions for the integrals to make sense.) Notice that this definition intertwines the adjoint and coadjoint action of G — provided that G-invariant Lebesgue measures have been chosen on $\mathfrak{g}$ and $\mathfrak{g}^*$, we have

$$\widehat{f}(g\cdot\beta) = (gf)\widehat{\ }(\beta) \qquad \widehat{\psi}(gX) = (g\psi)\widehat{\ }(X).$$

Thus, in very general terms, if the Fourier transform of an adjoint orbit exists, then it should be able to be written as a combination of coadjoint orbits. (And the same can be said interchanging "adjoint" and "coadjoint".) In this section, we aim to show that the distributional Fourier transforms of adjoint orbits and inverse Fourier transforms of coadjoint orbits both exist, and to find formulae for them.

Of course, unlike the compact case, these orbits are in general non-compact sets, so we cannot expect to be dealing with functions: the associated distributions need to be defined as "principal value at infinity" integrals. Recall that in the case of a compact group, the Fourier transform $\widehat{\mu_{\mathcal{O}}}(X)$ of a coadjoint orbit is a generalized Bessel function $\mathbf{J}_{\mathcal{O}}(X)$. (See [7].)

However, our orbits are fibred over compact orbits. Now the Fourier transform of the constant function $\mathbf{1}$ on $\mathbb{R}$ is δ_0 (in a suitable p.v. sense) and similarly, the Fourier transform of a subspace P in $\mathbb{R}^n$ "is" the orthogonal $P^\perp$ of P. We begin by recalling how this works in the $\mathbb{R}^n$ case.

Let P be a vector subspace of $\mathbb{R}^n$. Choose a Lebesgue dx measure on P. For each $R > 0$, let B_R be a ball of radius R in $\mathbb{R}^n$, and for $\phi \in C_c^\infty(\mathbb{R}^n)$, let

$$\langle \chi_P, \phi 1\rangle = \lim_{R\to\infty} \int_{B_R\cap P} \phi(x)dx. \tag{27}$$

This defines χ_P as a p.v. distribution. Then it is a simple exercise to show that for every $u \in C_c^\infty(\mathbb{R}^n)$

$$\langle \widehat{\chi}_P, u\rangle = \lim_{R\to\infty} \int \int_{B_R\cap P} e^{i\xi(x)}dx\, u(\xi)d\xi \tag{28}$$

exists and equals $\langle \chi_{P^\perp}, u\rangle$, where $P^\perp$ is the subspace of $\mathbb{R}^n$ given by

$$P^\perp = \{\xi : \xi(x) = 0 \ \forall x \in P\}. \tag{29}$$

Thus $\widehat{\chi}_P = \chi_{P^\perp}$.
We now use these ideas to define and compute the Fourier transform of an adjoint orbit. Let $B_R \subseteq V$ be a ball of radius $R > 0$ and let $B'_R \subset \mathfrak{k}$ be a ball of radius R with respect to the negative Killing form on $\mathfrak{k}$.

We define $\widetilde{B_R}$ to be $B_R \times B'_R$ and consider the Radon measure $\mu_{\mathcal{O}}$, concentrated on the adjoint orbit $\mathcal{O}$, which is defined for $u \in C_c^\infty(\mathfrak{g})$ by

$$\langle \mu_{\mathcal{O}}, u\rangle \ = \textstyle\int_{\mathcal{O}} u(X)d\mu_{\mathcal{O}}(X) = \lim_{R\to\infty} \int_{\mathcal{O}\cap\widetilde{B_R}} u(X)d\mu_{\mathcal{O}}(X) \tag{30}$$

The following theorem is a direct result of the $\mathbb{R}^n$ case studied above.

Theorem 5.1. *Let $\phi \in C_c^\infty(\mathfrak{g}^*)$ be a K-invariant function, and let $\mathcal{O}$ be an adjoint orbit. The limit*

$$\langle \widehat{\mu}_{\mathcal{O}}, \phi\rangle = \lim_{R\to\infty} \int_{\mathfrak{g}^*} \left\{ \int_{\mathcal{O}\cap\widetilde{B_R}} e^{i\beta(X)}d\mu_{\mathcal{O}}(X)\right\}\phi(\beta)d\beta \tag{31}$$

exists and defines a distribution in $\mathcal{D}(\mathfrak{g}^*)$.

It turns out that the Fourier transform can be given as a locally integrable function. We now give an explicit formula for this function in terms of generalised Bessel functions.

Definition 5.2. For $X \in \mathfrak{g}$ and $\beta \in \mathfrak{g}^*$, we define

$$J_X(\beta) = \kappa_2(H) \int_K e^{i\beta(\mathrm{Ad}(k)X)}dk.$$

where X is conjugate under the K-action to $H \in \mathfrak{t}$, and $\kappa_2(H)$ is the constant defined in (15). The functions J_X will be known as *generalised Bessel functions.*

These expressions are to be compared with the expressions for the generalized Bessel function for a compact Lie group given in Clerc [4].

Notice that we have, as with the standard Bessel functions, for all $k \in K$, $J_X(\beta) = J_{\mathrm{Ad}(k)X}(\beta) = J_X(\mathrm{Ad}^*(k^{-1})\beta)$.

Notice also that if $X = (a_H, H)$ with $H \in \mathfrak{t}$ and $a_H \in V_H$, the integrand is invariant under the action of K_H, as is the coefficient κ_2; so we can write

$$J_X(\beta) = \kappa_2(H) \int_{K/K_H} e^{i\beta(\mathrm{Ad}(k)X)} d\dot{k}.$$

where $d\dot{k}$ denotes normalised invariant measure on K/K_H.

Proposition 5.3. *Let $\mathcal{O}$ be an adjoint orbit. Then $\widehat{\mu_{\mathcal{O}}}$ is an* Ad^**- invariant locally integrable function on $\mathfrak{g}^*$ defined as follows. Choose $(a_H, H) \in \mathcal{O}$, where $H \in \mathfrak{t}^+$ and $a_H \in V_H$ (c.f. the discussion preceding Proposition (2.7)). Then*

$$\widehat{\mu_{\mathcal{O}}}(\beta) = J_{(a_H,H)}(\varphi, f)$$

where $\beta = (\varphi, f)$ with $f \in \mathfrak{k}_\varphi$.

Proof. Let $\phi \in C_c^\infty(\mathfrak{g}^*)$.

$$\langle \widehat{\mu}_{\mathcal{O}}, \phi \rangle = \lim_{R\to\infty} \int_{\mathfrak{g}^*} \int_{\mathcal{O}\cap\widetilde{B}_R} e^{i\beta X} d\mu_{\mathcal{O}}(X)\phi(\beta)d\beta.$$

Note that, since ϕ has compact support, we may exchange the limit and the outer integral. We calculate the inner integral using Proposition (2.7).

$$\begin{aligned}
\textstyle\int_{\mathcal{O}\cap\widetilde{B}_R} e^{i\beta X} d\mu_{\mathcal{O}}(X) &= \kappa_2(H) \textstyle\int_{K/K_H} \int_{V^H\cap B_R} e^{i\beta(k\cdot(a_H+a_1),\mathrm{Ad}(k)H)} da_1 d\dot{k} \\
&= \kappa_2(H) \textstyle\int_{K/K_H} \int_{V^H\cap B_R} e^{i\varphi(k\cdot(a_H+a_1))} e^{if(\mathrm{Ad}(k)H)} da_1 d\dot{k} \\
&= \kappa_2(H) \textstyle\int_{K/K_H} \int_{V^H\cap B_R} e^{ik\cdot\varphi(a_H+a_1)} da_1 e^{i\mathrm{Ad}^*(k)f(H)} d\dot{k}
\end{aligned}$$

Taking the limit as $R \to \infty$, we may first pass the limit inside the K/K_H integral, which is compact. The limit of the inner integral, which is purely Euclidean, may then be evaluated by (27), obtaining $e^{ik\cdot\varphi(a_H)}$, since V_H is orthogonal to V^H. Thus

$$\widehat{\mu}_{\mathcal{O}}(\beta) = \kappa_2(H) \int_{K/K_H} e^{ik\cdot\varphi(a_H)} e^{i\mathrm{Ad}^*(k)f(H)} d\dot{k}$$

The right hand side of this equation coincides with the definition of the generalised Bessel function $J_{(a_H,H)}(\varphi, f)$.

We can thus write

$$\widehat{\mu}_{\mathcal{O}}(\beta) = J_{(a_H,H)}(\varphi, f)$$

as claimed. ∎

The above formula serves as a model for the introduction of spaces of invariant distributions in the next section.

6. Convolution structures

In order to prove our main theorem — an analogue of Theorem 1 of [5] — we need to define a suitable notion of convolution. This must be defined at least on the conjugacy classes, the adjoint orbits and the coadjoint orbits of G. In fact, we will define a notion of convolution for spaces of distributions invariant under conjugation, the adjoint representation or the coadjoint representation respectively, and show that our spaces include the orbital measures defined in sections 2. and 3..

In the case of compact Lie groups, where the orbits are of compact support, G-invariant distributions of compact support are the appropriate space to use. Here, however, the definition of the space of distributions is more delicate, and is achieved by the use of a space of test functions $\mathcal{AP}_G$ which is an amalgam of C^∞ and the space $\mathcal{AP}$ of almost periodic functions. We denote this space of G-almost periodic functions by $\mathcal{AP}_G$. Our G-invariant distributions live in the duals of these spaces.

Recall that in $\mathbb{R}^n$ a function is said to be **almost periodic** if it is the uniform limit of trigonometric polynomials.

Our spaces of test functions will be defined for each of conjugation action, the adjoint action and the coadjoint action. Thus, will define three spaces: $\mathcal{AP}_G(G)$, $\mathcal{AP}_G(\mathfrak{g})$ and $\mathcal{AP}_G(\mathfrak{g}^*)$, and develop their properties.

Definition 6.1. For $G = V \rtimes K$, let

$$\begin{aligned}
\mathcal{AP}_G(G) &= \{h \in C^\infty(V \rtimes K) : (\forall k \in K)\, v \mapsto h(v,k) \text{ is almost periodic on } V\} \\
\mathcal{AP}_G(\mathfrak{g}) &= \{h \in C^\infty(V \oplus \mathfrak{k}) : (\forall A \in \mathfrak{k})\, v \mapsto h(v,A) \text{ is almost periodic on } V\} \\
\mathcal{AP}_G(\mathfrak{g}^*) &= \{h \in C^\infty(V^* \oplus \mathfrak{k}^*) : (\forall \varphi \in V^*) f \mapsto h(\varphi,f) \text{ is almost periodic on } \mathfrak{k}^*\}
\end{aligned}$$

We shall refer to these spaces of functions as *G-almost periodic functions on G*, $\mathfrak{g}$ and $\mathfrak{g}^*$ respectively.

