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Lie Algebras of Simple Hypersurface Singularities

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Abstract. We investigate structural properties and numerical invariants of the finite-dimensional solvable Lie algebras naturally associated with simple hypersurface singularities. In particular, we establish that the analytic isomorphism class of a simple hypersurface singularity is determined by the Lie algebra of derivations of its moduli algebra if the dimension of the latter algebra is not less than 6. We also describe natural gradings on the Lie algebras of simple singularities and show that all roots of their Poincaré polynomials lie on the unit circle. Moreover, the indices of those Lie algebras are calculated and existence of maximal commutative polarizations is established.
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Introduction

The aim of the present paper is to present a number of results about the finite-dimensional Lie algebras which can be naturally associated with the germs of isolated hypersurface singularities (IHS). Recall that following S. S.-T. Yau [28], for any IHS germ $X = X(f) = \{f = 0\}$, one considers the Lie algebra of derivations $L(X) = \mathrm{Der}_{\mathbb{C}}(A(X), A(X))$ of the factor-algebra $A(X) = O_n/(f, df)$, where O_n is the algebra of convergent power series in n indeterminates, $f \in O_n$, and (f, df) is the ideal in O_n generated by f and all of its partial derivatives $\partial_i f = \frac{\partial f}{\partial x_i}$. According to S. S.-T. Yau, $L(X)$ is a finite-dimensional solvable Lie algebra called the *Lie algebra of singularity* X [31]. It should be noticed that such algebras are also called *Yau algebras* [32].

One of our main goals is to show that singularities of certain types can be classified by their Lie algebras. Since derivations of function algebras are analogs of vector fields on smooth manifolds, such direction of research is in the spirit of the classical theorem of L. Pursell and M. Shanks stating that the Lie algebra of smooth vector fields on a smooth manifold determines the diffeomorphism type of the manifold [23]. Theorem 3.1 below yields a similar result for the so-called *simple singularities* which play significant rôle in singularity theory [5]. We also show that the Lie algebras associated with simple singularities possess a number of

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curious algebraic properties. Our results suggest further developments and open problems some of which are briefly discussed in the last section.

It should be noted that the systematic study of Lie algebras of isolated hypersurface singularities was undertaken by S. S.-T. Yau and his collaborators in eighties (see, e. g., [28], [29], [22], [7], [25]). In particular, Lie algebras of simple singularities and simple elliptic singularities were computed and a number of elaborate applications to deformation theory were presented in [7] and [25]. A detailed survey of results obtained in that period can be found in [7].

Our initial motivation was to investigate the structure of arising finite-dimensional Lie algebras and find out, for which classes of singularities those Lie algebras determine the analytic or topological structure of singularity by analogy with the mentioned result of L. Pursell and M. Shanks. In this context we aimed at computing such Lie algebras for some natural classes of singularities beyond simple singularities, in order to reveal the properties specific for simple singularities.

Along these lines, we obtain rather detailed information about associated Lie algebras for two series of isolated hypersurface singularities chosen in such way that each simple singularity belongs to one of these series. Thus our results can be considered as extensions of those presented in [7]. It should be noted that in course of this research we were also able to explicate the previously known results for simple singularities and establish some properties which apparently have not been mentioned neither in [7] nor in other related papers.

The structure of the paper is as follows. After recalling a few related concepts and general results in section 1, in the second section we develop some auxiliary technical tools which enable us to describe the associated Lie algebras for two infinite series of singularities containing all simple singularities. More precisely, we deal with the *Pham singularities* $X(P_\kappa)$ defined by polynomials $P_\kappa = \sum x_j^{k_j+1}, \kappa = (k_1, \dots, k_n)$, and D_{k_1,k_2} series defined by polynomials $D_{k_1,k_2} = x_1^{k_1}x_2 + x_2^{k_2}$, where k_i are arbitrary natural numbers bigger than 1. The calculations presented in section 2 serve as a natural background for the further discussion and, in particular, provide a source of examples illustrating some curious phenomena concerned with Lie algebras considered.

In the third section we show that, except for just one pair, simple singularities can be distinguished by their Lie algebras (Theorem 3.1). This follows directly from the computations performed in section 2 by a direct analysis of arising Lie algebras. The same conclusion follows from the computations presented in [7] but this fact is not mentioned in [7]. Moreover, our approach seems to be more general and direct than in [7], so we believe that the discussion in sections 2 and 3 contains some novelties which deserve to be presented in some detail. We also show that singularities of Pham and D_{**} series are classified by their Lie algebras (Theorem 3.2).

In section 4 we discuss natural gradings on Lie algebras of quasihomogeneous IHS and properties of the corresponding Poincaré polynomials. In particular, we show that the Poincaré polynomials of Lie algebras of simple singularities are palindromic (recurrent) and all of their roots lie on the unit circle (Theorem 4.3). We also present a generalization of this result to the direct sums of certain simple singularities.

In the next section we establish that all derivations of Lie algebras of simple singularities with the Milnor number greater than 8 are inner (Theorem 5.1). In other words, those Lie algebras are complete in the sense of [18]. We give examples showing that, in general, derivation Lie algebras need not be complete (see the tables in Section 5). In section 6 we discuss further algebraic properties of derivation Lie algebras. In particular, we compute the indices of Lie algebras of simple singularities and show that they possess so-called *maximal commutative polarizations* defined in [14]. In conclusion we briefly discuss several conjectures and problems suggested by our results.

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1. Generalities on singularities and Lie algebras

We present here necessary definitions and auxiliary results about hypersurface germs with isolated singularities and derivations of Lie algebras. To give a consistent description of the background and setting we begin with recalling necessary concepts and constructions from singularity theory.

Let $\mathbb{C}_n$ be the algebra of complex polynomials in n variables. Denote by O_n the algebra of germs of holomorphic functions in n variables at the origin which is naturally identified with the algebra of convergent power series in n indeterminates with complex coefficients. For a polynomial $f \in \mathbb{C}_n$, denote by X the germ at the origin of $\mathbb{C}^n$ of hypersurface $X = \{f = 0\} \subset \mathbb{C}^n$.

We say that X is a germ of isolated hypersurface singularity if the origin is an isolated zero of the gradient of f. The local (function) algebra of X is defined as the (commutative associative) algebra $F(X) \cong O_n/(f)$, where (f) is the principal ideal generated by the germ of f at the origin. Further, denote by (f, df) the ideal in O_n generated by f and all of its partial derivatives. Recall that, for an isolated singularity $X = X(f) = \{f = 0\}$ as above, from the Hilbert's Nullstellensatz immediately follows that the factor-algebra $A(X) \cong O_n/(f, df)$ is finite dimensional. This factor-algebra is called the *moduli algebra* of X. An important result of J. Mather and S. S.-T. Yau states that the analytic isomorphism type of an isolated hypersurface singularity is determined by the isomorphism class of its moduli algebra [22].

Remark 1.1. As is well known, the moduli algebra $A(X)$ can serve as the base space of versal deformation of singularity X [5]. Its (complex) dimension $\tau(X)$ is often called the Tyurina number of X [5].

In many problems it is necessary to have an explicit basis of $A(X)$. It is well known and easy to prove that there always exist bases consisting of monomials.

Such bases are called monomial bases and will be often used in the sequel. We are basically interested in the so-called simple singularities [5] which consist of two series $A_k : \{x^{k+1} = 0\} \subset \mathbb{C}$, $D_k : \{x^2y + y^{k-1} = 0\} \subset \mathbb{C}^2$ and three exceptional singularities E_6, E_7, E_8 defined in $\mathbb{C}^2$ by polynomials $x^3 + y^4, x^3 + xy^3, x^3 + y^5$, respectively. Monomial bases in moduli algebras of simple singularities are given in [5]. In order to be able to compare singularities defined by polynomials of different number of variables, several equivalence relations are used in singularity theory.

Two IHS are called (analytically) equivalent if they are isomorphic as germs of algebraic varieties [5]. It is often convenient to use another equivalence relation between IHS. If $f \in \mathbb{C}_n$ defines an IHS $X = X(f)$ then it is obvious that $g = f + x_{n+1}^2$ also defines an IHS in $\mathbb{C}^{n+1}$ which is called stabilization of X. Two singularities are called stably equivalent if they can be obtained as iterated stabilizations of the same IHS. It is easy to see that the moduli algebra is not changed under taking suspensions so stably equivalent singularities have isomorphic moduli algebras [5].

As was already mentioned, for our purposes it is sufficient to deal with homogeneous and quasihomogeneous polynomials. Recall that a polynomial $f \in \mathbb{C}_n$ is called quasihomogeneous (qh) if there exist positive rational numbers $w_1, \dots, w_n$ (called weights of indeterminates x_j) and d such that, for each monomial $\prod x_j^{k_j}$ appearing in f with nonzero coefficient, one has $\sum w_j k_j = d$. The number d is called the quasihomogeneous degree (w-degree) of f with respect to weights w_j and denoted $w \deg f$. Obviously, without loss of generality one can assume that $w \deg f = 1$ and we will often do so in the sequel. The collection $(\mathbf{w}; d) = (w_1, \dots, w_n; d)$ is called the quasihomogeneity type (qh-type) of f. As is well known, for such an f, one has $f = \frac{1}{d} \sum w_j x_j \partial_j f$ (Euler formula). Hence in this case $f \in (df)$. Moreover, the w-degree defines natural gradings on $F(X)$ and $A(X)$ called qh-gradings. Thus one can introduce the Poincaré polynomials with respect to these gradings and in many cases they can be explicitly computed in terms of qh-type (see, e. g., [16], [5]).

If singularity X is defined by quasihomogeneous polynomial f, then the Tyurina number $\tau(X)$ coincides with the *Milnor number* $\mu(X)$ which is defined as $\dim_{\mathbb{C}} M(X)$, where $M(X) = O_n/(df)$ is the so-called *Milnor algebra* of X [22]. The equality $\tau(X) = \mu(X)$ for quasihomogeneous polynomial f immediately follows from the aforementioned fact that f belongs to the ideal (df) generated by its derivatives [5]. Thus in the quasihomogeneous case $A(X) \cong M(X)$. As is well known, $\mu(X)$ is a topological invariant of germ X which plays important rôle in many problems of singularity theory [5]. In quasihomogeneous case the Milnor number can be computed by a simple formula which will be repeatedly used in the sequel.

Proposition 1.2. *For an isolated hypersurface singularity X defined by a quasihomogeneous polynomial of $(\mathbf{w}; d)$ type, one has*

$$\tau(X) = \mu(X) = \prod_{i=1}^{n} \frac{d - w_i}{w_i}. \tag{1}$$

Remark 1.3. We wish to emphasize that throughout the whole paper we only deal with singularities defined by quasihomogeneous polynomials. Thus in our setting there is no difference between the moduli algebra and Milnor algebra, and the Milnor number $\mu(X)$ can be computed by the formula (1). Also, we often write $A(f)$ and $M(f)$ instead of $A(X)$ and $M(X)$ where this cannot cause misunderstanding.

