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Inner Ideals of Finitary Simple Lie Algebras

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Abstract. Inner ideals of infinite dimensional finitary simple Lie algebras over a field of characteristic zero are described in geometric terms. We also study when these inner ideals are principal or minimal, and characterize those elements which are von Neumann regular. As a consequence we prove that any finitary central simple Lie algebra over a field of characteristic zero satisfies the descending chain condition on principal inner ideals. We also characterize when these algebras are Artinian, proving in particular that a finitary simple Lie algebra over an algebraically closed field of characteristic zero is Artinian if and only if it is finite dimensional. Because it is useful for our approach, we provide a characterization of the trace of a finite rank operator on a vector space over a division algebra which is intrinsic in the sense that it avoids imbeddings into finite matrices
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Introduction

A submodule I of a Lie algebra L is an *inner ideal* of L if $[I,[I,L]] \subset I$. The initial motivation to study inner ideals in Lie algebras can be found in a paper [3] by G. Benkart. The inner ideals of a Lie algebra are closely related to the ad-nilpotent elements, and certain restrictions of the ad-nilpotent elements yield an elementary criterion for distinguishing the nonclassical from classical (finite dimensional) simple Lie algebras over algebraically closed fields of characteristic $p > 5$.

In [2], G. Benkart examines the Lie inner ideal structure of semiprime associative rings, R, and of the skew elements of prime rings with involution,

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$K = \mathrm{Skew}(R, *)$. In the special case that R is a simple Artinian ring of characteristic not 2 or 3 with center Z, proper inner ideals of $[R, R]/[R, R] \cap Z$ are of the form eRf for e, f idempotents of R such that $fe = 0$. If R has an involution and the dimension of R over Z is greater than 16, then there are two possible kinds of proper inner ideals of $[K, K]/[K, K] \cap Z$. Moreover, both simple Lie algebras $[R, R]/[R, R] \cap Z$ and $[K, K]/[K, K] \cap Z$ satisfy the ascending and descending chain conditions on inner ideals.

A similar study has been carried over in the Jordan setting by K. McCrimmon [9]. He describes the Jordan inner ideal structure of R and $\mathrm{Sym}(R, *)$, where R is again a simple Artinian ring and $*$ is an involution. The Jordan algebras $R^{(+)}$ and $\mathrm{Sym}(R, *)$ have the ascending and descending chain conditions on inner ideals.

By the Wedderburn-Artin Theorem, any simple Artinian ring is isomorphic to the ring of linear operators $\mathcal{L}(X)$ on a finite dimensional left vector space X over a division ring Δ. The infinite dimensional analogues of $\mathcal{L}(X)$ are the simple algebras (over the center of Δ) of finite rank linear operators that are continuous with respect to an infinite dimensional pair of dual vector spaces $\mathcal{P} = (X, Y, g)$ over Δ. These algebras $\mathcal{F}(X)$ are precisely the simple associative algebras satisfying the descending chain condition on principal left ideals (equivalently, coinciding with their socles), but not the ascending chain condition on principal left ideals. Moreover, if such an algebra has an involution $*$, then $\mathcal{P}$ comes from a self-dual vector space over a division ring with involution $(\Delta, -)$, and $*$ is now the corresponding adjoint involution of $\mathcal{F}(X)$ [8]. In [4], A. Fernández López and E. García Rus determined the inner ideals of $\mathcal{F}(X)^{(+)}$ and $\mathrm{Sym}(\mathcal{F}(X), *)$. These Jordan algebras are simple and satisfy the descending chain condition on principal inner ideals.

The aim of the present paper is to study the inner ideals of Lie algebras of the form $\mathcal{F}(X)^{(-)}$, defined by pair of dual vector spaces $\mathcal{P} = (X, Y, g)$ over a division ring Δ, and of $\mathrm{Skew}(\mathcal{F}(X), *)$, relative to a nonsingular Hermitian or skew-Hermitian inner product space (X, h) over a division ring with involution $(\Delta, -)$. In the special case that Δ is finite dimensional over its center, such Lie algebras are known as finitary Lie algebras. We use the classification given by A. A. Baranov in [1] to describe the proper inner ideals of any infinite dimensional finitary central simple Lie algebra over a field F of characteristic zero, and prove that such algebras satisfy the descending chain condition on principal inner ideals. We also characterize when these algebras are Artinian, and prove that if F is algebraically closed, then there are no infinite dimensional Artinian finitary simple Lie algebras over F.

Our main reason for studying inner ideals in finitary Lie algebras is that most of the examples of Lie algebras coinciding with their Jordan socles [5, 6] occur in the class of finitary Lie algebras, so the study of the inner ideal structure of such algebras could be useful to develop a general socle theory for Lie algebras.

The paper is divided into 3 sections. In the first one we provide an intrinsic proof of the fact that the trace defines a homomorphism of the general linear algebra of finite rank operators $\mathfrak{fgl}(\mathcal{P})$, given by a pair $\mathcal{P} = (X, Y, g)$ of dual vector spaces over a division algebra Δ, onto the Lie algebra $\Delta^{(-)}/[\Delta, \Delta]$, the kernel of this homomorphism being precisely the special linear algebra of finite rank operators $\mathfrak{fsl}(\mathcal{P})$.

Let F be a field of characteristic zero. We summarize the classification of the infinite dimensional central finitary simple Lie algebras over F given in [1] by distinguishing between two cases: (1) finitary special linear algebras $\mathfrak{fsl}(\mathcal{P})$, for $\mathcal{P}$ an infinite dimensional pair of dual vector spaces over a finite dimensional central division algebra over F, and (2) Lie algebras of skew-operators $[K,K]$, for $K = \mathrm{Skew}(\mathcal{F}(X),*)$ relative to a nonsingular Hermitian or skew-Hermitian inner product space (X,h) over a finite dimensional central division algebra with involution $(\Delta,-)$, $F = \mathrm{Sym}(Z(\Delta),-)$. Because it is useful for our purposes, we make the following distinction in case (2): either h is a symmetric bilinear form (Δ is the field F with the identity as involution) and then $[K,K] = K$ is the finitary orthogonal algebra $\mathfrak{fo}(X,h)$, or h is skew-Hermitian.

The second section of the paper is devoted to the study of inner ideals of finitary special linear algebras $\mathfrak{fsl}(\mathcal{P})$. Proper inner ideals of $\mathfrak{fsl}(\mathcal{P})$ are described in terms of pairs of orthogonal subspaces $V \le X$ and $W \le Y$ of $\mathcal{P} = (X,Y,g)$. Principal inner ideals are those for which $\dim_\Delta V = \dim_\Delta W < \infty$, while minimal inner ideals are those where the dimensions of V and W are equal to one. As a consequence we get that $\mathfrak{fsl}(\mathcal{P})$ satisfies the descending chain condition on principal inner ideals.

In the third section we study the inner ideal structure of finitary Lie algebras of skew-operators $[K,K]$, for $K = \mathrm{Skew}(\mathcal{F}(X),*)$ and (X,h) being a nonsingular (skew-Hermitian or orthogonal) inner product space over a finite dimensional central division algebra with involution $(\Delta,-)$. When h is skew-Hermitian, proper inner ideals are described in terms of a totally isotropic subspace V of X, principal inner ideals occur when V is finite dimensional, and minimal inner ideals correspond to the case when V is one-dimensional. When h is symmetric, proper inner ideals of $[K,K] = \mathfrak{fo}(X,h)$ only occur in the isotropic case and they are either described in terms of a totally isotropic subspace, or determined by a hyperbolic plane. In any case, Lie algebras of skew-operators $[K,K]$ satisfy the descending chain condition on principal inner ideals.

Von Neumann regular elements (cf. (2.3)) were introduced in [3] in connection with the definition of $*$-Lie algebra (although without giving them this denomination). As a consequence of our study of the inner ideal structure, we determine the von Neumann regular elements of each one of the types of infinite dimensional finitary central simple Lie algebras over a field of characteristic zero.

1. The trace of a finite rank operator

In this section we provide a characterization of the trace for finite rank linear operators on a left vector space over a division algebra Δ. This characterization of the trace is intrinsic, in the sense that it avoids imbeddings into finite matrices, elementary, since it can be easily computed in terms of any expression of operators as a sum of rank one transformations, and consistent with the usual notion of trace for matrices. However, unlike the commutative case (endomorphisms of free modules of finite rank), the trace of a finite rank linear operator is not an element of Δ, but a residue class modulo $[\Delta,\Delta]$.