We define topologies on these spaces as follows. Choose any basis $X_1, \dots X_N$ for $\mathfrak{g}$ and its dual basis $X_1^*, \dots X_N^*$ for $\mathfrak{g}^*$. We will use the usual multi-index notation, so that $D^\alpha h = X_1^{\alpha_1} \dots X_n^{\alpha_n} h$ for $h \in C^\infty(G)$, and similarly for $h \in C^\infty(\mathfrak{g})$ or $D^\alpha h = (X_1^*)^{\alpha_1} \dots (X_n^*)^{\alpha_n} h$ for $h \in C^\infty(\mathfrak{g}^*)$.

For $AP_G(G)$, we use the seminorms

$$\rho^N_{V\times B}(h) = \sup_{|\alpha| \le N,\, x \in V\times B} |D^\alpha h(x)|,$$

where B runs through all compact subsets of K and $N \in \mathbb{N}$.

For $\mathcal{AP}_G(\mathfrak{g})$ we use

$$\rho^N_{V\times B}(h) = \sup_{|\alpha| \le N,\, x \in V\times B} |D^\alpha h(x)|,$$

where B runs through all compact subsets of $\mathfrak{k}$ and $N \in \mathbb{N}$.

For $\mathcal{AP}_G(\mathfrak{g}^*)$, we use

$$\rho^N_{V\times B}(h) = \sup_{|\alpha| \le N,\, x \in V\times B} |D^\alpha h(x)|,$$

where B runs through all compact subsets of $\mathfrak{k}^*$ and $N \in \mathbb{N}$.

An equivalent way of defining these seminorms is to compactify V (respectively $\mathfrak{k}^*$) and define the usual C^∞ topology by taking local convergence of all derivatives. It is clear from the definition together with the fact that a uniform limit of almost periodic functions on $\mathbb{R}^n$ is again almost periodic, that the spaces are complete in the topology defined by these semi-norms.

The basic properties of G-almost periodic functions which we shall need are contained in the following propositions.

Lemma 6.2. (i) *$\mathcal{AP}_G(G)$ is a closed linear subspace of $C^\infty(G)$ when equipped with the above topology.*

(ii) *If $h \in \mathcal{AP}_G(G)$, then for all $g \in G$, both ${}_gh : x \mapsto h(gx)$ and $h_g : x \mapsto h(xg)$ belong to $\mathcal{AP}_G(G)$; and hence so does $x \mapsto h(g^{-1}xg)$.*

Proof. The first statement follows from the fact that the almost periodic functions are a linear space, and that a uniform limit of almost periodic functions is almost periodic ([14] Ch 12, ex.29).

For the second, let $g = (v_0, k_0)$ and notice that

$$ {}_gh(v,k) = h((v_0,k_0)(v,k)) = h(v_0 + k \cdot v, k_0k). $$

Now the space of almost periodic functions on V is translation invariant and rotationally invariant, which proves the result.

The statement for right translations follows similarly. ■

Lemma 6.3. (i) *$\mathcal{AP}_G(\mathfrak{g})$ is a closed linear subspace of $C^\infty(\mathfrak{g})$ when equipped with the above topology.*

(ii) *If $h \in \mathcal{AP}_G(\mathfrak{g})$, then for all $X_0 \in \mathfrak{g}$*

$$ X \mapsto h(X + X_0) \in \mathcal{AP}_G(\mathfrak{g}). $$

(iii) *If $h \in \mathcal{AP}_G(\mathfrak{g})$ then for all $g \in G$*

$$ X \mapsto h(\mathrm{Ad}(g)X) \in \mathcal{AP}_G(\mathfrak{g}). $$

Proof. (i) follows as in the previous lemma, as does (ii). For (iii), use Lemma 2.2 to write

$$ h(\mathrm{Ad}(v,k)(a,A)) = h(k \cdot a - (\mathrm{Ad}(k)A) \cdot v, \mathrm{Ad}(k)A) $$

and again notice that in the first variable we have a translation and a rotation. ■

Lemma 6.4. (i) *$\mathcal{AP}_G(\mathfrak{g}^*)$ is a closed linear subspace of $C^\infty(\mathfrak{g}^*)$.*

(ii) *If $h \in \mathcal{AP}_G(\mathfrak{g}^*)$ and $\alpha \in \mathfrak{g}^*$ then*

$$ \beta \mapsto h(\alpha + \beta) \in \mathcal{AP}_G(\mathfrak{g}^*). $$

(iii) *If $h \in \mathcal{AP}_G(\mathfrak{g}^*)$ then for all $g \in G$*

$$ \beta \mapsto h(\mathrm{Ad}^*(g)\beta) \in \mathcal{AP}_G(\mathfrak{g}^*). $$

Proof. (i) and (ii) follow as before. For (iii), we use (17) to see that

$$h(\mathrm{Ad}^*(v,k)(\varphi,f)) = h(k\cdot\varphi, k.f + v\times(k\cdot\varphi)),$$

and the result follows as before. ■

Furthermore, the spaces $\mathcal{AP}_G(G)$ and $\mathcal{AP}_G(\mathfrak{g}^*)$ are related by the exponential map.

Lemma 6.5. *Let $h\in\mathcal{AP}_G(G)$. Then $h\circ\exp\in\mathcal{AP}_G(\mathfrak{g})$.*

Proof. It is clear that $h\circ\exp\in C^\infty(\mathfrak{g})$.

It remains to check the almost periodicity. By (5), we have

$$h(\exp(a,A)) = h((\frac{I-e^{-A}}{A})\cdot a, A).$$

Since $a\mapsto(\frac{I-e^{-A}}{A})\cdot a$ is a linear map on V, and since composition of linear transformations with almost periodic functions are almost periodic, the result is true. ■

We now begin consideration of the natural analogues of G-invariant distributions of compact support, which were important in the compact case. These will be defined in each of the three cases as above.

Definition 6.6. (i) Let $\mathcal{AP}'_G(G)$ denote the set of continuous linear functionals ϕ on $\mathcal{AP}_G(G)$, where the pairing is denoted $\langle\phi,h\rangle$. We shall say that ϕ is G-invariant if for all $g\in G$,

$$\langle\phi,h^{(g)}\rangle = \langle\phi,h\rangle,$$

where $h^{(g)}(x) = h(g^{-1}xg)$.

Denote the set of G-invariant elements of $\mathcal{AP}'_G(G)$ by $\mathcal{I}(G)$.

(ii) Let $\mathcal{AP}'_G(\mathfrak{g})$ denote the set of continuous linear functionals ϕ on $\mathcal{AP}_G(\mathfrak{g})$, where the pairing is also denoted $\langle\phi,h\rangle$. We shall say that ϕ is G-invariant if for all $g\in G$,

$$\langle\phi,h\circ\mathrm{Ad}(g)\rangle = \langle\phi,h\rangle.$$

Denote the set of G-invariant elements of $\mathcal{AP}'_G(\mathfrak{g})$ by $\mathcal{I}(\mathfrak{g})$.

(iii) Let $\mathcal{AP}'_G(\mathfrak{g}^*)$ denote the set of continuous linear functionals ϕ on $\mathcal{AP}_G(\mathfrak{g}^*)$, where the pairing is also denoted $\langle\phi,h\rangle$. We shall say that ϕ is *G-invariant* if for all $g\in G$,

$$\langle\phi,h\circ\mathrm{Ad}^*(g)\rangle = \langle\phi,h\rangle.$$

Denote the set of G-invariant elements of $\mathcal{AP}'_G(\mathfrak{g}^*)$ by $\mathcal{I}(\mathfrak{g}^*)$.

In the case of compact groups, the orbital measures, that is, G-invariant measures concentrated on conjugacy classes, and on adjoint and coadjoint orbits and suitably normalised, play an important role, as the simplest invariant distributions. Before going on to develop the general theory, we give the analogues of orbital measures for the case of $G = V\rtimes K$. These are elements of $\mathcal{I}(G)$.

Definition 6.7. (i) For $h \in \mathcal{AP}_G(G)$, $k \in K$ and $a \in V$, we let

$$(Mh)(a,k) = \lim_{R\to\infty} \frac{1}{\lambda(B_R \cap V^k)} \int_{B_R \cap V^k} h(a+b,k)\, d\lambda(b), \tag{32}$$

where λ is any Lebesgue measure on the Euclidean space V^k, and B_R is the ball of radius R in V (c.f. the discussion preceding Theorem 5.1).

(ii) Let $h \in \mathcal{AP}_G(\mathfrak{g})$. For $A \in \mathfrak{k}$, and $a \in V_A$ define

$$(Mh)(a,A) = \lim_{R\to\infty} \frac{1}{\lambda(B_R \cap V^A)} \int_{B_R \cap V^A} h(a+b,A)\, d\lambda(b). \tag{33}$$

Here, λ denotes any Lebesgue measure on the Euclidean space V^A and B_R is as above.

(iii) For $h \in \mathcal{AP}_G(\mathfrak{g}^*)$, $\varphi \in V^*$, $f_0 \in \mathfrak{k}^*_\varphi$,

$$(Mh)(\varphi,f) = \lim_{R\to\infty} \frac{1}{\lambda(B_R \cap V \times \varphi)} \int_{B_R \cap V \times \varphi} h(\varphi, f+f_1)\, d\lambda(f_1), \tag{34}$$

where λ is any Lebesgue measure on $V \times \varphi \subseteq \mathfrak{k}$.

The fact that the limits exists in each of the three cases follows from the fact that we may take the invariant mean of an almost periodic function. These functions are pointwise limits of measurable functions and so are (jointly) measurable. Note also that the limit is in each case independent of the normalisation of λ.

Proposition 6.8. *Let $h \in \mathcal{AP}_G(G)$, $t \in T$, $a \in V$. If $\mathcal{C}$ denotes the conjugacy class of (a,t), let*

$$\langle \mu_{\mathcal{C}}, h\rangle = \kappa_1(t) \int_{K/K_t} (Mh)(k\cdot a, ktk^{-1})\, d\dot{k} \tag{35}$$

Then $\mu_{\mathcal{C}} \in \mathcal{I}(G)$.

Proof. This follows readily from the definition of $\mathcal{AP}_G(G)$. ∎

We shall refer to $\mu_{\mathcal{C}}$ as the *orbital distribution* of $\mathcal{C}$.

Proposition 6.9. *For $h \in \mathcal{AP}_G(\mathfrak{g})$, for $A \in \mathfrak{k}$ and for $a \in V_A$, we have $(\widetilde{Mh})(a,A) = \int_{K/K_A} (Mh)(k\cdot a, \mathrm{Ad}(k)A) d\dot{k}$ is an Ad-invariant function which is bounded and Borel measurable.*

Proof. We observed above that the integrand is measurable. It is clearly bounded since elements of $\mathcal{AP}_G$ are bounded. Since we are integrating over a compact set, the result is again bounded and measurable. By construction, it is constant on adjoint orbits. ∎

If $\mathcal{O}$ is the adjoint orbit of (a,A), we denote by $\mu_{\mathcal{O}}$ the associated orbital distribution in $\mathcal{I}(\mathfrak{g})$ defined by

$$\langle \mu_{\mathcal{O}}, h\rangle = \kappa_2(A)\widetilde{M}h(a,A). \tag{36}$$

It is also clear how to define orbital distributions on coadjoint orbits:

Proposition 6.10. *Let $\varphi \in V^*$, $f \in \mathfrak{k}_\varphi$, and $\mathcal{O}^*$ the coadjoint orbit through (φ, f). Let*

$$\langle \mu_{\mathcal{O}^*}, h\rangle = \kappa_3(\varphi, f)\int_{K/K_\varphi}(Mh)(\mathrm{Ad}^*(k)\varphi, k\cdot f)d\dot{k}. \tag{37}$$

Then $\mu_{\mathcal{O}^*} \in \mathcal{I}(\mathfrak{g}^*)$.