Recall that a derivation of commutative associative algebra A is defined as a linear endomorphism D of A satisfying the Leibniz rule: $D(ab) = D(a)b + aD(b)$. Thus for such an algebra A one can consider the Lie algebra of its derivations $\operatorname{Der} A$ with the bracket defined by the commutator of linear endomorphisms. In particular, for a singularity X as above, one can consider the Lie algebras $DF(X) = \operatorname{Der} F(X)$ and $DA(X) = \operatorname{Der} A(X)$. Since by the aforementioned result of L. Pursell and M. Shanks the algebra $C^\infty(M)$ of smooth functions on a smooth manifold M is completely determined by the Lie algebra of its derivations, one can wonder if the same holds for algebras $F(X)$ and $A(X)$. Notice that if this is the case, then by the mentioned result of J. Mather and S. S.-T. Yau the corresponding Lie algebra determines the analytic isomorphism type of the singularity considered. As was shown by H. Hauser and G. Müller, for an isolated hypersurface singularity X, the Lie algebra $DF(X)$ indeed determines the analytic type of X [17].

Elegant as it is, this result is not quite effective because $DF(X)$ is an infinite-dimensional Lie algebra which is difficult to investigate and work with. At the same time $DA(X)$ is typically a finite-dimensional Lie algebra and its structural constants may be found in an algorithmic way. Moreover, S. S.-T. Yau showed that, for any isolated hypersurface singularity X, $DA(X)$ is a solvable Lie algebra [31]. Thus one may hope to identify such Lie algebras in concrete cases using a wealth of existing results on classification of solvable and nilpotent Lie algebras. Moreover, some natural numerical invariants of such Lie algebras can be effectively computed and it is natural to try to relate them to the numerical invariants of the singularity considered.

These are the two main directions of research which we pursue in this paper. Notice at once that there are no a priori reasons that an analog of the result of H. Hauser and G. Müller may hold for $DA(X)$ because this is a much smaller algebra than $DF(X)$. In fact, we indicate below two simple singularities X and Y such that $DA(X) \cong DA(Y)$ but X is not analytically isomorphic to Y. So it came as a sort of surprise for us when it turned out that $DA(X)$ is a complete invariant for all simple singularities with Milnor number bigger than 6. Actually, this fact served as an impetus for our research. For this reason in the present paper we concentrate on investigation of Lie algebras of the form $DA(X)$. For clarity and convenience, it seems appropriate to explicitly present the main concept and related terminology in a separate definition.

Definition 1.4. Let $X = \{f = 0\}$ be a germ of isolated hypersurface singularity at the origin of $\mathbb{C}^n$ defined by complex polynomial $f \in \mathbb{C}_n$. The Lie algebra $\operatorname{Der} A(X)$ of derivations of the moduli algebra $A(X) = O_n/(f, df)$ is called the Lie algebra of X and denoted $L(X)$. Its dimension will be denoted $\lambda(X)$ and called the Yau number of X.

Two technical remarks are now in order. Firstly, elements of $L(X)$ can be represented as holomorphic vector fields $V = \sum h_i \partial_i, h_i \in O_n$, considered with the standard action on $O_n : Vg = \sum h_i \partial_i g$. Such a vector field V defines an element of $L(X)$ if and only if it leaves the ideal (f, df) invariant, i. e., for each $g \in (f, df)$, one has $Vg \in (f, df)$. It is obvious that in such case the standard action V can be pulled down to $A(X)$ and defines a derivation $\hat{V}$ of $A(X)$ since the Leibniz rule is trivially fulfilled for $\hat{V}$. We often omit the "hat" and denote the corresponding element of $L(X)$ simply by V.

Secondly, the coefficients h_i in vector field presentation of an element $V \in L(X)$ can be reduced modulo the ideal (f, df) which implies that one can always construct a basis $V_j = \sum_{i=1}^{n} h_j^i \partial_i, j = 1, \ldots, \lambda(X)$, in $L(X)$ such that all coefficients h_j^i are monomials. However an important caveat is appropriate here: there need not always exist a basis consisting of elements of the form "monomial $\times\ \partial_i$" so, in general, one needs to take linear combinations of elementary vector fields ∂_i with monomial coefficients.

In order to describe the structural properties of Lie algebras of hypersurface singularities in an appropriate way, we will make use of various notions and results from the theory of Lie algebras. All algebraic definitions and results used in the sequel can be found, e. g., in [9]. For convenience of the reader some of the most frequently used concepts and results are collected in the rest of this section.

Recall that a Cartan subalgebra C in Lie algebra L is defined as a maximal commutative subalgebra consisting of semi-simple elements. If L is the Lie algebra of an algebraic Lie group then all Cartan subalgebras are pairwise conjugated, hence of the same dimension r which is called the rank $\operatorname{rk} L$ of Lie algebra L [9]. The index $\operatorname{ind} L$ is defined as the minimal codimension of subspaces of the form $\{\mathrm{ad}^*(l)(f), l \in L\}, f \in L^*$, in its coadjoint representation $\mathrm{ad}^* : L \to \mathrm{End}(L^*), (\mathrm{ad}^*(x)f)(y) := f([x, y]), f \in L^*, x, y \in L$, where L^* is the space dual to L [10]. It is well known that $\operatorname{ind} L$ can be also defined as follows. For $f \in L^*$, define a skew-symmetric bilinear form B_f on L by the formula $B_f(x, y) = f([x, y])$. Then $\operatorname{ind} L = \min_{f \in L^*} (\dim(\ker B_f))$ [10]. A Lie algebra is called a Frobenius Lie algebra if its coadjoint representation has an open dense orbit [14], which is obviously equivalent to $\operatorname{ind} L = 0$. It is well known that $\dim L + \operatorname{ind} L$ is always even and the maximal dimension of commutative subalgebras in a Lie algebra does not exceed $\frac{1}{2}(\dim L + \operatorname{ind} L)$ [10]. Lie algebra L is said to possess a maximal commutative polarization if it has a commutative subalgebra of dimension equal to $\frac{1}{2}(\dim L + \operatorname{ind} L)$ [14].

We will basically deal with solvable and nilpotent Lie algebras so for completeness we recall the corresponding definitions. Given a Lie algebra L, introduce two series of ideals: $L_{(*)} = \{L_{(i)}\}, L^{(*)} = \{L^{(i)}\}, L_{(0)} = L^{(0)} = L, L_{(1)} = L^{(1)} = [L, L], L_{(i)} = [L, L_{(i-1)}], L^{(i)} = [L^{(i-1)}, L^{(i-1)}], i = 2, 3, \ldots$. Lie algebra is called nilpotent if the series $L_{(*)}$ (the lower central series of L) contains only a finite number of non-zero ideals. Lie algebra is called solvable if the series $L^{(*)}$ contains a finite number of non-zero ideals. According to Engel's theorem, Lie algebra is nilpotent if and only if all operators $\mathrm{ad}\, a : L \to L$ are nilpotent for $a \in L$ [9]. Another general result states that a solvable algebraic Lie algebra can be decomposed into semi-direct sum of a Cartan subalgebra and maximal nilpotent ideal $N(L)$ (the latter is called the nilpotent radical of L).

The following concepts and results enable one to compute the Lie algebras of many concrete singularities we are going to deal with. Let A, B be associative algebras over a field F of characteristic zero which in the sequel will be either $\mathbb{R}$ or $\mathbb{C}$. Recall that the multiplication algebra $M(A)$ of A is defined as the subalgebra of endomorphisms of A generated by the identity element and left and right multiplications by elements of A. The centroid $C(A)$ is the set of endomorphisms of A which commute with all elements of $M(A)$. Clearly, $C(A)$ is a unital subalgebra of $\mathrm{End}A$. The following statement is a particular case of a general result from [8].

Proposition 1.5. cf. [8], p.438 *Let $S = A \otimes_F B$ be a tensor product of finite-dimensional associative algebras with units. Then*

$$\operatorname{Der} S \cong (\operatorname{Der} A) \otimes C(B) + C(A) \otimes (\operatorname{Der} B). \tag{2}$$

We will only use this result for commutative associative algebras with unit, in which case the centroid coincides with the algebra itself. Thus for commutative associative algebras A, B one has:

$$\operatorname{Der}(A \otimes B) \cong (\operatorname{Der} A) \otimes B + A \otimes (\operatorname{Der} B). \tag{3}$$

The latter formula will be repeatedly used in the sequel.

Finally, we wish to notice that, for a quasihomogeneous IHS, one can obtain a natural grading on $L(X)$ by putting the weight of ∂_j equal to $-w_j$ [16]. Thus the weight of a vector field of the form $x_k^m \partial_j$ is equal to $mw_k - w_j$, which obviously defines a grading on $L(X)$ compatible with the standard one on $A(X)$ in the sense that the action of $L(X)$ becomes the action of graded Lie algebra. This grading is called the *quasihomogeneous grading* on $L(X)$. The Poincaré polynomial of $A(X)$ with respect to this grading was computed in [16] and this could be used for computing the Poincaré polynomial of $L(X)$ with respect to the above grading. However, for our purposes it appears more convenient to use certain other natural gradings on $L(X)$ introduced in Section 4.

2. Auxiliary constructions and computations

A number of results presented in this paper is based on a few basic computations which are described in this section. We begin with introducing certain formalism which facilitates further considerations.

Let n be a natural number and $\kappa = (k_1 + 1, \dots, k_n + 1)$ a non-negative multi-index with $k_i \geqslant 2, 1 \leqslant i \leqslant n$. Denote by $V = V_\kappa$ the vector space spanned by a basis indexed by collections $(a_1, \dots, a_n; i)$, where $1 \leqslant i \leqslant n$, $0 \leqslant a_j \leqslant k_j - 1$ for $j \neq i$, $1 \leqslant j \leqslant n$, and $1 \leqslant a_i \leqslant k_i - 1$. It is easy to see that the dimension of this space is equal to $n\sigma_n(k_1, \dots, k_n) - \sigma_{n-1}(k_1, \dots, k_n)$, where σ_j denotes the j-th symmetric function of n variables. Indeed, this immediately follows from the identity

$$\sum_{i=1}^{n} (x_i - 1) \prod_{j \neq i} x_j = n\sigma_n(x_1, \dots, x_n) - \sigma_{n-1}(x_1, \dots, x_n).$$

Identifying each basis vector with its index, introduce a bilinear operation on V by

$$\begin{aligned} &[(a_1, \dots, a_n; i), (b_1, \dots, b_n; j)] = \\ &- a_j(a_1 + b_1, \dots, a_{j-1} + b_{j-1}, a_j + b_j - 1, a_{j+1} + b_{j+1}, \dots, a_n + b_n; i) \\ &+ b_i(a_1 + b_1, \dots, a_{i-1} + b_{i-1}, a_i + b_i - 1, a_{i+1} + b_{i+1}, \dots, a_n + b_n; j). \end{aligned} \tag{$*$}$$

We show now that this operation actually defines a Lie algebra structure on V. The skew-symmetry is obvious so only Jacobi identity should be verified. To this end we relate V with the Lie algebra of Pham singularity defined by polynomial $P_\kappa = \sum x_j^{k_j+1}$. Since this is a direct sum of simple singularities A_{k_j}, the moduli algebra $A(P_\kappa)$ is isomorphic to the tensor product of moduli algebras of A_{k_j} singularities which are well known [5].