Since it does not mean a major difficulty and at the same time is more suitable for our purposes, we shall deal with finite rank operators which are continuous with respect to a pair of dual vector spaces over a division algebra. Recall that a pair of dual vector spaces $\mathcal{P} = (X, Y, g)$ consists of a left vector space X, a right vector space Y (both over the same division Φ-algebra Δ) and a nondegenerate bilinear form $g : X \times Y \to \Delta$. Any left vector space X over Δ gives rise to the canonical pair (X, X^*, g), where X^* stands for the dual of X and where $g(x, \phi) = \phi(x)$ for $x \in X$, $\phi \in X^*$.

1.1. Let $\mathcal{P} = (X, Y, g)$ be a pair of dual vector spaces over Δ. A linear operator $a : X \to X$ is *continuous* (relative to $\mathcal{P}$) if there exists $a^\# : Y \to Y$, necessarily unique, such that $g(ax, y) = g(x, a^\# y)$ for all $x \in X$, $y \in Y$. Denote by $\mathcal{L}(X)$ the Φ-algebra of all continuous operators on X, and by $\mathcal{F}(X)$ the ideal of $\mathcal{L}(X)$ consisting of all finite rank continuous operators. Note that every $a \in \mathrm{End}_\Delta(X)$ is continuous with respect to the canonical pair (X, X^*) : $a^\# \phi = \phi a$ for all $\phi \in X^*$.

1.2. Let $\mathcal{P} = (X, Y, g)$ be as above. For $x \in X$, $y \in Y$, write $y^* x$ to denote the linear operator defined by $y^* x(x') = g(x', y)x$ for all $x' \in X$. Note that $y^* x$ is continuous with adjoint yx^* given by $yx^*(y') = yg(x, y')$ for all $y' \in Y$. Every $a \in \mathcal{F}(X)$ can be expressed as a sum $a = \sum y_i^* x_i$ of these rank one operators. The following facts can be easily verified:

(i) The mapping $(y, x) \mapsto y^* x$ is a balanced product of Y_Δ and ${}_\Delta X$ to the abelian group of $\mathcal{F}(X)$.

(ii) $a(y^* x) = y^*(ax)$ and $(y^* x)a = (a^\# y)^* x$ for $x \in X$, $y \in Y$ and $a \in \mathcal{L}(X)$.

(iii) $(y_2^* x_2)(y_1^* x_1) = y_1^* g(x_1, y_2) x_2$, for $x_1, x_2 \in X$, $y_1, y_2 \in Y$.

An elementary but very useful result in duality theory is the following:

1.3. Let $\mathcal{P} = (X, Y, g)$ be a pair of dual vector spaces. If $V \leq X$ and $W \leq Y$ are finite dimensional subspaces, then (V, W) can be imbedded in a finite dimensional subpair of dual vector spaces (X_0, Y_0) of $\mathcal{P}$.

Lemma 1.4. *Let $\mathcal{P} = (X, Y, g)$ be a pair of dual vector spaces over Δ. For any $a \in \mathcal{F}(X)$, the element $\sum_i g(u_i, w_i) \in \Delta$ is independent up to congruence modulo $[\Delta, \Delta]$ of the chosen representation $a = \sum_i w_i^* u_i$, where $\{w_i\} \subset Y$, $\{u_i\} \subset X$ and $1 \leq i \leq r$.*

Proof. Let $a = \sum z_j^* v_j$ $(1 \leq j \leq s)$ be another expression of the operator a as a sum of rank one operators. If $V \leq X$ denotes the linear span of the set $\{u_i\} \cup \{v_j\}$ and $W \leq Y$ is the linear span of $\{w_i\} \cup \{z_j\}$, we have by (1.3) that (V, W) can be imbedded in a finite dimensional subpair of dual vector spaces (X_0, Y_0) of $\mathcal{P}$. Let $\{x_k\} \subset X_0$, $\{y_k\} \subset Y_0$ $(1 \leq k \leq n)$ be dual bases. Then, for each $1 \leq i \leq r$,

$$u_i = \sum_k \alpha_{ik} x_k \text{ and } w_i = \sum_k y_k \beta_{ki} \tag{1}$$

where $\alpha_{ik}, \beta_{ki} \in \Delta$, $1 \leq k \leq n$. Similarly, for each $1 \leq j \leq s$, there exist $\lambda_{jk}, \mu_{kj} \in \Delta$ such that

$$v_j = \sum_k \lambda_{jk} x_k \text{ and } z_j = \sum_k y_k \mu_{kj}. \tag{2}$$

Fix $1 \leq l \leq n$ and compute ax_l using (1). We have

$$ax_l = (\sum_i w_i^* u_i) x_l = \sum_i g(x_l, w_i) u_i =$$
$$\sum_i g(x_l, \sum_k y_k \beta_{ki})(\sum_k \alpha_{ik} x_k) = \sum_k (\sum_i \beta_{li} \alpha_{ik}) x_k.$$

In particular, the lth-coordinate of ax_l with respect to the basis $\{x_k\}$ is given by $\sum_i \beta_{li} \alpha_{il}$. Taking the sum of all these, we get the element $\sum_l (\sum_i \beta_{li} \alpha_{il})$ of Δ, which only depends on a and the chosen dual bases $\{x_k\}, \{y_k\}$ of (V, W). Therefore,

$$\sum_{i,l} \beta_{li} \alpha_{il} = \sum_{j,l} \mu_{lj} \lambda_{jl} \tag{3}$$

for $1 \leq i \leq r$, $1 \leq l \leq n$ and $1 \leq j \leq s$.

Set $\tau_1(a) = \sum_i g(u_i, w_i)$ and use (1) to compute it. We have

$$\tau_1(a) = \sum_i g(u_i, w_i) = \sum_i g(\sum_k \alpha_{ik} x_k, \sum_l y_l \beta_{li}) = \sum_{i,l} \alpha_{il} \beta_{li}. \tag{4}$$

Similarly, by using (2) instead of (1), we get

$$\tau_2(a) = \sum_j g(v_j, z_j) = \sum_{j,l} \lambda_{jl} \mu_{lj}. \tag{5}$$

Then it follows from (3), (4) and (5) that

$$\tau_1(a) - \tau_2(a) = \sum_{i,l} \alpha_{il} \beta_{li} - \sum_{i,l} \beta_{li} \alpha_{il} + \sum_{j,l} \mu_{lj} \lambda_{jl} -$$
$$\sum_{j,l} \lambda_{jl} \mu_{lj} = \sum_{i,l} [\alpha_{il}, \beta_{li}] + \sum_{j,l} [\mu_{lj}, \lambda_{jl}] \in [\Delta, \Delta].$$

■

1.5. For $a = \sum_i y_i^* x_i \in \mathcal{F}(X)$, set

$$\operatorname{tr}(a) := \overline{\sum_i g(x_i, y_i)} := \sum_i g(x_i, y_i) + [\Delta, \Delta]. \tag{6}$$

Note that by Lemma 1.4, the mapping $a \mapsto \operatorname{tr}(a)$, which associates to any linear operator $a \in \mathcal{F}(X)$ its *trace*, $\operatorname{tr}(a) \in \Delta/[\Delta, \Delta]$, is well defined. Moreover, if we take a *matrix representation* for a: $a = \sum_{i,j} y_i^* \alpha_{ij} x_j$ $(1 \leq i, j \leq n)$, with $\alpha_{ij} \in \Delta$ and where the $\{x_i\}$, $\{y_i\}$ are *biorthonormal*: $g(x_i, y_j) = \delta_{ij}$ (such a representation always exists in virtue of (1.3)), we have the following determination of the trace

$$\operatorname{tr}(a) = \sum_i \alpha_{ii} + [\Delta, \Delta], \tag{7}$$

where $\sum_i \alpha_{ii}$ coincides with the usual trace of the matrix $(\alpha_{ij}) \in M_n(\Delta)$.

1.6. Given an associative algebra A, we denote by $A^{(-)}$ the corresponding Lie algebra under the usual Lie commutator. Associated to a pair of dual vector spaces $\mathcal{P} = (X, Y, g)$ over Δ, we have the following Lie algebras:

(i) The *general linear algebra* $\mathfrak{gl}(\mathcal{P}) := \mathcal{L}(X)^{(-)}$.

(ii) The *general linear algebra of finite rank operators* $\mathfrak{fgl}(\mathcal{P}) := \mathcal{F}(X)^{(-)}$.

(iii) The *special linear algebra* $\mathfrak{fsl}(\mathcal{P}) := [\mathfrak{fgl}(\mathcal{P}), \mathfrak{fgl}(\mathcal{P})]$.