In section Proposition 5.3, we calculated the Fourier transform of $\mu_{\mathcal{O}}$ as a generalised Bessel function. It is not too hard to see that the Fourier transform belongs to $\mathcal{I}(\mathfrak{g}^*)$.

We now need to make the observation that each of these spaces of distributions is closed under convolution. Convolution will be defined in the usual fashion (Definition 6.12 below). To ensure that it is well-defined, we will need the following result.

Proposition 6.11. (i) *Let $h \in \mathcal{AP}_G(G)$ and $\phi \in \mathcal{AP}'_G(G)$. Then the function $g \mapsto \langle \phi(\,\cdot\,),\ h(g\,\cdot\,)\rangle$ belongs to $\mathcal{AP}_G(G)$.*

(ii) *Let $h \in \mathcal{AP}_G(\mathfrak{g})$ and $\phi \in \mathcal{AP}'_G(\mathfrak{g})$. Then the function $X \mapsto \langle \phi(\,\cdot\,),\ h(\,\cdot + X)\rangle$ belongs to $\mathcal{AP}_G(\mathfrak{g})$.*

(iii) *Let $h \in \mathcal{AP}_G(\mathfrak{g}^*)$. Then the function $\beta \mapsto \langle \phi(\,\cdot\,),\ h(\,\cdot\, + \beta)\rangle$ belongs to $\mathcal{AP}_G(\mathfrak{g}^*)$.*

Proof. We shall prove (ii) only — the proofs of (i) and (iii) are sufficiently similar to be left to the reader.

Recall that the dual space of $C^\infty(\mathbb{R}^n)$ is the space $\mathcal{E}(\mathbb{R}^n)$ of distributions of compact support.

Our theorem, of course, both extends and uses in its proof, the standard result that for $h \in C^\infty(\mathbb{R}^n)$ and $\phi \in \mathcal{E}(\mathbb{R}^n)$, $x \mapsto \langle \phi(\,\cdot\,), h(\,\cdot\, + x)\rangle$ belongs to $C^\infty(\mathbb{R}^n)$. (See [14] 6.35.) In this statement (and in the sequel) we use the conventional abuse of notation, writing ϕ as if it were a function.

We know that $h \in C^\infty(V \times K)$, and that for each fixed $k \in K$, $h(\cdot, k)$ is almost periodic on V.

Consider the function

$$\zeta : (v_0, k_0) \mapsto \langle \phi(v, k), h(v_0 + k_0 \cdot v, k_0 k)\rangle.$$

By the standard result quoted above, applied to the compactification in the V-direction, ζ belongs to $C^\infty(V \times K)$. Now fix $k_0 \in K$. The function $h(\cdot, k_0 k)$ may be approximated uniformly by trigonometric polynomials on V. Since the translation by $k_0 \cdot v$ of such a function may again be approximated uniformly by trigonometric polynomials, and since the $\mathcal{AP}_G(\mathfrak{g})$ topology is stronger than the uniform topology in the V direction, we see that $\zeta(\cdot, k)$ may also be uniformly approximated by trigonometric polynomials. Hence, $\zeta \in \mathcal{AP}(\mathfrak{g})$.

This completes the proof. ∎

Proposition 6.11 now allows us to define convolution of our spaces of distributions. This coincides with the usual definition

Definition 6.12. (i) Suppose that $\eta, \phi \in \mathcal{AP}'_G(G)$. We define $\eta * \phi$ by

$$\langle \eta * \phi, h \rangle = \langle \eta(g), \langle \phi(g_0), h(gg_0) \rangle \rangle$$

for all $h \in \mathcal{AP}_G(G)$.

(ii) Suppose that $\eta, \phi \in \mathcal{AP}'_G(\mathfrak{g})$. We define $\eta * \phi$ by

$$\langle \eta * \phi, h \rangle = \langle \eta(X), \langle \phi(Y), h(X+Y) \rangle \rangle$$

for all $h \in \mathcal{AP}_G(\mathfrak{g})$.

(iii) Suppose that $\eta, \phi \in \mathcal{AP}'_G(\mathfrak{g}^*)$. We define $\eta * \phi$ by

$$\langle \eta * \phi, h \rangle = \langle \eta(\beta), \langle \phi(\gamma), h(\beta+\gamma) \rangle \rangle$$

for all $h \in \mathcal{AP}_G(\mathfrak{g}^*)$.

It follows from Proposition 6.11 that the above convolutions are all well-defined. Furthermore, the convolution of two invariant distributions is again invariant. It follows from this proposition that convolutions of orbital distributions exist as elements of $\mathcal{I}$. We shall see how to compute these in some specific examples in section 8.

We now give the definition of the wrapping map for our spaces of distributions. This is a direct analogue of the definition from [5].

Definition 6.13. Let $\phi \in \mathcal{AP}'_G(\mathfrak{g})$. We define $\Phi\phi \in \mathcal{AP}'_G(G)$, the *wrap of* ϕ by

$$\langle \Phi\phi, h \rangle = \langle \phi, j \cdot h \circ \exp \rangle \tag{38}$$

for all $h \in \mathcal{AP}_G(G)$.

Φ is called the *wrapping map.*

The results we have proved so far show that this definition is valid. By Lemma 6.5, $h \in \mathcal{AP}_G(G)$ implies that $h \circ \exp \in \mathcal{AP}_G(\mathfrak{g})$, and hence, since j is Ad-invariant and bounded, $j \cdot h \circ \exp$ also belongs to $\mathcal{AP}_G(\mathfrak{g})$. Hence we have defined $\langle \Phi\phi, h \rangle$ for each $h \in \mathcal{AP}_G(G)$. It follows that $\Phi\phi \in \mathcal{AP}'_G(G)$.

It is clear that the wrap of an element of $\mathcal{I}(\mathfrak{g})$ belongs to $\mathcal{I}(G)$.

We may now state our main theorem. It is a direct analogue of Theorem 1 of [5].

Theorem 6.14. *Let* $\eta, \phi \in \mathcal{I}(\mathfrak{g})$. *Then*

$$\Phi(\eta * \phi) = \Phi(\eta) * \Phi(\phi). \tag{39}$$

(The convolution on the left-hand side is on $\mathfrak{g}$ and that on the right-hand side is on G.)

The proof of the theorem will occupy the remainder of this section. It will be preceded by a sequence of lemmas which make a systematic study of the spaces $\mathcal{I}$.

In the case of a compact group, every conjugacy class of G meets the maximal torus in a unique Weyl group orbit. Thus, the central distributions can be analysed in terms of $\mathcal{W}$-invariant distributions on the torus. Similar statements are true for both adjoint and coadjoint orbits. These facts were used in [5] and [6] in our proofs of the basic properties of the wrapping maps.

We now develop an analgous theory for the semi-direct product case — again, we will need to do this for each of the three cases; G, $\mathfrak{g}$ and $\mathfrak{g}^*$.

We start with the conjugation action.

If $f \in \mathcal{AP}(G)$, then $\widetilde{Mf}(a,k) = \int_K Mf(k_1 \cdot a, k_1^{-1}kk_1)dk_1$ is a central invariant function on G.

Note that for t, $t_0 \in T$ and for $v \in V$, we have

$$\widetilde{Mf}(t_0 \cdot v, t_0^{-1}tt_0) = \widetilde{Mf}(v,t). \tag{40}$$

Recall that the Weyl group $\mathcal{W}$ of (K,T) is the quotient $N_K(T)/T$ of the normaliser of T in K by its centraliser (which is equal to T itself).

By (40) the action $\widetilde{Mf}(w\cdot v, w^{-1}tw)$ is well-defined for all $w \in \mathcal{W}$, and one must have $\widetilde{Mf}(w \cdot v, w^{-1}tw) = \widetilde{Mf}(v,t)$.

Note that the function $\kappa_1(t)$ is also $\mathcal{W}$-invariant. Thus

$$(a,t) \to \kappa_1(t)\widetilde{Mf}(a,t)$$

is a $\mathcal{W}$-invariant function on $V \times T$ which, in addition, is constant on V^t, as a function of a, for any fixed t.

Note that every element of K is conjugate to an element of T, and to a unique element of T^+. Thus, the values of $\widetilde{Mf}$ on $V \times T^+$ define its values everywhere.

Thus $\widetilde{Mf}\mid_{V\times T}$ belongs to

$$\begin{aligned} \mathcal{AP}_{\mathcal{W}}(V\times T) = \{g \in C^\infty(V\times T) : \quad & \forall t \in T \text{ and } \forall v \in V, \\ (i) \quad & \forall w \in \mathcal{W},\ g(w\cdot v, w^{-1}tw) = g(v,t); \\ (ii) \quad & \forall a \in V^t,\ g(v+a,t) = g(v,t)\}. \end{aligned}$$

Note that by equations (12) and (13), we have

$$V_t = \oplus_{\{\lambda:\lambda(\log t)=0\}}V_\lambda \quad \text{and} \quad V^t = \oplus_{\{\lambda:\lambda(\log t)\neq 0\}}V_\lambda.$$

In fact, for a finite set $F \subseteq \Lambda$, let $V_F = \oplus_{\{\lambda:\lambda\in F\}}V_\lambda$,

$$V^F = \oplus_{\{\lambda:\lambda\notin F\}}V_\lambda,$$

and $T_F = \{t \in T : \exp(i\lambda t) = 1 \text{ if and only if } \lambda \in F\}$, a closed subset of T.

Then for $t \in T_F$, $V_t = V_F$ and $V^t = V^F$. Note that for $w \in \mathcal{W}$, $wT_F = T_{wF}$ and for $t \in T_F$, $V_{wt} = V_{wF}$ and $V^{wt} = V^{wF}$.

If $\phi \in \mathcal{I}(G)$ and $f \in \mathcal{AP}_G(G)$ then $< \phi, f > = < \phi, \widetilde{Mf} >$. Thus, we see that, dual to the above restriction of $\mathcal{AP}_G(G)$ to $\mathcal{AP}_{\mathcal{W}}(T)$, there is a notion of extension of a distribution of compact support in $\mathcal{E}(V_F \times T_F)$ to an element of $\mathcal{I}(G)$. This extension is given by the exact analogue of Definition 3.4 of [6].