Using Proposition 1.5, the tensor structure of the moduli algebra for P_κ, and the above description of $L(A_k)$ we can now identify the Lie algebra of P_κ with the vector space V introduced above. Using the vector field notation for elements of $L(P_\kappa)$ it is easy to check that an explicit isomorphism between $L(P_\kappa)$ and $V = V_\kappa$ is established by the correspondence:

$$\prod x_k^{a_k} \partial_j \mapsto (a_1, \dots, a_n; j).$$

Comparing the commutators in $L(P_\kappa)$ and in V we see that they coincide, which immediately implies that the bilinear operation introduced above satisfies Jacobi identity and defines thus a Lie algebra structure on V_κ. As a by-product we obtain a formula for the Yau number of Pham singularity.

Proposition 2.1.

$$\lambda(P_\kappa) = n\sigma_n(k_1, \dots, k_n) - \sigma_{n-1}(k_1, \dots, k_n). \tag{4}$$

Consider now the Lie algebra of D_{k_1,k_2}-singularity defined by the polynomial $f = x_1^{k_1} x_2 + x_2^{k_2}$. As is well known (see, e. g., [5]), its moduli algebra A is of dimension $k_2(k_1 - 1) + 1$ and has monomial basis of the form

$$\{X_1^{a_1} X_2^{a_2}, 0 \leqslant a_1 \leqslant k_1 - 2; 0 \leqslant a_2 \leqslant k_2 - 1; X_1^{k_1-1}\}. \tag{**}$$

Notice that here and in the sequel the class of a function g in the moduli algebra $A(f)$ we often denote by the corresponding capital letter G.

Then it is easy to verify the following identities in the moduli algebra:

$$X_1^{k_1-1} X_2 = 0, \tag{5}$$

$$X_1^{k_1} + k_2 X_2^{k_2-1} = 0. \tag{6}$$

¿From the formulæ (5, 6) we get:

$$X_1^{k_1+i} = -k_2 X_1^i X_2^{k_2-1}, \ 0 \leqslant i \leqslant k_1 - 2, \tag{7}$$

$$X_1^m = 0, \ m \geqslant 2k_1 - 1, \tag{8}$$

$$X_2^m = 0, \ m \geqslant k_2. \tag{9}$$

As usual, in order to define a derivation d of A it suffices to indicate its values on the generators X_1, X_2 which can be written in the basis (**). Thus using the Einstein notation we can write

$$dX_j = d^j_{i_1,i_2} X_1^{i_1} X_2^{i_2} + d^j_{k_1-1,0} X_1^{k_1-1}, \ j = 1, 2.$$

Using the relations (5 - 9) one now easily finds conditions defining a derivation of A.

Lemma 2.2. *In order that a linear transformation d defines a derivation of $A(f)$ it is necessary and sufficient that*

$$d^1_{0,0} = d^1_{0,1} = \ldots = d^1_{0,k_2-3} = 0;$$

$$d^2_{0,0} = d^2_{1,0} = \ldots = d^2_{k_1-2,0} = 0;$$

$$k_1 d^1_{1,0} = (k_2-1)d^2_{0,1}, k_1 d^1_{2,0} = (k_2-1)d^2_{1,1}, \ldots, k_1 d^1_{k_1-1,0} = (k_2-1)d^2_{k_1-2,1},$$

$$(k_1-1)d^1_{0,k_2-2} = k_2 d^2_{k_1-1,0}.$$

Using this lemma we easily obtain the following description of the Lie algebra in question.

Proposition 2.3. *The dimension of Lie algebra $L(D_{k_1,k_2})$ is equal to*

$$\lambda(D_{k_1,k_2}) = 2k_1k_2 - 2k_1 - 3k_2 + 5. \tag{10}$$

The derivations represented by the following vector fields form a basis in $L(D_{k_1,k_2})$:

$$(k_2-1)x_1\partial_1 + k_1x_2\partial_2, (k_2-1)x_1^2\partial_1 + k_1x_1x_2\partial_2, \ldots, (k_2-1)x_1^{k_1-1}\partial_1 + k_1x_1^{k_1-2}x_2\partial_2,$$

$$k_2x_2^{k_2-2}\partial_1 + (k_1-1)x_1^{k_1-1}\partial_2;$$

$$x_1^{a_1}x_2^{a_2}\partial_1, x_2^{k_2-1}\partial_1, 1 \leqslant a_1 \leqslant k_1-2,\ 1 \leqslant a_2 \leqslant k_2-1, x_1^{b_1}x_2^{b_2}\partial_2,\ 0 \leqslant b_1 \leqslant k_1-2,$$

$$2 \leqslant b_2 \leqslant k_2-1.$$

Having this done, we are able to clarify an interesting issue concerned with the Lie algebras of quasihomogeneous singularities. Proposition 1 shows that the dimension of moduli algebra of such a singularity X is determined by its quasihomogeneity type. It is thus natural to wonder if the same holds for the dimension of $L(X)$. We are now able to give a simple example showing that this is not always true.

To this end consider the two singularities defined by polynomials $P_{a_1,b_1} = x_1^{a_1} + x_2^{b_1}$ and $D_{a_2,b_2} = x_1^{a_2}x_2 + x_2^{b_2}$. As indicated in [5] their quasihomogeneity types are $(\frac{1}{a_1}, \frac{1}{b_1})$ and $(\frac{b_2-1}{a_2b_2}, \frac{1}{b_2})$, respectively. Taking any natural b, q and putting $b_2 = b_1 = b,\ a_1 = qb,\ a_2 = q(b-1)$ we obtain that the two above polynomials have the same quasihomogeneity type $(\frac{1}{qb}, \frac{1}{b})$.

¿From the formulae for the dimension of $L(X)$ presented above we get that $\dim L(P_{a_1,b_1}) = 2qb^2 - 3b(q+1) + 4$ and $\dim L(D_{a_2,b_2}) = 2qb^2 - 2q(2b-1) - 3b + 5$. The difference between the two dimensions is equal to $q(b-2) - 1$ so we see that they only coincide for $q = 1, b = 3$. Thus we see that the quasihomogeneity type does not determine the dimension of associated Lie algebra.

This conclusion suggests a number of natural questions. First of all, one may wonder if there is a typical value of $\dim L(X)$ for a given quasihomogeneity type, i. e. such value which is obtained by all polynomials except a finite number of those. If so, one could hope to express this typical value in terms of the quasihomogeneity type or, more specifically, in terms of the corresponding Newton diagram.

It is also interesting to find the maximal and minimal values of $\dim L(X)$ within a given quasihomogeneity type and characterize singularities for which the extremal values are attained. In many cases we have verified that the Pham singularity has the maximal value of $\dim L(X)$ within its quasihomogeneity type.

However this is definitely not always so. For example, the two singularities $X_1 = X(x_1^4+x_2^2)$ (stabilization of A_3 singularity) and $X_2 = X(x_1^2 x_2+x_2^2)$ (D_3 singularity) are both quasihomogeneous of the type $(1/4, 1/2; 1)$. At the same time $\lambda(X_i)$ can be computed by formulæ (4), (10) and we get that $\dim L(A_3) = 2 < \dim L(D_2) = 3$. Thus $\lambda(A_3)$ is not maximal in the qh-type $(1/4, 1/2; 1)$. Actually in this case $\lambda(A_3)$ realizes the minimal value while $\lambda(D_2)$ gives the maximal value.

Nevertheless the formulæ (4), (10) suggest that this phenomenon takes place due to the fact that the value of Milnor number is rather small. Taking all this into account we believe that a plausible conjecture is that the Pham singularity with $\mu(X) > 5$ has the maximal value of $\dim L(X)$ in its qh-type. The following simple considerations show that this conjecture is at least "asymptotically correct".

Proposition 2.4. *Let X be a quasihomogeneous isolated hypersurface singularity in $\mathbb{C}^n$ with Milnor number μ. Then one has* $\dim L(X) \leqslant n\mu$.

Proof. As was explained in section 2 each element of $L(X)$ can be written as a vector field $V = \sum v_i \partial_i$. Denote by I the ideal (f, df) in O_n which in our case coincides with (df) and choose a monomial basis $\{e_j, j = 1, \ldots, \mu = \mu(X)\}$ in $A(X)$. Then each coefficient v_i can be written as $v_i = u_i + w_i$, where $u_i = \sum_{j=1}^{\mu} u_i^j e_j, u_i^j \in \mathbb{C}, w_i \in I, i = 1, \ldots, n$. Put $\tilde{V} = \sum u_i \partial_i$ and notice that such vector fields form a vector space $\mathcal{V}$ of dimension $n\mu$. Moreover, it is obvious that the action of V on $A(X) = O_n/I$ coincides with the action of $\tilde{V}$. Thus the correspondence $V \mapsto \tilde{V}$ defines an embedding of $L(X)$ in $\mathcal{V}$, which immediately implies the desired estimate. ∎

Simple as it is, the above estimate is asymptotically exact. To show this, it is sufficient to express the Milnor number and Yau number of a Pham singularity in terms of parameters k_i.

Proposition 2.5. *For a Pham singularity X of the type $(n; k_1, \ldots, k_n)$, one has*

$$\mu(X) = \prod_{i=1}^{n} k_i, \ \lambda(X) = n \prod_{i=1}^{n} k_i - \sigma_{n-1}(k_1, \ldots, k_n).$$

Fixing n and putting $k_i = k$, one concludes that, as $k \to \infty$, the Yau number $\lambda(P_\kappa)$ grows exactly as $n\mu(P_\kappa)$. It would be interesting to find a reasonable lower bound for $\lambda(X)$ in terms of $\mu(X)$ and other numerical invariants of $A(X)$. Notice that certain lower estimates were given in [27].

In line with a general principle of singularity theory (see [5]) one may hope that, for a typical function with a given Newton diagram, the Yau number can be computed in terms of the Newton diagram. It would be interesting to find the spectrum of all possible values of $\dim L(X)$ for each quasihomogeneous type. An analogous problem can be posed for singularities with a fixed Milnor number.

3. Classifying simple singularities by Lie algebras

We pass now to a precise formulation and outline of the proof of the first main result. Recall that a singularity X is called simple if only a finite number of singularity types appear as small deformations of X. According to V. I. Arnol'd, the list of such singularities is analogous to the classification of simple Lie groups and consists of two infinite series A_k, D_k and three exceptional singularities E_6, E_7, E_8 which were described in Section 2 [5]. It is well known that simple singularities are pairwise non-isomorphic. In particular, singularities within each series can be classified by their Milnor numbers. We will show that simple singularities can be essentially classified by their Lie algebras and this result serves for us as a starting point and pattern.

Theorem 3.1. *If X and Y are two simple hypersurface singularities except A_6 and D_5, then $L(X) \cong L(Y)$ as Lie algebras if and only if X and Y are analytically isomorphic.*

Proof. The proof is obtained by computing the structural constants of derivation Lie algebras and checking that those are pairwise non-isomorphic for all singularities satisfying the conditions of the theorem. Such computations can be done using a monomial bases in $A(X)$ which are well known for all simple singularities [5]. Then a basis in $L(X)$ can be found using merely some linear algebra and one becomes able to compute and compare basic numerical invariants of arising Lie algebras, which directly leads to the desired conclusion. Let us present the argument in some detail for A_k singularities defined by polynomials x^{k+1}.