Theorem 1.7. *Let $\mathcal{P} = (X, Y, g)$ be a pair of dual vector spaces over Δ. The mapping $a \mapsto \mathrm{tr}(a)$ is a homomorphism of $\mathfrak{fgl}(\mathcal{P})$ onto $\Delta^{(-)}/[\Delta, \Delta]$, with kernel equal to $\mathfrak{fsl}(\mathcal{P})$.*

Proof. As noted above, the mapping $a \mapsto \mathrm{tr}(a)$ is well defined, and clearly it is Φ-linear and onto. So it will be enough to show that an operator $a \in \mathfrak{fgl}(\mathcal{P})$ is traceless if and only if it is a sum of commutators $[b, c]$, with $b, c \in \mathcal{F}(X)$.

Since the commutator is a bilinear mapping, to prove the part *if* we only need to consider commutators of rank one operators. Actually we will prove something more:

$$\text{If } b = y^*x \in \mathcal{F}(X) \text{ and } a \in \mathcal{L}(X), \text{ then } \mathrm{tr}([a, b]) = \overline{0}.$$

Indeed, by 1.2(ii), $[a, b] = a(y^*x) - (y^*x)a = y^*(ax) - (a^{\#}y)^*x$. Hence $\mathrm{tr}([a, b]) = \overline{0}$, since $g(ax, y) = g(x, a^{\#}y)$. Conversely, let $a \in \mathcal{F}(X)$ be such that $\mathrm{tr}(a) = \overline{0}$. By taking a matrix representation $a = \sum_{i,j} y_i^* \alpha_{ij} x_j$, we have, by (7), that $\alpha := \sum_i \alpha_{ii} \in [\Delta, \Delta]$, i.e., $\alpha = \sum_l [\lambda_l, \mu_l]$ is a sum of commutators. It is not difficult to see that whenever $i \neq j$, $y_i^* \alpha_{ij} x_j = [y_i^* \alpha_{ij} x_j, y_i^* x_i]$. So, for $a' := \sum_i y_i^* \alpha_{ii} x_i$, we have that

$$a \equiv a' \ \mathrm{mod}([\mathfrak{fgl}(\mathcal{P}), \mathfrak{fgl}(\mathcal{P})]).$$

Set $b = y_1^* \alpha x_1$. Then $a' - b = \sum_{i \geq 2} (y_i^* \alpha_{ii} x_i - y_1^* \alpha_{ii} x_1)$. But

$$b = y_1^* (\sum_l [\lambda_l, \mu_l]) x_1 = \sum_l [y_1^* \mu_l x_1, y_1^* \lambda_l x_1] \in [\mathfrak{fgl}(\mathcal{P}), \mathfrak{fgl}(\mathcal{P})],$$

and the same is true for each $y_i^* \alpha_{ii} x_i - y_1^* \alpha_{ii} x_1 = [y_1^* x_i, y_i^* \alpha_{ii} x_1]$. Thus, $a = (a - a') + a' \in [\mathfrak{fgl}(\mathcal{P}), \mathfrak{fgl}(\mathcal{P})]$, which completes the proof. ■

If $\Delta = \mathrm{Z}(\Delta) \oplus [\Delta, \Delta]$ (for instance, Δ is a field or a division quaternion algebra over its center), then we can give a *central determination* of the trace.

Corollary 1.8. *If $\Delta = \mathrm{Z}(\Delta) \oplus [\Delta, \Delta]$, then for any $a \in \mathcal{F}(X)$ the exists a unique element $\mathrm{tr}_c(a) \in \mathrm{Z}(\Delta)$ such that $\mathrm{tr}(a) = \mathrm{tr}_c(a) + [\Delta, \Delta]$. Moreover, $\mathrm{tr}_c(ab) = \mathrm{tr}_c(ba)$ for all $a, b \in \mathcal{F}(X)$.* ■

2. Inner ideals of $\mathfrak{fsl}(\mathcal{P})$

Recall that a Lie algebra over a field F is said to be *finitary* if it is isomorphic to a subalgebra of the Lie algebra $\mathfrak{fgl}(X)$ of all finite rank operators on a vector space X over F. As a refinement of [7, Th. 1.12] we get

Proposition 2.1. *Let Δ be a finite dimensional central division algebra over a field F, and let $\mathcal{P} = (X, Y, g)$ be a pair of dual vector spaces over Δ,* $\dim_\Delta X > 1$. *If F has characteristic zero or X is infinite dimensional over Δ, then $\mathfrak{fsl}(\mathcal{P})$ is a finitary central simple Lie algebra over F.* ■

2.2. A submodule I of a Lie algebra L is an *inner ideal* of L if $[I, [I, L]] \subset I$. An inner ideal I of a Lie algebra L is called *principal* if $I = \mathrm{ad}_a^2(L)$ for some $a \in I$, where ad_a denotes the adjoint mapping determined by a.

2.3. Following [3], a Lie algebra L is called a $*$-Lie algebra if there exists an element $e \in L$ such that $\mathrm{ad}_e^3 = 0$ and $e \in \mathrm{ad}_e^2(L)$. An element e satisfying these two conditions will be called *von Neumann regular.* As shown below, there are good reasons for using this terminology. Recall that an element a in an associative algebra A is called von Neumann regular if $a = aba$ for some $b \in A$. Note that, by [3, Lemma 1.8], if L is 3-torsion free, then any von Neumann regular element $e \in L$ generates the principal inner ideal $\mathrm{ad}_e^2(L)$, which is also abelian.

Proposition 2.4. *Let A be an associative algebra over a ring of scalars Φ such that $\frac{1}{2}, \frac{1}{3} \in \Phi$, and let L be a subalgebra of the Lie algebra $A^{(-)}$. For any $x, y \in L$, we have:*

(i) $\mathrm{ad}_x^2(y) = [x, [x, y]] = x^2y - 2xyx + yx^2$.
(ii) $\mathrm{ad}_x^3(y) = x^3y - 3x^2yx + 3xyx^2 - yx^3$.
Thus, if $x^2 = 0$, then
(iii) $\mathrm{ad}_x^3(L) = 0$ *and* $\mathrm{ad}_x^2(L) = xLx$.
(iv) *Set $L = [A, A]$ and let $a \in L$ be such that $a^2 = 0$. Then a is von Neumann regular in A if and only if it is von Neumann regular in L.*
(v) *Suppose that A has an involution $*$, set $K =$ $\mathrm{Skew}(A, *)$, and let $a \in [K, K]$ be such that $a^2 = 0$. Then a is von Neumann regular in A if and only if it is von Neumann regular in $[K, K]$.*

Proof. (i), (ii) and (iii) are straightforward.

(iv) By (iii), $a^2 = 0$ implies $\mathrm{ad}_a^3(A^{(-)}) = 0$. Suppose first that a is von Neumann regular in A. Then, again by (iii), $a = \mathrm{ad}_a^2(b)$ for some $b \in A$. Now using [3, Lemma 1.7(iii)], we can write

$$a \in \mathrm{ad}_a^2(A^{(-)}) = \mathrm{ad}^2_{\mathrm{ad}_a^2(b)}(A^{(-)}) = \mathrm{ad}_a^2\mathrm{ad}_b^2\mathrm{ad}_a^2(A^{(-)}) \subset \mathrm{ad}_a^2(L),$$

which proves that a is von Neumann regular in $[K, K]$. The converse is clear since $\mathrm{ad}_a^2(L) \subset aAa$, again by (iii).

(v) It follows as (iv) taking account that if $a \in K$ is von Neumann regular in A, then we can write $a = aca$ for some $c \in K$: set $a = aba$ for some $b \in A$ and take $c = \frac{1}{2}(b - b^*)$. ■

In this section we shall describe the inner ideals, the principal inner ideals, and the von Neumann regular elements of a finitary simple Lie algebra of type $\mathfrak{fsl}(\mathcal{P})$, for $\mathcal{P} = (X, Y, g)$ a pair of dual vector spaces over a finite dimensional central division algebra Δ over a field F of characteristic zero. A different (algebraic) description of the inner ideals of $\mathfrak{fsl}(\mathcal{P})$ for the case that X is finite dimensional was given by Benkart in [2, Theorem 5.1].

Given two subspaces $V \leq X$ and $W \leq Y$, we will denote by W^*V the span of all elements of the form w^*v, for $w \in W$ and $v \in V$ (see (1.2)).