Definition 6.15. Let $F \subseteq \Lambda$, and let $\psi \in \mathcal{E}(V_F \times T_F)$. Define the *extension* of ψ to an element $e(\psi)$ of $\mathcal{I}(G)$ by

$$\langle e(\psi), f\rangle = \frac{1}{|\mathcal{W}|}\sum_{w\in\mathcal{W}}\langle \psi, \kappa_1(t)\widetilde{Mf}\mid_{V_F\times T_F}(w\cdot a, w^{-1}tw)\rangle.$$

Following the proof of Lemma 3.5 of [6], we have

Lemma 6.16. *If ψ is $\mathcal{W}$ -invariant of compact support, then $e(\psi) \in \mathcal{I}(G)$. Furthermore, if $\psi \in L^1_{\mathrm{loc}}(G)$ is Ad-invariant then $e(\psi\,|_{V\times T}) = \psi$ a.e.*

The following results are now obvious.

Lemma 6.17. *$g \in \mathcal{AP}_{\mathcal{W}}(V \times T)$ if and only if there exists $f \in \mathcal{AP}_G(G)$ such that $\widetilde{Mf}\,|_{V\times T}= g$.*

Proof: The "only if" direction needs a proof. It is clear how to extend an element $g \in \mathcal{AP}_{\mathcal{W}}(V \times T)$ to a K- invariant function $\widehat{g}$ on $V \times K$. Now it suffices to choose an element of $\mathcal{AP}_G(G)$ whose invariant means in the V-directions coincide with the values of $\widehat{g}$ on the corresponding conjugacy classes. ∎

Corollary 6.18. (i) *For all $\psi \in \mathcal{AP}'_{\mathcal{W}}(V\times T)$ there exists $e(\psi) \in \mathcal{I}(G)$ such that*

$$\langle e(\psi), f\rangle = \langle \psi, \widetilde{Mf}\,|_{V\times T}\rangle.$$

(ii) *$\psi \to e(\psi)$ is a one-one correspondence.*

Given a distribution $\psi \in \mathcal{E}(V_F \times T_F)$ we can define $\overline{\psi} \in \mathcal{AP}'_{\mathcal{W}}(V \times T)$ by, for $g \in C^\infty(V \times T)$,

$$\langle\overline{\psi}, g\rangle = \frac{1}{|\mathcal{W}|}\sum_{w\in\mathcal{W}}\langle \psi, (wg)\,|_{V_F\times T_F}\rangle. \tag{41}$$

Then $\overline{\psi}$ is $\mathcal{W}$-invariant and supported on $\cup_{w\in\mathcal{W}} V_{wF}\times T_{wF}$. By Lemma 6.16, $e(\overline{\psi}) \in \mathcal{I}(G)$. We have therefore defined a one-to-one correspondence between the $\mathcal{W}$- invariant elements of $\cup_{F\subseteq\Lambda}\mathcal{E}(V_F \times T_F)$ and $\mathcal{I}(G)$.

More precisely, we have

Proposition 6.19. *Suppose that $\phi \in \mathcal{I}(G)$. Then for each $F \subseteq \Lambda$ there is $\phi_F \in \mathcal{E}(V_F \times T_F)$, with the property that for all $w \in \mathcal{W}$, $\phi_{wF} = w\cdot\phi_F$, and such that*

$$\phi = \sum_{F\subseteq\Lambda} e(\phi_F).$$

We may now treat the Adjoint representation in a similar fashion.

For $H \in \mathfrak{t}$, we have $V_H = \oplus_{\{\lambda:\lambda(H)=0\}}V_\lambda$ and $V^H = \oplus_{\{\lambda:\lambda(H)\neq0\}}V_\lambda$. Given a subset F of Λ, we let $\mathfrak{t}_F = \{H \in \mathfrak{t} : \lambda(H) = 0 \text{ iff } \lambda \in F\}$.

We shall denote as above the space of distributions of compact support on $V_F \times \mathfrak{t}_F$ by $\mathcal{E}(V_F \times \mathfrak{t}_F)$. For such a distribution ψ, let $\overline{\psi}$ be the $\mathcal{W}$-invariant distribution on $V \times \mathfrak{t}$ given by

$$\langle\overline{\psi}, h\rangle = \frac{1}{|\mathcal{W}|}\sum_{w\in\mathcal{W}}\langle \psi, (wh)\,|_{V_F\times \mathfrak{t}_F}\rangle \tag{42}$$

and define $e(\psi) \in \mathcal{I}(\mathfrak{g})$ by letting, for $h \in \mathcal{AP}_G(\mathfrak{g})$,

$$\langle e(\psi), h\rangle = \langle\overline{\psi}(a, H),\ \kappa_2(H)\int_K (Mh)(k\cdot a, \mathrm{Ad}(k)H)dk\rangle. \tag{43}$$

Proposition 6.20. (i) *For any choice of* $\psi \in \mathcal{E}(V_F \times \mathfrak{t}_F)$, $e(\psi) \in \mathcal{I}(\mathfrak{g})$.

(ii) *Every element of* $\mathcal{I}(\mathfrak{g})$ *has the form* $\sum_{F \subseteq \Lambda} e(\phi_F)$, *for suitable choices of distributions* $\phi_F \in \mathcal{E}(V_F \times \mathfrak{t}_F)$, $F \subseteq \Lambda$, *such that* $\phi_{wF} = w\phi_F$ *for all* $w \in \mathcal{W}$.

The above proposition establishes a linear mapping from a space of the $\mathcal{W}$-invariant elements of $\mathcal{E}(\mathfrak{t}_F \times V_F)$ into $\mathcal{I}(\mathfrak{g})$. Furthermore, the sum of these images consists of all of $\mathcal{I}(\mathfrak{g})$.

Finally, we show how to carry out the analogue of this procedure for $\mathcal{I}(\mathfrak{g}^*)$. Choose a basis $\{\psi_\lambda : \lambda \in \Lambda\}$ for V^* which is dual to $\{v_\lambda\}$, i.e. $\varphi_\lambda(v_\mu) = \delta_{\mu\lambda}$. For $F \subseteq \Lambda$, let

$$V_F^* = \{\varphi \in V^* : \varphi(V_\lambda) = \{0\} \text{ for all } \lambda \notin F\}.$$

Let $\mathfrak{t}_F^* = sp\{\alpha \in \Phi : \langle \alpha, \lambda \rangle = 0 \; \forall \lambda \in F\}$.

Then the coadjoint orbits associated with F are indexed by (φ, f), with $\varphi \in V_F^*$, $f \in \mathfrak{t}_F^*$.

Definition 6.21. For a distribution τ of compact support on $V_F^* \times \mathfrak{t}_F^*$, and for $h \in \mathcal{AP}_G(\mathfrak{g}^*)$ let

$$\langle e(\tau), h \rangle = \langle \tau(\varphi, \beta),\ \kappa_3(C, a) \int_K Mh(k \cdot \varphi, \mathrm{Ad}^*(k)\beta) dk \rangle. \tag{44}$$

Given this definition, we now have a direct analogue of the previous propositions.

Proposition 6.22. (i) *For any finite subset* F *of* Λ, *let* $\tau \in \mathcal{E}(V_F^* \times \mathfrak{t}_F^*)$. *Then* $e(\tau) \in \mathcal{I}(\mathfrak{g}^*)$.

(ii) *Every element of* $\mathcal{I}(\mathfrak{g}^*)$ *has the form* $\sum_{F \subseteq \Lambda} e(\tau_F)$, *for suitable choices of* $\tau_F \in \mathcal{E}(V_F^* \times \mathfrak{t}_F^*)$, *with* $\tau_{wF} = w\tau_F \; \forall w \in \mathcal{W}$.

Proposition 6.23. *Let* $\phi \in \mathcal{I}(\mathfrak{g})$. *Then for all* $g \in \mathcal{AP}_G(G)$,

$$\langle \Phi(\phi), g \rangle = \langle \Phi_K(\phi), Mg \rangle,$$

where Φ_K *denotes the wrapping map for the compact group* K, *acting in the second variable only.*

Proof. We may assume that $\phi = e(\overline{\phi_F})$, by Proposition 6.20. By (43), for $h \in \mathcal{AP}_G(\mathfrak{g})$,

$$\langle e(\overline{\phi}_F),\ h \rangle = \frac{1}{|\mathcal{W}|} \sum_{w \in \mathcal{W}} \langle w \cdot \phi_F(a, H),\ \kappa_2(H) \int_K Mh(k \cdot a, \mathrm{Ad}(k)H) dk \rangle. \tag{45}$$

Thus for $g \in \mathcal{AP}_G(G)$,

$$\langle \Phi(\phi),\ g \rangle = \langle \overline{\phi_F}(a, H),\ \kappa_2(H) j(a, H) \int_K Mg(k \cdot \frac{e^H - 1}{H} \cdot a, k^{-1} \exp_K(H) k) dk \rangle. \tag{46}$$

Now if $H \in \mathfrak{t}_F$, and $a \in V_F$, then $H \cdot a = 0$, so that $\frac{e^H - 1}{H} \cdot a = a$. Also, by (10) and (15),

$$j(a, H) = j_K(H) \prod_{\lambda \notin F} \left(\frac{2 \sin \frac{\lambda(H)}{2}}{\lambda(H)} \right)^{m_\lambda}, \quad \text{and} \tag{47}$$

$$\kappa_2(H) = \prod_{\{\alpha \in \Phi^+ : \alpha(H) \neq 0\}} \alpha(H)^2 \prod_{\{\lambda \notin F\}} \lambda(H)^{m_\lambda}. \tag{48}$$

Substituting (47) and (48) in (46), we obtain $\langle \Phi(\phi),\, g \rangle$

$$\begin{aligned} &= \langle \overline{\phi_F}(a, H),\ j_K(H) \textstyle\prod_{\{\alpha \in \Phi^+ : \alpha(H) \neq 0\}} \alpha(H)^2 \kappa_1(\exp_K(H)) \times \\ &\times \textstyle\int_K Mg(k \cdot a, k^{-1} \exp_K(H) k) dk \rangle \\ &= \langle \Phi_K(\overline{\phi_F})(a,\, \cdot)(t) \,|_{V \times T},\ \kappa_1(t) Mg(a,\, t\,) \rangle. \end{aligned} \tag{49}$$

This follows by Lemma 3.5 of [6]. Hence we finally obtain

$$\langle \Phi(\phi),\, g \rangle = \langle \Phi_K(\phi),\ Mg \rangle, \tag{50}$$

as claimed. ∎

Proof of Theorem 6.14.
By Proposition 6.20, we may assume that $\phi = e(\overline{\phi}_E)$ and $\psi = e(\overline{\phi}_F)$, where E and F are $\mathcal{W}$-invariant subsets of Λ, $\phi_E \in \mathcal{E}(V_E \times \mathfrak{t}_E)$ and $\phi_F \in \mathcal{E}(V_F \times \mathfrak{t}_F)$.