First of all, one easily checks that the classes of monomials $1, x, \ldots, x^{k-1}$ are linearly independent and span $A(A_k)$. This means that $\{1, x, \ldots, x^{k-1}\}$ is a monomial basis in the moduli algebra $A(A_k) = O_1/(x^k)$. In other words, $A(A_k)$ is a truncated algebra of polynomials in one variable with the identity $x^k = 0$. Using the vector field notation explained in Section 2, it is now easy to verify that a basis for $L(X)$ is given by derivations defined by vector fields $x^j \partial x$ with $j = 1, \ldots, k-1$. Indeed, they all preserve the jacobian ideal $(x^k) \subset O_1$ of f and so define derivations $e_j, j = 1, \ldots, k-1$, of $L(A_k)$. The commutation relations for the basis derivations are: $[e_j, e_p] = (p-j)e_{j+p-1}$, where $e_s = 0$ for $s > k-1$.

Other simple singularities are treated in a completely similar way. In particular, from Propositions 2.1 and 2.2. we immediately get $\dim L(D_k) = k$, $\dim L(E_6) = 7, \dim L(E_7) = 8, \dim L(E_8) = 10$. Notice that the results of Section 2 also yield explicit monomial bases in the moduli algebras. Using the vector field notation, it is then easy to verify that $\operatorname{rk} L(A_k) = \operatorname{rk} L(D_k) = \operatorname{rk} L(E_7) = 1$ while $\operatorname{rk} L(E_6) = \operatorname{rk} L(E_8) = 2$. The structure constants of Lie algebras for D and E series can be found by elementary calculations (those structure constants are also given in [7]).

Comparing the dimensions it becomes clear that the Lie algebras $L(A_k)$ are indeed pairwise non-isomorphic for the singularities from A_k series. Further, the minimal number of generators for the maximal nilpotent ideal of $L(A_k)$ is two. For $k \geqslant 6$, the minimal number of generators for the maximal nilpotent ideal of $L(D_k)$ is three. Thus, despite $\dim L(A_{k+1}) = \dim L(D_k) = k$, these two algebras are non-isomorphic for $k \geqslant 6$. Lie algebra $L(D_4)$ is a direct sum of two two-dimensional non-commutative Lie algebras. At the same time, $L(A_k)$ for all k is indecomposable. We add that $L(D_k)$ is indecomposable for $k \geqslant 5$.

Singularities from D_k series and exceptional singularities are treated similarly. The cases of E_6 and E_8 are distinguished from each other because $7 = \lambda(E_6) \neq \lambda(E_8) = 10$ and from all other simple singularities by the ranks of their

Lie algebras. It remains to treat the three algebras $L(A_9), L(D_8), L(E_7)$ which all have the same dimension 8 and the same rank 1. However we manage to show that they are pairwise non-isomorphic by considering the sequences of dimensions of the upper central series [9]. Indeed, this sequence is $(1,2,3,4,5,7)$ for $L(A_9)$; $(1,3,5,7)$ for $L(D_8)$; $(1,2,4,5,7)$ for $L(E_7)$, which implies the statement of the theorem. ■

Now we indicate an exceptional pair of derivation Lie algebras. Namely, although D_5 is not analytically isomorphic to A_6, one has $L(D_5) \cong L(A_6)$. This can be easily seen from the multiplication tables of the latter algebras given in [7]. In fact, the multiplication table for $L(A_6)$ in the basis $e_1, ..., e_5$ is given in the proof of Theorem 3.1, whereas for $L(D_5)$ we can use Proposition 2.3: note that D_5 is the same as $D_{2,4}$, so from Proposition 2.3 we get a basis $d_1, ..., d_5$ with the multiplication table

$$[d_1, d_k] = (k-1)d_k, \quad k = 1, ..., 5; \ [d_2, d_3] = -8d_4; \ [d_2, d_4] = -d_5.$$

It is then easy to see that the linear isomorphism given by

$$e_1 \mapsto d_1, \ e_2 \mapsto d_2, \ e_3 \mapsto -\frac{1}{8}d_3, \ e_4 \mapsto d_4, \ e_5 \mapsto -\frac{1}{2}d_5$$

respects the Lie bracket (hence also the Euler grading).

Thus we have established that the Lie algebras of simple singularities are pairwise non-isomorphic except the pair $L(A_6) \cong L(D_5)$. Up to our knowledge the latter circumstance was not mentioned in the literature. For example, in the paper [7] which contains the description of Lie algebras $L(X)$ for all simple singularities this fact was not noticed. The example given above explains the condition imposed on Milnor numbers in Theorem 3.1.

It is now natural to look for wider classes of singularities which can be classified by their Lie algebras. The ultimate goal would be to find reasonable sufficient conditions which guarantee that a given singularity is determined by its Lie algebra up to isomorphism. As a step towards solving this problem, we establish an analog of Theorem 3.1 for Pham singularities and D_{**} series.

Theorem 3.2. *If X and Y are two singularities from P_κ and D_{**} series and if $\mu(X)$, $\mu(Y) > 6$, then $L(X) \cong L(Y)$ as Lie algebras if and only if X and Y are analytically isomorphic.*

Before giving the proof let us outline the scheme of argument. First we show that the isomorphism type of $L(P_\kappa)$ determines values of parameters k_i up to the order. Next we verify the same for D_{**} series. Thus singularities within each series can be classified by their Lie algebras. Finally, we show that, for $\mu > 6$, no Pham singularity can be isomorphic to a D_{k_1,k_2} singularity. Correspondingly, proof consists of three parts.

Proof of theorem 3.2. (1) We begin with considering the Pham series. As was shown above, if $X = X(P_\kappa)$ is defined by a polynomial in n variables then $\mathrm{rk}L(X) = n$. This implies that two such Lie algebras can be only isomorphic if they are defined by polynomials of the same number of variables. So without restricting generality we may consider two Pham singularities of the type $(n;\kappa)$. We use induction in n. The appropriate inductive hypothesis sounds as follows: for all Pham singularities with the number of variables not exceeding n the exponents k_j are determined by the Lie algebra $L(X)$ up to the order. It is obvious for $n = 1$ because in this case k_1 is equal to $\dim L(X) + 1$.

Suppose now that this holds for $n \leqslant k$. Without restricting generality we may suppose that the exponents are written in the order of decreasing. We will show then that k_1 is an invariant of $L(X)$. Notice that k_1 has the biggest modulus (absolute value) among the eigenvalues of all basic operators $E_i x_i \partial_i$. Notice that the spectra of all those operators are real and nonnegative and the smallest eigenvalue of all E_i is equal to one. These data can be used to obtain a numerical invariant of algebra $L(X)$ as follows.

Introduce a norm on $A(X)$. This induces the operator norm on $L(X)$ and we may consider the unit sphere $S \subset L(X)$. For each derivation $T \in L(X)$ denote by $\Lambda(T)$ (respectively, $\lambda(T)$) the maximal (respectively, minimal nonzero) modulus of eigenvalues of T. Consider now the maximum M of the ratio $r(T) = \Lambda(T)/\lambda(T)$ for all $T \in S$ (this maximum is attained since S is compact). Obviously, M being defined as a ratio does not depend on the choice of norm on $A(X)$. Thus it is an invariant of Lie algebra $L(P_k)$. Moreover, since all minimal moduli of eigenvalues of basic operators E_i are equal it follows that $\lambda(T)$ is constant on S. Thus the maximum of the ratio $r(T)$ is achieved on the operator which an eigenvalue with the maximal modulus over S. From the structure of spectra of the basic operators E_i described above and well known interlacing property of eigenvalues of linear combinations of operators [6] it is clear that this maximum is equal to $k_1 - 1$. Thus $k_1 - 1$ is an invariant of $L(X)$ and the rest of the proof runs is obtained by the following algebraic considerations.

Using the description of $L(P_k)$ given in Section 2, consider the set $I_+ = \{(a;j) : a_1 > 0\}$. It is easy to verify that I_+ is an ideal and the factor of $L(P_k)/I_+$ is isomorphic to $L(\tilde{P}_k)$ given by the Pham polynomial in $n-1$ variables. By the inductive hypothesis the isomorphism class of the factor determines the exponents $(k_2, \ldots, k_n)$ up to the order. Thus the whole collection κ is determined by $L(P_\kappa)$ up to the order and the first part of the proof is completed.

(2) In order to deal with D_{**} series let us introduce a numerical invariant of Lie algebras of the form $L(D_{k_1,k_2})$. As was mentioned on $L(D_{k_1,k_2})$ there exists a $\mathbb{Z}$-grading defined by the vector field $E = (k_2 - 1)x_1\partial_1 + k_1 x_2 \partial_2$ (Euler field). Instead of E we could take any other semi-simple element of $L(D_{k_1,k_2})$. Since the set of semi-simple elements in $L(D_{k_1,k_2})$ is one-dimensional, any two such gradings coincide up to a (complex) multiple. Notice that E is regular, in the sense that its orbit in the adjoint representation has the maximal dimension. The invariant we are after is now defined as follows.

Take a homogeneous element $a \in L(D_{k_1,k_2})$ of positive degree. Then the operator $\operatorname{ad} a$ is nilpotent. Let $n(a)$ denote its nilpotency index with respect to this grading. It is clear that $n(a)$ does not change if the grading is multiplied by a complex number. Define $n(L)$ as the maximal number among $n(a)$ over the set of all (non-zero) homogeneous nilpotent elements. Then it is obvious that $n(L)$ is an invariant of graded Lie algebras. We will compute this invariant for $L(D_{k_1,k_2})$ and show that it distinguishes such algebras.

For computing $n(L(D_{k_1,k_2}))$ it is useful to notice that there is a natural Lie-Rinehart [24] structure on the pair $A(D_{k_1,k_2}), L(D_{k_1,k_2})$ and use some well-known properties of such structures [9]. Consider the elements $S = (k_2-1)x_1^2\partial_1 + k_1 x_1 x_2 \partial_2$

and $T = x_2^2\partial_2$. By direct computation it is not difficult to show that $n(S) = k_2 - 2$ and $n(T) = 2k_1 - 5$. Moreover, from the description of the homogeneous components for Euler grading given in Section 2 it follows that no other element can have nilpotency index bigger than $N = \max(n(S), n(T))$. Thus $n(D_{k_1,k_2}) = k_2 - 2$ if $k_2 \geqslant 2k_1 - 5$ and $n(D_{k_1,k_2}) = 2k_1 - 5$ if $2k_1 - 5 > a_2 - 2$.