Theorem 2.5. *Let $\mathcal{P} = (X, Y, g)$ be a pair of dual vector spaces over a division algebra Δ, $\dim_\Delta X > 1$.*

(i) *If $V \leq X$, $W \leq Y$ are mutually orthogonal subspaces, i.e., $g(V, W) = 0$, then $I = W^*V$ is an inner ideal of $\mathfrak{gl}(\mathcal{P})$ strictly contained in $\mathfrak{fsl}(\mathcal{P})$.*

(ii) *For an inner ideal I of $\mathfrak{fsl}(\mathcal{P})$ the following conditions are equivalent:*
(α) $I = e\mathcal{F}(X)f$, where e, f are idempotents of $\mathcal{F}(X)$ such that $fe = 0$;
(β) $I = W^*V$, where $V \leq X$, $W \leq Y$ are mutually orthogonal and finite dimensional.

(iii) *Let $I = W^*V$ be as in* (i). *Then $V = IX = \{ax : a \in I, x \in X\}$ and $W = I^\# Y = \{a^\# y : a \in I, y \in Y\}$. Hence $W_1^*V_1 \subset W_2^*V_2$ if and only if $V_1 \subset V_2$ and $W_1 \subset W_2$, and $I = W^*V$ is minimal if and only if $\dim_\Delta V = \dim_\Delta W = 1$.*

(iv) *Let $I = W^*V$ be as in* (i). *Then I is a principal inner ideal of $\mathfrak{fsl}(P)$ if and only if both V and W are finite dimensional and have the same dimension over Δ.*

Suppose now that Δ is a finite dimensional central division algebra over a field F of characteristic zero.

(v) *Every proper inner ideal I of $\mathfrak{fsl}(\mathcal{P})$ is of the form $I = W^*V$, where $W \leq Y$, $V \leq X$ are mutually orthogonal, therefore I is an abelian subalgebra.*

(vi) *$\mathfrak{fsl}(\mathcal{P})$ satisfies the descending chain condition on principal inner ideals.*

(vii) *$a \in \mathfrak{fsl}(\mathcal{P})$ is von Neumann regular if and only if $a^2 = 0$.*

Proof. (i) Let $V \leq X$ and $W \leq Y$ be mutually orthogonal vector subspaces. For any $w \in W$, $v \in V$ the operator w^*v is traceless, and hence W^*V is contained in $\mathfrak{fsl}(\mathcal{P})$. Moreover, $(W^*V)(W^*V) = 0$ and we have for $v_1, v_2 \in V$, $w_1, w_2 \in W$ and $a \in \mathfrak{gl}(\mathcal{P})$:

$$\begin{aligned}[][w_1^*v_1, [a, w_2^*v_2]] &= (w_1^*v_1)a(w_2^*v_2) + (w_2^*v_2)a(w_1^*v_1) = (w_1^*v_1)(w_2^*av_2) \\ &+ (w_2^*v_2)(w_1^*av_1) = w_2^*g(av_2, w_1)v_1 + w_1^*g(av_1, w_2)v_2 \in W^*V,\end{aligned} \tag{8}$$

where we have applied the formulas (ii)-(iii) of (1.2). This proves that W^*V is an inner ideal of $\mathfrak{gl}(\mathcal{P})$, and since it is contained in $\mathfrak{fsl}(\mathcal{P})$, also an inner ideal of $\mathfrak{fsl}(\mathcal{P})$. Note also that $W^*V \neq \mathfrak{fsl}(\mathcal{P})$, since $\dim_\Delta X \geq 2$.

(ii) By (1.2)(ii), for idempotents e, f of $\mathcal{F}(X)$ such that $fe = 0$, we have

$$e\mathcal{F}(X)f = e(Y^*X)f = (f^\# Y)^*(eX),$$

where the subspaces $V := eX$, $W := f^\# Y$ satisfy $g(V, W) = g(feX, Y) = 0$. Conversely, let $V \leq X$, $W \leq Y$ be finite dimensional and mutually orthogonal subspaces, with bases $\mathcal{B}_1 = \{v_1, \ldots, v_n\}$ and $\mathcal{B}_2 = \{w_1, \ldots, w_m\}$ respectively. Take $\{y_1, \ldots, y_n\} \subset Y$ and $\{x_1, \ldots, x_m\} \subset X$ dual bases to $\mathcal{B}_1$ and $\mathcal{B}_2$ respectively. Then $e := \sum y_i^* v_i$ and $f := \sum w_j^* x_j$ are idempotents of $\mathcal{F}(X)$ such that $e\mathcal{F}(X)f = W^*V$.

(iii) It follows directly from the definition that

$$V = IX = \{ax : a \in I, x \in X\} \quad \text{and} \quad W = I^{\#}Y = \{a^{\#}y : a \in I, y \in Y\}.$$

This clearly implies that $W_1^* V_1 \subset W_2^* V_2$ if and only if $V_1 \subset V_2$ and $W_1 \subset W_2$, and therefore, $I = W^*V$ is minimal if and only if $\dim_\Delta V = \dim_\Delta W = 1$.

(iv) Let $I = W^*V$ be as in (i), suppose that I is principal, i.e., $I = \mathrm{ad}_a^2(\mathfrak{fsl}(\mathcal{P}))$ for some $a \in I$. Since $g(V,W) = 0$, $a^2 = 0$ and hence, by (2.4(iii)), we have $I = \mathrm{ad}_a^2(\mathfrak{fsl}(\mathcal{P})) = a\mathfrak{fsl}(\mathcal{P})a$, which implies that $V = IX = a\mathfrak{fsl}(\mathcal{P})aX = aX$ and $W = I^{\#}Y = a^{\#}\mathfrak{fsl}(\mathcal{P})^{\#}a^{\#}Y = a^{\#}Y$, since $a \in a\mathfrak{fsl}(\mathcal{P})a = I$; hence both V and W have the same finite dimension. Conversely, any proper inner ideal $I = W^*V$, with W and V having the same finite dimension is principal: take $a = \sum_i y_i^* x_i$, with $\{y_i\}$, $\{x_i\}$ bases of W and V respectively.

(v) Assume now that F has characteristic zero and Δ is a finite dimensional central algebra over F. Then $\mathfrak{fsl}(\mathcal{P})$ is a simple Lie algebra by (2.1). We distinguish between two cases: if $\mathcal{P}$ is finite dimensional, then we have by [2, Theorem 5.1] that any proper inner ideal I of $\mathfrak{fsl}(\mathcal{P})$ is of the form $I = e\mathcal{F}(X)f$, for e, f idempotents in $\mathcal{F}(X) = \mathcal{L}(X)$ such that $fe = 0$. Then, by (ii), $I = W^*V$, for $W \le Y$, $V \le X$ such that $g(V,W) = 0$. Assume then that $\mathcal{P}$ is infinite dimensional. Then $\mathcal{P}$ is a direct limit of finite dimensional subpairs of dual vector spaces $\mathcal{P}_\lambda = (X_\lambda, Y_\lambda)$ of $\mathcal{P}$. Hence $L = \mathfrak{fsl}(\mathcal{P})$ is the direct limit of the subalgebras $L_\lambda = [Y_\lambda^* X_\lambda, Y_\lambda^* X_\lambda]$, each of which is isomorphic to $\mathfrak{sl}_{n_\lambda}(\Delta)$, with n_λ being the dimension of X_λ. Moreover, we can always assume that no L_λ is contained in I. Then $I_\lambda := I \cap L_\lambda$ is a proper inner ideal of L_λ for every index λ. By the finite dimensional case we have just seen, $I_\lambda = W_\lambda^* V_\lambda$, with $W_\lambda \le Y_\lambda$, $V_\lambda \le X_\lambda$ mutually orthogonal. Since both the W_λ and the V_λ form directed sets, $W = \cup W_\lambda$ and $V = \cup V_\lambda$ are vector subspaces of Y and X respectively. It is clear that $g(V,W) = 0$ and $I = W^*V$. Note also that $[I,I] = 0$, and therefore any proper inner ideal of $\mathfrak{fsl}(\mathcal{P})$ is an abelian subalgebra.

(vi) It is a direct consequence of (iv) and (v).