By definition, we have, for $h \in \mathcal{AP}_G(\mathfrak{g})$,

$$\langle \phi * \psi,\, h \rangle = \langle e(\overline{\phi}_E)(X),\, \langle e(\overline{\phi}_F)(Y),\, h(X + Y) \rangle \,\rangle. \tag{51}$$

Thus for $g \in \mathcal{AP}_G(G)$, we may use Proposition 6.23 to calculate

$$\begin{aligned} \langle \Phi(\phi) * \psi,\, g \rangle &= \langle \phi(X), \langle \psi(Y),\, j.g \circ \exp(X + Y) \rangle \,\rangle \\ &= \langle \Phi_K(\phi))(a, k_1), \langle \Phi_K(\psi)(b, k_2), M_1 M_2 g(a + b, k_1 k_2) \rangle \,\rangle \\ &= \langle \Phi_K(\phi)(a, k_1),\, \langle \Phi_K(\psi)(b, k_2),\, M_3 g(a + k_1 \cdot b\,, k_1 k_2) \rangle \,\rangle. \end{aligned} \tag{52}$$

Here, we denote by M_1 the average in the variable a over the space V^{k_1}, by M_2 the average in the variable a over the space V^{k_2} and by M_3 the average over the space $V^{k_1 k_2}$ in the first variable of g.

To see the last equality, notice that for any almost periodic function p on V, and for subspaces W_1, W_2 of V, we have (in the obvious notation),

$$M_a^{W_1} M_b^{W_2} p(a + b) = M_c^{W_1 + W_2} p(c).$$

Furthermore, we have $V^k = (I - k) \cdot V$, and for all $v \in V$, $(I - k_1 k_2) \cdot v = (I - k_1) \cdot v + k_1 \cdot (I - k_2) \cdot v$. Thus $V^{k_1 k_2} = V^{k_1} + k_1 \cdot V^{k_2}$.

It follows that $M_1 M_2 p(a + b) = M_3 p(a + k_1 \cdot b)$.

Since $(a, k_1)(b, k_2) = (a + k_1 \cdot b, k_1 k_2)$ and since by [5] Theorem 2.1, we know the wrapping formula for distributions of compact support on $\mathfrak{k}$, we can now simplify (52) to

$$\begin{aligned} \langle \Phi(\phi * \psi),\, g \rangle &= \langle \Phi(\phi)(a, k_1), \langle \Phi(\psi)(b, k_2),\, g((a, k_1)(b, k_2)) \rangle \,\rangle \\ &= \langle \Phi(\phi * \psi),\, g \rangle. \end{aligned}$$

∎

7. Representation Theory and the Character Formula

In the previous section we saw that the spaces $\mathcal{AP}_G(\mathfrak{g})$, $\mathcal{AP}_G(\mathfrak{g}^*)$ and $\mathcal{AP}_G(G)$ have natural dual spaces of distributions which carry convolution structures. Furthermore, the convolutions of G-invariant distributions in these spaces is commutative. In this section, we establish that the Gelfand spaces — the set of characters — of these commutative convolution algebras may be naturally identified respectively with the set of coadjoint orbits in $\mathfrak{g}^*$, the set of adjoint orbits in $\mathfrak{g}$, and the set of characters of irreducible representations of G. Theorem 6.14 then leads to a natural correspondence between the irreducible characters of G and coadjoint orbits, and hence to a version of the Kirillov character formula. Whereas both the original Kirillov statement [11] and Lipsman's formulation [12] have a local character, our Theorem will be stated as an equality of distributions in $\mathcal{I}(\mathfrak{g})$, and hence holds globally. We first establish some lemmas.

Lemma 7.1. *Let $e(\overline{\phi}_F) \in \mathcal{I}(\mathfrak{g})$ where $\phi_F \in \mathcal{E}(\mathfrak{t}_F \times V_F)$, and let $\beta = (\varphi, f) \in \mathfrak{g}^*$. Then $X \mapsto e^{i\beta X} \in \mathcal{AP}_G(\mathfrak{g})$ and*

$$\langle e(\overline{\phi}_F),\, e^{i\beta}\rangle = \langle \phi_F(H,a),\; e^{if|_{\mathfrak{t}_F}(H)} e^{i\varphi|_{V_F}(a)}\rangle. \tag{53}$$

Proof. By Lemma 3.2 and the fact that $e(\overline{\phi}_F)$ is Ad-invariant, $\langle e(\overline{\phi}_F),\, e^{i\beta}\rangle = \langle e(\overline{\phi}_F),\, e^{i\beta_0}\rangle$, where $\beta_0 = (\varphi, f_0)$ with $f_0 = f\mid_{\mathfrak{k}_\varphi} \in \mathfrak{k}_\varphi^*$. Then $e^{i\beta_0(X,v)} = e^{i\varphi(X)} e^{if_0(V)}$ is clearly in $\mathcal{AP}_G(\mathfrak{g})$ as exponential functions are periodic. Further $\langle e(\overline{\phi}_F), e^{i\beta_0}\rangle = \langle \phi_F, e^{if|_{\mathfrak{t}_F}} e^{i\varphi|_{V_F}}\rangle$. The Lemma follows. ■

Now $V_F \times \mathfrak{t}_F$ is a Euclidean space, and so there is a one-to-one correspondence between its characters and $V_F^* \times \mathfrak{t}_F^*$. Thus we get immediately

Proposition 7.2. *The space of characters of $\mathcal{I}(\mathfrak{g})$ is in one-one correspondence with the set of coadjoint orbits in $\mathfrak{g}^*$.*

An entirely similar proof, which is omitted, gives

Proposition 7.3. *The space of characters of $\mathcal{I}(\mathfrak{g}^*)$ is in one-one correspondence with the set of adjoint orbits in $\mathfrak{g}$.*

We now consider the space $\mathcal{I}(G)$. The characters of this space will turn out to be in one-to-one correspondence with the set of traces of irreducible representations of G, in a sense which we shall make precise. We will need three Lemmas to do this.

Lemma 7.4. *Let ρ be an irreducible representation of G. For each $\xi, \xi' \in \mathcal{H}_\rho$, the matrix entry*

$$t^{\rho}_{\xi,\xi'}(g) = \langle \rho(g)\xi, \xi'\rangle$$

belongs to $\mathcal{AP}_G(G)$.

Proof. This follows immediately from the expression for ρ as an induced representation given in (4.). ■

Lemma 7.5. *Let* $\rho = \rho_{(\varphi,\eta)} \in \widehat{G}$ *and* $e(\overline{\phi}_F) \in \mathcal{I}(G)$. *Then*

$$\langle e(\overline{\phi}_F), t^{\rho}_{\xi,\xi'}\rangle = \langle \phi_F(t,a), \kappa_2(a) e^{2\pi i\varphi|_{V_F}(a)} \int_K \langle \eta(t)\xi,\, \eta(k_1)\xi'\rangle dk_1\rangle. \tag{54}$$

Proof. By the remarks at the end of Section 4, we have

$$t^{\rho}_{\xi,\xi'}(v,k_1) = \int_{K/K\varphi} e^{2\pi i\varphi(k^{-1}v)}\xi(k_1^{-1}k)\overline{\xi'(k)}\, d\dot{k}.$$

Now by Definition (6.21), we have

$$\langle e(\overline{\phi}_F), t^{\rho}_{\xi,\xi'}\rangle = \langle \phi_F(t,a), \kappa_1(a) \int_K M t^{\rho}_{\xi,\xi'}(k\cdot a, kt)\, dk\rangle \tag{55}$$

$$= \langle \phi_F(t,a), \kappa_1(a) \int_K \int_K M e^{2\pi i\varphi(ka)}\xi(t^{-1}k_1^{-1}k)\overline{\xi'(k)}\, dk_1 dk\rangle \tag{56}$$

In the sense of equation (29), the mean of an exponential function is the delta function in the perpendicular direction. Hence we obtain $M^{kt}e^{2\pi i\varphi(ka)} = e^{2\pi i\varphi|_{V_F}(a)}$. Thus the double integral (56) simplifies to give

$$e^{2\pi i\varphi|_{V_F}(a)} \int_K \int_K \xi(t^{-1}k_1^{-1}k)\overline{\xi'(k)}\, dk_1 dk = e^{2\pi i\varphi|_{V_F}(a)} \int_K \langle \eta(k_1 t)\xi, \xi'\rangle_{\mathcal{H}_\eta} dk_1.$$

Substituting this in (55), we obtain (54). ∎

Now we may choose an orthonormal basis $\{\xi_i^\eta\}_{i=1}^{d_\eta}$ basis of weight vectors for $\mathcal{H}_\eta$, so that $\eta(t)\xi_i^\eta$ is a multiple of $\alpha_i(t) \in \mathbb{C}$ of ξ_i^η. We then have, for all $\xi' \in \mathcal{H}_\eta$,

$$\langle e(\overline{\phi}_F), t_{\xi_i^\eta \xi'}\rangle = \langle \phi_F(a,t), \kappa_1(a) e^{2\pi i\varphi|_{V_F}(a)}\alpha_i(t)\rangle \langle \xi_i^\eta, \xi'\rangle_{\mathcal{H}_\eta}.$$

Putting these together, we have an orthonormal basis $\{\xi_i^\eta : \eta \in \widehat{K},\ i = 1, \dots, d_\eta\}$ for $L^2(K)$ and with respect to this basis,

$$\mathrm{Tr}\langle e(\overline{\phi}_F), \rho\rangle = \sum_{\mu\in\widehat{K}} d_\mu \langle \phi_F(a,t), \kappa_1(a) e^{2\pi i\varphi|_{V_F}(a)}\chi_\mu(t)\rangle.$$

Now $\sum_{\mu\in\widehat{K}} d_\mu\chi_\mu(t)$ converges as a distribution on K to $\delta_0(t)$. Thus

$$\begin{aligned}\mathrm{Tr}\langle e(\overline{\phi}_F), \rho\rangle &= \langle \phi_F(a,t), \kappa_1(a) e^{2\pi i\varphi|_{V_F}(a)}\delta_0(t)\rangle \\ &= \langle \phi_F(a,e),\, \kappa_1(a) e^{2\pi i\varphi|_{V_F}(a)}\rangle.\end{aligned} \tag{57}$$

Since this is clearly finite, we have shown

Lemma 7.6. $\langle e(\overline{\phi}_F), \rho\rangle$ *is a diagonal matrix on* $\mathcal{H}_\rho$, *and* $Tr\langle e(\overline{\phi}_F), \rho\rangle < \infty$. ∎

Theorem 7.7. *The mappings $\psi \mapsto Tr\langle\psi,\rho\rangle$ are $*$-homomorphisms $\mathcal{I}(G) \to \mathbb{C}$. Furthermore, every such $*$-homomorphism arises in this way for a suitable choice of ρ.*

Proof. We have shown that for $\psi \in \mathcal{I}(G)$, the operator-valued function $\psi \mapsto \langle\psi,\rho\rangle$ is well-defined into $\mathcal{B}(\mathcal{H}_\rho)$, and that it is trace class. We have by standard arguments $\langle\overline{\phi} * \overline{\psi},\rho\rangle = \langle\overline{\phi},\rho\rangle\langle\overline{\psi},\rho\rangle$. Now since both $\overline{\phi}$ and $\overline{\psi}$ are G-invariant, both $\langle\overline{\phi},\rho\rangle$ and $\langle\overline{\psi},\rho\rangle$ are, in fact, multiples of the identity on $\mathcal{AP}(G)$. The first statement follows.