Suppose now that $L(D_{k_1,k_2}) \cong L(D_{m_1,m_2})$ for some pairs $(k_1, k_2), (m_1, m_2) \in \mathbb{Z}_+^2$. In order to show that $(k_1, k_2) = (m_1, m_2)$ we proceed as follows. First of all, in such a case we have $\dim L(D_{k_1,k_2}) = \dim L(D_{m_1,m_2})$ and using the explicit formula for the Yau number of D_{k_1,k_2} singularity we get equation

$$2k_1k_2 - 2k_1 - 3k_2 = 2m_1m_2 - 2m_1 - 3m_2. \tag{11}$$

Moreover, we have $n(D_{k_1,k_2}) = n(D_{m_1,m_2})$. Taking into account the above formulæ for $n(D_{k_1,k_2})$ it is obvious that there are four logically possible relations between the parameters k_1, k_2, m_1, m_2: 1) $k_2 = m_2$; 2) $k_1 = m_1$; 3) $k_2 - 2 = 2m_1 - 5$; 4) $m_2 - 2 = 2k_1 - 5$. In the first two cases, substituting the relations in (11) we immediately get that the second parameters also coincide. The last two cases are symmetric so it is sufficient to prove the result when $2k_1 - 5 > k_2 - 2$ and $m_2 - 2 \geqslant 2m_1 - 5$. Thus in this case we have $m_2 - 2 = 2k_1 - 5$, hence $2k_1 = m_2 + 3$. Transforming the left hand side of (11) and substituting $m_2 + 3$ instead of $2k_1$ we get:

$$2k_1(k_2 - 1) - 3k_2 = (m_2 + 3)(k_2 - 1) - 3k_2 = m_2k_2 - m_2 - 3.$$

This obviously gives the equation

$$m_2k_2 - m_2 - 3 = 2m_1(m_2 - 1) - 3m_2.$$

Taking the number 3 to the right hand side and factoring the latter we get:

$$m_2(k_2 - 1) = (m_2 - 1)(2m_1 - 3).$$

Let us now rewrite the last relation in the form:

$$k_2 - 1 = \frac{m_2 - 1}{m_2}(2m_1 - 3).$$

Since in the left hand side we have an integer and $m_2 - 1$ is relatively prime with m_2, it follows that $2m_1 - 3$ is divisible by m_2. However by assumption $m_2 \geqslant 2m_1 - 3$ so we conclude that $2m_1 - 3 = m_2$, hence $k_2 = m_2$. The rest of the proof goes as in the first two cases. Thus we have shown that singularities of D_{**} series are classified by their Lie algebras. Actually, the invariant $n(L)$ could be also used in the first part of the proof concerned with Pham singularities. However there one should use $\mathbb{Z}^n$ gradings and the argument does not seem simpler than the one presented above.

(3) Suppose now that $L(P_\kappa) \cong L(D_{m_1,m_2})$ for certain values of parameters k_i, m_j. Then their ranks should be equal. Remembering that for a Pham singularity of the type $n; \kappa$ the rank of Lie algebra equals n and for D_{**} series this rank always equals one, we get that n necessarily equals one. Thus we should only compare D_{**} singularities with A_k singularities. Since the Cartan subalgebras are in both cases one-dimensional, one can arrive at the desired conclusion by comparing the spectra of semi-simple operators $\operatorname{ad} h$ in both cases. Notice that

the arithmetic structure of these spectra does not depend on the choice of such an element. Recall that, for A_k series, the spectrum consists of a single arithmetic progression. From the description of the homogeneous components of $L(D_{m_1,m_2})$ given in Section 2 it is clear that the spectrum of $\operatorname{ad} h$ reduces to a single arithmetic progression only if $m_1 = 2$. Thus we only have to compare A and D series which has already been done in the proof of preceding theorem. Hence the third step is accomplished and the proof of Theorem 3.2 is complete. ∎

Actually, the same considerations applied to singularities with $\mu \leqslant 6$ enable one to show that the only exceptions in the above theorem are again A_6 and D_5 singularities. Summing up, we have indicated a sufficiently wide class of singularities classified by their Lie algebras. On the other hand, the above discussion shows that $L(X)$ does not always determine the analytic isomorphism type of X. Thus it seems natural to have a closer look at the algebraic properties of Lie algebras associated to singularities of the said class and this will be our main occupation in the rest of the paper. Since all singularities considered above are defined by quasihomogeneous polynomials, we proceed by discussing the gradings on Lie algebras of quasihomogeneous IHS and properties of associated Poincaré polynomials.

4. Poincaré polynomials of quasihomogeneous singularities

By analogy with the results on Poincaré polynomials of moduli algebra of qh singularity obtained in [16], [1], we now wish to compute Poincaré polynomials of Lie algebras for all simple and Pham singularities. We first consider a Pham singularity $P_{\mathbf{k}} = \sum x_j^{k_j+1}$. Notice that its Lie algebra was described in Section 2.

There are two natural gradings on the associated Lie algebra which can be conveniently described using the notation of Section 2. The first one is a $\mathbb{Z}^n$-grading which can be called the Cartan grading, while the second one is a $\mathbb{Z}$-grading which can be called the Euler grading. In order to describe the Cartan grading we recall that a Cartan subalgebra is a maximal commmutative subalgebra H such that, for each $h \in H$, $\operatorname{ad} H$ is semi-simple.

¿From $(*)$ follows that a basis in Cartan subalgebra is given by the elements of the following form $h_i = (0, \ldots, 1, \ldots, 0; i)$ with 1 staying on the ith place. Obviously it is n-dimensional. The basis vectors introduced above are eigenvectors for each of operators $\operatorname{ad} h_i$. The eigenvalues of those operators define the Cartan $\mathbb{Z}^n$-grading. Thus a basis element of the form $(a; j)$ has $\mathbb{Z}^n$-grading $(a_1, \ldots, a_{j-1}, a_j - 1, a_{j+1}, \ldots, a_n)$.

The Euler grading is defined as follows. For an element of the form $(a; j)$, define $h(a, j) = -1 + \sum a_i$ and call it the Euler height of $(a; j)$. The product formula implies that the correspondence $(a; j) \mapsto h(a; j)$ defines a Lie algebra $\mathbb{Z}$-grading called the Euler grading. Thus the Euler grading is equal to the sum of components of the Cartan grading. For a homogeneous Pham polynomial (i.e., when all k_j are equal), it is easy to see that the Euler grading is just a multiple of the quasihomogeneous grading mentioned in Section 1. However, in general these two gradings are not proportional. As was mentioned in Section 1, it is more suitable for our purposes to use the Cartan and Euler gradings.

It turns out that the Lie algebra $L(P)$ satisfies certain duality with respect to the Euler grading. In order to formulate this property we first compute the dimension of the lowest and highest homogeneous components. It is easily seen that they both are equal to n. The basis in the zero height component is given by $h_i = (0, \ldots, 1, \ldots, i)$. The maximal height is $M = -1 + \sum(k_i - 1)$ and the basis is $(k_1 - 1, \ldots, k_n - 1; i)$, $i = 1, \ldots, n$.

In fact, each component of height k is isomorphic to the one of the height $M - k$. This can be easily proved using an involution I on $L(V)$ defined by the formula

$$I(a_1, \ldots, a_n; i) := (b_1, \ldots, b_n; j),$$

where

$$b_i = k_i - 1 - a_i, i \neq j; b_j = k_j - a_j.$$

It is easy to see that I acts as declared. However I is neither an algebra isomorphism nor derivation. The basis of h_i is transformed by I into the basis of z_i.

Moreover, using the tensor structure of algebra described above one can also compute the generating function of $L(P_\kappa)$. For simplicity, we only consider the case of two variables. The general case is completely similar.

We first compute the generating function for $n = 1$ in which case P_{k_1} coincides with A_{k_1}. Thus we can use the description of $L(A_k)$ obtained in the previous section. In this way we get:

$$g_1(t) = \frac{1 - t^{k_1}}{1 - t}.$$

For the algebra of derivations we get:

$$g_2(t) = \frac{1 - t^{k_1 - 1}}{1 - t}.$$

Consider now the polynomial $P_{k_1,k_2} = x_1^{k_1+1} + x_2^{k_2+1}$. In its moduli algebra and derivation algebra we have natural bigradings and the generating functions with respect to those bigradings are, respectively:

$$g_1(t) = \frac{1 - t_1^{k_1} - t_2^{k_2} + t_1^{k_1} t_2^{k_2}}{(1 - t_1)(1 - t_2)}\ ,$$

$$g_2(t) = \frac{2 - t_1^{k_1-1} - t_1^{k_1} - t_2^{k_2-1} - t_2^{k_2} + t_1^{k_1-1} t_2^{k_2} + t_1^{k_1} t_2^{k_2-1}}{(1 - t_1)(1 - t_2)}\ .$$

Specializing the latter formula by putting $t_1 = t_2 = t$, we get the generating function for Cartan grading of $L(P_\kappa)$ in the form:

$$g_3(t) = \frac{2 - t^{k_1-1} - t^{k_1} - t^{k_2-1} - t^{k_2} + 2t^{k_1+k_2-1}}{(1 - t)^2}\ .$$

Using this formula we can explicitly write down the dimensions d_r of graded components of $L(P_\kappa)$. Namely, one has:

$$d_r = 2(r + 1), \text{ for } 0 \leqslant r \leqslant k_1 - 2;$$
$$d_r = 2k_1 - 1, \text{ for } k_1 - 1 \leqslant r \leqslant k_2 - 2;$$
$$d_r = 2(k_1 + k_2 - r - 2), \text{ for } k_2 - 1 \leqslant r \leqslant k_1 + k_2 - 3.$$

The above calculations of the dimensions of homogeneous components for the Euler grading for two variables suggest that, for n variables, the explicit formulæ would be quite complicated. For this reason we prefer to deal with the generating functions. As usual, in case when the generating function is a polynomial it is called the Poincaré polynomial of the corresponding algebra.

Recall that a real polynomial $P = \sum a_i t^i$ is called unimodal if, for some i, the coefficients a_k monotonously increase up to $k = i$ and monotonously decrease for $k > i$ [4]. We say that a polynomial $P = \sum a_i t^i$ is palindromic if, for each i, one has $a_i = a_{n-i}$ (some authors use the term "recurrent polynomial" [4] but we prefer to avoid the word "recurrent" which has a number of other meanings as well). Correspondingly, a real polynomial $P = \sum a_i t^i$ is called unipalindromic if it is palindromic and unimodal simultaneously. Finally, a polynomial is called unimodular if all of its roots lie on the unit circle.

Proposition 4.1. *For a Pham singularity P_κ, the Poincaré polynomial for Euler grading on $L(P_\kappa)$ is unipalindromic.*

Proof. ¿From Proposition 1.5 it follows that the generating function for the Cartan $\mathbb{Z}^n$-grading has the form

$$G_1(t_1, \ldots, t_n) = \sum_j \frac{1 - t_j^{k_j - 1}}{1 - t_j} \prod_{1 \leqslant i \neq j \leqslant n} \frac{1 - t_i^{k_i}}{1 - t_i}.$$

It is easy to realize that in order to get the Euler grading it is sufficient to put $t_i = t$, for all i, in this formula. A direct check shows that each summand in this formula is a unipalindromic polynomial in the above sense and they all have the same degree equal to $-1 + \sum k_i$. It becomes now obvious that the sum is also a unipalindromic polynomial, which completes the proof. ■

The following observation together with its proof presented below belongs to M. Jibladze.

Proposition 4.2. *All roots of polynomial G_1 lie on the unit circle.*

Proof. First of all, it is known that any palindromic polynomial f can be represented in the form $x^k g(x + \frac{1}{x})$ for some k and g. Clearly, if the leading coefficient of f is positive then the same holds for g, which is useful to keep in mind since the generating functions we will deal with, will have positive coefficients.