(vii) Let $a \in \mathfrak{fsl}(\mathcal{P})$ be von Neumann regular. Then $a \in \mathrm{ad}_a^2(\mathfrak{fsl}(\mathcal{P}))$, with $\mathrm{ad}_a^2(\mathfrak{fsl}(\mathcal{P}))$ being a *proper* inner ideal of $\mathfrak{fsl}(\mathcal{P})$, (2.3). Hence, by (v), $\mathrm{ad}_a^2(\mathfrak{fsl}(\mathcal{P})) = W^*V$, with $g(V,W) = 0$, and therefore $a^2 \in (W^*V)(W^*V) = 0$. The converse follows from (2.4)(iv)) since the associative algebra $\mathcal{F}(X)$ is von Neumann regular. ■

3. Inner ideals of Lie algebras of skew-symmetric operators

Let $(A, *)$ be a simple associative algebra with involution containing a minimal left ideal. We study in this section inner ideals of Lie algebras of the form $[\mathrm{Skew}(A,*), \mathrm{Skew}(A,*)]$. We will be particularly interested in the case that the division algebra Δ associated to A is finite dimensional over its center, equivalently, A satisfies a generalized polynomial identity. In this case, both Lie algebras $\mathrm{Skew}(A,*)$ and $[\mathrm{Skew}(A,*), \mathrm{Skew}(A,*)]$ are finitary over the field $Z(\Delta)$.

3.1. Let (X,h) be a nonsingular Hermitian or skew-Hermitian inner product ($h(y,x)=\epsilon\overline{h(x,y)}$, $\epsilon=\pm 1$) vector space over a division algebra with involution $(\Delta,-)$. Denote by F the field of those elements of $Z(\Delta)$ fixed by $-$. Then $\mathcal{F}(X)$ is a simple associative F-algebra with minimal left ideals and involution $*$ given by the adjoint involution: $h(ax,y)=h(x,a^*y)$, for all $x,y\in X$. In fact (see [8]), any simple associative algebra with involution and containing minimal left ideals is $*$-isomorphic to one of those $(\mathcal{F}(X),*)$ described above. We can also consider the Lie F-algebras $\mathrm{Skew}(\mathcal{F}(X),*)\subset\mathrm{Skew}(\mathcal{L}(X),*)$.

3.2. Inner ideals of $[K,K]/Z\cap[K,K]$, where $K=\mathrm{Skew}(\mathcal{F}(X),*)$ and Z denotes the center of $\mathcal{F}(X)$, were described by Benkart in the finite dimensional case [2, Theorem 5.5]. In what follows we extend her results to the infinite dimensional case.

3.3. Let $(\Delta,-)$ be a division algebra with involution over a field of characteristic not 2, and let (X,h) be a nonsingular Hermitian or skew-Hermitian inner product space over $(\Delta,-)$, $h(y,x)=\epsilon\overline{h(x,y)}$, $\epsilon=\pm 1$. For any $x,y\in X$, $\alpha\in\Delta$, we have

(i) $(\alpha x)^*y=x^*(\overline{\alpha}y)$ and $(x^*y)^*=\epsilon y^*x$. Hence,

(ii) the operator defined by $[x,y]:=x^*y-\epsilon y^*x$, with ϵ as above, belongs to $\mathrm{Skew}(\mathcal{F}(X),*)$ and it will be called a *skew-trace.*

If V,W are subspaces of X, we shall write $[V,W]$ to denote the set of all finite sums of skew-traces $[v_i,w_i]$, $v_i\in V$, $w_i\in W$. With this notation, we have

(iii) $\mathrm{Skew}(\mathcal{F}(X),*)=[X,X]$.

(iv) If h is skew-Hermitian, then $x^*x=[(1/2)x,x]$ is a skew-trace for any $x\in X$. In fact, for any skew-trace $[x,y]$ we have, $[x,y]=(x+y)^*(x+y)-x^*x-y^*y$.

(v) If h is Hermitian, then $(\alpha x)^*x$ is a skew-trace if and only if $\overline{\alpha}=-\alpha$.

3.4. If there exists $0\neq\xi\in\mathrm{Skew}(\Delta,-)$, the involution $-$ in Δ can be replaced by $\tilde{}$, defined as $\widetilde{\alpha}=\xi^{-1}\overline{\alpha}\xi$ for all $\alpha\in\Delta$, and the Hermitian form (respectively the skew-Hermitian form) h over $(\Delta,-)$ can be replaced by h^ξ, where $h^\xi(x,y):=h(x,y)\xi$ is a skew-Hermitian form (respectively Hermitian form) over $(\Delta,\tilde{})$, without changing the adjoint involution. So, when working with Lie algebras of the form $\mathrm{Skew}(\mathcal{F}(X),*)$, we can consider two types of inner products: symmetric (in this case, Δ is necessarily a field with the identity as involution) or skew-Hermitian.

Assume that Δ is a field F of characteristic not 2 with the identity as involution, and that $\dim_F X>2$.

(i) If $\epsilon=1$, i.e., h is a symmetric bilinear form, then $\mathrm{Skew}(\mathcal{L}(X),*)$ is the *orthogonal algebra* $\mathfrak{o}(X,h)$, and $\mathrm{Skew}(\mathcal{F}(X),*)=[\mathrm{Skew}(\mathcal{F}(X),*),\mathrm{Skew}(\mathcal{F}(X),*)]$ is the *finitary orthogonal algebra* $\mathfrak{fo}(X,h)$ [1].

(ii) If $\epsilon=-1$, i.e., h is alternate, then $\mathrm{Skew}(\mathcal{L}(X),*)$ is the *symplectic algebra* $\mathfrak{sp}(X,h)$, and $\mathrm{Skew}(\mathcal{F}(X),*)=[\mathrm{Skew}(\mathcal{F}(X),*),\mathrm{Skew}(\mathcal{F}(X),*)]$ is the *finitary symplectic algebra* $\mathfrak{fsp}(X,h)$ [1].

If $\dim_F X > 4$ (possibly infinite), both $\mathfrak{fo}(X,h)$ and $\mathfrak{fsp}(X,h)$ are simple by [7, Th. 2.15], since in both cases $\mathrm{Skew}(\mathcal{F}(X),*) \cap Z = 0$, for Z the center of $\mathcal{F}(X)$.

An idempotent e of a ring with involution $(R,*)$ will be called *isotropic* if $e^*e = 0$. Isotropic idempotents are related to totally isotropic subspaces.

Lemma 3.5. *Let (X,h) be a nonsingular symmetric or skew-Hermitian inner product space over $(\Delta,-)$.*

(i) *An idempotent $e \in \mathcal{L}(X)$ is isotropic if and only if the subspace $eX \le X$ is totally isotropic.*

(ii) *$V \le X$ is totally isotropic and finite dimensional if and only if $V = eX$ for some isotropic idempotent $e \in \mathcal{F}(X)$.*

Proof. (i) By nonsingularity of h, $h(eX,eX) = h(X,e^*eX) = 0 \Leftrightarrow e^*e = 0$.

(ii) Let V be a totally isotropic subspace of X of finite dimension, say n. Let $\{v_i\}$ be a basis of V and take $\{x_i\} \subset X$ dual to $\{v_i\}$. Then $e := \sum_i x_i^* v_i$ is an idempotent in $\mathcal{F}(X)$ and it is isotropic by (i) because eX is totally isotropic as a subspace of V. Moreover, for every $v \in V$, $v = \sum \alpha_j v_j$, we have

$$ev = (\sum_i x_i^* v_i)(\sum \alpha_j v_j) = \sum h(\alpha_j v_j, x_i) v_i = \sum \alpha_j v_j = v.$$

which implies that $V = eX$. ∎

Proposition 3.6. *Let (X,h) be a nonsingular symmetric or skew-Hermitian inner product space over $(\Delta,-)$, and set $K = \mathrm{Skew}(\mathcal{F}(X),*)$, where $*$ denotes the adjoint involution.*

(i) *If $V \le X$ is a totally isotropic subspace, then $I = [V,V]$ is an inner ideal of $\mathrm{Skew}(\mathcal{L}(X),*)$ strictly contained in $[K,K]$, and therefore also a proper inner ideal of $[K,K]$. Moreover, $IX = V$.*

(ii) *Let V be a totally isotropic subspace of X. Then V is finite dimensional if and only if $[V,V] = eKe^*$ for an isotropic idempotent $e \in \mathcal{F}(X)$ (satisfying $V = eX$).*

(iii) *Given a totally isotropic subspace V of X, $[V,V]$ is a principal inner ideal of $[K,K]$ if and only if*

(a) when h is skew-Hermitian, V is finite dimensional over Δ;

(b) when h is symmetric (so Δ is a field with the identity as involution), the dimension of V over Δ is even.

Suppose further that Δ is finite dimensional over its center, has characteristic zero, and $\dim_{Z(\Delta)} X$ is greater than 4.

(iv) *If I is a proper inner ideal of $[K,K]$ such that $a^2 = 0$ for all $a \in I$, then $I = [V,V]$ for some totally isotropic subspace V of X. Furthermore, I is minimal if and only if*

(a) when h is skew-Hermitian, $I = x^*(\Delta x) (= [\Delta x, \Delta x])$ for an isotropic vector $0 \ne x \in X$;

(b) when h is symmetric, $I = \Delta[x,y]$, for two orthogonal isotropic linearly independent vectors $x, y \in X$.