Now, elements of $\mathcal{I}(G)$ are constant on conjugacy classes of G, and hence are defined by their values on character of unitary representations of G. ∎

We may now deduce our version of the Kirillov character formula.

Corollary 7.8. *We have the following equality as elements of $\mathcal{I}(\mathfrak{g})$*

$$j(X)\,Tr(\rho(\exp X)) = \int_{\mathcal{O}_\rho} e^{i\beta(X)}d\mu_{\mathcal{O}_\rho}(\beta).$$

Proof. By Theorem 6.16, Φ is a homomorphism of commutative algebras from $\mathcal{I}(\mathfrak{g})$ to $\mathcal{I}(G)$. Now Proposition 7.2 identifies the set of characters of the first as the set of coadjoint orbits and Theorem 7.7 identifies the set of characters of the second as the set of traces of irreducible representations. The theorem now follows. ∎

Of course, versions of this theorem have been given before. However, all of these have had some restrictions on the support of X. Over many years, and in a number of papers, Kirillov has shown a local version of it for some Lie groups and conjectured that his version holds for all groups (see [11] for a history). Lipsman [12] gave a version of the character formula for semidirect products with co-compact nilradical — this includes our classes of groups (though again, his version was local).

There are two novel aspects in our version of the formula. Firstly, our formula is globally valid — with no restrictions such as that the support of $\widetilde{\phi}$ must lie close to $\{0\}$. This is a consequence of our proof via $(\mathcal{AP}_G, \mathcal{AP}'_G)$ duality.

Secondly, this duality also gives a more precise statement of the equality than in other versions. We believe that this will allow for greater use of the formulae in analysis on motion groups.

In the case where G is a compact Lie group we have [7] given a very explicit formula for the convolution of coadjoint orbits, proving a continuous version of the Kostant multiplicity formula. Knowledge of the compact case, together with the notion of convolution in the space $\mathcal{I}$ allows one to develop similar formulae for the semi-direct product case.

A different feature present here is the fact that, unlike the compact case, there is no identification of the adjoint and coadjoint pictures, as the Killing form is now indefinite. Nevertheless, the coadjoint hypergroup is the dual of the adjoint hypergroup. This fact is expressed by Proposition 7.3. In fact, the duality of the two hypergroups is realized by the Fourier transform, and may be conveniently represented by the generalised Bessel functions introduced in Definition 5.2. In future work, we plan to extend these ideas to other Lie groups.

8. Examples

In this section, we provide two worked examples: the Euclidean motion groups of degree two and three. We believe that these two examples form a good illustration of the theory developed in this paper.

Let $K = SO(n)$ act in $\mathbb{R}^n$ by rotations, and form the semi-direct product $G = M(n) = \mathbb{R}^n \rtimes SO(n)$. This is the n-dimensional Euclidean motion group, with product

$$(v, k)(v_1, k_1) = (v + kv_1, kk_1).$$

Its Lie algebra is $\mathfrak{g} = \mathbb{R}^n \oplus \mathfrak{so}(n)$.

A maximal torus in $SO(n)$ is given by

$$T = \{T_{\theta_1 \cdots \theta_k} : \theta_i \in [0, 2\pi),\ i = 1 \cdots k\},$$

where $k = \left[\frac{n}{2}\right]$, $T_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ and $T_{\theta_1 \cdots \theta_k}$ is the $n \times n$ matrix with blocks $T_{\theta_1, \cdots}, T_{\theta_n}$ on the diagonal, finishing with a 1 in the bottom right hand corner when n is odd.

Note that $T_{\theta_1 \cdots \theta_k} = \exp\left(\sum_{i=1}^k \theta_i H_i\right)$, where H_i is the matrix with

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

in the ith block. Let $e_1 \cdots e_n$ be the standard basis for $\mathbb{R}^n$. We have

$$H_i e_j = \begin{cases} 0 & \text{if } \ j \neq 2i-1, 2i \\ e_{2_i} & \text{if } \ j = 2i-1 \\ -e_{2i-1} & \text{if } \ j = 2i \end{cases}$$

Thus, for the weight $\lambda_j(\Sigma\theta_i H_i) = \theta_j$ on $\mathfrak{so}(n)$, we have the weight space $sp\{e_{2i-1}, e_{2i}\}$. If n is odd, we also have the weight $\lambda_0(H) = 0$, with weight space $\{e_n\}$. Then

$$V = \begin{cases} \oplus_{j=1}^k V_{\lambda_j} & \text{if } \ n \text{ even} \\ \oplus_{j=0}^k V_{\lambda_j} & \text{if } \ n \text{ odd.} \end{cases}$$

We will consider the two cases $n = 2$ and $n = 3$ separately, as they illustrate our theorems.

8.1. The Group $M(2)$.

In this case, $SO(2) = \mathbb{T} = \{T_\theta : 0 \leq \theta < 2\pi\}$.

Fix a Haar measure on $\mathbb{R}^2$. Note that, if $\theta = 0$, $V_{T_\theta} = V$ and $V^{T_\theta} = \{0\}$, while if $\theta \neq 0$, $V_{T_\theta} = \{0\}$ and $V^{T_\theta} = V$.

The **conjugacy classes** through (v, T_θ) corresponding to $\theta = 0$ are circles of radius $|v| \geq 0$. Thus, we have the single point $(0, 0)$ with orbital measure δ_0, and circles of radius $R > 0$, $\{(u, 0) : |u| = R\}$ with orbital measure $Rd\theta$. These have dimension 0 and 1 respectively. The conjugacy class though (v, T_θ) corresponding to $\theta \neq 0$ are $\{(u, T_\theta) : u \in \mathbb{R}^2\}$ a two dimensional fibre over a single point $(0, T_\theta)$, a trivial K-orbit. The orbital measure on this orbit is $\sin\theta dx$.

The **adjoint orbits** through (v, sH_1) are likewise of two types. If $s = 0$, there is the single point $(0, 0)$ with measure δ_0 when $|v| = 0$, and the circle

$\{(v, xH_1) : |u| = R\}$ with measure $Rd\theta$ when $|v| = R$. If $s \neq 0$, we have two dimensional orbits $\{(u, sH_1) : u \in \mathbb{R}^2\}$ with measure sdx.

The **coadjoint orbits** are as follows. Let $\varphi \in \mathbb{R}^{2*} = \mathbb{R}^2$ and $f = f_\lambda \in \mathfrak{so}(2)^* = \mathbb{R}$ where$\langle f_\lambda, H_1\rangle = -\lambda$. Then for $v \in \mathbb{R}^2$, we calculate

$$(v \times \varphi)(sH_1) = s\varphi(H_1 \cdot v) = s(\varphi_1 \varphi_2)\begin{pmatrix} v_2 \\ -v_1 \end{pmatrix} = s(\varphi_1 v_2 - \varphi_2 v_1).$$

Thus $v \times \varphi = f_{v_1\varphi_2 - \varphi_1 v_2}$ indeed corresponds to the cross product of vector calculus.

There are again two types of coadjoint orbit. In the case where $\varphi = 0$, the orbit of $(0, f_\lambda)$ is the single point with Liouville measure $\delta_{(0,f_\lambda)}$. By contrast, in the case $\varphi \neq 0$ the space is fibred over the circle $\{T_\theta\varphi : 0 \leq \theta < 2\pi\}$, the fibre being $\{f_\mu : \mu \in \mathbb{R}\}$. These are cylinders of radius $|\varphi| > 0$. The Liouville measure on such a cylinder is $Rd\theta \times d\mu$, where $d\mu$ is Lebesgue measure on $\mathbb{R}$. For $\varphi \neq 0,\ K_\varphi = \{f_0\}$ and so the condition $f_\lambda \in \mathfrak{k}_\varphi$ is simply $\lambda = 0$.

The spaces $\mathcal{AP}$ are as follows:

$$\begin{aligned}
\mathcal{AP}(G) &= \{f \in C^\infty(\mathbb{R}^2 \times \mathbb{T}) : \text{ for all } \theta,\ v \mapsto f(v, T_\theta) \in \mathcal{AP}(\mathbb{R}^2)\} \\
\mathcal{AP}(\mathfrak{g}) &= \{f \in C^\infty(\mathbb{R}^2 \times \mathbb{R}) : \text{ for all } s, v \mapsto f(v, sH_1) \in \mathcal{AP}(\mathbb{R}^2)\} \\
\mathcal{AP}(\mathfrak{g}^*) &= \{f \in C^\infty(\mathbb{R}^2 \times \mathbb{R}) : \text{ for all } \varphi \in \mathbb{R}^2,\ \lambda \mapsto f(\varphi, f_\lambda) \in \mathcal{AP}(\mathbb{R})\}.
\end{aligned}$$

We now describe the spaces of invariant distributions $\mathcal{I}$ and their convolution structures. There is just one weight λ, of $\mathfrak{t}$ on V. Thus, the possible values for F are:
$F = \emptyset$, corresponding to $V_F = \{0\}$, $V^F = V$, $T_F = T$, $\mathfrak{t}_F^* = \mathfrak{t}$, $V_F^* = \{0\}$ and $\mathfrak{t}_F^* = \mathfrak{t}^*$, furthermore,
$F = \{\lambda\}$ corresponding to $V_F = V$, $V^F = \{0\}$, $T_F = \{I\}$, $\mathfrak{t}_F = \{0\}$, $V_F^* = V^*$ and $\mathfrak{t}_F^* = \mathfrak{t}^*$.

The structure of $\mathcal{I}(\mathbf{G})$ is as follows.

Corresponding to $F = \emptyset$, we choose $\varphi_0 \in \mathcal{E}(\{0\} \times T) = \mathcal{E}(T)$.

For $f \in \mathcal{AP}(G)$, let $\tilde{f}(T_\theta)$ be the invariant mean of the function $v \mapsto f(v, T_\theta)$. Then for $\tau_0 = e(\varphi_0)$, we have

$$\langle \tau_0, f\rangle = \langle \varphi_0(T_\theta),\ \tilde{f}(T_\theta)\rangle,$$

If τ_0, τ_0' are two such distributions arising from φ_0, φ_0', (say) then $\langle \tau_0 * \tau_0', f\rangle = \langle \varphi_0 * \varphi_0'(T_\theta),\ \tilde{f}(T_\theta)\rangle$.

Corresponding to $F = \{\lambda\}$, we choose $\varphi_1 \in \mathcal{E}(V \times \{I\}) = \mathcal{E}(V)$. Setting $\tau_1 = e(\varphi_1)$, we have

$$\begin{aligned}
\langle \tau_1, f\rangle &= \langle \varphi_1(v),\ \textstyle\int_0^{2\pi} f(T_\theta v, I)d\theta\rangle \\
&= \langle \varphi_1(v), f(v, I)\rangle \text{ if } \varphi_1 \text{ is } K\text{-invariant on } V.
\end{aligned}$$

If φ_1, φ_1' are two such K-invariant distributions with extensions τ_1, τ_1', then

$$\langle \tau_1 * \tau_1', f\rangle = \langle \varphi_1 *_V \varphi_1', f(v, I)\rangle.$$

If τ_0, τ_1 arise from φ_0, φ_1 as above, then

$$\begin{aligned}\langle \tau_1 * \tau_0,\ f\rangle &= \langle \varphi_1(v), \langle \tau_0(T_\theta),\ \tilde{f}(T_\theta)\rangle\rangle \\ &= \langle \tau_1, 1\rangle\langle \tau_0, f\rangle\end{aligned}$$

where 1 denotes the constant function.