Next, we notice that if f has real coefficients, then all of its roots lie on the unit circle if and only if all the roots of g are real and their absolute values do not exceed 2. Indeed, if f is real then the roots come in complex conjugate pairs, and if z and z' is such a pair of roots then $z' = \frac{1}{z}$, and $z + \frac{1}{z} = 2\Re(z)$ is a real number in the segment $[-2, 2]$. It is obvious that in this way we get all roots of g. And vice versa, the roots of g are the same as the numbers of the form $z + \frac{1}{z}$, where z is a root of f. However, if z is not real while $z + \frac{1}{z}$ is real, then z necessarily lies on the unit circle. In fact, one has:

$$x + iy + \frac{1}{x + iy} = x + \frac{x}{x^2 + y^2} + i(y - \frac{y}{x^2 + y^2}),$$

so that reality of $z+\frac{1}{z}$ means that $\frac{y}{x^2+y^2}=y$, i. e., $x^2+y^2=1$ as $y\neq 0$ since z is not real. Thus if all roots of g are real then the roots of f either lie on the unit circle or they are real. And if z is a real root of f not on the unit circle then $z+\frac{1}{z}$ is a root of g with the absolute value exceeding 2.

Notice now that our generating functions are unipalindromic and all of their coefficients are positive. Thus the corresponding polynomials g have the desired properties and it remains to show that if we take the sum of even or odd polynomials of the same degree with positive leading coefficients and all of their roots real in the segment $[-2,2]$, then all roots of the sum are also real and lie in the segment $[-2,2]$.

For proving the latter claim it is sufficient to consider the sum of two such polynomials, say, g and h. Let x be the biggest root of g and y be the biggest root of h, and, for example, $x<y$. Since the leading coefficients are positive we have $h(z)>0$ for $z>y$ and $g(z)>0$ for $z>x$, so that $g(z)>0$ for $z>y$. Thus $g(z)+h(z)>0$ for $z>y$. Analogously, $g(z)+h(z)$ is non-zero for $z=t$ where t is the smallest root of h (in fact, $t=-y$). This obviously completes the proof of proposition. ∎

The above properties are apparently very specific. They need not hold in general even for singularities from D_{**} series. In order to show this we need a closer look at the algebraic structure of $L(D_{**})$. In fact, taking into account identities (5 - 9) in algebra A, it is easy to explicitly write down the multiplication table of $L(D_{k_1,k_2})$ which appears similar to that of $L(P_k)$ (see Section 2). It follows that the derivation $h=(k_2-1)x_1+k_1x_2\partial_2$ has diagonal form in the above basis and the eigenvalues of $\operatorname{ad} h$ on the above basis vectors listed in the same order are

$$0, k_2-1, \dots, (i-1)(k_2-1), \dots, (k_1-2)(k_2-1), (k_1-1)(k_2-1)-k_1;$$
$$(a_1-1)(k_2-1)+a_2k_1, (k_1-1)(k_2-1), a_1(k_2-1)+(a_2-1)k_1.$$

Since all these eigenvalues are non-zero, it follows that the Lie algebra H generated by h is a Cartan subalgebra.

We are already able to give an example of isolated singularity with non-unimodal Poincaré polynomial. Consider the $D_{3,5}$ singularity defined by the polynomial $f=x_1^3x_2+x_2^5$. Here $\dim A(f)=11$ and $\dim L(A)=14$. The weights of the homogeneous components of $L(A)$ in the Euler grading are: 0, 3, 4, 5, 6, 7, 8, 9, 10, 12 while the dimensions of those components are 1, 2, 1, 1, 2, 1, 1, 3, 1, 1. Thus we see that the generating function of $D_{3,5}$ is neither unimodal nor palindromic.

We present now the structural constants of derivation algebras for D_k and E_7 singularities which will be used in the sequel. Notice first that the series of D_k singularities (resp. E_7 singularity) is obtained from D_{k_1,k_2} for $k_1=2, k_2=k-1$ (resp. $k_1=k_2=3$). Specifying the above basis in case of D_k, we obtain that a basis in $L(D_k)$ is formed by vector fields:

$$h=(k-2)x_1\partial_1+2x_2\partial_2, x_2\partial_2, x_2^2\partial_2, \dots, x_2^{k-2}\partial_2, (k-1)x_2^{k-3}\partial_1+x_1\partial_2, x_2^{k-2}\partial_1.$$

Denote now $d_0^*=h, d_{2i}^*=x_2^{i+1}\partial_2, 1\leqslant i\leqslant k-3, d_{k-4,1}^*=(k-1)x_2^{k-3}\partial_1+x_1\partial_2, d_{k-2,1}^*=x_2^{k-2}\partial_1$. Then direct calculation shows that the only nonzero commutators are as follows:

$$[d_0^*, d_{2i}^*]=2id_{2i}^*,\ 1\leqslant i\leqslant k-3,\ \ [d_0^*, d_{k-4,1}^*]=(k-4)d_{k-4,1}^*,$$

$$[d_0^*, d_{k-2,1}^*] = (k-2)d_{k-2,1}^*,\ \ [d_{2i}^*, d_{2j}^*] = (j-i)d_{2(i+j)}^*,\ 2 \leqslant i+j \leqslant k-3,$$
$$[d_2^*, d_{k-4,1}^*] = (k-3)(k-1)d_{k-2,1}^*,\ \ [d_{k-4,1}^*, d_{k-2,1}^*] = -d_{2k-6}^*.$$

It follows that, for $k > 4$, Cartan subalgebras are one-dimensional and one of them is generated by $h = d_0$. The eigenvalues of $\operatorname{ad} h$ are $2, 4, \ldots, 2k-6, k-4, k-2$ so they are pairwise different for odd k.

If $k = 2l$ is even, we take as the basis of Cartan subalgebra the element $d_0 = (l-1)x_1\partial_1 + x_2\partial_2$ and denote $d_i = d_{2i}^*, d_{l-2,1} = d_{k-4,1}^*, d_{l-1,1} = d_{k-2,1}^*$. Then we have:

$$[d_0, d_i] = id_i,\ 1 \leqslant i \leqslant 2l-3,\ \ [d_0, d_{l-2,1}] = (l-2)d_{l-2,1},$$
$$[d_0, d_{l-1,1}] = (l-1)d_{l-1,1},\ \ [d_i, d_j] = (j-i)d_{i+j},\ 2 \leqslant i+j \leqslant 2l-3,$$
$$[d_1, d_{l-2,1}] = (2l-1)(2l-3)d_{l-1,1},\ \ [d_{l-2,1}, d_{l-1,1}] = -d_{2l-3}.$$

We see that the eigenspaces of $\operatorname{ad} d_0 = \operatorname{ad} h$ are one-dimensional except for eigenvalues $l-2, l-1$ in which cases they are two-dimensional. The eigenspace with eigenvalue $k-2$ (resp. $k-1$) is spanned by vectors $d_{k-2}, d_{k-2,1}$ (resp. $d_{k-1}, d_{k-1,1}$).

One can now investigate the same properties for all simple singularities. The Poincaré polynomials can be written down explicitly and the symmetry of their coefficients becomes apparent.

Theorem 4.3. *The Poincaré polynomial of $L(X)$ is palindromic and unimodular for any simple hypersurface singularity X.*

Proof. Since in preceding sections we have presented the explicit homogeneous bases in derivation algebras of all simple singularities, it is easy to write down the Poincaré polynomials. Namely, the Poincaré polynomials look as follows:

$$A_k : \sum_{i=0}^{k-2} t^i,$$

$$D_k : \sum_{i=0}^{k-3} t^{2i} + t^{k-4} + t^{k-2},$$
$$E_6 : 2 + 3t + 2t^2,$$
$$E_7 : 1 + t + t^2 + 2t^3 + t^4 + t^5 + t^6,$$
$$E_8 : 2 + 3t + 3t^2 + 2t^3 = (t+1)(2t^2 + t + 2).$$

Unimodularity follows by some simple algebraic transformations of these polynomials. For E_6 and E_8, this is obvious. In case A_k we just have:

$$\sum_{i=0}^{k-2} t^i = \frac{1 - t^{k-1}}{1-t}\ .$$

For D_k singularity, one can use the formula for the sum of geometric progression to get:

$$\sum_{i=0}^{k-3} t^{2i} + t^{k-4} + t^{k-2} = \frac{1 - t^{2k-4}}{1-t^2} + \frac{t^{k-4}(1-t^4)}{1-t^2} =$$

$$\frac{1+t^{k-4}-t^{2k-4}-t^k}{1-t^2}=\frac{(1-t^k)(1+t^{k-4})}{1-t^2}\ ,$$

which obviously yields unimodularity. Finally, for E_7, we have:

$$1+t+t^2+2t^3+t^4+t^5+t^6=(t^3+1)(1+t+t^2+t^3),$$

and the theorem is proved. ∎

For further use let us notice that, for an even $k=2l$, the gradings for $A(D_k)$ and $L(D_k)$ can be normalized by dividing by 2. After this manipulation the Poincaré polynomials for $A(D_k)$ and $L(D_k)$ look as follows:

$$\frac{(1-t^l)(1+t^{l-1})}{1-t}\,,\ \frac{(1-t^l)(1+t^{l-2})}{1-t}\,,$$

and then all the resulting Poincaré polynomials occurring in Theorem 4.3 except for D_k with k odd become unipalindromic.

In relation with the above considerations we introduce now a natural class of singularities with some special properties of generating function.

Definition 4.4. An isolated hypersurface singularity f is called semisimple if in some coordinate system the function f is representable as a sum of functions f_j depending on disjoint subsets of variables and such that each f_j defines a simple hypersurface singularity.

It is convenient to say that the type of such a singularity is the sum of the simple types participating in its decomposition. If some of the participating singularities are just Morse points then the above concept reduces to the concept of stabilization [5].

For example, Pham singularity P_k is a semisimple singularity of the type $A_{k_1}+\ldots+A_{k_n}$. Obviously, the moduli algebra of a semisimple singularity is the tensor product of the moduli algebras of its simple summands. Thus one can use Block's theorem (Proposition 1.5 above) for computation of the derivation algebra of semisimple singularity. This enables one to describe the derivation algebra as a graded vector space and compute its generating function and examine its analytic properties. In this way one arrives at the following generalization of Theorem 4.3.

The above proof of Proposition 4.2 is applicable for many semisimple singularities. For example, this holds for some classes of semisimple singularities.

We say that a semisimple singularity is of the first class if all of its simple summands are of A_k type or D_{2l} type. A semisimple singularity is of the second class if all of its simple summands are of D_{2l+1} or E_7 type.

Theorem 4.5. *For a semisimple singularity of the first or second class, the Poincaré polynomial is unipalindromic and unimodular.*

Proof. The proof is based on repeated use of the Block theorem. Notice that, for every semisimple singularity X of the first class, the degree of Poincaré polynomial of $L(X)$ is by one less than that of the Poincaré polynomial of $A(X)$. From the Block theorem and our Theorem 4.1 it follows that the Poincaré polynomial of a semisimple singularity of the first class is a sum of unipalindromic and unimodular polynomials of the same algebraic degree. Then it is quite easy to see that the sum is also unipalindromic. To show that the sum is unimodular, one uses the same transformation as the proof of Proposition 4.2.