Proof. (i) Let $V \leq X$ be totally isotropic. We observe that $ab = 0$ for all $a, b \in [V, V]$, and hence $[a, [c, b]] = acb + bca$ for any $c \in \mathrm{End}(X)$. Moreover, $[V, V] \subset [K, K]$. Indeed, let $a = [v_1, v_2]$, for $v_1, v_2 \in V$. Since the $\mathcal{F}(X)$ is von Neumann regular, there exists $c \in \mathcal{F}(X)$ such that $aca = a$. Set $b := c - c^*$. Then $b^* = -b$ and $aba = aca - ac^*a = aca - (aca)^* = a - a^* = 2a$. But then $\frac{1}{4}[a, [b, a]] = \frac{1}{2}aba = a$ implies that $a \in [K, K]$. Therefore, $[V, V] \subset [K, K]$, with the inclusion being strict because $V \leq X$ is a proper subspace.

Let us now prove that $[V, V]$ is an inner ideal of $\mathrm{Skew}(\mathcal{L}(X), *)$. By our initial observation, it suffices to verify that $[a, [c, b]] = acb + bca \in [V, V]$ for any $c \in \mathrm{Skew}(\mathcal{L}(X), *)$ and any skew-traces $a = [v_1, v_2]$, $b = [w_1, w_2]$, $(v_i, w_i \in V)$. Indeed,

$$\begin{aligned} acb &= (v_1^* v_2 - \epsilon v_2^* v_1) c (w_1^* w_2 - \epsilon w_2^* w_1) = (v_1^* v_2 - \epsilon v_2^* v_1)(w_1^* c w_2 - \epsilon w_2^* c w_1) \\ &= w_1^* h(cw_2, v_1) v_2 - \epsilon w_2^* h(cw_1, v_1) v_2 - \epsilon w_1^* h(cw_2, v_2) v_1 + w_2^* h(cw_1, v_2) v_1. \end{aligned} \tag{9}$$

since $\epsilon^2 = 1$. By symmetry,

$$bca = v_1^* h(cv_2, w_1) w_2 - \epsilon v_2^* h(cv_1, w_1) w_2 - \epsilon v_1^* h(cv_2, w_2) w_1 + v_2^* h(cv_1, w_2) w_1. \tag{10}$$

Then, by (9) and (10),

$$\begin{aligned} &[a, [c, b]] = acb + bca = \\ &w_1^* h(cw_2, v_1) v_2 - \epsilon w_2^* h(cw_1, v_1) v_2 - \epsilon w_1^* h(cw_2, v_2) v_1 + w_2^* h(cw_1, v_2) v_1 \\ &+ v_1^* h(cv_2, w_1) w_2 - \epsilon v_2^* h(cv_1, w_1) w_2 - \epsilon v_1^* h(cv_2, w_2) w_1 + v_2^* h(cv_1, w_2) w_1 \\ &= (w_1^* h(cw_2, v_1) v_2 + v_2^* h(cv_1, w_2) w_1) - \epsilon (w_2^* h(cw_1, v_1) v_2 + v_2^* h(cv_1, w_1) w_2) \\ &- \epsilon (w_1^* h(cw_2, v_2) v_1 + v_1^* h(cv_2, w_2) w_1) + (w_2^* h(cw_1, v_2) v_1 + v_1^* h(cv_2, w_1) w_2) \\ &= [w_1, h(cw_2, v_1) v_2] - \epsilon [w_2^*, h(cw_1, v_1) v_2] \\ &- \epsilon [w_1^*, h(cw_2, v_2) v_1] + [w_2^* h(cw_1, v_2), v_1] \end{aligned}$$

since

$$v_2^* h(cv_1, w_2) w_1 = -v_2^* h(v_1, cw_2) w_1 = -\epsilon v_2^* \overline{h(cw_2, v_1)} w_1 = -\epsilon (h(cw_2, v_1) v_2)^* w_1,$$

and similarly for the other summands. Therefore, $[a, [c, b]] \in [V, V]$.

Given $a \in I = [V, V]$, we have by definition that $aX \subset V$. Conversely, if h is skew-Hermitian, I is generated by operators of the form $x^* x$, $x \in V$, and given $0 \neq x \in V$, we can always find some $z \in X$ such that $h(z, x) = 1$. Hence $x = h(z, x) x = x^* x(z) \in IX$. When h is symmetric and I is nonzero, V has dimension at least 2. Let $x, y \in V$ be linearly independent vectors. We can take $z \in X$ such that $h(z, x) = 0$ and $h(z, y) = 1$. Hence, $x = [y, x](z) \in IX$.

(ii) For any idempotent $e \in \mathcal{F}(X)$ and any skew-trace $[x, y]$, we have

$$e[x, y]e^* = e(x^* y - \epsilon y^* x) e^* = (ex)^*(ey) - \epsilon (ey)^*(ex) = [ex, ey], \tag{11}$$

which proves that $eKe^* = [eX, eX]$, since $K = \mathrm{Skew}(\mathcal{F}(X), *)$ is additively generated by skew-traces, i.e., elements of the form $[x, y] := x^* y - \epsilon y^* x$ for $x, y \in X$. Conversely, by (3.5), $V \leq X$ is totally isotropic and finite dimensional if and only if $V = eX$ for some isotropic idempotent $e \in \mathcal{F}(X)$, which implies by (11) that $[V,V] = eKe^*$.

(iii) If $[V,V]$ is a principal inner ideal generated by an element $a \in [V,V]$, then $[V,V] = [a,[a,K]] = aKa$ since $a^2 = 0$, and we can use the formula

$$a[x,y]a = -[ay,ax], \quad x,y \in X \text{ (since } a = -a^* \in K\text{)},$$

to show that $[V,V] = aKa = a[X,X]a = [aX,aX]$, i.e., $V = aX$ is finite dimensional over Δ.

Furthermore, if h is skew-Hermitian, given a finite dimensional subspace V of X with basis $\mathcal{B} = \{x_1,\dots,x_n\}$, we can define $a = \sum_{i=1}^n [x_i,x_i]$, and this element generates the principal inner ideal $[V,V]$ because $aX = V$ (notice that the image under a of a dual basis of $\mathcal{B}$ gives all the elements of $\mathcal{B}$).

If h is symmetric (so Δ is a field with the identity as involution), every $a \in K$ has even rank so the dimension of V is necessarily even (it is easy to check that given an element $a \in K$, the alternate form $\varphi : X \times X \to \Delta$ defined by $\varphi(x,y) = h(ax,y)$ induces a nondegenerate alternate form on $(X/\mathrm{Ker}a) \times (X/\mathrm{Ker}a)$, implying that $\dim(X/\mathrm{Ker}a) = \dim \Im a$ is an even number). Conversely, in this case, if V has even dimension we can consider a basis $\{x_1,\dots,x_{2n}\}$ of V and the element $a = \sum_{i=1}^n [x_i, x_{2n-i+1}]$ generates the principal inner ideal $[V,V]$.

(iv) We shall use a direct limit argument (see [1, p. 318]) to reduce the question to the case that X is finite dimensional. Let X_α be a finite dimensional subspace of X such that $\dim_{Z(\Delta)} X_\alpha$ is bigger than 4, and the restriction h_α of h to X_α is nonsingular. Then $X = X_\alpha \oplus X_\alpha^\perp$. Set $A_\alpha := \mathcal{L}_{X_\alpha}(X_\alpha)$ and denote by L_α the set of all $a \in L$ such that $aX_\alpha^\perp = 0$. Then L_α is a subalgebra of L isomorphic to $[K_\alpha, K_\alpha]$, $K_\alpha = \mathrm{Skew}(A_\alpha,*)$. Moreover, L_α is simple by [7, Th. 2.15]. Given a proper inner ideal I of L, take the local system of L consisting of those L_α not contained in I. Then, $I_\alpha := I \cap L_\alpha$ is a proper inner ideal of L_α. Hence we have by [2, Theorem 5.5] that $I_\alpha = e_\alpha K e_\alpha^*$ for some idempotent $e_\alpha \in A_\alpha$ such that $e_\alpha^* e_\alpha = 0$, equivalently, by (ii), $I_\alpha = [V_\alpha, V_\alpha]$, with $V_\alpha = e_\alpha X$ being a totally isotropic subspace of X. Since the L_α form a directed set, so do the V_α. Thus, $V := \cup V_\alpha$ is a totally subspace of X, and $I = [V,V]$.