Notice that every element of $\mathcal{I}(G)$ is a sum of two distributions $\tau = \tau_0 + \tau_1, \tau_i$ as above.

The structure of $\mathcal{I}(\mathfrak{g})$ is similar. We can write $\tau \in \mathcal{I}(\mathfrak{g})$ as $\tau = \tau_0 + \tau_1$, where τ_0 is the extension of $\varphi_0 \in \mathcal{E}(\mathfrak{t})$ and τ_1 is the extension of $\varphi_1 \in \mathcal{E}(v)$, which is K-invariant.

If $\tau_0, \tau_0',\ \tau_1, \tau_1'$ have this form, then

$$\begin{aligned}\langle \tau_0 * \tau_0', f\rangle \quad &= \langle \varphi_0 * \varphi_0'(sH_1), \tilde{f}(sH_1)\rangle \\ \langle \tau_1 * \tau_1', f\rangle \quad &= \langle \varphi_1 * \varphi_1'(v), f(v,0)\rangle \\ \text{and } \langle \tau_1 * \tau_0, f\rangle \quad &= \langle \varphi_1, 1\rangle\ \langle \varphi_0(sH_1), \tilde{f}(sH_1)\rangle.\end{aligned}$$

The picture for $\mathcal{I}(\mathfrak{g}^*)$ is slightly different. Corresponding to $F = \emptyset$, $\varphi_0 \in \mathcal{E}(\mathfrak{t}^*) = \mathcal{E}(\{0\} \times \mathfrak{t}^*)$ we obtain $\tau_0 = e(\varphi_0)$ as follows. Let $f \in \mathcal{AP}(\mathfrak{g}^*)$ and denote by $\tilde{f}(v^*)$ the mean of the function $\beta \mapsto f(v^*, \beta)$. Then $\langle \tau_0, f\rangle = \langle \varphi_0(B), f(0,\beta)\rangle$, and for τ_0, τ_0' of this form,

$$\langle \tau_0 * \tau_0', f\rangle = \langle \varphi_0 * \varphi_0'(\beta), f(0,\beta)\rangle.$$

On the other hand, corresponding to $\varphi_1 \in \mathcal{E}(V^*)$, we have

$$\begin{aligned}\langle \tau_1, f\rangle \quad &= \langle \varphi_1(v^*), \textstyle\int \tilde{f}(T_\theta v^*) d\theta\rangle \\ &= \langle \varphi_1(v^*), \tilde{f}(v^*)\rangle\end{aligned}$$

provided that φ_1 is K-invariant. As above, we obtain

$$\begin{aligned}\langle \tau_1 * \tau_1', f\rangle \quad &= \langle \varphi_1 * \varphi_1'(v^*), \tilde{f}(v^*)\rangle, \text{ and} \\ \langle \tau_0 * \tau_1, f\rangle \quad &= \langle \varphi_0, 1\rangle\langle \varphi_1, \tilde{f}\rangle.\end{aligned}$$

These notions of convolution give the natural convolution on the conjugacy classes, adjoint orbits and coadjoint units. For example, the convolution of two cylinders (coadjoint orbits) of radius R_1, R_2 can be written

$$Cyl(R_1) * Cyl(R_2) = \int N_{R_1R_2}(R) Cyl(R) dR$$

where $N_{R_1R_2}(R)$ is the density for convolutions of circles in the plane.

8.2. The Group $M(3)$**.**

The group $SO(3)$ has maximal torus

$$T = \{\tilde{T}_\theta = \mathrm{diag}(T_\theta, 1) : 0 \leq \theta < 2\pi\},$$

and one positive root α. The weights on $\mathbb{R}^3$ are $\lambda_0 = 0$ corresponding to the z-axis, $sp\{e_3\}$, and one non-zero weight λ, with weight space $sp\{e_1, e_2\}$, defined by $\lambda(sH) = s$ where

$$H = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Recall that each non-zero element A of $SO(3)$ has an axis of rotation in $\mathbb{R}^3$. A matrix which rotates this axis to the z-axis will conjugate A into T.

If $\theta = 0$, then $V_{\tilde{T}_\theta} = V$ and $V^{\tilde{T}_\theta} = \{0\}$. If $\theta \neq 0$, then $V_{\tilde{T}_\theta} = sp\{e_3\}$ $V^{\tilde{T}_\theta} = sp\{e_1, e_2\}$.

The **conjugacy class** of G through the point (v, A) is as follows:

- If $A = I, |v| = 0$, we have the single point $(0, I)$. The orbital measure is $\delta_{(0,I)}$.
- If $A = I, |v| = R > 0$, we have a two-dimensional compact orbit $S_R \times \{0\}$, where S_R is the sphere of radius R in $\mathbb{R}^3$. The orbital measure is $Rd\sigma, d\sigma$ being surface measure on the sphere.
- If $A \neq I, |v| = 0$ then the orbit is fibred over the compact orbit $\{0\} \times C_A$, where $C_A = \{B^{-1}AB : B \in SO(3)\}$ is the conjugacy class of A. The fibre at $B^{-1}AB$ is the two-dimensional space $V^{B^{-1}AB}$. This orbit is four dimensional with measure $d\mathcal{O}_c \times dx$.
- If $A \neq I, |v| = R > 0$, then the orbit is fibred over $S_R \times C_A$, with fibre at $(Bv, B^{-1}AB)$ being $V^{B^{-1}AB}$. This orbit is six dimensional, and the orbital measure is $Rd\sigma \times d\mathcal{O}_c \times dx$.

The **adjoint orbit** through $(v, X) \in \mathbb{R}^3 \oplus \mathfrak{so}(3)$ is similar:

- If $X = 0, v = 0$, we have the single point $(0, 0)$.
- If $X = 0, |v| = R > 0$, we have $S_R \times \{0\}$.
- If $X \neq 0, v = 0$ we have the compact orbit $\{0\} \times \mathcal{O}_X, \mathcal{O}_X$ being the $SO(3)$-adjoint orbit of X, with fibre at $(0, y)$ being the two-dimensional space V^y.
- If $X \neq 0, |v| = R > 0$, we have $S_R \times \mathcal{O}_X$ with fibre at (u, y) being V^y.

These orbits have dimension $0, 2, 4$ and 6 respectively.

For the **coadjoint orbits**, let $\varphi \in \mathbb{R}^{3*} = \mathbb{R}^3, v \in \mathbb{R}^3$ and calculate

$$(v \times \varphi)A = \varphi(sH_* v) = s(\varphi_1, \varphi_2, \varphi_3) \begin{pmatrix} v_2 \\ -v_1 \\ 0 \end{pmatrix} = s(\varphi_1 v_2 - \varphi_2 v_1).$$

For $\varphi \neq 0, \mathfrak{k}_\varphi$ is the set of rotations with axis φ.

Then for (φ, f) with $f \in \mathfrak{k}_\theta$, we have

- If $\varphi = 0, f = 0$ the single point $(0, 0)$.

- If $\varphi = 0, f \neq 0$, we get $\{0\} \times \mathcal{O}_f$, where $\mathcal{O}_f$ is the (two dimensional) coadjoint orbit in $\mathfrak{k}^*$ through f.
- If $\varphi \neq 0, f = 0, V \times \varphi$ has dimension 2 in $\mathfrak{k}$. It is the orthogonal complement of the rotations which preserve φ. The orbit is then $S_{|\varphi|} \times (V \times \varphi)$.
- If $\varphi \neq 0, f \neq 0$, we get the compact orbit $S_{|\varphi|} \times \mathcal{O}_f$ with fibre at $(k\varphi, \mathrm{Ad}^*(k)f)$ being $V \times k\varphi$.

The spaces $\mathcal{AP}$ are as follows:

$$\begin{aligned}
\mathcal{AP}(G) &= \{f \in C^\infty(V \times SO(3)) : \forall A \in SO(3),\ f(\cdot, A) \text{ is almost periodic on } \mathbb{R}^3\} \\
\mathcal{AP}(\mathfrak{g}) &= \{f \in C^\infty(V \oplus \mathfrak{so}(3)) : \forall x \in \mathfrak{so}(3), f(\cdot, x) \text{ is almost periodic in } \mathbb{R}^3\} \\
\mathcal{AP}(\mathfrak{g}^*) &= \{f \in C^\infty(V^* \oplus \mathfrak{so}(3)^*) : \forall \varphi \in V^*, f(\varphi, \cdot) \text{ is almost periodic on } \mathfrak{so}(3)\}.
\end{aligned}$$

For $f \in \mathcal{AP}(G)$, $\mathbf{i} \subseteq \{1, 2, 3\}$, let $M_{\mathbf{i}}f$ be the function on $\mathbb{R}^{3-|\mathbf{i}|} \times SO(3)$ which is the invariant mean in the variables $x_i, i \in \mathbf{i}$ of the function $x_{\mathbf{i}} \mapsto f(x_1, x_2, x_3, A)$. Thus $M_{12}f(x_3, A)$ denotes the result of taking the mean in x_1 and x_2 of f. We use similar notation for $\mathcal{AP}(\mathfrak{g})$.

Given that $\lambda = \{0, \lambda\}$ contains two elements, there are four possible subsets: $F = \emptyset,\ F = \{0\},\ F = \{\lambda\},\ F = \Lambda$. Each of our invariant distributions is a sum of four distributions defined by these four sets. We will denote them by $\tau_0, \tau_1, \tau_2, \tau_3$ respectively.

Note that for

- $F = \emptyset,\ V_F = \{0\},\ V^F = V,\ T_F = T,\ \mathfrak{t}_T = \mathfrak{t},\ V_F^* = \{0\}$ and $\mathfrak{t}_F^* = \{0\}$.
- $F = \{0\},\ V_F = sp\{e_3\},\ V^F = sp\{e_1, e_2\},\ T_F = T,\ \mathfrak{t}_F = \mathfrak{t},\ V_F^* = \{0\}$ and $t_F^* = \mathfrak{t}$.
- $F = \{\lambda\},\ V_F = sp\{e_1, e_2\},\ V^F = sp\{e_3\},\ T_F = \{I\},\ \mathfrak{t}_F = \{0\},\ V_F^* = sp\{e_1^*, e_2^*\},\ \mathfrak{t}_F^* = \{0\}$.
- $F = \lambda,\ V_F = V,\ V^F = \{0\},\ T_F = \{I\},\ \mathfrak{t}_F = \{0\},\ V_F^* = V^*,\ \mathfrak{t}_F^* = \{0\}$.