For a singularity X of the second class, the degree of Poincaré polynomial of $L(X)$ is by two less than that of the Poincaré polynomial of $A(X)$. This means that the same mode of reasoning is applicable in this case as well. ∎

In general, for a semisimple singularity X which has summands of different classes, the Poincaré polynomial of $L(X)$ need not be neither unipalindromic, nor unimodular. For example, consider semisimple singularity X of the type D_5+D_6. Then by Block's theorem we get that

$$P(L(X)) = (2+t^2+t^3)\frac{(1-t^5)(1-t^3)}{(1-t)^2} = (1+t+t^2)(1+t+t^2+t^3+t^4)(2+t^2+t^3).$$

By elementary considerations, one concludes that the polynomial $2+t^2+t^3$ has a real root in the segment $(-2,-1)$, hence it is not unimodular.

5. Completeness of Lie algebras of simple singularities

We now turn to the structural properties of Lie algebras of isolated hypersurface singularities. One of the most natural concepts to study in this relation is the completeness in the sense of [18]. Recall that a Lie algebra is called complete if its center is trivial and all of its derivations are inner [18]. It turns out that, typically, the Lie algebras of simple singularities have only inner derivations. A precise formulation is given below and this is the main result of the section.

Theorem 5.1. *For a simple hypersurface singularity X with $\mu(X) \geqslant 8$, the Lie algebra $L(X)$ is complete.*

To prove it we need the following lemma which we could not find in the literature. Recall that the *centralizer* of a subalgebra H in Lie algebra L is defined as

$$C_L(H) = \{c \in L : [c,h] = 0 \text{ for all } h \in H\}.$$

Lemma 5.2. *Let L be an algebraic solvable finite-dimensional Lie algebra over an algebraically closed field of characteristic zero with the trivial center. Let H be its Cartan subalgebra and $D = \operatorname{Der} L$ ($\operatorname{Inn} L$) be the Lie algebra of all (resp. inner) derivations of L, $C_D(H)$ the centralizer of H in D. Then*

$$\operatorname{Der}(L)/\operatorname{Inn}(L) \cong C_D(H)/H.$$

Before giving its proof let us notice that since the Lie algebra in question has trivial center, its adjoint representation is exact and so $\operatorname{ad} L \cong \operatorname{Inn} L$. Thus the above lemma yields that

$$\operatorname{Der}(L)/L \cong C_D(H)/H.$$

Proof of lemma 5.2. First of all, the Cartan subalgebra H being the Lie algebra of a torus acts in completely reducible way in the adjoint representation of D.

Hence the whole algebra D can be represented as the direct sum $D = D^H \oplus D_{\neq 0}$, where D^H is the subspace of invariants of H (in other words the centralizer of $D^H = C_D(H)$), while $D_{\neq 0}$ is the sum of eigenspaces of H corresponding to nonzero weights.

Notice now that the subspace $D_{\neq 0}$ is contained in Inn$L$. Indeed, for each $v \in D_{\neq 0}$ there exists a nonzero form $\alpha \in H^*$ such that $hv = \alpha(h)v$ for any $h \in H$. Since in our case $hv = [\operatorname{ad} h, v]$, $\operatorname{ad} h$ is an inner derivation and all inner derivations form an ideal in D, we conclude that $v \in \operatorname{Inn} L$. This completes the proof of lemma. ∎

Proof of theorem 5.1. In order to apply lemma 5.2 it is necessary to find out the structure of Cartan subalgebra. In case of the derivation algebra of a simple singularity, for the whole A_k-series and D_k-series with $k > 4$, Cartan subalgebra H is one-dimensional, say, spanned by $h \in L(X)$. Moreover, all eigenvalues of $\operatorname{ad} H$ have multiplicity one. For this reason, the centralizer appearing in the lemma consists of diagonal matrices of the order equal to the dimension of $L(X)$.

Suppose now that $\mu \geqslant 8$. Using the multiplication table for $L(A_k)$ given in the proof of Theorem 3.1 one can write down the relations satisfied by a diagonal matrix in order to be a derivation of $L(A_k)$. ¿From those relations it is readily seen that the set of diagonal matrices which define derivations consists only of the matrices having eigenvalues of h on the diagonal. This obviously implies completeness of $L(A_k)$ for $k \geqslant 8$.

The proof for a D_k-singularity with odd $k > 8$ is absolutely similar to the case of A_k. For D_k-singularity with an even k we need additional considerations since in this case there are eigenvalues of $\operatorname{ad} h$ of multiplicity two.

For $k = 2l$, the structural constants of $L(D_{2l})$ were found in the previous section. Let D be a derivation of $L(D_{2l})$ for $l \geqslant 4$. Using Einstein convention we represent D in basis $\{d_i, 0 \leqslant i \leqslant k-3, d_{l-2,1}, d_{l-1,1}\}$ as follows (as above the indices i, j lie between 0 and $k-3$):

$$Dd_i = D_i^j d_j + D_i^{l-2,1} d_{l-2,1} + D_i^{l-1,1} d_{l-1,1},$$

$$Dd_{l-2,1} = D_{l-2,1}^j d_j + D_{l-2,1}^{l-2,1} d_{l-2,1} + D_{l-2,1}^{l-1,1} d_{l-1,1},$$

$$Dd_{l-1,1} = D_{l-1,1}^j d_j + D_{l-1,1}^{l-2,1} d_{l-2,1} + D_{l-1,1}^{l-1,1} d_{l-1,1}.$$

We consider now the four commutators in Lie algebra $L(D_{2l})$, namely:

$$[d_1, d_{l-2}] = (l-3)d_{l-1},\ [d_1, d_{l-2,1}] = (2l-3)(2l-1)d_{l-1,1},$$

$$[d_1, d_{l-1,1}] = 0,\ [d_{l-2,1}, d_{l-1}] = 0.$$

Applying D to the third of these relations and projecting the result on the basis vector d_l we get $(l-2)D_{l-1,1}^{l-1} = 0$, hence $D_{l-1,1}^{l-1} = 0$. Applying D to the second commutator and projecting it on d_{l-1} we get the relation $(2l-3)(2l-1)D_{l-2}^{l-2,1} = (l-3)D_{l-1}^{l-1,1}$, hence $D_{l-2}^{l-2,1} = 0$.

Analogously, applying D to the fourth and second commutators and performing appropriate projections we derive that $D_{l-2}^{l-2,1} = 0$ and $D_{l-1,1}^{l-1} = 0$. Thus an arbitrary derivation D has diagonal form in this basis. Now, using the same argument as for A_k singularity one proves that D coincides with $\operatorname{ad} h$, which obviously implies the desired result. ∎

The following table contains information on factor $\operatorname{Der} L(A_k)/\mathrm{Inn}L(A_k)$ for all values of k.

k	$\dim A_k$	$\dim L(A_k)$	$\dim \operatorname{Der} L(A_k)/\mathrm{Inn}L(A_k)$
2	2	1	0
3	3	2	0
4	4	3	1
5	5	4	1
6	6	5	1
$k \geqslant 7$	k	$k-1$	0

For D_k-series, the table looks as follows:

k	$\dim D_k$	$\dim L(D_k)$	$\dim \operatorname{Der} L(D_k)/\mathrm{Inn}L(D_k)$
4	4	4	2
5	5	5	1
6	6	6	2
7	7	7	1
$k \geqslant 8$	k	k	0

Remark 5.3. It should be noted that [7] contains description of Cartan subalgebras of derivation algebras of $L(F)$ for all simple singularities. The named subalgebra does not always coincide with a Cartan subalgebra of $L(F)$. In case when they do not coincide it is clear that completeness fails. However their coincidence does not in general automatically imply completeness. The fact that it is so for simple singularities, looks thus a bit miraculous. It should be added that the issue of completeness of $L(F)$ is not addressed in [7] at all. Neither do they describe the factor $\operatorname{Der} L(F)/\mathrm{Inn}L(F)$ which appears in the above tables.

In order to investigate this issue for exceptional simple singularities we present first the multiplication table for E_6. Notice that E_6 is in fact a Pham singularity $P_{2,3}$ and hence its multiplication table can be obtained from the calculations performed in section 3. We write the result in a more explicit form and to this end we denote $h_1 = (1,0;1) = x_1\partial_1$, $h_2 = (0,1;2) = x_2\partial_2$, $d_1 = (1,1;2) = x_1x_2\partial_2$, $d_2 = (0,2;2) = x_2^2\partial_2$, $d_3 = (1,1;1) = x_1x_2\partial_1$, $d_4 = (1,2;2) = x_1x_2^2\partial_2$, $d_5 = (1,2;1) = x_1x_2^2\partial_1$.

The nonzero brackets are as follows:

$$[h_1, d_1] = d_1,\ [h_1, d_4] = d_4,\ [h_2, d_2] = d_2,\ [h_2, d_3] = d_3,$$

$$[h_2, d_4] = d_4,\ [h_2, d_5] = 2d_5,\ [d_1, d_2] = d_4,\ [d_1, d_3] = -d_4,\ [d_2, d_3] = d_5.$$

¿From the multiplication table it is clear that a Cartan subalgebra is spanned by h_1, h_2. The Cartan multi-gradings of the basis vectors written in the same order are $(0,0),(0,0),(1,0),(0,1),(0,1),(1,1),(0,2)$. The degrees in Euler grading are $0,1,2$ and the dimensions of the corresponding homogeneous components are $2,3,2$. A direct calculation (which is left to the reader) shows that $\operatorname{Der}(L(E_6))/\mathrm{Inn}L(E_6) \cong V_2$, where V_2 is the two-dimensional non-abelian Lie algebra. Hence the Lie algebra of E_6 is not complete. Moreover, one readily sees that rk $\operatorname{Der}(L(E_6)) = 3$ as it was indicated in [7].

Lie algebras $L(E_7)$ and $L(E_8)$ are also complete, which can be verified by direct computations in the bases constructed in Section 2 for Pham and D_{**} singularities. Notice that $L(E_7) = L(D_{3,3})$ and $L(E_8) = L(P_{2,4})$, so their dimensions are 8 and 10 respectively. Since these dimensions are not too big, the verification of completeness can be done by means of existing computer algorithm packages for Lie algebras. For example, one can use the Lie algebra package of GAP (available at `www.groups.dcs.st-and.ac.uk/gap`) as did the authors. These remarks complete the proof of Theorem 5.2.

Notice that the case of E_7 singularity formally is not covered by the formulation of Theorem 5.2 because of the condition $\mu > 8$.

6. Indices and maximal commutative polarizations

Let X be a simple hypersurface singularity in its canonical form defined by polynomial f and $L(f)$ be the Lie algebra of derivations of $A(f)$ as above. Put $m = \lambda(f)$. Recall that Lie algebra L is said to possess a maximal commutative polarization if it has a commutative subalgebra of dimension equal to $\frac{1}{2}(\dim L + \operatorname{ind} L)$ [14].