By (ii), given two totally isotropic subspaces V and W of X, the associated inner ideals $I = [V,V]$ and $I' = [W,W]$ satisfy $I \subset I'$ if and only if $V \subset W$. So $I = [V,V]$ is minimal for the dimension of V being the smallest such that $[V,V] \neq 0$. It is then clear that $I = [V,V]$ is minimal if and only if $V = \Delta x \oplus \Delta y$ when h is symmetric, so $I = [\Delta x, \Delta y]$, or $V = \Delta x$ when h is skew-Hermitian, so $I = [\Delta x, \Delta x]$. ∎

What about the proper inner ideals I of $[K,K]$ such that $a^2 \neq 0$ for some $a \in I$? Let X be a vector space over a field F ($\mathrm{char}F \neq 2$), and let h be a nonsingular symmetric bilinear form on X. Assume that (X,h) contains a nonzero isotropic vector, say x. Then X contains a hyperbolic plane, $H = Fx \oplus Fy$ ($h(y,y) = 0$, $h(x,y) = 1$) and we have $X = H \oplus H^\perp$. Denote by $[x, H^\perp]$ the set of all skew-traces $[x,z] = x^*z - z^*x$, $z \in H^\perp$.

Lemma 3.7. *Let (X,h) be a nonsigular inner vector space (over a field F of* *characteristic not* 2*) containing a hyperbolic plane,* $H = Fx \oplus Fy$ *as above. Then*

(i) *The set* $[x, H^\perp]$ *is an abelian inner ideal of* $\mathfrak{o}(X,q)$ *contained in* $\mathfrak{fo}(X,h)$*, and therefore a proper inner ideal of* $\mathfrak{fo}(X,h)$*.*

(ii) $[x,z]^3 = 0$ *for any* $z \in H^\perp$*, and* $[x,z]^2 = 0$ *if and only if* z *is isotropic. Hence, if* $\dim_F X > 2$*, then there exists* $b \in [x, H^\perp]$ *such that* $b^2 \neq 0$*.*

(iii) $[x, H^\perp] = \mathrm{ad}^2_{[x,z]}(\mathfrak{fo}(X,h))$ *for any nonisotropic vector* $z \in H^\perp$*.*

(iv) $I = [x, H^\perp]$ *is minimal if and only if* $H^\perp$ *has no nonzero isotropic vectors.*

Proof. (i) Let $a \in \mathfrak{o}(X,h)$ and $z \in H^\perp$. We have

$$\begin{aligned} &\mathrm{ad}^2_{[x,z]}(a) = [[x,z],[[x,z],a]] = 2h(ax,z)[x,z] - h(z,z)[x,ax] \\ &+ h(x,x)[az,z] + h(z,x)[z,ax] + h(x,z)[x,az] \\ &= 2h(ax,z)[x,z] - h(z,z)[x,ax], \end{aligned} \tag{12}$$

since $h(x,x) = h(x,z) = 0$. Write $ax = \alpha x + \beta y + w$ ($\alpha, \beta \in F$ and $w \in H^\perp$) according to the decomposition $X = Fx \oplus Fy \oplus H^\perp$, and take into account that $h(ax,x) = 0$, because $a^* = -a$. Then $\beta = 0$ and $[x,ax] = [x, \alpha x + w] = [x,w] \in [x, H^\perp]$. Therefore, $\mathrm{ad}^2_{[x,z]} a \in [x, H^\perp]$. This proves that $[x, H^\perp]$ is an inner ideal of $\mathfrak{o}(X,q)$. Clearly, $[x, H^\perp]$ is contained in $\mathfrak{fo}(X,h)$. Moreover, for any $z, v \in H^\perp$, we have $[[x,v],[x,z]] = h(x,x)[v,z] + h(v,z)[x,x] = 0$, which proves that $[x, H^\perp]$ is an abelian subalgebra.

(ii) For any $z \in H^\perp$, $[x,z]^2 = (x^*z - z^*x)(x^*z - z^*x) = -x^*h(z,z)x$, and therefore $[x,z]^2 = 0$ if and only if $h(z,z) = 0$. We also have that

$$[x,z]^3 = -x^*h(z,z)x(x^*z - z^*x) = -h(z,z)h(z,x)x^*x + h(z,z)h(x,x)z^*x = 0$$

for any $z \in H^\perp$.

(iii) Let $z \in H^\perp$ be nonisotropic. We must prove that for any $v \in H^\perp$ there exists $a \in H^\perp$ such that $\mathrm{ad}^2_{[x,z]}(a) = [x,v]$. First we observe that for $a = [y,u]$, $u \in H^\perp$, we have

$$ax = (y^*u - u^*y)x = h(x,y)u - h(x,u)y = u. \tag{13}$$

Take, now, $u = \alpha z + \beta v$ for $\alpha, \beta \in F$, and use (12) to compute $\mathrm{ad}^2_{[x,z]}(a)$, $a = [y,u]$. We get, by (13),

$$\begin{aligned} &\mathrm{ad}^2_{[x,z]}(a) = 2h(u,z)[x,z] - h(z,z)[x,u] = (2\alpha h(z,z) + 2\beta h(v,z))[x,z] \\ &- h(z,z)\alpha[x,z] - h(z,z)\beta[x,v] = (\alpha h(z,z) + 2\beta h(v,z))[x,z] - h(z,z)\beta[x,v]. \end{aligned}$$

Since $h(z,z) \neq 0$, we can take $\alpha = 2h(z,z)^{-2}h(v,z)$ and $\beta = -h(z,z)^{-1}$ so that $\mathrm{ad}^2_{[x,z]}(a) = [x,v]$.

(iv) If $I = [x, H^\perp]$ is a minimal inner ideal, then $H^\perp$ has no nonzero isotropic vectors: if $w \in H^\perp$ is isotropic, then $F[x,w]$ is a minimal inner ideal of K strictly contained in I (notice that $\dim H^\perp > 2$), leading to a contradiction. Conversely, if $H^\perp$ has no nonzero isotropic vectors, it follows from (iii) that $I = \mathrm{ad}^2_{[x,z]}(\mathfrak{fo}(X,h))$ for any nonzero $z \in H^\perp$, which proves that I is minimal.■

Proposition 3.8. *Let (X,h) be a nonsingular symmetric or skew-Hermitian inner product space over a division algebra with involution $(\Delta,-)$, and set $K=\mathrm{Skew}(\mathcal{F}(X),*)$. Suppose that Δ is finite dimensional over its center, has characteristic zero, and $\dim_{\mathrm{Z}(\Delta)} X$ is greater than 4. If I is a proper inner ideal of $[K,K]$ such that $a^2\neq 0$ for some $a\in I$, then*

(i) *h is symmetric (so Δ is a field, set $\Delta=F$, with the identity as involution) and $K=[K,K]=\mathfrak{fo}(X,h)$.*

(ii) *$I=[x,H^\perp]$, where H is a hyperbolic plane and $0\neq x\in H$ is isotropic.*

Proof. Assume first that X is finite dimensional over Δ. Then we have by [2, Theorem 5.5] that Δ is a field, say $\Delta=F$, with the identity as involution. Moreover, there exists a basis $\{x_i\}$ of X, $1\le i\le n$, so that I is the F-span of $e_{1j}-e_{j2}$, $j\ge 3$. Let us translate the latter statement into geometric terms. Set $a_j:=e_{1j}-e_{j2}$ and observe that $a_j^2=-e_{12}$ and $a_ja_k=0$ for $j\neq k$. Then, for $j\neq k$,

$$h(x_1,x_1)=h(a_jx_j,a_kx_k)=h(x_j,a_j^*a_kx_k)=-h(x_j,a_ja_kx_k)=0. \tag{14}$$

Similarly we obtain

$$h(x_1,x_k)=-h(a_jx_j,a_kx_2)=0 \text{ for all } k\ge 3 \tag{15}$$

and $h(x_k,x_2)=-h(a_kx_2,x_2)=h(x_2,a_kx_2)=-h(x_2,x_k)$, which implies

$$h(x_k,x_2)=0 \text{ for all } k\ge 3. \tag{16}$$

Summarizing, we have that $H=Fx_1\oplus Fx_2$ is a hyperbolic plane, with x_1 isotropic, and $H^\perp$ is the F-span of the x_j, $j\ge 3$. Therefore, $I=[x_1,H^\perp]$.