Using this notation, we now describe our spaces of invariant distributions $\mathcal{I}(G)$ consists of $\tau = \tau_0 + \tau_1 + \tau_2 + \tau_3$, where there exist

$$\varphi_0 \in \mathcal{E}(T), \varphi_1 \in \mathcal{E}(sp\{e_3\} \times T),\ \varphi_2 \in \mathcal{I}(sp\{e_1, e_2\}), \varphi_3 \in \mathcal{E}(V),$$

with

- $\langle \tau_0, f \rangle = \langle e(\varphi_0), f \rangle = \langle \tilde{\varphi}_0,\ M_{123}f \rangle$.

  Here $\langle \tilde{\varphi}_0(x), f \rangle = \langle \varphi_0(t), \prod_{\alpha \in \Phi_+} \sin(\log \alpha(t)) \int_{SO(3)} f(ktk^{-1}) dk \rangle$.

Given τ_0, τ_0' of this form, we have $\tau_0 \times \tau_0' = e(\tilde{\varphi}_0 * \tilde{\varphi}_0')$, the usual convolution on K.

- $\langle \tau_1, f \rangle = \langle e(\varphi_1), f \rangle = \langle \tilde{\varphi}_1(x_3, A), M_{12}f(x_3, A) \rangle$.

Given τ_1, τ_1' of this form, we have

$$\langle \tau_1 * \tau_1', f\rangle = \langle \tilde{\varphi}_1 * \tilde{\varphi}_1'(x_3, A),\ M_{12}f(x_3, A)\rangle.$$

Further, we have

$$\langle \tau_1 * \tau_2, f\rangle = \langle \tilde{\varphi}_1(x_3, \cdot) *_G \tilde{\varphi}_0(A), M_{12}f(x_3, A)\rangle.$$

- $\langle \tau_2, f\rangle = \langle e(\varphi_2), f\rangle = \langle \varphi_2(x_1, x_2),\ M_3 f(x_1, x_2, I)\rangle$

$$\begin{aligned} \text{with} \qquad \tau_2 * \tau_2' &= e(\varphi_2 * \varphi_2') \quad \text{(convolution in } \mathcal{E}(\mathbb{R}^2)) \\ \tau_2 * \tau_0 &= \langle \varphi_0, 1\rangle \tau_2, \quad \text{where 1 is the constant function.} \\ \tau_2 * \tau_1 &= \langle \varphi_0(x_1, x_2), \langle \tilde{\varphi}_1(x_3, A),\ f(x, x_2, x_3, I)\rangle\rangle. \end{aligned}$$

- $\langle \tau_3, f\rangle = \langle e(\varphi_3), f\rangle = \langle \varphi_3(x_1, x_2, x_3), f(x_1, x_2, x_3, I)\rangle$ where

$$\begin{aligned} &\tau_3 * \tau_3' \text{ is the usual convolution in } \mathbb{R}^3 \\ &\langle \tau_3 * \tau_2, f\rangle = \langle\langle \varphi_3, 1_{x_3}\rangle *_{x_1x_2} \varphi_2(x_1, x_2), M_3 f(x_1, x_2, I)\rangle \\ &\langle \tau_3 * \tau_1, f\rangle = \langle\langle \varphi_3, 1_{x_1x_2}\rangle *_{x_3} \tilde{\varphi}_1(x_3, A), M_{12}f(x_3, A)\rangle\rangle. \end{aligned}$$

$\mathcal{I}(\mathfrak{g})$ consists of distributions $\tau = \tau_0 + \tau_1 + \tau_2 + \tau_3$, extensions of $\varphi_0 \in \mathcal{E}(\mathfrak{t}), \varphi_1 \in \mathcal{E}(\mathbb{R} \times \mathfrak{t}), \varphi_2 \in \mathcal{E}(\mathbb{R}^2),\ \varphi_3 \in \mathcal{E}(\mathbb{R}^3)$ such that

$$\begin{aligned} \langle \tau_0, f\rangle &= \langle \tilde{\varphi}_0(x),\ M_{23}f(X)\rangle \\ \langle \varphi_1, f\rangle &= \langle \tilde{\varphi}_1(x_3, X),\ M_{12}f(x_3, X)\rangle \\ \langle \tau_2, f\rangle &= \langle \varphi_2(x_1, x_2) M_3 f(x_1, x_2, 0)\rangle \\ \langle \tau_3, f\rangle &= \langle \varphi_3(x_1x_2x_3), f(x_1x_2x_3, 0)\rangle. \end{aligned}$$

The convolution structure is given by

$$\begin{aligned} \langle \tau_0 * \tau_0', f\rangle &= \langle \tilde{\varphi}_0 *_X \tilde{\varphi}_0'(X),\ M_{123}f(X)\rangle, \text{ usual convolution in } \mathcal{E}(\mathfrak{g}) \\ \langle \tau_1 * \tau_1', f\rangle &= \langle \tilde{\varphi}_1 *_{x_3,X} \tilde{\varphi}_1'(x_3, X),\ M_{12}f(x_3, X)\rangle \\ \langle \tau_1 * \tau_0, f\rangle &= \langle (\tilde{\varphi}_1(x_3, \cdot) *_X \tilde{\varphi}_0)(X),\ M_{12}f(x_3, X)\rangle \\ \langle \tau_2 * \tau_2', f\rangle &= \langle \varphi_2 *_{x_1,x_2} \varphi_2'(x_1, x_2),\ M_{x_3}f(x_1, x_2, 0) \\ \langle \tau_2 * \tau_1, f\rangle &= \langle \varphi_2(x_1, x_2)\tilde{\varphi}_1(x_3, X), f(x_1, x_2, x_3, X)\rangle \\ \langle \tau_2 * \tau_0, f\rangle &= \langle \varphi_2, 1_{x_1,x_2}\rangle \langle \tilde{\varphi}_0(X),\ M_{123}f(X)\rangle \\ \langle \tau_3 * \tau_3', f\rangle &= e(\varphi_3 * \varphi_3') \quad \text{is the usual } \mathbb{R}^3 \text{ convolution} \\ \langle \tau_3 * \tau_2, f\rangle &= \langle\langle \varphi_3, 1_{x_3}\rangle *_{x_1,x_2} \varphi_2(x_1, x_2),\ M_3 f(x_1, x_2, 0)\rangle \\ \langle \tau_3 * \tau_1, f\rangle &= \langle\langle \varphi_3, 1_{x_1,x_2}\rangle *_{x_3} \tilde{\varphi}_1(x_3, X),\ M_{12}f(x_3, X)\rangle \\ \langle \tau_3 * \tau_0, f\rangle &= \langle \varphi_3, 1_{x_1,x_2,x_3}\rangle \langle \tilde{\varphi}_0(X),\ M_{123}f(X)\rangle. \end{aligned}$$

$\mathcal{I}(\mathfrak{g}^*)$ may be described as follows. Let $e_1^*, e_2^* e_3^*$ be the basis in V^* dual to $\{e_1, e_2, e_3\}$. For $f \in \mathcal{AP}(\mathfrak{g}^*)$, let $Mf(V)$ be the mean of the function $\beta \mapsto f(v,\beta)$. Again, we have $\tau = \tau_0 + \tau_1 + \tau_2 + \tau_3$, where $\tau_i = e(\varphi_i), \varphi_0 \in \mathcal{E}(\mathfrak{t})$, $\varphi_1 \in \mathcal{E}(\mathbb{R} \times \mathfrak{t}^*)$, $\varphi_2 \in \mathcal{E}(\mathbb{R}^2)$, $\varphi_3 \in \mathcal{E}(\mathbb{R}^3)$ and

$$\langle \tau_0, f\rangle = \langle \tilde{\varphi}_0(\beta), f(0,\beta)\rangle,$$

where $\tilde{\varphi}_0$ is the radial extension of φ_0 from $\mathcal{E}(\mathfrak{t}^*)$ to $\mathcal{E}(\mathfrak{k}^*)$.

$$\begin{aligned}
\langle \tau_1, f\rangle &= \langle \tilde{\varphi}_1(x_3^*, \beta),\ f(0,0,x_3^*,\beta)\rangle \\
\langle \tau_2, f\rangle &= \langle \varphi_2(x_1^*, x_2^*),\ (Mf)(x_1^*, x_2^*, 0)\rangle \\
\langle \tau_3, f\rangle &= \langle \varphi_3, Mf\rangle.
\end{aligned}$$

The convolution relationships between the distributions τ_i are given as follows:

$$\begin{aligned}
\langle \tau_0 * \tau_0', f\rangle &= \langle \tilde{\varphi}_0 * \tilde{\varphi}_0'(\beta),\ f(0,\beta)\rangle \\
\langle \tau_1 * \tau_1', f\rangle &= \langle \tilde{\varphi}_1 *_{x_3^*,\beta} \varphi_1'(x_3^*,\beta),\ f(0,0,x_3^*,\beta)\rangle \\
\langle \tau_1 * \tau_0, f\rangle &= \langle \tilde{\varphi}_1(x_3^*,0) *_\beta \tilde{\varphi}_0(\beta),\ f(0,0,x_3^*,\beta)\rangle \\
\langle \tau_2 * \tau_2', f\rangle &= \langle \varphi_2 *_{x_1^*,x_2^*} \varphi_2'(x_1^*,x_2^*),\ (Mf)(x_1^*,x_2^*,0)\rangle \\
\langle \tau_2 * \tau_1, f\rangle &= \langle \varphi_2(x_1^*,x_2^*), \langle \tilde{\varphi}_1(x_3^*,\beta),\ f(x_1^*,x_2^*,x_3^*,\beta\rangle\rangle \\
\langle \tau_2 * \tau_0, f\rangle &= \langle \varphi_2(x_1^*,x_2^*), \langle \tilde{\varphi}_0(\beta),\ f(x_1^*,x_2^*,0,\beta\rangle\rangle \\
\langle \tau_3 * \tau_3', f\rangle &= \langle \varphi_3 * \varphi_3', Mf\rangle \\
\langle \tau_3 * \tau_2, f\rangle &= \langle \langle \varphi_3(\cdot,\cdot,x_3^*), 1_{x_3^*}\rangle *_{x_1^*,x_2^*} \varphi_2(x_1^*,x_2^*),\ Mf(x_1,x_2,\beta)\rangle \\
\langle \tau_3 * \tau_1, f\rangle &= \langle \langle \varphi_3(x_1^*,x_2^*,\cdot), 1_{x_1^*,x_2^*}\rangle *_{x_3^*} \varphi_1(x_3^*,\beta),\ f(0,0,x_3,\beta)\rangle \\
\langle \tau_3 * \tau_0, f\rangle &= \langle \varphi_3(x_1^*,x_2^*,x_3^*) \langle \varphi_0(\beta),\ f(x_1^*,x_2^*,x_3^*,\beta\rangle\rangle.
\end{aligned}$$

Again, it can be readily seen that this gives a natural convolution structure on the orbits, and that the coadjoint orbits are the natural "dual hypergroups" with respect to this convolution.

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