Theorem 6.1. *If m is even then* $\operatorname{ind} L(f) = 0$*, and if m is odd then* $\operatorname{ind} L(f) = 1$*. Moreover, the Lie algebra $L(f)$ has commutative polarizations.*

Theorem 6.1 can be proved by direct verification using the structural constants obtained in the proof of Theorem 3.1 and methods of [7]. In the sequel we will work with the Euler grading on $L(f)$ defined above. The main construction uses the homogeneous component of maximal degree and it runs especially simply in cases when all the homogeneous components are one-dimensional. For this reason for the proof it is sufficient to consider separately several cases. To this end it is useful to list the maximal degree $j(f)$ of homogeneous components in all cases. So we recall that for A_k one has $j(f) = k-2$, for D_k with odd k, one has $j(f) = 2k-6$, for D_k with even k, one has $j(f) = k-3$. Moreover, $j(E_6) = 2$, $j(E_7) = 6$, $j(E_8) = 3$. For our argument it is also important to notice that the homogeneous component of maximal degree is one-dimensional in all cases except E_6 and E_8. In the latter two cases it is two-dimensional.

Proof of theorem 6.1. We construct first a linear form ϕ_J on $L(f)$ as follows. We fix a homogeneous basis in $L(f)$ and put ϕ_J equal to one on the basis elements of degree $j(f)$ and equal to zero on all other basis elements. Then we define a skew-symmetric bilinear form on $L(f)$ by the formula:

$$B_J(a, b) = \phi_J([a, b]), \ a, b \in L(f).$$

This is actually a Lie algebra counterpart of the Gorenstein quadratic form appearing in the algebraic formula for the topological degree obtained in [12], [19]. The following simple lemma can be derived from the multiplication table for $L(f)$ presented above, as well as from the results of [7].

Lemma 6.2. *One has* $\dim\ker B_J = 1$ *if* $\dim L(f)$ *is odd and* $\dim\ker B_J = 0$ *if* $\dim L(f)$ *is even.*

Referring to a well-known definition of index in terms of B_J, we obtain the first part of the Theorem 6.1. The second part is proved using case-by-case considerations.

First, we consider polynomial $f(x) = x^{k+1}$ defining a germ of A_k singularity. Then from the commutation relations $[e_j, e_p] = (p-j)e_{j+p-1}$ it follows that a maximal commutative subalgebra is spanned by the derivations e_p with $2p > k-1$, which gives a maximal commutative polarization. For all other cases, except E_6, the proof is completely analogous. This follows from the explicit values of dimensions of the homogeneous components which can be derived from the generating function calculated above.

In case of E_6 the homogeneous components have degrees $0, 1, 2$ and their dimensions are $2, 3, 2$. Thus in order to show existence of commutative polarizations we need to construct a commutative subalgebra of dimension $4 = (7+1)/2$. Since there are no homogeneous components of degree greater than 2, it is obvious that the highest degree component is commutative and commutes with any element of degree 1. Elements of degree one need not be all pairwise commuting (and there are indeed non-commuting elements of degree one). However, in order to obtain a commutative polarization, it suffices to find two commuting elements of degree one. ¿From the multiplication table presented at the end of the previous section it is easy to see that one can take, for example, the elements $d_1, d_2 + d_3$. Thus the subalgebra generated by $d_1, d_2 + d_3, d_4, d_5$ yields a commutative polarization for $L(E_6)$, which completes the proof of Theorem 6.1. ∎

If one omits the requirement that X is a simple singularity, the statement is no longer true. We present now in some detail a typical example of such kind.

Consider the Pham singularity $P_{2,2,2}$ defined by the polynomial

$$f(x_1, x_2, x_3) = x_1^3 + x_2^3 + x_3^3.$$

Using the notation and formalism developed in section 3 for Pham singularities, one readily checks that, with respect to Euler grading, $L(f)$ has three homogeneous components of degrees $0, 1, 2$ and the dimensions of those components are $3, 6, 3$.

Let us compute the index of $L(f)$. To this end we first compute the determinant of matrix representing the multiplication table which appears equal to zero. This means that the rank of this matrix over the symmetric algebra of $L(f)^*$ is less or equal to ten. Consider now a linear form ϕ on $L(f)$ which is equal to one on all basis vectors of maximal degree and vanishes on all other basis vectors. By direct computation one finds that the centralizer of the linear ϕ in $L(f)$ is two-dimensional and spanned by the two vectors $e_1 = (1,1,0;1)+(0,1,1;2)+(1,0,1;3)$ and $e_2 = (1,0,1;1) + (1,1,0;2) + (0,1,1;3)$. Hence this form is regular and the two vectors e_1, e_2 belong to the so-called *Frobenius ideal* $F = F(L(f))$ of $L(f)$ introduced in [14].

Since F is an ideal it follows that all summands in the representation of e_1 and e_2 lie in F. Notice now that $[(1,1,0;1),(0,1,1;2)] = -(1,1,1;1)$ hence the Frobenius ideal F is not commutative. Recall now that, as was shown in [14], if a Lie algebra L has a commutative polarization then its Frobenius ideal $F(L)$ is necessarily commutative. As we have just shown $F(L(f))$ is not commutative and so we conclude that the algebra $L(f)$ has no commutative polarizations, as was claimed.

We would like to emphasize that, in fact, existence of commutative polarizations does not seem to be a typical phenomenon outside the class of simple singularities. However there also exist series of singularities for which there exist analogs of Theorem 6.1. For example, this is so for D_{k_1,k_2}-series but we omit the proof of this claim.

7. Concluding remarks

In conclusion we briefly mention some related results and open problems. First of all, there is a bulk of natural problems concerned with the Lie algebras of IHSs defined by quasihomogeneous polynomials. As was explained in Section 2, the qh-type of singularity does not determine the dimension of $L(X)$. In fact, the infinite series of such examples presented in Section 2 shows that the variation of values of $\dim L(X)$ within a given qh-type can become arbitrarily big. An interesting problem is to estimate the modulus of variation of $\lambda(X)$ in terms of qh-weights. Clearly, this is closely related to the problem of finding the exact upper and lower bounds for the Yau number within a given qh-type. A still more difficult general problem is to describe the whole spectrum of possible values of $\lambda(X)$ within a given qh-type (notice that, according to Proposition 2.4 such a spectrum is always finite). For simple qh-types these problems can be successfully attacked using case-by-case considerations, which may help to reveal some general phenomena and conjectures.

As was mentioned, in the quasihomogeneous case one has a natural grading on $L(X)$ defined by putting the weight of ∂_j equal to $-w_j$ [5]. Thus the weight of a vector field of the form $x_k\partial_j$ is equal to $w_k - w_j$, which obviously defines a grading on $L(X)$ compatible with the standard w-grading on $A(X)$. These and other natural gradings on Lie algebras of quasihomogeneous singularities suggest a number of further topics. For example, using the Euler grading described in Section 4 it is possible to indicate some sufficient conditions on the qh-type of a quasihomogeneous IHS X which guarantee that its Lie algebra $L(X)$ possesses a maximal commutative polarization.

In fact, under these conditions it can be shown that the index vanishes exactly when the dimension of $L(X)$ is even so one can indicate an extensive list of isolated singularities for which $L(X)$ is a Frobenius Lie algebra. For example, from the computations presented in [7] it follows that an analog of Theorem 6.1 holds for the so-called parabolic unimodal singularities X_8, P_9, J_{10} [5]. There is good evidence that the same holds for all unimodal isolated singularities [5] and a natural problem is to look for other classes of singularities possessing this property.

Furthermore, one may wish to give estimates for the number of basic derivations of each weight in $L(f)$ valid for any polynomial f of fixed qh-type. It would also be interesting to investigate the properties of Euler gradings within various series of IHS. For example, a natural problem is to find out, for which values of (k_1, k_2) the Poincaré polynomial of $L(D_{k_1,k_2})$ is palindromic and unimodular. This problem is meaningful since from our Theorem 3.1 follows that both these properties take place if $k_2 = 2$, while, as we have shown in Section 4, none of them holds for $D_{3,5}$ singularity.

There is good evidence that some developments in the above topics are possible if one restricts attention to polynomials with a fixed Newton diagram. Then one may hope to express various invariants of the Lie algebra of a typical function with a given Newton diagram P in terms of the geometry of P. Results of such type concerned with computing Milnor numbers are well known in singularity theory [5] and their analogs may exist for Yau numbers. Specifically, the expressions for the Yau numbers given in Propositions 4, 10 look very much like the mixed volume of some polytope associated with the Newton diagram. Thus an intriguing open problem is to express the Yau numbers in terms of mixed volumes.

Next, it is tempting to investigate which IHSs are completely determined by their derivation algebras. Let us say that an IHS X is Der-detectable if, for any other IHS Y, existence of an isomorphism $L(Y) \cong L(X)$ implies that Y is analytically isomorphic to X. A natural problem is to describe some classes of singularities within which one has Der-detectability. In these terms, our Theorem 3.1 may be reformulated by saying that Der-detectability holds in the class of simple hypersurface singularities.

To provide a natural broader context for this problem, notice that the simple singularities and the whole D_{**} series are defined by binomials in two variables, while Pham polynomials in n variables contain n monomials. In order to refer to those examples in a uniform way, let us say that a polynomial $P \in \mathbb{C}_n$ is a *fewnomial* if the number of monomials in P does not exceed n. It is known that a homogeneous fewnomial of degree bigger than two and with the number of monomials less than n, cannot have an isolated critical point at the origin (see, e.g., [1]). At the same time, Pham polynomials give evident examples of n-nomials with isolated critical point at the origin of $\mathbb{C}^n$.

Let us say that an IHS in $\mathbb{C}^n$ is *economial* if it can be defined by an n-nomial in n variables. Using the normal forms presented in [5] (Section 13, p.179), it is not difficult to show that Der-detectability holds for binomials in two variables. On the other hand, in [7] one can find examples of trinomials in two variables which cannot be classified by their Lie algebras. Hence there is no hope for Der-detectability if we drop the condition that singularities are defined by fewnomials. Thus a reasonable problem is to investigate which economial IHSs are Der-detectable. Our conjecture is that this does not hold in general even for economial IHSs and so it is interesting to find out what is the smallest dimension where one can construct counterexamples.

An important conceptual problem of general nature is to investigate the behavior of derivation Lie algebra $L(X)$ under deformations of germs. Recall that a germ X is said to be adjacent to Y if Y appears among arbitrarily small deformations of X (exact definitions and complete list of adjacencies of simple singularities are given in [5]). By analogy with the behavior of Dynkin diagrams of simple singularities with respect to adjacencies, it seems plausible that if X is adjacent to Y then the algebra $L(X)$ can be embedded in $L(Y)$ as a Lie subalgebra. This is true in all examples we were able to analyze but we do not possess a general proof.

Finally, most of the concepts and problems discussed above are meaningful for singularities of zero-dimensional complete intersections (cf. [2], [3]). An intriguing problem is to find out if the analogs of our main results hold for the simple singularities of zero-dimensional complete intersections classified by M. Giusti [15].

Summing up, the topics discussed in the present paper give rise to a variety of natural problems and the authors are determined to continue discussing them in forthcoming publications.

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