Suppose now that X is infinite dimensional (over Δ), and fix $a\in I$ such that $a^2\neq 0$. As in the proof of (3.6)(iv), take a Lie subalgebra $L_\alpha=[K_\alpha,K_\alpha]$ of $[K,K]$ ($K_\alpha=\mathrm{Skew}(\mathcal{F}_{X_\alpha}(X_\alpha),*)$, with $\dim_{\mathrm{Z}(\Delta)} X_\alpha>4$, and with nonsingular restriction h_α) such that $a\in L_\alpha$ and L_α is not contained in I. Then $I_\alpha:=I\cap L_\alpha$ is a proper inner ideal of L_α containing an element of nonzero square. Then, by the previous finite dimensional case, Δ is a field, say F, with the identity as involution, and $I_\alpha=[x,H^\perp\cap X_\alpha]$, with H a hyperbolic plane contained in X_α and $0\neq x\in H$ isotropic. We claim that $I=[x,H^\perp]$.

Take a nonisotropic vector $z\in H^\perp\cap X_\alpha$. Then $[x,z]\in I_\alpha\subset I$, and by (3.7)(iii), $[x,H^\perp]=\mathrm{ad}^2_{[x,z]}(\mathfrak{fo}(X,h))\subset I$, since I is an inner ideal of $\mathfrak{fo}(X,h)$. To get the reverse inclusion it is enough to see that $[x,H^\perp]$ is a *maximal* inner ideal.

Suppose then that $[x,H^\perp]$ is strictly contained in some inner ideal, say I, of $\mathfrak{fo}(X,h)$. Then I contains some nonzero operator of the form

$$b=\alpha[x,y]+[y,z]+[u,v], \text{ where } \alpha\in F \text{ and } z,u,v\in H^\perp, \tag{17}$$

since $X=H\oplus H^\perp=Fx\oplus Fy\oplus H^\perp$, and hence $\mathfrak{fo}(X,h)=[X,X]=F[x,y]\oplus[x,H^\perp]+[y,H^\perp]+[H^\perp,H^\perp]$.

A case by case analysis will prove that we may always assume that $[x,y]\in I$. First we display the following formulae which can be easily checked and will be used in what follows. For $z,u,v\in H^\perp$, we have

$$[[x,y],[y,z]] = [y,z] \tag{18}$$

$$[[x,u],[y,v]] = [u,v] + h(u,v)[x,y] \tag{19}$$

$$[[u,v],[y,z]] = h(z,u)[y,v] + h(z,v)[u,y]. \tag{20}$$

Case 1: $\alpha = 1$. Take $w \in H^\perp$ such that $w \in \{z,u,v\}^\perp$ and $h(w,w) = 1$. By (18) and (19), we get

$$[[x,w],[b,[y,w]]] = [[x,w],[[x,y],[y,w]]] = [[x,w],[y,w]] = [x,y] \in I.$$

Case 2: $b = [y,z] + [u,v]$, *with* $z \neq 0$. Take $w \in H^\perp$ such that $h(w,z) = 1$. By (18) and (19), we get

$$[[x,w],[[x,y],b]] = [[x,w],[[x,y],[y,z]]] = [[x,w],[y,z]] = [w,z] + [x,y] \in I,$$

and we are in Case 1.

Case 3: $b = [u,v]$, *where* $u,v \in H^\perp$ *are linearly independent.* Take $z \in H^\perp$ such that $h(z,u) = 1$ and $h(z,v) = 0$. By (20),

$$[b,[y,z]] = [[u,v],[y,z]] = h(z,u)[y,v] + h(z,v)[u,y] = [y,v]. \tag{21}$$

Now take $w \in H^\perp$ such that $h(w,v) = 1$. By (21) and (19),

$$[[x,w],[b,[y,z]]] = [[x,w],[y,v]] = [w,v] + [x,y] \in I,$$

and again we are in Case 1.

Thus, we may assume that

$$[x,H^\perp] \subset I \text{ and } [x,y] \in I. \tag{22}$$

Hence, for any $z \in H^\perp$, we have by (18) that

$$[[x,y],[[x,y],[y,z]]] = [y,z] \in I,$$

and therefore,

$$[y,H^\perp] \subset I. \tag{23}$$

Finally, let $u,v \in H^\perp$. Using (18) and (19), we get

$$[[x,v],[[x,y],[y,u]]] = [[x,v],[y,u]] = [v,u] + h(v,u)[x,y] \in I,$$

which implies that $[v,u] \in I$. Then

$$[H^\perp,H^\perp] \subset I. \tag{24}$$

It follows from (22)-(24) that $I = [X,X] = \mathfrak{so}(X,h)$, as required. ■

Proposition 3.9. *Let* (X,h) *be a nonsingular symmetric or skew-Hermitian inner product space over a division algebra with involution* $(\Delta,-)$, *and set* $K=\mathrm{Skew}(\mathcal{F}(X),*)$. *Suppose that* Δ *is finite dimensional over its center, has characteristic zero, and* $\dim_{Z(\Delta)} X$ *is greater than* 4. *Then*

(i) $[K,K]$ *satisfies the descending chain condition on principal inner ideals.*

(ii) *A nonzero element* $a\in[K,K]$ *is von Neumann regular if either* $a^2=0$ *or* $a=[x,z]$, *where in the last case* h *is symmetric,* x *is a nonzero isotropic vector of a hyperbolic plane* H *of* (X,h), *and* $z\in H^\perp$ *is not isotropic.*

Proof. (i) Let $I_1\supset I_2\supset\dots$ be a descending chain of principal inner ideals of $[K,K]$. If there exists $a\in I_1$ such that $a^2\neq 0$, we have by (3.8)(i) that h is symmetric and $[K,K]=K=\mathfrak{fo}(X,h)$. Moreover, (3.8)(ii) and (iii), $I_1=[x,H^\perp]$, and, for any $b\in I_1$, either $\mathrm{ad}_b^2(\mathfrak{fo}(X,h))=I_1$ or it is a minimal inner ideal, which proves that the chain is stationary in this case. Suppose otherwise that for every element a of I_1, $a^2=0$. Then, by (3.6)(iii),(iv), $I_1=[V,V]$ for a finite dimensional totally isotropic subspace V of X over Δ, and again the chain becomes stable.

(ii) Let $0\neq a\in[K,K]$ be von Neumann regular. Then $\mathrm{ad}_a^2([K,K])$ is an abelian, and therefore proper, inner ideal of $[K,K]$ (2.3). Hence, by (3.8), either $a^2=0$ or $a=[x,z]$, where x is an isotropic vector of a hyperbolic plane H of X, and z is a nonisotropic vector of $H^\perp$. Conversely, let $0\neq a\in[K,K]$. If $a^2=0$, then a is von Neumann regular in $[K,K]$, by (2.4)(v) and *associative* regularity of $a\in\mathcal{F}(X)$. Suppose then that $a=[x,z]$, where h is symmetric, $[K,K]=\mathfrak{fo}(X,h)$, x is a nonzero isotropic vector of a hyperbolic plane H of (X,h), and $z\in H^\perp$ is not isotropic. By (3.8)(ii), $a=[x,z]\in\mathrm{ad}^2_{[x,z]}(\mathfrak{fo}(X,h))=[x,H^\perp]$, and since $[x,H^\perp]$ is abelian (3.7), we also have that $\mathrm{ad}^3_{[x,z]}=0$, which proves that a is also von Neumann regular in this case. ∎

Baranov's classification of infinite dimensional finitary central simple Lie algebras over a field F of characteristic zero (see [1, Theorem 1.1]) admits the following reformulation: such a Lie algebra is either (i) a finitary special linear algebra $\mathfrak{fsl}(\mathcal{P})$, for $\mathcal{P}$ an infinite dimensional pair of dual vector spaces over a finite dimensional division F-algebra Δ; or (ii) $[\mathrm{Skew}(\mathcal{F}(X),*),\mathrm{Skew}(\mathcal{F}(X),*)]$, relative to a nonsingular (skew-Hermitian or symmetric) and infinite dimensional inner product space (X,h) over a division algebra with involution $(\Delta,-)$ which is finite dimensional over F.

Corollary 3.10. (i) *Infinite dimensional finitary central simple Lie algebras over a field* F *of characteristic zero satisfy the descending chain condition on principal inner ideals.*

(ii) *Such an algebra is Artinian, i.e., it satisfies the descending chain condition on all inner ideals if and only if it is of type (ii) above with* (X,h) *not containing infinite dimensional totally isotropic subspaces.*

(iii) *A finitary simple Lie algebra over an algebraically closed field of characteristic zero is Artinian if and only if it is finite dimensional.* ∎

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